# The Ising Model 

Michael Siepmann

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#### Abstract

The Ising Model is one of best developed models in statistical physics. Although it has been solved in one and two dimensions without the presence of an external magnetic field already 60 years ago there is still no solution to the three remaining cases: the two-dimensional model with an external magnetic field and the three-dimensional model with and without an external magnetic field. The aim of this report is not to study the physical relevance of the model (this has been done in Ref. [10]), but to review the famous approaches to the model and summarize some mathematical developments that should eventually lead to a solution of the 3 -dimensional model. In the first section, the origins of the Ising Model [1] and some statistical physics are reviewed. In the second section, the Ising model will be introduced and the classical one-dimensional solution to it [4]. The third section is concerned with the two-dimensional Ising model: Peierls' proof of the existence of the phase transition, a quick review of the Onsager solution [6], and two combinatorial derivations of the Onsager formula $[5,8]$. Finally, in the fourth section, we will report briefly on the development of the mathematical foundation for the three-dimensional model as proposed in References [9,11].


## 1 Introduction

The aim of statistical physics is to understand the macroscopic behavior of a system formed by a number of particles from information about how these interact with each other. The number of particles is usually very large so that "classical" mechanics fails.
A common tool to describe this behavior is a simplified model. This model can often be handled mathematically and can be reduced in its features if one is concerned with only one characteristic behavior. For example one can neglect certain details that one does not assume to have any influence on the overall behavior. This approach is of course disputable and it is always a question whether the simplifications made lower the degree of realism to a level where it is not physically useful anymore, especially when it turns out that certain key assumptions that were made are not true or contradict newer theories. An example of such a model is the Ising Model.

## 2 Phase Transition etc.

The phenomenon of a phase transition is very familiar. An example is the following: when the temperature of water in a pot (normal pressure etc.) is increased from $99^{\circ} \mathrm{C}$ to $101{ }^{\circ} \mathrm{C}$ the density decreases by a factor of 1600 which is obviously a dramatic change on a very short range, a discontinuity so to say. Although this phase transition is so well known as a physical phenomenon it is not completely understood, i.e., predictable by the interaction of the molecules etc. Another example of a phase transition is that of spontaneous magnetization. An iron bar is placed in a strong magnetic field and will be almost completely magnetized. Now the field is slowly turned off: one expects the magnetization to be zero again; this is true for high enough temperatures, but below a certain critical point $T_{c}$ it is not. There remains a spontaneous magnetization $M_{0}$, which depends on the direction of the field, i.e. the magnetization thought of as a function of the external magnetic field has a discontinuity in 0 . So if we think of the magnetization as a function of the external field and the temperature, it turns out that it is analytical on the half plane (external field $\times$ temperature) except of a cut going from $(0,0)$ to $\left(0, T_{c}\right)$. See the figure below.
The main interest is in the behavior of the iron bar close to critical point. So in 1920 Lenz proposed a model to describe this phenomenon in an article in which he reviewed recent models. But since he never did actual calculations on this model, it was not named after him but after his student Ernst Ising who took up his teacher's task in his PhD thesis.


Figure 1: The half plane with a cut from the origin to the critical point $T_{c}$

## 3 The Ising model

The following section is mainly a review of Ref. [4].

### 3.1 Some General Observations

The basic idea of the Ising model is that microscopic magnets (molecules) are ordered in a lattice (each magnet is constrained to lie on one lattice site). We assume that each magnet $i$ has two possible directions to point along: along some preferred axis or in the opposite direction, i.e., either spin +1 or spin -1 ., up or down. The totality of all "spins" determines the state of a system $s, s=\left\{\sigma_{1}, \ldots, \sigma_{N}\right\}$, where $N=n^{d}$ is the number of lattice sites, $\sigma_{i} \in\{ \pm 1\}$, and $d$ the dimension. One also expects the magnets to interact with each other (this interaction is represented by bonds between the interacting lattice sites which gives us a graph of the lattice with vertices and edges).
Now one can give the Hamiltonian of the system which depends only on the state of the system. That is were another severe assumption is made: we require that the Hamiltonian depends only on interactions of the external field and the lattice sites and so-called nearest neighbor interaction which means that

$$
\begin{equation*}
E(s)=-\sum_{i=1}^{N} \mathcal{E} \sigma_{i}-\sum_{<i, j>} J \sigma_{i} \sigma_{j}, \tag{1}
\end{equation*}
$$

where $\mathcal{E}$ and $J$ are parameters that depend on the external field and the strength of the nearest neighbor interaction, respectively. In reality there is of course also interaction between other pairs of sites and there are other models which include this interaction: for example the Mean Field Model which includes interactions between all lattice sites. In this case one takes the average over all interactions. But here we assume that that the interaction is equal for every pair of sites which is also unrealistic. A next nearest neighbor model would be the next less trivial model: now the lattice and the bonds cannot be drawn in a planar graph (for $d=2$ ) which makes things more complicated. So we start with the simplest model we can think of (which is the Ising Model) and see if it produces a phase transition. But first some preliminaries:
A central object in statistical physics is the partition function:

$$
\begin{equation*}
Z_{I}(\beta, \mathcal{E}, J, N)=\sum_{s \in S} \exp (-\beta E(s)) \tag{2}
\end{equation*}
$$

where $\beta$ is usually $\frac{1}{k_{B} T}$ and $k_{B}$ the Boltzmann constant. The partition function determines the probability of the system to be in the state $s$ :

$$
\begin{equation*}
P(\sigma=s)=Z^{-1} \exp (-\beta E(s)) \tag{3}
\end{equation*}
$$

It turns out that what the calculations on the Ising model are all about is a handy description of this partition function. Finally we define the free energy per lattice site to be the thermodynamic limit

$$
\begin{equation*}
F(\beta, \mathcal{E}, J)=\lim _{N \rightarrow \infty} \frac{1}{N} \log Z(\beta, \mathcal{E}, J, N) \tag{4}
\end{equation*}
$$

It is not obvious how this limit is taken in dimensions higher than one and if it exists. But for the moment we will assume it does not depend on the way it is taken as long as the lattice stretches in every dimension and that the limit does exist. The phase transition at a critical point should show up as a discontinuity in some derivative of $F$. We have for the magnetization

$$
\begin{equation*}
M(\mathcal{E}, \beta)=-\frac{\partial}{\partial \mathcal{E}} F(\mathcal{E}, \beta) \tag{5}
\end{equation*}
$$

In a first step, we will make some general observations of the Ising model. For simplicity one identifies the boundaries of the lattice so one can deal with a circle, a torus or a wrapped up cube to get rid of boundary influences. That is from a physicist's point of view highly problematic, but if $N$ is very large the boundary interactions which now arise can be neglected (at least that is what is assumed). Observe that the partition function can now be written as

$$
\begin{equation*}
Z=\left(2 \cosh ^{d}(\beta J) \cosh (\beta \mathcal{E})\right)^{N} \frac{1}{2^{N}} \sum_{s \in S}\left(\prod_{<i, j>}\left(1+\sigma_{i} \sigma_{j} T\right)\right)\left(\prod_{i}\left(1+\sigma_{i} U\right)\right) \tag{6}
\end{equation*}
$$

where $U=\tanh (\beta \mathcal{E}), T=\tanh (\beta J)$, and $d$ is the dimension. Now the free energy per lattice site splits into two parts

$$
\begin{equation*}
F=\log \left(2 \cosh ^{d}(\beta J) \cosh (\beta \mathcal{E})\right)+\lim _{N \rightarrow \infty} \frac{1}{N} \log \underbrace{\left(\frac{1}{2^{N}} \sum_{s \in S}\left(\prod_{<i, j>}\left(1+\sigma_{i} \sigma_{j} T\right)\right)\left(\prod_{i}\left(1+\sigma_{i} U\right)\right)\right)}_{=Z^{\prime}} . \tag{7}
\end{equation*}
$$

A closer look at the $Z^{\prime}$ reveals that (since $\sigma_{i}^{2}=1$ )
$Z^{\prime}=\frac{1}{2^{N}} \sum_{\sigma \in S}\left(P(T, U)+\sigma_{1} P_{1}\left(T, U, \sigma_{2}, \ldots, \sigma_{N}\right)+\ldots+\sigma_{k} P_{k}\left(T, U, \sigma_{k+1}, \ldots, \sigma_{N}\right)+\ldots+\sigma_{N} P_{n}(T, U)\right)=P(T, U)$
each $P_{i}$ being a polynomial in the given variables. In the zero field case, we have only a (observe that after summation over $\sigma$ all other polynomials vanish) polynomial depending on T left

$$
\begin{equation*}
P(T, 0)=\sum_{k=1}^{d N} c_{d}(k) T^{k} \tag{9}
\end{equation*}
$$

It is easy (it is actually a common trick in combinatorics) to see that the coefficients $c_{d}(n)$ count the number of even subgraphs (consisting of bonds and lattice sites), i.e. each vertex (lattice site) in the graph has positive even degree, of length $n$ i.e. it contains $n$ bonds (so $P(T, 0)$ is a generating function for the number of even subgraphs of equal length). It follows that this is exactly the number of graphs of length $n$ whose components consist of closed paths. We can start to calculate the coefficients in two dimensions:

- $c_{2}(k)$ is obviously zero for $k$ odd or $k=2$ and $c_{d}(0)=1$.
- We have on the torus $N$ squares of length 4 which means that $c_{2}(4)=N$.
- For paths of length 6 there is only one congruence class of paths with two orientations which sums up to a number of $c_{2}(6)=2 N$.
- Now for subgraphs with 8 edges we have first of the disconnected pairs of squares which are in total $\frac{N(N-9)}{2}$. Secondly we have the four other possibilities shown below with $1+4+2+2=9$ orientations which means that we have $c_{2}(8)=\frac{N(N-21)}{2}+9 N=\frac{N(N+9)}{2}$
and so on.
In the three-dimensional case we have:
- $c_{3}(k)$ is obviously zero for $k$ odd or $k=2$ and $c_{d}(0)=1$.
- For paths of length 6 we have of course the cases from the two dimensional case with now two additional orientations, i.e. 6 N possibilities and also the bent pairs of squares shown below which cover twelve cases, i.e. $12 N$ possibilities and finally the $4 N$ twisted rectangles which means that $c_{3}(6)=22 N$.
- And we get $c_{3}(8)=\frac{3 N(3 N-33)}{2}+231 N$ form the graphs below.


Figure 2: One possible case of a closed path of length 4 for $d=2$


Figure 3: Four possible cases of closed paths of length 6 for $d=2$


Figure 4: Four possible cases of closed paths of length 8 for $d=2$


Figure 5: Four possible cases of closed paths of length 8 for $d=3$

One can see that it is not very reasonable to continue in this manner except for the one-dimensional case (circle) where there is only one closed path of length $N$ which means that

$$
\begin{equation*}
P(T, 0)=1+T^{N} \tag{10}
\end{equation*}
$$

so we have, since $T$ is smaller than 1 ,

$$
\begin{equation*}
F=\log \left(2 \cosh ^{2}(\beta J)\right) \tag{11}
\end{equation*}
$$

Before continuing, the reader is encouraged to find a formula to count the number of such paths for two or even higher dimensions.

### 3.2 The Lattice Gas and the Binary Alloy

The Ising model can also be seen as a model for a lattice gas which is of relevance in the investigation of gas-liquid and liquid-solid transitions: now we deal with holes (empty sites) and occupied sites. The question is: how do the molecules condensate below a certain temperature? Yang and Lee showed that there is a mapping between this model and the Ising model. For binary alloys, one tries to describe the clustering of two kinds of atoms in a mixture below a certain temperature. This can also be related to the Ising Model.

### 3.3 The Exact Solution

Now to the exact solution of the one-dimensional model (first the line): In a first step one observes that the modified partition function $Z^{\prime}$ can be split into two parts: one with $\sigma_{N}=+1$ and one with $\sigma_{N}=-1$, i.e. $Z_{N}^{\prime+}$ and $Z_{N}^{\prime-}$. Now using the product representation, it is not very difficult to write the two parts as linear combinations of $Z_{N-1}^{\prime+}$ and $Z_{N-1}^{\prime-}$. One obtains a recursive formula in two dimensions which can be written as a matrix multiplication. It turns out that

$$
Z^{\prime}(N)=\left(\begin{array}{ll}
1 & 1
\end{array}\right) \underbrace{\left(\begin{array}{rr}
(1+U)(1+T) & (1+U)(1-T)  \tag{12}\\
(1-U)(1-T) & (1-U)(1+T)
\end{array}\right)^{(N-1)}}_{=M^{N-1}}\binom{1}{0}
$$

whereas for the wrapped up model, i.e. the circle, we obtain

$$
\begin{equation*}
Z_{\mathrm{wu}}^{\prime}(N)=\operatorname{tr}\left(M^{N-1}\right) \tag{13}
\end{equation*}
$$

By calculating the largest eigenvalue of the matrix, we obtain

$$
\begin{equation*}
F_{1}=\log \left(\frac{1+T+\left((1+T)^{2}-4 T\left(1-U^{2}\right)^{\frac{1}{2}}\right)}{2}\right) \tag{14}
\end{equation*}
$$

which is, as a function of $\beta$, analytic on the positive real axis. So the Ising model fails to exhibit any phase transition in form of a discontinuity in a derivative of $F$. Ising assumed the same for the higherdimensional models and abandoned his approach. Actually he abandoned also his physics research. He fled Nazi Germany and continued to work as teacher in the USA after the war, but not as a researcher. The only post-war publication we know of is a two page article about Goethe's Farbenlehre.

### 3.4 The Phase Transition in the Two-Dimensional Isinig Model

It was Peierls who recovered the Ising model in 1936 when he was able to prove that there is a phase transition in the two-dimensional case. The proof follows a rather easy argument that will be presented briefly: Turning the magnetic field slowly off is is equivalent to setting the boundary (which now exists), i.e. all lattice sites on the boundary, spin up and then letting $N$ go to infinity, i.e. letting the boundary move to infinity. The question is: What is the probability that we find a spin up deep inside the lattice afterwards, i.e. is there a probability not equal to $\frac{1}{2}$ which would mean that we actually have spontaneous magnetization? Since the spins of the boundary sites of the lattice are all +1 , it is clear that we obtain "islands" of spin ( -1 ) sites. These islands have shorelines (lines that connect the midpoints of adjacent squares) and possible lakes. Now to the probability that $\sigma_{0}=-1$ where 0 is a lattice site deep inside: We have to sum over all possible configurations that contain $\sigma_{0}=-1$ which will be denoted by $\Omega_{0}$. In a first step, we take a certain shoreline surrounding 0 and give the probability of a configuration with that shoreline which is the sum over $\Omega_{S} \subseteq \Omega_{0}$. If we change the signs in configuration $\sigma \in \Omega_{S}$ of all sites inside the shoreline (which is an injective mapping $\sigma \mapsto \sigma^{\prime} \subset \Omega_{S^{\prime}}$ ), we obtain

$$
\begin{equation*}
\sum_{<i, j>\notin S} \sigma_{i} \sigma_{j}=\sum_{<i, j>} \sigma_{i}^{\prime} \sigma_{j}^{\prime}-n(S), \tag{15}
\end{equation*}
$$

where $n(S)$ is the length of the shoreline $S$. The $-n(s)$ results from the fact that each line segment of the shoreline relates to a pair of lattice sites of opposite spin. It follows for the probability that

$$
\begin{align*}
P\left(\Omega_{S}\right) & =e^{-\beta J n(S)} \frac{1}{Z} \sum_{s \in \Omega_{S}} e^{\beta J \sum_{<i, j>\notin S} \sigma_{i} \sigma_{j}} \\
& <e^{-\beta J n(S)} \frac{1}{Z} \sum_{\sigma \in \Omega_{S}^{\prime}} e^{-\beta E\left(\sigma^{\prime}\right)}  \tag{16}\\
& <e^{-\beta J n(S)} \frac{1}{Z} \sum_{\sigma \in \Omega} e^{-\beta E(\sigma)}=e^{-\beta J n(S)} .
\end{align*}
$$

So we have $P\left(\sigma_{0}=-1\right)=\sum_{n \geq 4} s(n) e^{-\beta J n}$, where $s(n)$ is the number of shorelines surrounding 0 of length $n$. One can estimate this number by $s(n)<\frac{1}{2} n 4^{n}$ and obtain a geometric series for the probability which finally gives:

$$
\begin{equation*}
P\left(\sigma_{0}=-1\right)<\frac{1}{2}\left(\frac{4 e^{-\beta J}}{\left(1-4 e^{-\beta J}\right)^{2}}\right) \tag{17}
\end{equation*}
$$

which becomes arbitrarily small for $\beta$ sufficiently large, i.e. the temperature sufficiently low. So we have indeed a spontaneous magnetization for low temperatures. In a similar combinatorial argumentation, Kramers and Wannier were able to calculate the critical point $T_{c}$ as being equal to $\sqrt{2}-1 \approx 0.4142$ in 1941.

## 4 The Exact Solution to the Two-Dimensional Ising Model

Only one year later, Onsager announced another breakthrough in the calculation of the Ising model. On February 28, 1942 at the end of a meeting of the New York Academy of Sciences, Onsager claimed to have solved the two-dimensional Ising model without an external magnetic field. Ironically it was neither a physicist nor a mathematician, but a chemist (at least that is what he considered himself. However some people called him one of the last true universalists of our time) to find a method to handle this difficult problem. He obtained the following famous formula (here written in two equivalent ways) [4,5]:

$$
\begin{align*}
F_{2}(T, 0) & =\log \left(2 \cosh ^{2}(\beta J)\right)+\frac{1}{2} \int_{0}^{1} \int_{0}^{1} \log \left(\left(T^{2}+1\right)^{2}-2 T\left(1-T^{2}\right)(\cos (2 \pi x)+\cos (2 \pi y))\right) d x d y  \tag{18}\\
& =\log (2 \cosh (2 \beta J))+\frac{1}{2 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} d \epsilon d \eta \log (1-2 \kappa(\cos (\eta)+\cos (\epsilon)))
\end{align*}
$$

where $2 \kappa=\frac{\tanh (2 \beta J)}{\cosh (2 \beta J)}$. In his own words, Onsager summarized his solution:

> The special properties of the operators involved in this problem allow their expansion as linear combinations of the generating basis elements of an algebra which can be decomposed into direct products of quaternion algebras. The representation of the operators in question can be reduced accordingly to a sum of direct products of two dimensional representations, and the roots of the secular equation for the problem in hand are obtained as product of the roots of certain quadratic equations. To find all the roots requires complete reduction, which is best performed by the explicit construction of a transforming matrix, with valuable by-products of identities useful for the computation of averages pertaining to the crystal. It so happens that the representations of maximal dimension, which contain the two largest roots, are identified with ease from simple general properties of the operators and their representative matrices. The largest roots whose eigenvectors satisfy certain special conditions can be found by a moderate elaboration of the procedure; these results will suffice for a qualitative investigation of the spectrum. To determine the thermodynamic properties of the model it suffices to compute the largest root of the secular equation as a function of temperature.
> The passage to the limiting case of an infinite base involves merely the substitution of integrals for sums. The integrals are simplified by elliptic substitutions... [3]

Onsager's solution was considered quite "obscure" [2] and the simplification of the solution by Buria Kaufman seems to be better understandable.

### 4.1 B. Kaufman's treatment of the Onsager solution

This section is a summary of one chapter of Ref. [6]. Again we look at the wrapped up model, i.e. the lattice is embedded in a torus.

In the two-dimensional model, we have $n$ rows and $n$ columns. Each row is in a state $\mu$ and two neighboring rows interact as described before. Define $E(\mu)$ to be the interaction energy of a row in the state $\mu$ viewed as a one dimensional model. Further $E\left(\mu, \mu^{\prime}\right)$ as the interaction energy of two neighboring rows. Finally one constructs a $2^{n} \times 2^{n}$-matrix $P$ such that

$$
\begin{equation*}
\langle\mu| P\left|\mu^{\prime}\right\rangle=\exp \left(-\beta\left(E\left(\mu, \mu^{\prime}\right)+E(\mu)\right)\right) \tag{19}
\end{equation*}
$$

Then the partition function reads

$$
\begin{equation*}
Z_{n^{2}}(0, \beta)=\operatorname{tr}\left(P^{n}\right) \tag{20}
\end{equation*}
$$

If one can diagonalize $P$ and thus find the largest eigenvalue one can (of course the existence and uniqueness has to be established) show that

$$
\begin{equation*}
\lim _{n^{2} \rightarrow \infty} \frac{1}{n^{2}} Z_{n^{2}}(0, \beta)=\lim _{n \rightarrow \infty} \frac{1}{n} \lambda_{\max }(n) \tag{21}
\end{equation*}
$$

To diagonalize $P$ it is first decomposed into two matrices $V_{1}$ and $V_{2}$ which can each be written as a direct product of exponentials of the two-dimensional spinor matrices (Pauli matrices). We split $V=V_{1} V_{2}$ into $V^{+}$and $V^{-}$where $P=f V, f$ being some scalar function, such that $V^{+}$and $V^{-}$can be seen as $2^{n} \times 2^{n}$ representatives of rotations in a $2 n$-dimensional space of $2 n$ generators of a subalgebra of $2^{n} \times 2^{n}$-matrices. So $V^{ \pm}$relates to rotations $\Omega^{ \pm}$. The eigenvalues of this matrix can be evaluated by using again the relation between rotations and $V^{ \pm}$. After calculating the eigenvalues of $\Omega^{ \pm}$the eigenvalues of $V^{ \pm}$and $V$, respectively, follow. Inserting the eigenvalues to calculate the trace of $P^{n}$ one obtains a Riemannian sum which leads to the integral which is found in the Onsager formula.

### 4.2 The Combinatorial Approach

### 4.2.1 The Kac and Ward approach

This is mainly a review of Ref. [5]. The combinatorial setting has roughly been outlined in first section: An admissible graph shall be the subset of the bond in the lattice whose sites have even valence:

$$
\begin{equation*}
I_{G}(u)=\prod_{i \in G} u=u^{L} \tag{22}
\end{equation*}
$$

Furthermore, we define a path to be a closed ordered sequence of bonds each starting at the site where the previous has ended and the sign of a path:

$$
\begin{equation*}
s(p)=(-1)^{1+t} \tag{23}
\end{equation*}
$$

where $t$ is the number of $2 \pi$-angles turned by a tangent vector while traversing the path $p$. Finally the amplitude of a path is

$$
\begin{equation*}
W_{p}(u)=s(p) I_{p}(u) \tag{24}
\end{equation*}
$$

where $I_{p}(u)=u^{l}, l=m_{1}+. .+m_{k}, k$ being the length of $p$ and $m_{i}$ the number of times the bond $i$ is covered by p. We remember that:

$$
\begin{align*}
Z_{N}(T) & =\sum_{\sigma \in \Omega}\left(1-T^{2}\right)^{-\frac{x}{2}} \prod_{<i, j>}\left(1+\sigma_{i} \sigma_{j} T\right) \\
& =2^{N}\left(1-T^{2}\right)^{-\frac{x}{2}}\left(1+\sum_{G \in \mathcal{A}} I_{G}(T)\right), \tag{25}
\end{align*}
$$

where $x$ is the number of bonds. Now a central theorem in the derivation of the Onsager formula was first conjectured by Feynman and was later proven by Sherman. It is actually only a special case of a more general relation. We omit the prove here though. However, it is really nice since it is almost completely self-contained:
Theorem 1 For $|T|<1$ the following equation holds:

$$
\begin{equation*}
1+\sum_{G \in \mathcal{A}} I_{G}(T)=\prod_{[p]}\left(1+W_{p}(T)\right) \tag{26}
\end{equation*}
$$

where $[p]$ denotes the equivalence classes of paths (two paths are considered equivalent if they are the same as subgraphs). The above theorem means that we have to calculate the product on the lefthand side to obtain the partition function. By studying the directions, a closed path takes at each site especially at arrival, one can obtain the following formula:

$$
\begin{equation*}
\sum_{p\left(n, P_{1}\right)} W_{p}(T)=-\frac{1}{8 \pi^{2}} \int_{0}^{2 \pi} \int_{0}^{2 \pi} d \epsilon d \eta \operatorname{tr}(u M)^{n} \tag{27}
\end{equation*}
$$



Figure 6: Orientation on the chess board. If an arrow points form $i$ to $j$ then $a(i, j)=1$ and $a(j, i)=1$.
where $M(u)$ is a complex $4 \times 4$ matrix which is obtained by the recursive formulation of the "Turn"functions. See Ref. [5] for details. Since

$$
\begin{equation*}
\sum_{[p]} \log \left(1+W_{p}(T)\right)=\log \left(\prod_{[p]}\left(1+W_{p}(T)\right)\right)=\sum_{n=1}^{\infty} \frac{1}{n} \sum_{p(n)} W_{p}(T), \tag{28}
\end{equation*}
$$

we find that

$$
\begin{align*}
\frac{\log \left(Z_{N}(T)\right)}{N}=\log (2)+2\left(1-\frac{1}{\sqrt{N}}\right) \log (\cosh (\beta J))+ & \underbrace{\log \left(\prod_{[p]}\left(1+W_{p}(T)\right)\right)}_{=\frac{1}{8 \pi^{2}} \int_{0}^{2 \pi} \int_{0}^{2 \pi} d \epsilon d \eta \log (\operatorname{det}(1-u M))} \tag{29}
\end{align*}
$$

which yields the Onsager formula. Unfortunately the integral in the Onsager formula cannot be expressed in terms of simple functions. So the solution leads to the theory of elliptic function which should not be too surprising.
The solution by Kac and Ward gave new insights and conjectures in the field of combinatorics and algebra. It also showed that the Ising model can be related to combinatorial problems which after a mathematical theory has been developed on them might help to tackle the three-dimensional Ising model.

### 4.2.2 The Kasteleyn Approach

For example another related combinatorial problem closely associated to the Kac and Ward method is the dimer problem which was first applied to the Ising model by Kasteleyn [8]. Dimer statistics is roughly speaking concerned with the following (dimer problem) question: how many configurations of dominoes on a chessboard, each domino filling two squares, are there such that the whole board is covered? (The answer is 12988816 [1] - this can easily be shown by constructing a $64 \times 64$ skew symmetric matrix $A$ with $a(i, j)= \pm 1$ iff $i$ and $j$ are adjacent squares where the signs are chosen according to an orientation given on the related square lattice as in the figure above. Then by taking the square root of the determinant of $A$ we obtain the number of configurations.) In case of the 2D-Ising model, one studies the number of dimer configurations on a decorated lattice which is obtained from the square lattice by replacing each lattice site by four new sites each connected with a bond. This method involves the use of a Pfaffian which is a central object in the approach by Hurst and Green [8]. The Pfaffian is an object somewhat similar to the determinant of a skew-symmetric matrix. If we look at the number of dimer configurations (assume $N$ to be even), this number can be written as:

$$
\begin{equation*}
Z=\left(\left(\frac{N}{2}\right)!\right)^{-1} \sum_{P} b\left(p_{1}, p_{2}\right) b\left(p_{3}, p_{4}\right) \ldots b\left(p_{N-1}, p_{N}\right) \tag{30}
\end{equation*}
$$

where the sums is over all permutations and $b(i, j)=1$ if $i$ and $j$ are adjacent and 0 otherwise. The Pfaffian is

$$
\begin{equation*}
\operatorname{Pf}(A)=\left(\left(\frac{N}{2}\right)!\right)^{-1} \sum_{P} \epsilon_{P} a\left(p_{1}, p_{2}\right) a\left(p_{3}, p_{4}\right) \ldots a\left(p_{N-1}, p_{N}\right) \tag{31}
\end{equation*}
$$

where $A$ is a skew-symmetric $N \times N$-matrix and $\epsilon_{P}$ the sign of the permutation. So the Pfaffian is just the square-root of the determinant of $A$. The main task in the solution to the dimer problem is the choice of the signs in $A$ such that the first equation really coincides with the second. This is in general not possible, but for some special cases (e.g. the planar lattice) it is.

## 5 The Three-Dimensional Ising Model

We have seen the development of different mathematical approaches to the two-dimensional Ising model in the preceding two sections. These approaches gave insight how the topological properties of the Ising model influence the solution to it. From these methods mathematical formalisms for the 3D model have been developed. One of them is a extension of the dimer problem which is described in Ref. [11]. It uses the combinatoric topology of the cube lattice to introduce a homology theory which finally classifies a number of Pfaffians which the Ising partition function consists of. A second approach [9] has been developed for the Ising model of arbitrary dimensions with an external magnetic field. It shows that the Ising model in presence of an external magnetic field is isomorphic to the model of localized particles satisfying the Fermi statistics. This isomorphism can be used to construct a general solution of the Ising model.

## 6 Summary

What has started as a seemingly simple problem has produced interesting problems in highly advanced mathematics on the borderline of algebra and combinatorics. It should be clear now that the Ising model is of high mathematical interest: The new approaches give insight why it is so difficult to find a solution and why we cannot use certain approaches in higher dimensions. For example if the lattice is embedded in a torus the the partition function can be expressed in terms of four determinants with the approach by Kac and Ward. Or more generally: the number of determinants needed in this approach depends on the genus $g$ of the surface by the formula $4^{g}[8]$ which means that the three-dimensional wrapped-up cube lattice model is impossible to solve in this manner. We have not mentioned the physical relevance (which was doubted but now widely accepted), especially the change in reception from a theory on ferromagnetism to cooperative phenomena in general, and physical implications that follow the solutions of Onsager etc. Also approximate models would be worth being discussed, but would go beyond the scope of this report. It remains to mention that the Ising model has also high relevance in biology, economics, and sociology. One example can be found in Ref. [7], where the Ising model is applied to the stock market as a model for the influence external effects on the stock prices of companies in similar fields of production.

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