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Fuzzy plane geometry I: Points and lines

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Abstract

We introduce a comprehensive study of fuzzy geometry in this paper by first defining a fuzzy point and a fuzzy line in fuzzy plane geometry. We consider the fuzzy distance between fuzzy points and show it is a (weak) fuzzy metric. We study various definitions of a fuzzy line, develop their basic properties, and investigate parallel fuzzy lines. © 1997 Elsevier Science B.V.

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1. Introduction

This paper initiates the study of fuzzy geometry by first studying two basic ideas, fuzzy points and fuzzy lines, in fuzzy plane geometry. Further research will be concerned with fuzzy circles, fuzzy rectangles, etc. in fuzzy plane geometry [3]. Then one can extend these results to fuzzy geometry in R^n, $n > 2$.

As an application of fuzzy plane geometry we will propose, in future publications, superimposing objects from fuzzy geometry onto data bases to obtain a fuzzy landscape over the data base. A soft query could be a fuzzy probe into the fuzzy landscape with the system's response the number data points in an α-cut of the interaction of the fuzzy probe and the fuzzy landscape.

Certain ideas in fuzzy plane geometry have been previously introduced and studied in a series of papers [1, 8–11]. In [8, 11] the author considered the area, height, width, diameter and perimeter of a fuzzy subset of the plane. In this paper we will not be concerned with these measures (area, height, etc.) of fuzzy subsets of R^2 but instead we study the basic properties of fuzzy points and lines. The concept of the perimeter of a fuzzy set was further studied in [1]. We note that all these measures of fuzzy subsets of R^2 are real valued. That is, in [1, 8, 11] the perimeter of a fuzzy subset of the plane is a real number. In future research papers when we study fuzzy circles, rectangles, etc. in R^2, our measures of area, perimeter, etc. will all be real fuzzy numbers. For example, in the second section of this paper the fuzzy distance between two fuzzy points in the plane turns out to be a real fuzzy number.

In [9, 10] the author introduces the ideas of fuzzy rectangles, fuzzy half-plane, fuzzy polygons and fuzzy triangles. Again, the area and perimeter of a fuzzy triangle is a real number. In future research papers when

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we do study fuzzy rectangles and fuzzy triangles we will compare our results to those in [9, 10]. However, we do plan to have the area and perimeter of a fuzzy triangle to be a real fuzzy number.

Now let us introduce the notation that will be used in the rest of this paper. We will place a "bar" over a capital letter to denote a fuzzy subset of R or R^2. So, $\bar{X}, \bar{Y}, \bar{A}, \bar{B}, \bar{C}, \ldots$ all represent fuzzy subsets of R^n, $n = 1, 2$. Any fuzzy set is defined by its membership function. If $\bar{A}$ is a fuzzy subset of R, we write its membership function as $\mu(x\,|\,\bar{A})$, x in R, with $\mu(x\,|\,\bar{A})$ in $[0, 1]$ for all x. If $\bar{P}$ is a fuzzy subset of R^2 we write $\mu((x, y)\,|\,\bar{P})$ for its membership function with (x, y) in R^2. The α-cut of any fuzzy set $\bar{X}$ of R, written $\bar{X}(\alpha)$, is defined as $\{x : \mu(x\,|\,\bar{X}) \geqslant \alpha\}$, $0 < \alpha \leqslant 1$. $\bar{X}(0)$ is the closure of the union of $\bar{X}(\alpha)$, $0 < \alpha \leqslant 1$. Similar definitions exist for α-cuts of fuzzy subsets of R^2.

We will adopt the definition of a real fuzzy number given in [6, 7]. $\bar{N}$ is a (real) fuzzy number if and only if:
1. $\mu(x\,|\,\bar{N})$ is upper semi-continuous;
2. $\mu(x\,|\,\bar{N}) = 0$ outside some interval $[c, d]$; and
3. there are real numbers a and b so that $c \leqslant a \leqslant b \leqslant d$ and $\mu(x\,|\,\bar{N})$ is increasing on $[c, a]$, $\mu(x\,|\,\bar{N})$ is decreasing on $[b, d]$, $\mu(x\,|\,\bar{N}) = 1$ on $[a, b]$.

It is well-known that $\bar{N}(\alpha)$ is a (bounded) closed interval for all α, when $\bar{N}$ is a fuzzy number. A special type of fuzzy number is a triangular fuzzy number. A triangular fuzzy number $\bar{N}$ is defined by three numbers a, b, c so that: (1) $a < b < c$; and (2) the graph of $y = \mu(x\,|\,\bar{N})$ is a triangle with base on $[a, c]$ and vertex at $(b, 1)$. We denote triangular fuzzy numbers as $\bar{N} = (a, b, c)$.

The next section is concerned with fuzzy points in the plane. The third section investigates fuzzy lines in R^2. The final section briefly presents future directions for research in fuzzy geometry.

2. Fuzzy points

We see that there are two natural ways to define a fuzzy point in the plane.

Method 1. A fuzzy point is a pair $(\bar{X}, \bar{Y})$ where $\bar{X}$ and $\bar{Y}$ are real fuzzy numbers.

Method 2. A fuzzy point at (a, b) in R^2, written $\bar{P}(a, b)$, is defined by its membership function:
1. $\mu((x, y)\,|\,\bar{P}(a, b))$ is upper semi-continuous;
2. $\mu((x, y)\,|\,\bar{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$; and
3. $\bar{P}(\alpha)$ is a compact, convex, subset of R^2 for all α, $0 \leqslant \alpha \leqslant 1$.

We will adopt Method 2 in this paper for defining a fuzzy point for two main reasons: (1) We can visualize $\bar{P}(a, b)$ as a surface in R^3 (the graph of $z = \mu((x, y)\,|\,\bar{P}(a, b))$) but we cannot construct pictures of $(\bar{X}, \bar{Y})$ in Method 1; and (2) the basic idea behind Method 1 does not give good results for fuzzy lines (Methods 1 and 2 in the next section). So, for the rest of this paper a fuzzy point at a point in the plane means it satisfies the definition in Method 2.

Example 1. Let $\bar{X}$ and $\bar{Y}$ be two real fuzzy numbers so that: $\mu(x\,|\,\bar{X}) = 1$ if and only if $x = a$, $\mu(y\,|\,\bar{Y}) = 1$ if and only if $y = b$. Then $\min(\mu(x\,|\,\bar{X}), \mu(y\,|\,\bar{Y})) = \mu((x, y)\,|\,\bar{P}(a, b))$ is a fuzzy point at (a, b).

In the definition of a fuzzy point the constraint that α-cuts must be convex subsets of R^2 may be too strong. Future researchers may wish to consider α-cuts of $\bar{P}(a, b)$ to be only connected and simply connected. However, in this initial paper we will stick with α-cuts of fuzzy points convex. The concept of fuzzy point (Method 2) is based on the idea of a fuzzy vector in R^n, $n \geqslant 2$ [2, 4].

Let $\bar{F}$ be a fuzzy subset of the plane. We would say that $\bar{F}$ is fuzzy convex if and only if $\mu(v\,|\,\bar{F}) \geqslant \min(\mu(u\,|\,\bar{F}), \mu(w\,|\,\bar{F}))$ where u, w are any two points in R^2 and v is any point on the line segment joining

u and w. It is not too difficult to see that $\bar{P}(a,b)$ is fuzzy convex if and only if $\bar{P}(a,b)(\alpha)$ is convex for all α. So, assuming $\bar{P}(a,b)$ has convex α-cuts is equivalent to assuming $\bar{P}$ is fuzzy convex.

Next we define the fuzzy distance between fuzzy points. Let $d(u,v)$ be the usual Euclidean distance metric between points u and v in R^2. We define the fuzzy distance $\bar{D}$ between two fuzzy points $\bar{P}(a_1,b_1)$ and $\bar{P}(a_2,b_2)$ in terms of its membership function $\mu(d\,|\,\bar{D})$. See [2,4,5].

Definition 1. $\Omega(\alpha) = \{d(u,v) : u$ is in $\bar{P}(a_1,b_1)(\alpha)$ and v is in $\bar{P}(a_2,b_2)(\alpha)\}$, $0 \leqslant \alpha \leqslant 1$. Then $\mu(d\,|\,\bar{D}) = \sup\{\alpha : d \in \Omega(\alpha)\}$.

Theorem 1. $\bar{D}(\alpha) = \Omega(\alpha)$, $0 \leqslant \alpha \leqslant 1$, *and* $\bar{D}$ *is a real fuzzy number.*

Proof. 1. First we show that $\bar{D}(\alpha) = \Omega(\alpha)$, $0 \leqslant \alpha \leqslant 1$. Let $d \in \Omega(\alpha)$. Then $\mu(d\,|\,\bar{D}) \geqslant \alpha$ and $\Omega(\alpha)$ is a subset of $\bar{D}(\alpha)$.

Now we argue that $\bar{D}(\alpha)$ is a subset of $\Omega(\alpha)$. Let $d \in \bar{D}(\alpha)$. Then $\mu(d\,|\,\bar{D}) \geqslant \alpha$. Set $\mu(d\,|\,\bar{D}) = \beta$. We consider two cases: (a) $\beta > \alpha$; and (b) $\beta = \alpha$.

(a) We assume that $\beta > \alpha$. There is a γ, $\alpha < \gamma \leqslant \beta$, with d in $\Omega(\gamma)$. Since $\Omega(\gamma)$ is a subset of $\Omega(\alpha)$ we have d in $\Omega(\alpha)$. Hence $\bar{D}(\alpha)$ is a subset of $\Omega(\alpha)$.

(b) Now $\beta = \alpha$. Let $K = \{\Delta : d \in \Omega(\Delta)\}$. Then $\sup K = \beta = \alpha = \mu(d\,|\,\bar{D})$. There is a sequence γ_n in K so that $\gamma_n \uparrow \alpha$. Given $\varepsilon > 0$ there is a positive integer N so that $\alpha - \varepsilon < \gamma_n$, $n \geqslant N$. Now d in $\Omega(\gamma_n)$, all n, implies d is also in $\Omega(\alpha - \varepsilon)$, any $\varepsilon > 0$. So $d = d(u,v)$ for some u in $\bar{P}(a_1,b_1)(\alpha - \varepsilon)$ and v in $\bar{P}(a_2,b_2)(\alpha - \varepsilon)$. This implies that $\mu(u\,|\,\bar{P}(a_1,b_1)) \geqslant \alpha - \varepsilon$ and $\mu(v\,|\,\bar{P}(a_2,b_2)) \geqslant \alpha - \varepsilon$. Since $\varepsilon > 0$ was arbitrary we see that $\mu(u\,|\,\bar{P}(a_1,b_1)) \geqslant \alpha$ and $\mu(v\,|\,\bar{P}(a_2,b_2)) \geqslant \alpha$. This means that $d \in \Omega(\alpha)$ and $\bar{D}(\alpha)$ is a subset of $\Omega(\alpha)$.

This concludes the proof that $\bar{D}(\alpha) = \Omega(\alpha)$ for $0 < \alpha \leqslant 1$. It follows that $\bar{D}(0) = \Omega(0)$ and the first part of the proof is complete.

2. We now argue that $\bar{D}$ is a fuzzy number.

(a) Since α-cuts of $\bar{P}(a_1,b_1)$ and $\bar{P}(a_2,b_2)$ are compact it is easily seen that $\Omega(\alpha)$ is a bounded closed interval for all α. Let $\Omega(\alpha) = [l(\alpha), r(\alpha)]$, $0 \leqslant \alpha \leqslant 1$. It is also known that if α-cuts of a fuzzy number are closed sets, then its membership function is upper semi-continuous [4]. But $\bar{D}(\alpha) = \Omega(\alpha)$ is a closed interval for all α. Hence, $\mu(d\,|\,\bar{D})$ is upper semi-continuous.

(b) Let $\Omega(0) = [c,d]$. Then $\mu(d\,|\,\bar{D}) = 0$ outside $[c,d]$.

(c) Let $\Omega(1) = a$, where $a = d((a_1,b_1),(a_2,b_2))$. Now since $\bar{D}(\alpha) = [l(\alpha), r(\alpha)]$ for all α with $l(\alpha)$ is increasing from c to a and $r(\alpha)$ decreasing from d to a we obtain $\mu(d\,|\,\bar{D})$ is increasing on $[c,a]$ and decreasing on $[a,d]$ with $\mu(d\,|\,\bar{D}) = 1$ at $d = a$.

This concludes our argument that $\bar{D}$ is a fuzzy number. □

A fuzzy point $\bar{P}(a,b)$ reduces to a crisp point at (a,b) when $\mu((x,y)\,|\,\bar{D}(a,b)) = 0$ for $(x,y) \neq (a,b)$ and equals 1 at $(x,y) = (a,b)$. Clearly, $\bar{D}$ reduces to d (Euclidean metric on R^2) when $\bar{P}(a_1,b_1)$ and $\bar{P}(a_2,b_2)$ are crisp points at (a_1,b_1) and (a_2,b_2), respectively.

The final concept in this section is that of a fuzzy metric.

Definition 2. A fuzzy metric $\bar{M}$ is a mapping from pairs of fuzzy points $(\overline{P_1}, \overline{P_2})$ into fuzzy numbers so that:

1. $\bar{M}(\overline{P_1}, \overline{P_2}) = \bar{M}(\overline{P_2}, \overline{P_1})$;
2. $\bar{M}(\overline{P_1}, \overline{P_2}) = \bar{0}$ if and only if $\overline{P_1}$ and $\overline{P_2}$ are both fuzzy points at (a,b); and
3. $\bar{M}(\overline{P_1}, \overline{P_2}) \leqslant \bar{M}(\overline{P_1}, \overline{P_3}) + \bar{M}(\overline{P_3}, \overline{P_2})$ for any fuzzy points $\overline{P_1}$, $\overline{P_2}$, $\overline{P_3}$.

To completely specify $\bar{M}$ in Definition 2 we need to do three things: (1) define $\bar{0}$ in (2); (2) define $\leqslant$ in (3); and (3) define $+$ in (3). $\bar{0}$ will be any fuzzy subset of R with the following properties: (1) $\mu(x\,|\,\bar{0}) = 0$ for $x < 0$; (2) $\mu(x\,|\,\bar{0}) = 1$ if and only if $x = 0$; and (3) $\mu(x\,|\,\bar{0})$ is decreasing for $0 < x < d$, for some $d > 0$, and $\mu(x\,|\,\bar{0}) = 0$ for $x \geqslant d$.

Since $\bar{M}(\overline{P_1},\overline{P_2})$ is a fuzzy number its α-cuts will be closed intervals. We will define two orderings ($\leqslant$) of fuzzy numbers in terms of their α-cuts.

Definition 3. Let $\bar{A}$ and $\bar{B}$ be two fuzzy numbers and set $\bar{A}(\alpha) = [a_1(\alpha), a_2(\alpha)]$, $\bar{B}(\alpha) = [b_1(\alpha), b_2(\alpha)]$ for all α. We write $\bar{A} \leqslant_s \bar{B}$ if and only if $a_1(\alpha) \leqslant b_1(\alpha)$ and $a_2(\alpha) \leqslant b_2(\alpha)$ for all α. We write $\bar{A} \leqslant_w \bar{B}$ if and only if $a_2(\alpha) \leqslant b_2(\alpha)$ for all α.

In Definition 3 "$\leqslant_s$" stands for a strong ordering and "$\leqslant_w$" denotes a weak ordering. The $\leqslant$ in (3) of Definition 2 can be $\leqslant_s$ or $\leqslant_w$. If we use $\leqslant_s$ ($\leqslant_w$) we will say that $\bar{M}$ is a strong (weak) fuzzy metric.

The addition (+) of fuzzy numbers in (3) of Definition 2 is to be done using interval arithmetic. That is, we just add the two intervals $\bar{M}(\overline{P_1},\overline{P_2})(\alpha)$ and $\bar{M}(\overline{P_3},\overline{P_2})(\alpha)$ for all α to obtain the fuzzy number for the sum.

Theorem 2. *The relation $\leqslant_s$ is a partial order (reflexive, transitive, antisymmetric) on the set of fuzzy numbers. The relation $\leqslant_w$ is reflexive and transitive.*

Proof. Obvious. □

Theorem 3. *$\bar{D}$ is a weak fuzzy metric.*

Proof. 1. Clearly, $\bar{D}(\overline{P_1},\overline{P_2}) = \bar{D}(\overline{P_2},\overline{P_1})$.

2. Let $\overline{P_1} = \bar{P}(a_1,b_1)$, $\overline{P_2} = \bar{P}(a_2,b_2)$. First suppose that $\bar{D}(\overline{P_1},\overline{P_2}) = \bar{0}$. This implies 0 is in $\bar{D}(\overline{P_1},\overline{P_2})(1)$. But $\bar{D}(\overline{P_1},\overline{P_2})(1)$ is the set of all $d(u,v)$ where $u \in \overline{P_1}(1) = \{(a_1,b_1)\}$, $v \in \overline{P_2}(1) = \{(a_2,b_2)\}$. Hence $d((a_1,b_1),(a_2,b_2)) = 0$ implies $(a_1,b_1) = (a_2,b_2)$. Now suppose that $\overline{P_1}$ and $\overline{P_2}$ are fuzzy points at (a,b). It follows that $\bar{D}(\overline{P_1},\overline{P_2})(1) = \{0\}$ and $\bar{D}(\overline{P_1},\overline{P_2})$ has the correct shape to be called an $\bar{O}$.

3. Let $\overline{P_3} = \bar{P}(a_3,b_3)$, $\bar{A} = \bar{D}(\overline{P_1},\overline{P_2})$, $\bar{B} = \bar{D}(\overline{P_1},\overline{P_3})$, $\bar{C} = \bar{D}(\overline{P_3},\overline{P_2})$, $\bar{A}(\alpha) = [a_1(\alpha),a_2(\alpha)]$, $\bar{B}(\alpha) = [b_1(\alpha),b_2(\alpha)]$, $\bar{C}(\alpha) = [c_1(\alpha),c_2(\alpha)]$. We need to show that $a_2(\alpha) \leqslant b_2(\alpha) + c_2(\alpha)$ for all α.

We know that, from Theorem 1

$$a_2(\alpha) = \sup\{d(u,v) : u \in P_1(\alpha),\ v \in P_2(\alpha)\}, \tag{1}$$

$$b_2(\alpha) = \sup\{d(u,v) : u \in P_1(\alpha),\ v \in P_3(\alpha)\}, \tag{2}$$

$$c_2(\alpha) = \sup\{d(u,v) : u \in P_3(\alpha),\ v \in P_2(\alpha)\}. \tag{3}$$

Therefore

$$\begin{aligned} a_2(\alpha) &\leqslant \sup_{(u,v)}\{d(u,w) + d(w,v) : u \in P_1(\alpha),\ w \in P_3(\alpha),\ v \in P_2(\alpha)\} \\ &\leqslant \sup_u\{d(u,w) : u \in P_1(\alpha),\ w \in P_3(\alpha)\} + \sup_v\{d(w,v) : w \in P_3(\alpha),\ v \in P_2(\alpha)\} \\ &\leqslant b_2(\alpha) + c_2(\alpha). \end{aligned}$$

The following example shows that $\bar{D}$ is not a strong fuzzy metric. □

Example 2. $\overline{P_1}$, $\overline{P_2}$, and $\overline{P_3}$ are fuzzy points at (1,0), (3,0), and (2,0), respectively. The shape of each $\overline{P_i}$ is a right circular cone. For example, $\overline{P_1}$ is a right circular cone with base $(x-1)^2 + y^2 \leqslant (1/2)^2$ and vertex at (1,0). The base of $\overline{P_2}$ ($\overline{P_3}$) is $(x-3)^2 + y^2 \leqslant 1/4$ ($(x-2)^2 + y^2 \leqslant 1/4$). Then $\bar{D}(\overline{P_1},\overline{P_2})(0) = [1,3]$, $\bar{D}(\overline{P_1},\overline{P_3})(0) = \bar{D}(\overline{P_3},\overline{P_2})(0) = [0,2]$ so that $\bar{D}(\overline{P_1},\overline{P_3})(0) + \bar{D}(\overline{P_3},\overline{P_2})(0) = [0,4]$ and $[1,3]$ is not $\leqslant_s [0,4]$.

3. Fuzzy lines

In this section we: (1) first consider various possible definitions of a fuzzy line; (2) look at a few examples of fuzzy lines; (3) show how to calculate α-cuts of a fuzzy line; (4) discuss basic properties of our fuzzy lines; and (5) consider parallel fuzzy lines and the intersection of fuzzy lines.

3.1. Types of fuzzy lines

Method 1. Given fuzzy numbers $\bar{A}$, $\bar{B}$, $\bar{C}$ a fuzzy line is the set $(\bar{X}, \bar{Y})$ of all fuzzy number solutions to

$$\bar{A}\bar{X} + \bar{B}\bar{Y} = \bar{C}. \tag{4}$$

However, we know [4] that too often Eq. (4) has no solution (using standard fuzzy arithmetic) for $\bar{X}$ and $\bar{Y}$. Therefore, we will not be able to use Method 1 to define a fuzzy line.

Method 2. Given fuzzy numbers $\bar{M}$, $\bar{B}$ a fuzzy line is the set $(\bar{X}, \bar{Y})$ of all fuzzy number solutions to

$$\bar{Y} = \bar{M}\bar{X} + \bar{B}. \tag{5}$$

Using Method 2 a fuzzy line will be $(\bar{X}, \bar{Y})$, $\bar{X}$ any fuzzy number and $\bar{Y}$ is calculated by Eq. (5). The main reason we will not employ Method 2 to define a fuzzy line is that we cannot construct any pictures (graphs) of this type of fuzzy line. The following methods lend themselves more easily to visualization. We believe it is important to be able to get pictures of fuzzy objects in fuzzy plane geometry.

Method 3. Let $\bar{A}$, $\bar{B}$, $\bar{C}$ be fuzzy numbers. If $\bar{A}(1) = \{a\}$ and $\bar{B}(1) = \{b\}$ we assume that a and b are not both zero. Let

$$\Omega_{11}(\alpha) = \{(x, y) : ax + by = c,\ a \in \bar{A}(\alpha),\ b \in \bar{B}(\alpha),\ c \in \bar{C}(\alpha)\}, \quad 0 \leqslant \alpha \leqslant 1. \tag{6}$$

The fuzzy line $\overline{L_{11}}$ is defined by its membership function

$$\mu((x, y) \mid \overline{L_{11}}) = \sup\{\alpha : (x, y) \in \Omega_{11}(\alpha)\}. \tag{7}$$

If $\bar{A}(1) = \{0\}$ and $\bar{B}(1) = \{0\}$, then $\Omega_{11}(1)$ can be empty because we end up with an equation $0x + 0y = c$, $c \in \bar{C}(1)$, which will have no solution when c is not zero.

We could also use the equation $y = mx + b$ to define a fuzzy line.

Method 4. Given fuzzy numbers $\bar{M}$ and $\bar{B}$ let

$$\Omega_{12}(\alpha) = \{(x, y) : y = mx + b,\ m \in \bar{M}(\alpha),\ b \in \bar{B}(\alpha)\} \quad \text{for } 0 \leqslant \alpha \leqslant 1. \tag{8}$$

Then we define $\overline{L_{12}}$ as

$$\mu((x, y) \mid \overline{L_{12}}) = \sup\{\alpha : (x, y) \in \Omega_{12}(\alpha)\}. \tag{9}$$

Method 5 (*Point-slope form*). Let $\bar{K}$ be a fuzzy point in R^2 and let $\bar{M}$ be a fuzzy number. Define, for $0 \leqslant \alpha \leqslant 1$,

$$\Omega_2(\alpha) = \{(x, y) : y - v = m(x - u), (u, v) \in \bar{K}(\alpha), m \in \bar{M}(\alpha)\}. \tag{10}$$

Fuzzy line $\overline{L_2}$ is

$$\mu((x, y) \mid \overline{L_2}) = \sup\{\alpha : (x, y) \in \Omega_2(\alpha)\}. \tag{11}$$

Method 6 (*Two-point form*). Let $\overline{P_1}$ and $\overline{P_2}$ be two fuzzy points in the plane. Define

$$\Omega_3(\alpha) = \left\{(x,y) : \frac{y-v_1}{x-u_1} = \frac{v_2-v_1}{u_2-u_1}, (u_1,v_1) \in \overline{P_1}(\alpha), (u_2,v_2) \in \overline{P_2}(\alpha)\right\}, \quad \text{for } 0 \leqslant \alpha \leqslant 1. \tag{12}$$

$\overline{L_3}$ is

$$\mu((x,y) \mid \overline{L_3}) = \sup\{\alpha : (x,y) \in \Omega_3(\alpha)\}. \tag{13}$$

We will use Methods 3–6 to define our fuzzy lines. Later on in this section we will discuss basic properties of our fuzzy lines and relationships between them.

3.2. Examples

Example 3 (*A "fat" fuzzy line*). Let $\bar{A} = (-1/0/1)$, $\bar{B} = (-1/1/2)$, $\bar{C} = (0/1/2)$ all triangular fuzzy numbers. Then we see that the support of $\overline{L_{11}}$, $\overline{L_{11}}(0)$, is all of R^2. Theorem 4 below will show that $\overline{L_a}(\alpha) = \Omega_a(\alpha)$ for all a, $a = 11, 12, 2, 3$. Notice that $\overline{L_{11}}(1)$ is the crisp line $y = 1$.

Example 4 (*A "thin" fuzzy line*). Let $\overline{L_{12}}$ be defined by $y = 2x + \bar{B}$, $\bar{B} = (0/1/2)$, using Method 4. Here $\bar{M}$ is the crisp number two. The graph of $z = \mu((x,y) \mid \overline{L_2})$ is generated by placing $\bar{B}$ on the y-axis, base on the interval $[0,2]$, and "running" the triangle $\bar{B}$ along the crisp line $y = 2x + 1$.

Example 5 (*Another "thin" fuzzy line using Method 5*). Let $\bar{M} = 1$ (crisp) and $\bar{K}$ a fuzzy point at $(1,1)$. Then $\overline{L_2}(\alpha)$ will be all lines, slope one, through a point in $\bar{K}(\alpha)$. $\overline{L_2}(1)$ is the crisp line $y = x$. $\overline{L_2}$ is "thin" when $\bar{K}(0)$ is "small".

Example 6 (*A "thin" and "fat" fuzzy line*). Let $\overline{P_1}(0,0)$ and $\overline{P_2}(1,1)$ be two fuzzy points whose graph is a right circular cone. The base of $\overline{P_1}(0,0)$ is $B_1 = \{(x,y) : x^2 + y^2 \leqslant (1/3)^2\}$ and vertex at $(0,0)$. The base of $\overline{P_2}(1,1)$ is $B_2 = \{(x,y) : (x-1)^2 + (y-1)^2 \leqslant (1/3)^2\}$ and vertex at $(1,1)$. Then $\overline{L_3}(0)$, the support of $\overline{L_3}$ by Method 6, is all lines through a point in B_1 and a point in B_2. $\overline{L_3}(1)$ is the line $y = x$. The graph of $z = \mu((x,y) \mid \overline{L_3})$ is "thin" between B_1 and B_2, but gets wider and wider as we move along $y = x$ for $x > 1$ or for $x < 0$.

3.3. Alpha-cuts of fuzzy lines

Theorem 4. $\overline{L_a}(\alpha) = \Omega_a(\alpha)$, $0 \leqslant \alpha \leqslant 1$, *for* $a = 11, 12, 2, 3$.

Proof. Similar to the proof that $\bar{D}(\alpha) = \Omega(\alpha)$ in Theorem 1 and is omitted. □

3.4. Some basic properties

We first consider $\overline{L_2}$ and $\overline{L_3}$ and then develop relationships between $\overline{L_{11}}$, $\overline{L_{12}}$, $\overline{L_2}$, and $\overline{L_3}$.

3.4.1. $\overline{L_2}$

In the definition of $\overline{L_2}$ let $\bar{K}$ be the fuzzy point and $\bar{M}$ the fuzzy slope. If $\bar{A}$ and $\bar{B}$ are two fuzzy subsets of the plane, we write $\bar{A} \leqslant \bar{B}$ if and only if $\mu((x,y) \mid \bar{A}) \leqslant \mu((x,y) \mid \bar{B})$ for all (x,y) in R^2.

Definition 4. We say a fuzzy line $\bar{L}$ contains a fuzzy point $\bar{Q}$ if and only if $\bar{Q} \leqslant \bar{L}$.

Clearly, $\overline{L_2}$ contains $\bar{K}$ because one can easily check that $\bar{K} \leqslant \overline{L_2}$. If $\bar{P}(c,d)$ is a fuzzy point at (c,d) and $\overline{L_2}$ contains $\bar{P}(c,d)$, then it follows that (c,d) must be in $\Omega_2(1)$. Let $\bar{M}(1) = [m_1, m_2]$ be an interval. We also see that $\Omega_2(1)$ is all lines through (a,b) with slope m, $m_1 \leqslant m \leqslant m_2$. If $\bar{M}$ is a triangle fuzzy number, $\bar{M}(1) = \{m\}$, a singleton, the $\Omega_2(1)$ is the crisp line $y - b = m(x-a)$.

3.4.2. $\overline{L_3}$

Let $\overline{P_1} = \bar{P}(a_1,b_1)$, $\overline{P_2} = \bar{P}(a_2,b_2)$ be two fuzzy points which define $\overline{L_3}$ by Method 6. Clearly, $\overline{L_3}$ contains both $\overline{P_1}$ and $\overline{P_2}$. Also, $\overline{L_3}(1) = \Omega_3(1)$ will always be the crisp line through (a_1,b_1) and (a_2,b_2). If $\overline{L_3}$ contains some other fuzzy point $\bar{Q}$, then $\bar{Q}(1)$ must be on the line which is $\Omega_3(1)$.

3.5. Relationships

We show how, under certain conditions, $\overline{L_{11}}$ is an $\overline{L_{12}}$, $\overline{L_3}$ is an $\overline{L_2}$, $\overline{L_2}$ is an $\overline{L_{12}}$, and $\overline{L_{12}}$ is an $\overline{L_2}$.

3.5.1. $\overline{L_{11}}$ *is an* $\overline{L_{12}}$

Assume that zero does not belong to $\bar{B}(0)$. Define

$$\Omega_m(\alpha) = \{-a/b : a \in \bar{A}(\alpha),\ b \in \bar{B}(\alpha)\}, \quad 0 \leqslant \alpha \leqslant 1, \tag{14}$$

and define $\bar{M}$ as

$$\mu(x \mid \bar{M}) = \sup\{\alpha : x \in \Omega_m(\alpha)\}. \tag{15}$$

Next set

$$\Omega_b(\alpha) = \{c/d : b \in \bar{B}(\alpha),\ c \in \bar{C}(\alpha)\}, \quad 0 \leqslant \alpha \leqslant 1, \tag{16}$$

and define $\overline{B_0}$ as

$$\mu(x \mid \overline{B_0}) = \sup\{\alpha : x \in \Omega_b(\alpha)\}. \tag{17}$$

In the above definitions $\bar{A}$, $\bar{B}$, $\bar{C}$ are the fuzzy numbers in the definition of $\overline{L_{11}}$. It can be shown that $\bar{M}$ and $\overline{B_0}$ are also fuzzy numbers and $\bar{M}(\alpha) = \Omega_m(\alpha)$, $\overline{B_0}(\alpha) = \Omega_b(\alpha)$ for all α. So let $\bar{M}$ and $\overline{B_0}$ be the fuzzy numbers in the definition of $\overline{L_{12}}$.

Theorem 5. $\overline{L_{11}} = \overline{L_{12}}$.

Proof. We show that $\overline{L_{11}}(\alpha) = \Omega_{11}(\alpha)$ is the same as $\overline{L_{12}}(\alpha) = \Omega_{12}(\alpha)$, for all α.

If $(x,y) \in \Omega_{11}(\alpha)$, then $ax + by = c$ for some $a \in \bar{A}(\alpha)$, $b \in \bar{B}(\alpha)$, $c \in \bar{C}(\alpha)$. Then $y = mx + b_0$ for $m = -a/b$, $b_0 = c/b$. But $m \in \bar{M}(\alpha)$, $b_0 \in \overline{B_0}(\alpha)$ so that $(x,y) \in \Omega_{12}(\alpha)$. Hence, $\Omega_{11}(\alpha)$ is a subset of $\Omega_{12}(\alpha)$.

Similarly, we can show that $\Omega_{12}(\alpha)$ is a subset of $\Omega_{11}(\alpha)$. □

3.5.2. $\overline{L_3}$ *is an* $\overline{L_2}$

Let $\overline{P_1} = \bar{P}(a_1,b_1)$, $\overline{P_2} = \bar{P}(a_2,b_2)$ be two fuzzy points which define $\overline{L_3}$. Define Proj_x (Proj_y) to be the projection of a subset of the plane onto the x-axis (y-axis). We assume that $\mathrm{Proj}_x \overline{P_1}(0) \cap \mathrm{Proj}_x \overline{P_2}(0)$ is empty. This means that if $(u_1,v_1) \in \overline{P_1}(0)$ and $(u_2,v_2) \in \overline{P_2}(0)$, then $u_1 - u_2$ will never be zero. Define

$$\Omega_m(\alpha) = \left\{ m : m = \frac{v_2 - v_1}{u_2 - u_1},\ (u_1,v_1) \in \overline{P_1}(\alpha),\ (u_2,v_2) \in \overline{P_2}(\alpha) \right\}, \quad \text{for } 0 \leqslant \alpha \leqslant 1, \tag{18}$$

and set

$$\mu(x \mid \bar{M}) = \sup\{\alpha : x \in \Omega_m(\alpha)\}. \tag{19}$$

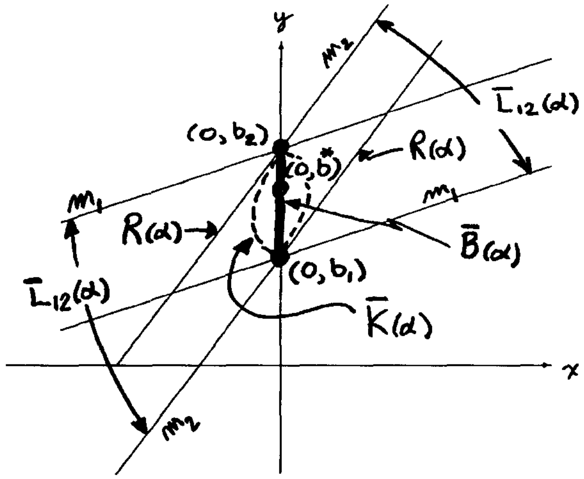

Fig. 1. Fuzzy lines $\overline{L_{12}}$ and $\overline{L_2}$ can be the same.

It can be shown that $\bar{M}$ is a fuzzy number and $\bar{M}(\alpha) = \Omega_m(\alpha)$ for all α. Now let $\bar{M}$ and $\overline{P_1}$ be fuzzy sets in the definition of $\overline{L_2}$.

Theorem 6. $\overline{L_3} = \overline{L_2}$.

Proof. We show $\Omega_3(\alpha) = \Omega_2(\alpha)$ for all α. Let $(x,y) \in \overline{L_3}(\alpha)$. Then $y - v_1 = m(x - u_1)$ for $m \in \bar{M}(\alpha)$, $(u_1, v_1) \in \overline{P_1}(\alpha)$. But then $(x,y) \in \Omega_2(\alpha)$ and $\Omega_3(\alpha)$ is a subset of $\Omega_2(\alpha)$. Similarly, we can show that $\Omega_2(\alpha)$ is a subset of $\Omega_3(\alpha)$. □

3.5.3. $\overline{L_{12}}$ versus $\overline{L_2}$

We first argue that given an $\overline{L_{12}}$, we can define an $\overline{L_2}$ so that $\overline{L_2} = \overline{L_{12}}$. Let $\bar{M}$, $\bar{B}$ be two fuzzy numbers that define $\overline{L_{12}}$. We will use the same $\bar{M}$ for $\overline{L_2}$. Fig. 1 shows a typical α-cut of an $\overline{L_{12}}$. Recall that $\overline{L_{12}}(\alpha) = \Omega_{12}(\alpha)$. We need to specify a fuzzy point $\bar{K}$ to completely define $\overline{L_2}$. Let $\bar{M}(\alpha) = [m_1, m_2]$ and $\bar{B}(\alpha) = [b_1, b_2]$. We now assume that $\mu(x \mid \bar{B}) = 1$ if and only if $x = b^*$ where $b_1 < b^* < b_2$. That is, $\bar{B}$ is normalized at only one point. Define $\bar{K}$ to be a fuzzy point at $(0, b^*)$ whose α-cut lies in region $R(\alpha)$ in Fig. 1. Then $\Omega_2(\alpha) = \Omega_{12}(\alpha)$ and $\overline{L_2}$ will be the same as $\overline{L_{12}}$.

Conversely, let $\bar{M}$ and $\bar{K}$ be the fuzzy sets in the definition of $\overline{L_2}$. $\bar{K}$ must be a fuzzy point on the y-axis. Let $\bar{K}$ be a fuzzy point at $(0, b^*)$ so that its α-cuts fit inside region $R(\alpha)$ of Fig. 1. $\mathrm{Proj}_y \bar{K}(\alpha)$ will be $[b_1, b_2]$ in Fig. 1. Define fuzzy number $\bar{B}$ so that $\bar{B}(\alpha) = \mathrm{Proj}_y \bar{K}(\alpha)$ for all α. We use the same $\bar{M}$ for $\overline{L_{12}}$. Then $\Omega_{12}(\alpha) = \Omega_2(\alpha)$ for all α so that $\overline{L_{12}} = \overline{L_2}$.

3.6. General properties

A fuzzy line $\bar{L}$ will be an $\overline{L_{11}}$, $\overline{L_{12}}$, $\overline{L_2}$, or an $\overline{L_3}$. We see in general that:

(1) α-cuts of $\bar{L}$ will be closed, connected and arcwise connected but not necessarily convex;
(2) $\mu((x,y) \mid \bar{L})$ will be upper semi-continuous (follows from α-cuts being closed); and
(3) $\bar{L}$ is normalized, or there is always at least one crisp line in $\bar{L}(1)$.

3.7. Parallel and intersecting fuzzy lines

Definition 5. Let $\overline{L_a}$, $\overline{L_b}$ be two fuzzy lines. A measure of parallelness (ρ) of $\overline{L_a}$ and $\overline{L_b}$ is defined to be $1-\lambda$ where

$$\lambda = \sup_{R^2}\{\min(\mu((x,y) \mid \overline{L_a}), \mu((x,y) \mid \overline{L_b}))\}. \tag{20}$$

In Eq. (20), λ is just the height of the intersection of $\overline{L_a}$ and $\overline{L_b}$. So, if $\overline{L_a} \cap \overline{L_b}$ is the empty set (completely parallel), then $\lambda = 0$ and $\rho = 1$. Let l_a (l_b) be a crisp line in $\overline{L_a}(1)$ ($\overline{L_b}(1)$). If l_a and l_b intersect, then $\lambda = 1$ and $\rho = 0$. So we see that ρ has some properties we would expect for a measure of parallelness.

Suppose $\overline{L_a}$ and $\overline{L_b}$ are both crisp lines. Then $\rho = 1$ if and only if $\overline{L_a}$ and $\overline{L_b}$ are parallel.

Now let $\overline{L_a}$ be a crisp line and $\overline{L_b}$ a fuzzy line. For example, let $\overline{L_b} = \overline{L_{12}}$ or $\overline{L_2}$. If $\bar{M}(1) = [m_1, m_2]$, $m_1 < m_2$, then $\overline{L_a}$ must intersect a crisp line in $\overline{L_b}(1)$ and $\rho = 0$. So we see that we may quite often obtain $\rho = 0$ for a crisp line and a fuzzy line.

Definition 6. Let $\overline{L_a}$ and $\overline{L_b}$ be two fuzzy lines. Assume $\rho < 1$. The fuzzy region $\bar{R}$ of intersection of $\overline{L_a}$ and $\overline{L_b}$ is $\bar{R} = \overline{L_a} \cap \overline{L_b}$. The membership function for $\bar{R}$ is

$$\mu((x,y) \mid \bar{R}) = \min(\mu((x,y) \mid \overline{L_a}), \mu((x,y) \mid \overline{L_b})). \tag{21}$$

Obviously, if $\rho = 1$, then the fuzzy region of intersection $\bar{R}$ is the empty set.

4. Future research

The next step in this research project on fuzzy geometry is to expand our results on fuzzy plane geometry. We will next consider fuzzy circles, fuzzy triangles, etc. We will define the fuzzy area and fuzzy perimeter of fuzzy circles, triangles, rectangles, etc. which we will show to be all fuzzy numbers. We will also investigate fuzzy measures of the angles of fuzzy triangles. Then the research effort will go into R^n, $n \geqslant 3$, and applications to fuzzy data base.

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