EPIREFLECTIONS AND SUPERCOMPACT CARDINALS

JOAN BAGARIA, CARLES CASACUBERTA, AND ADRIAN R. D. MATHIAS

Abstract. We prove that the existence of arbitrarily large supercompact cardinals implies that every absolute epireflective class of objects in a balanced accessible category is a small-orthogonality class. In other words, if L is a localization functor on a balanced accessible category such that the unit morphism $X \to LX$ is an epimorphism for all X and the class of L-local objects is defined by an absolute formula, then the existence of a sufficiently large supercompact cardinal implies that L is a localization with respect to some set of morphisms.

1. Introduction

The answers to many questions in infinite abelian group theory are known to depend on set theory. For example, the question whether torsion theories are necessarily singly generated or singly cogenerated was discussed in [6], where the existence or nonexistence of measurable cardinals played an important role.

In homotopy theory, it was asked around 1990 if every functor on simplicial sets which is idempotent up to homotopy is equivalent to f-localization for some map f (see [4] and [5] for terminology and details). Although this may not seem a set-theoretical question, the following counterexample was given in [3]: Under the assumption that measurable cardinals do not exist, the functor L defined as $LX = NP_{\mathcal{A}}(\pi X)$, where π denotes the fundamental groupoid, N denotes the nerve, and $P_{\mathcal{A}}$ denotes reduction with respect to the proper class $\mathcal{A}$ of groups of the form $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for all cardinals κ, is not equivalent to localization with respect to any set of maps. (Reduction with respect to a class $\mathcal{A}$ assigns to each groupoid G, in a universal way, a morphism $G \to P_{\mathcal{A}}G$ with $\mathrm{Hom}(A, P_{\mathcal{A}}G) = 0$ for all $A \in \mathcal{A}$.)

The statement that measurable cardinals do not exist is consistent with the Zermelo–Fraenkel axioms with the axiom of choice (ZFC), provided of course that ZFC is itself consistent. However, many large-cardinal assumptions, such as the existence of measurable cardinals, or bigger cardinals, are used in mathematical practice, leading to useful developments. Specifically, Vopěnka's principle (one of whose forms is the statement that between the members of every proper class of graphs there is at least one nonidentity map; cf. [2], [9]) implies that every homotopy idempotent functor on simplicial sets is an f-localization for some map f, as proved in [3]. Vopěnka's

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principle has many other similar consequences, such as the fact that all reflective classes in locally presentable categories are small-orthogonality classes [2].

In this article, we show that the existence of arbitrarily large supercompact cardinals (which is a much weaker assumption than Vopěnka's principle) implies that every epireflective class $\mathcal{L}$ is a small-orthogonality class, under mild conditions on the category and the given class. These conditions are fulfilled if the category is balanced and accessible [2] and $\mathcal{L}$ is defined by an absolute formula.

In order to explain the role played by absoluteness, we note that, if one assumes that measurable cardinals exist, then the reduction $P_{\mathcal{A}}$ mentioned above becomes the zero functor in the category of groups, since if λ is measurable then $\mathrm{Hom}(\mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}, \mathbb{Z}) \neq 0$ by [6], so in fact $P_{\mathcal{A}}\mathbb{Z} = 0$ and therefore $P_{\mathcal{A}}$ kills all groups. Remarkably, this example shows that one may define a functor $P_{\mathcal{A}}$, namely reduction with respect to a certain class of groups, and it happens that the conclusion of whether $P_{\mathcal{A}}$ is trivial or not depends on the set-theoretical model in which one is working. Thus, such a definition is not absolute in the sense of model theory, that is, there is no absolute formula in the usual language of set theory whose satisfaction determines precisely $P_{\mathcal{A}}$ or its image. A formula φ (with a free variable and possibly with parameters) is called *absolute* if, for any inner model M of set theory containing the parameters, a given set X in M satisfies φ in M if and only if X satisfies φ in the universe V of all sets. For instance, the statement "X is a module over a ring R" can be formalized by means of an absolute formula with R as a parameter. On the other hand, statements involving cardinals, unbounded quantifiers, or choices may fail to be absolute. An example of a definition which cannot be made absolute is that of a topological space, since a topology $\mathcal{T}$ on a set X in a set-theoretical model may fail to be closed under unions in a larger model.

We thank J. Rosický for his interest in this article and for showing us an example, described in Section 5, of an epireflective class of graphs which is not a small-orthogonality class under the negation of Vopěnka's principle, even if supercompact cardinals are assumed to exist. This is another instance of a class that cannot be defined by an absolute formula.

Analogous situations occur in other areas of mathematics. For example, if there exists a supercompact cardinal, then all sets of real numbers that are definable by formulas whose quantifiers range only over real numbers and ordinals, and have only real numbers and ordinals as parameters, are Lebesgue measurable [13]. In fact, in order to prove the existence of non-measurable sets of real numbers, one needs to use the axiom of choice, a device that produces nondefinable objects [14].

2. Preliminaries from category theory

Most of the material that we need from category theory can be found in the books [1], [2], and [11]. In this section we recall a number of notions and facts that are used in the article, and prove a new result (Theorem 2.6) which is a key ingredient of our main theorem in Section 4.

A category is called *balanced* if every morphism that is both a monomorphism and an epimorphism is an isomorphism. The category of rings and the category of graphs are important examples of nonbalanced categories. In this article, as in [2], a *graph* will be a set X equipped with a binary relation, where the elements of X are called vertices and there is a directed edge from x to y if and only if the pair (x, y) is in the binary relation. Each map of graphs is determined by the images of the vertices. Hence, the monomorphisms of graphs are the injective maps, and epimorphisms of graphs are maps that are surjective on vertices (but not necessarily surjective on edges).

A monomorphism $m\colon X \to Y$ in a category is *strong* if, given any commutative square

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "u"'] & Q \arrow[d, "v"] \\ X \arrow[r, "m"] & Y \end{tikzcd}$$

in which e is an epimorphism, there is a unique morphism $f\colon Q \to X$ such that $f \circ e = u$ and $m \circ f = v$. A monomorphism m is *extremal* if, whenever it factors as $m = v \circ e$ where e is an epimorphism, it follows that e is an isomorphism. Split monomorphisms are strong, and strong monomorphisms are extremal. If a morphism is both an extremal monomorphism and an epimorphism, then it is necessarily an isomorphism, and, if $\mathcal{C}$ is balanced, then all monomorphisms are extremal. The dual definitions and similar comments apply to epimorphisms.

A *subobject* of an object X in a category $\mathcal{C}$ is an equivalence class of monomorphisms $A \to X$, where $m\colon A \to X$ and $m'\colon A' \to X$ are declared equivalent if there are morphisms $u\colon A \to A'$ and $v\colon A' \to A$ such that $m = m' \circ u$ and $m' = m \circ v$. For simplicity, when we refer to a subobject A of X, we view A as an object equipped with a monomorphism $A \to X$. A subobject is called strong (or extremal) if the corresponding monomorphism is strong (or extremal). The notion of a *quotient* of an object X is defined dually. A category is *well-powered* if the subobjects of every object form a set, and *co-well-powered* if the quotients of every object form a set.

A *reflection* (also called a *localization*) on a category $\mathcal{C}$ is a pair (L, η) where $L\colon \mathcal{C} \to \mathcal{C}$ is a functor and $\eta\colon \mathrm{Id} \to L$ is a natural transformation, called *unit*, such that $\eta_{LX}\colon LX \to LLX$ is an isomorphism and $\eta_{LX} = L\eta_X$ for all X in $\mathcal{C}$. By abuse of terminology, we often say that the functor L is itself a reflection if the natural transformation η is clear from the context.

If L is a reflection, the objects X such that $\eta_X\colon X \to LX$ is an isomorphism are called *L-local objects*, and the morphisms f such that Lf is an isomorphism are called *L-equivalences*. By definition, η_X is an L-equivalence for all X. In fact, η_X is terminal among L-equivalences with domain X, and it is initial among morphisms from X to L-local objects.

A reflection L is called an *epireflection* if, for every X in $\mathcal{C}$, the unit morphism $\eta_X\colon X \to LX$ is an epimorphism. We say that L is a strong (or extremal) epireflection if η_X is a strong (or extremal) epimorphism for all X. A typical example of an epireflection is the abelianization functor on the category of groups. More generally, there is a bijective correspondence in the category of groups between epireflections and radicals; cf. [12, §2].

Since a full subcategory is determined by the class of its objects, the terms *reflective class* and *reflective full subcategory* are used indistinctly to denote the class of L-local objects for a reflection L or the full subcategory with these objects. If L is an epireflection, then the class of its local objects is called *epireflective*. It is called strongly epireflective or extremally epireflective if L is a strong or extremal epireflection.

The proof of the following facts is omitted. Similar statements can be found in [1, Theorem 16.8] and [12, Theorem 6].

Proposition 2.1. *Let (L, η) be a reflection on a category $\mathcal{C}$.*

(a) *If L is an epireflection, then the class of L-local objects is closed under strong subobjects, and it is closed under arbitrary subobjects if $\mathcal{C}$ is balanced.*
(b) *Suppose that $\eta_X\colon X \to LX$ can be factored as an epimorphism followed by a monomorphism for all X. If the class of L-local objects is closed under subobjects, then L is an epireflection.*
(c) *If η_X factors as an epimorphism followed by a strong monomorphism for all X and the class of L-local objects is closed under strong subobjects, then L is an epireflection.*
(d) *If η_X factors as a strong epimorphism followed by a monomorphism for all X and the class of L-local objects is closed under subobjects, then L is a strong epireflection.*

The claims in (c) and (d) also hold if strong is replaced by extremal.

A category $\mathcal{C}$ is called *complete* if all limits exist in $\mathcal{C}$, and *cocomplete* if all colimits exist in $\mathcal{C}$.

Proposition 2.2. *If a category $\mathcal{C}$ is complete, well-powered, and co-well-powered, then every class of objects $\mathcal{L}$ closed under products and extremal subobjects in $\mathcal{C}$ is epireflective, and if $\mathcal{L}$ is closed under products and subobjects then it is extremally epireflective.*

Proof. It follows from [1, Proposition 12.5 and Corollary 14.21] that, if $\mathcal{C}$ is complete and well-powered, then every morphism in $\mathcal{C}$ can be factored as an extremal epimorphism followed by a monomorphism, and also as an epimorphism followed by an extremal monomorphism. Thus, we may define a reflection by factoring, for each object X, the canonical morphism from X into the product of its quotients that are in $\mathcal{L}$ as an epimorphism η_X followed by an extremal monomorphism, or alternatively as an extremal epimorphism followed by a monomorphism if $\mathcal{L}$ is closed under subobjects. $\square$

A morphism $f\colon A \to B$ and an object X are called *orthogonal* in a category $\mathcal{C}$ if for each $g\colon A \to X$ there is a unique $g'\colon B \to X$ with $g' \circ f = g$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\forall g"'] & B \arrow[dl, dotted, "\exists!\, g'"] \\ X. & \end{tikzcd}$$

If L is any reflection, then an object is L-local if and only if it is orthogonal to all L-equivalences, and a morphism is an L-equivalence if and only if it is orthogonal to all L-local objects.

A *small-orthogonality class* in a category $\mathcal{C}$ is the class of objects that are orthogonal to some set of morphisms $\mathcal{F} = \{f_i \colon P_i \to Q_i \mid i \in I\}$. Such objects will be called *$\mathcal{F}$-local.* If a reflection L exists such that the class of L-local objects coincides with the class of $\mathcal{F}$-local objects for some set of morphisms $\mathcal{F}$, then L will be called an *$\mathcal{F}$-localization* (or an *f-localization* if $\mathcal{F}$ consists of one morphism f only).

Note that, if a coproduct $f = \coprod_{i\in I} f_i$ exists and all hom-sets $\mathcal{C}(X, Y)$ of $\mathcal{C}$ are nonempty, then an object is orthogonal to f if and only if it is orthogonal to f_i for all $i \in I$. More precisely, if X is orthogonal to all f_i then it is orthogonal to their coproduct, and the converse holds if $\mathcal{C}(P_i, X) \neq \emptyset$ for all $i \in I$, where P_i is the domain of f_i.

Proposition 2.3. *Let (L, η) be an $\mathcal{F}$-localization on a category $\mathcal{C}$, where $\mathcal{F}$ is a nonempty set of morphisms.*

(a) *Suppose that every morphism of $\mathcal{C}$ can be factored as an epimorphism followed by a strong monomorphism. If every $f \in \mathcal{F}$ is an epimorphism, then L is an epireflection.*
(b) *If L is an epireflection, then there is a set $\mathcal{E}$ of epimorphisms such that L is also an $\mathcal{E}$-localization.*

Proof. By part (c) of Proposition 2.1, in order to prove (a) it suffices to check that the class of L-local objects is closed under strong subobjects. Thus, let X be L-local and let $s\colon A \to X$ be a strong monomorphism. We need to show that A is orthogonal to every morphism $f\colon P \to Q$ in $\mathcal{F}$. For this, let $g\colon P \to A$ be any morphism. Since X is orthogonal to f, there is a unique morphism $g'\colon Q \to X$ such that $g' \circ f = s \circ g$. Since f is an epimorphism and s is strong, there is a morphism $g''\colon Q \to A$ such that $g'' \circ f = g$ and $s \circ g'' = g'$. Moreover, if $g'''\colon Q \to A$ also satisfies $g''' \circ f = g$, then $g''' = g''$ since f is an epimorphism. Hence, A is orthogonal to f.

Our argument for part (b) is based on a similar result in [12, Theorem 1]. Write $\mathcal{F} = \{f_i \colon P_i \to Q_i \mid i \in I\}$, and let

$$\mathcal{E} = \{\eta_{P_i} \colon P_i \to LP_i \mid i \in I\} \cup \{\eta_{Q_i} \colon Q_i \to LQ_i \mid i \in I\}.$$

Then every morphism in $\mathcal{E}$ is an epimorphism, and the class of $\mathcal{E}$-local objects coincides precisely with the class of $\mathcal{F}$-local objects. □

Example 2.4. In the category of graphs, let L be the functor assigning to every graph X the complete graph (i.e., containing all possible edges between its vertices) with the same set of vertices as X, and let $\eta_X \colon X \to LX$ be the inclusion. Then L is an epireflection. The class of L-local objects is the class of complete graphs, which is closed under strong subobjects, but not under arbitrary subobjects. In fact L is an f-localization, where f is the inclusion of the two-point graph $\{0, 1\}$ into $0 \to 1$, which is an epimorphism.

The following notion and the subsequent result are essential for our purposes in Section 4.

Definition 2.5. Let $\mathcal{A}$ be a class of objects in a category $\mathcal{C}$. A set $\mathcal{H}$ of objects of $\mathcal{C}$ will be called *transverse* to $\mathcal{A}$ if for every object $A \in \mathcal{A}$ there is an object $H \in \mathcal{H} \cap \mathcal{A}$ and a monomorphism $H \to A$.

That is, $\mathcal{H}$ is transverse to $\mathcal{A}$ if every object of $\mathcal{A}$ has a subobject in $\mathcal{H} \cap \mathcal{A}$.

Theorem 2.6. *Suppose that (L, η) is an epireflection on a category $\mathcal{C}$.*

(a) *If $\mathcal{C}$ is balanced and there exists a set $\mathcal{H}$ of objects in $\mathcal{C}$ transverse to the class of objects that are not L-local, then there is a set of morphisms $\mathcal{F}$ such that L is an $\mathcal{F}$-localization.*

(b) *If $\mathcal{C}$ is co-well-powered and every morphism can be factored as an epimorphism followed by a monomorphism, then the converse holds, that is, if L is an $\mathcal{F}$-localization for some set of morphisms $\mathcal{F}$, then there is a set $\mathcal{H}$ transverse to the class of objects that are not L-local.*

Proof. To prove (a), let $\mathcal{F} = \{\eta_A \colon A \to LA \mid A \in \mathcal{H}\}$. We claim that L is an $\mathcal{F}$-localization. To prove this, pick any object X of $\mathcal{C}$. If X is L-local, then X is $\mathcal{F}$-local, since all the morphisms in $\mathcal{F}$ are L-equivalences. Next, suppose that X is $\mathcal{F}$-local and suppose further, towards a contradiction, that X is not L-local. By assumption, in the set $\mathcal{H}$ there is a subobject A of X that is not L-local. Let $s \colon A \to X$ be a monomorphism. Since X is orthogonal to η_A, there is a morphism $t \colon LA \to X$ such that $s = t \circ \eta_A$. This implies that η_A is a monomorphism and hence an isomorphism, since $\mathcal{C}$ is balanced. However, this contradicts the fact that A is not isomorphic to LA. Hence, X is L-local, as needed.

For the converse, suppose that L is an $\mathcal{F}$-localization for some nonempty set of morphisms $\mathcal{F} = \{f_i \colon P_i \to Q_i \mid i \in I\}$. Since L is an epireflection, we may assume, by part (b) of Proposition 2.3, that each f_i is an epimorphism. Since we suppose that $\mathcal{C}$ is co-well-powered, we may consider the set $\mathcal{H}$ of all quotients of P_i for all $i \in I$ (that is, we choose a representative object of each isomorphism class). Let X be an object which is not L-local. Note that, if a morphism $P_i \to X$ can be factored through Q_i, then it can be factored in a unique way, since f_i is an epimorphism. Hence, if X is not L-local, then there is a morphism $g \colon P_i \to X$ for some $i \in I$ for which there is no morphism $h \colon Q_i \to X$ with $h \circ f_i = g$. Factor g as $g'' \circ g'$, where $g' \colon P_i \to X'$ is an epimorphism and $g'' \colon X' \to X$ is a monomorphism, in such a way that X' is in $\mathcal{H}$. Note finally that X' is not L-local, for if it were then there would exist a morphism $h' \colon Q_i \to X'$ such that $g'' \circ h' \circ f_i = g$, which, as we know, cannot happen. □

Remark 2.7. For the validity of part (a) of Theorem 2.6, the assumption that $\mathcal{C}$ is balanced can be weakened by assuming only that the epimorphisms η_A are extremal for $A \in \mathcal{H}$, so that they are isomorphisms whenever they are monomorphisms. This ensures the validity of the theorem in important categories that are not balanced, such as the category of graphs (see Section 5 below), provided that L is an extremal epireflection.

By Proposition 2.1, the condition that L is an extremal epireflection is satisfied if the class of L-local objects is closed under subobjects, and morphisms in $\mathcal{C}$ can be factored as an extremal epimorphism followed by a monomorphism. By [1, Corollary 14.21], the latter holds in complete well-powered categories.

We end this section by recalling the definition of locally presentable and accessible categories. For a regular cardinal λ, a partially ordered set is called λ-*directed* if every subset of cardinality smaller than λ has an upper bound. An object X of a category $\mathcal{C}$ is called λ-*presentable*, where λ is a regular cardinal, if the functor $\mathcal{C}(X, -)$ preserves λ-directed colimits, that is, colimits of diagrams indexed by λ-directed partially ordered sets. A category $\mathcal{C}$ is *locally presentable* if it is cocomplete and there is a regular cardinal λ and a set $\mathcal{X}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{X}$. Locally presentable categories are complete, well-powered and co-well-powered. The categories of groups, rings, modules over a ring, and many others are locally presentable; see [2, 1.B] for further details and more examples.

If the assumption of cocompleteness is weakened by requiring instead that λ-directed colimits exist in $\mathcal{C}$, then $\mathcal{C}$ is called *λ-accessible*. A category $\mathcal{C}$ is called *accessible* if it is λ-accessible for some regular cardinal λ. As shown in [2, Theorem 5.35], the accessible categories are precisely the categories equivalent to categories of models of basic theories. The definition of the latter is recalled at the end of the next section.

3. Preliminaries from set theory

The *universe* V of all sets is a proper class defined recursively on the class Ord of ordinals as follows: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ for all α, where $\mathcal{P}$ is the power-set operation, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal. Finally, $V = \cup_{\alpha\in \mathrm{Ord}} V_\alpha$. Transfinite induction shows that, if α is any ordinal, then $\alpha \subseteq V_\alpha$. The axiom of regularity, stating that every nonempty set has a minimal element with respect to the membership relation, implies that every set is an element of some V_α; see [8, Lemma 9.3]. The *rank* of a set X, denoted $\mathrm{rank}(X)$, is the least ordinal α such that $X \in V_{\alpha+1}$.

A set or a proper class X is called *transitive* if every element of an element of X is also an element of X. The universe V is transitive, and so is V_α for every ordinal α. The *transitive closure* of a set X, written $\mathrm{TC}(X)$, is the smallest transitive set containing X, that is, the intersection of all transitive sets that contain X. The elements of $\mathrm{TC}(X)$ are the elements of X, the elements of the elements of X, etc.

The *language of set theory* is the first-order language whose only nonlogical symbols are equality $=$ and the binary relation symbol $\in$. The language consists of *formulas* built up in finitely many steps from the *atomic formulas* $x = y$ and $x \in y$, where x and y are members of a set of variables, using the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and the quantifiers $\forall v$ and $\exists v$, where v is a variable. We use Greek letters to denote formulas. The variables that appear in a formula φ outside the scope of a quantifier are called *free*. The notation $\varphi(x_1, \ldots, x_n)$ means that $x_1, \ldots, x_n$ are the free variables in φ. A formula without free variables is called a *sentence*.

All axioms of ZFC can be formalized in the language of set theory. A *model* of ZFC is a set or a proper class M in which the formalized axioms of ZFC are true when the binary relation symbol $\in$ is interpreted as the membership relation. A model M is called *inner* if it is transitive and contains all the ordinals. Thus, inner models are not sets, but proper classes. Given a model M and a formula $\varphi(x_1, \ldots, x_n)$, and given an n-tuple $a_1, \ldots, a_n$ of elements of M, we say that $\varphi(a_1, \ldots, a_n)$ *is satisfied in* M if the formula is true in M when x_i is replaced by a_i for each $1 \leq i \leq n$ and all the quantifiers range over M.

For a model M, we say that a set or a proper class C is *definable in M* if there is a formula $\varphi(x, x_1, \ldots, x_n)$ of the language of set theory and elements $a_1, \ldots, a_n$ in M such that C is the class of elements $c \in M$ such that $\varphi(c, a_1, \ldots, a_n)$ is satisfied in M. We then say that C is *defined by φ in M with parameters* $a_1, \ldots, a_n$. Every set $a \in M$ is definable in M with a as a parameter, namely by the formula $x \in a$.

A formula $\varphi(x, x_1, \ldots, x_n)$ is *absolute between two models* $N \subseteq M$ with respect to a collection of parameters $a_1, \ldots, a_n$ in N if, for each $c \in N$, $\varphi(c, a_1, \ldots, a_n)$ is satisfied in N if and only if it is satisfied in M. For example, formulas in which all quantifiers are bounded (that is, of the form $\exists x \in a$ or $\forall x \in a$) are absolute between any two transitive models. A formula is called *absolute* with respect to $a_1, \ldots, a_n$ if it is absolute between any inner model M that contains $a_1, \ldots, a_n$ and the universe V. We call a set or a proper class X *absolute* if membership of X is defined by an absolute formula with respect to some parameters.

A submodel N of a model M is *elementary* if all formulas are absolute between N and M with respect to every set of parameters in N. An embedding of V into a model M is an *elementary embedding* if its image is an elementary submodel of M. If $j \colon V \to M$ is a nontrivial elementary embedding with M transitive, then M is inner, and induction on rank shows that there is a least ordinal κ moved by j, that is, $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. Such a κ is called the *critical point* of j, and is necessarily a measurable cardinal; see [8, Lemma 28.5].

For a set X and a cardinal κ, let $\mathcal{P}_\kappa(X)$ be the set of subsets of X of cardinality less than κ. A cardinal κ is called *λ-supercompact*, where λ is an ordinal, if the set $\mathcal{P}_\kappa(\lambda)$ admits a normal measure [8]. A cardinal κ is *supercompact* if it is λ-supercompact for every ordinal λ. Instead of recalling the definition of a normal measure, we recall from [8, Lemma 33.9] that a cardinal κ is λ-supercompact if and only if there is an elementary embedding $j \colon V \to M$ such that $j(\alpha) = \alpha$ for all $\alpha < \kappa$ and $j(\kappa) > \lambda$, where M is an inner model such that $\{f \mid f \colon \lambda \to M\} \subseteq M$, i.e., every λ-sequence of elements of M is an element of M. For more information on supercompact cardinals, see [9] or [10].

If $j \colon V \to M$ is an elementary embedding, then for every set X the *restriction* $j \restriction X \colon X \to j(X)$ is the function that sends each element $x \in X$ to $j(x)$. The statement that $j \restriction X \colon X \to j(X)$ is in M means that the set $\{(x, j(x)) \mid x \in X\}$ is an element of M.

Proposition 3.1. *A cardinal κ is supercompact if and only if for every set X there is an elementary embedding j of the universe V into an inner model M with critical point κ, such that $X \in M$, $j(\kappa) > \operatorname{rank}(X)$, and $j \restriction X \colon X \to j(X)$ is in M.*

Proof. Given any set X, let λ be the cardinality of the transitive closure of the set $\{X\}$, and consider the binary relation R on λ that corresponds to the membership relation on this transitive closure, that is, $(\mathrm{TC}(\{X\}), \in)$ and (λ, R) are isomorphic. By [9, (3.12)], the binary relation R embeds into λ. Therefore, the set X is encoded by a λ-sequence of ordinals. Now choose an elementary embedding $j \colon V \to M$ with M transitive and critical point κ, such that $j(\kappa) > \lambda$ and M contains all the λ-sequences of its elements.

From the latter it follows that $X \in M$. Finally, we use the fact that the restriction $j \restriction \lambda$ is in M if and only if $\{f \mid f\colon \lambda \to M\} \subseteq M$; see [10, Proposition 22.4]. $\square$

Infinitary languages allow infinite formulas. We recall the definitions that we need for this article, following [2, Chapter 5]. For a set S and a regular cardinal λ, a *λ-ary S-sorted signature* Σ consists of a set of *operation symbols*, each of which has a certain *arity* $\prod_{i\in I} s_i \to s$, where s and all s_i are in S and $|I| < \lambda$, and another set of *relation symbols*, each of which has also a certain arity of the form $\prod_{j\in J} s_j$, where all s_j are in S and $|J| < \lambda$. Given a signature Σ, a *Σ-structure* is a set $X = \{X_s \mid s \in S\}$ of nonempty sets together with a function

$$\sigma_X \colon \prod_{i\in I} X_{s_i} \longrightarrow X_s$$

for each operation symbol $\sigma\colon \prod_{i\in I} s_i \to s$, and a subset $\rho_X \subseteq \prod_{j\in J} X_{s_j}$ for each relation symbol ρ of arity $\prod_{j\in J} s_j$. A *homomorphism* of Σ-structures is a set $f = \{f_s \mid s \in S\}$ of functions preserving operations and relations. The category of Σ-structures and their homomorphisms is denoted by $\mathbf{Str}\,\Sigma$.

Given a λ-ary S-sorted signature Σ and a set $W = \{W_s \mid s \in S\}$ of sets of cardinality λ, where the elements of W_s are called *variables of sort s*, one defines the infinitary language L_λ corresponding to Σ as follows. *Terms* are defined by declaring that each variable is a term and, for each operation symbol $\sigma\colon \prod_{i\in I} s_i \to s$ and each collection of terms τ_i of sort s_i, the expression $\sigma(\tau_i)_{i\in I}$ is a term of sort s. *Formulas* are built up by means of logical connectives (allowing conjunctions and disjunctions of formulas indexed by sets of cardinality less than λ) and quantifiers (allowing quantification over arbitrary sets of less than λ variables) from the *atomic formulas* $\tau_1 = \tau_2$ and $\rho(\tau_j)_{j\in J}$, where ρ is a relation symbol and each τ_j is a term.

As in the finitary case, variables which appear unquantified in a formula are said to appear free, and a formula without free variables is called a sentence. A set of sentences is called a *theory* (with signature Σ). A *model* of a theory T is a Σ-structure satisfying each sentence of T. For each theory T, we denote by $\mathbf{Mod}\,T$ the full subcategory of all models of T in $\mathbf{Str}\,\Sigma$.

A formula in L_λ is called *basic* if it has the form $\forall x(\varphi(x) \to \psi(x))$, where φ and ψ are disjunctions of (less than λ) formulas of type $\exists y\ \zeta(x, y)$ in which ζ is a conjunction of (less than λ) atomic formulas. A *basic theory* is a theory of basic sentences.

4. Main results

In this section, categories will be equipped with an embedding into the category of sets. Hence, we will consider pairs $(\mathcal{C}, E)$ where $\mathcal{C}$ is a category and $E\colon \mathcal{C} \to \mathbf{Set}$ is a faithful functor (so $\mathcal{C}$ becomes *concrete* in the sense of [1]), and we assume in addition that E is injective on objects. Under these assumptions, every morphism $f \in \mathcal{C}(X, Y)$ has an *underlying function* $Ef\colon EX \to EY$, preserving composition and identities, and each function in the image of E underlies a unique morphism. From now on, the embedding E will be omitted from the notation.

Not every category can be embedded into the category of sets. For example, the homotopy category of topological spaces cannot be made concrete, as shown in [7].

If Σ is any signature, then the category of Σ-structures embeds canonically into sets. In what follows, we will implicitly assume that each subcategory of $\mathbf{Str}\,\Sigma$ is equipped with its canonical embedding.

As explained in [2], every accessible category $\mathcal{C}$ can be embedded into a category of relational structures (hence into sets) as follows. If $\mathcal{C}$ is λ-accessible for a regular cardinal λ, then there are full embeddings

$$\mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{A}} \longrightarrow \mathbf{Str}\,\Sigma, \tag{4.1}$$

where $\mathcal{A}$ is the opposite of the full subcategory of $\mathcal{C}$ having as objects a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$, and $\mathbf{Set}^{\mathcal{A}}$ denotes the category of functors $\mathcal{A} \to \mathbf{Set}$. The embedding of $\mathcal{C}$ into $\mathbf{Set}^{\mathcal{A}}$ is of Yoneda type; the fact that it is full is proved in [2, Proposition 2.8]. The signature Σ is chosen by picking the objects of $\mathcal{A}$ as sorts and the morphisms of $\mathcal{A}$ as relation symbols. The full embedding of $\mathbf{Set}^{\mathcal{A}}$ into $\mathbf{Str}\,\Sigma$ is described in [2, Example 1.41].

Definition 4.1. A category $\mathcal{C}$ equipped with an embedding into sets is called *absolute* if there is a formula $\varphi(x, x_1, \ldots, x_n)$ in the first-order language of set theory which is absolute with respect to some set of parameters $a_1, \ldots, a_n$ and such that, for any two sets A, B and any function $f\colon A \to B$, the sentence $\varphi(f, a_1, \ldots, a_n)$ is satisfied if and only if the function f underlies a morphism of $\mathcal{C}$.

We will say, for shortness, that a formula φ *defines* $\mathcal{C}$ if membership of a function in the class of morphisms of $\mathcal{C}$ is defined by φ. Hence, absolute categories are those defined by absolute formulas.

Note that a set X is in the class of objects of $\mathcal{C}$ if and only if the identity function $X \to X$ is in the class of morphisms. Hence, if $\mathcal{C}$ is defined by an absolute formula, then the class of objects of $\mathcal{C}$ also admits an absolute definition.

For example, the category of groups is absolute (without parameters) if we view it as the subcategory of sets whose objects are quadruples (G, μ, ι, e) where G is a set, $\mu\colon G \times G \to G$ is an associative operation, e is a neutral element, and $\iota\colon G \to G$ picks an inverse of each element, and whose morphisms are functions $G \to G$ preserving μ. The category of modules over a ring R is absolute in a similar way, with R as a parameter.

More generally, every category $\mathbf{Mod}\,T$ of models over a theory T is absolute. This can be seen as follows. For a given λ-ary S-sorted signature Σ, an object of $\mathbf{Mod}\,T$ is a Σ-structure in which all sentences of T are satisfied. The class of Σ-structures is defined by an absolute formula with Σ as a parameter, while the satisfaction of a sentence $\psi \in T$ by a Σ-structure X is defined recursively and depends solely on the transitive closure of $\{X\}$. Thus, if $M \subseteq N$ are transitive models of ZFC that contain X, then X satisfies ψ in M if and only if X satisfies ψ in N. Hence, objects of $\mathbf{Mod}\,T$ are defined by an absolute formula with Σ and T as parameters. Similarly, $f\colon X \to Y$ being a homomorphism depends solely on the transitive closure of $\{f, X, Y\}$, so we may argue in the same way with homomorphisms.

By [2, Theorem 5.35], if a category $\mathcal{C}$ is accessible, then the image of the full embedding (4.1) is precisely the category of models of a suitable basic theory. This proves that *all accessible categories are absolute*, assuming, as we do, that they embed into the category of sets by means of (4.1).

Definition 4.2. We say that a category $\mathcal{C}$ equipped with an embedding into sets *supports elementary embeddings* if, for every elementary embedding $j\colon V \to M$ and all objects X of $\mathcal{C}$, the restriction $j \restriction X\colon X \to j(X)$ underlies a morphism of $\mathcal{C}$.

Note that $j \restriction X\colon X \to j(X)$ is always injective, since $j(x) = j(y)$ implies that $x = y$. Hence, if $\mathcal{C}$ supports elementary embeddings, then $j \restriction X$ is a monomorphism in $\mathcal{C}$ for all X. (In a concrete category, every morphism whose underlying function is injective is a monomorphism; see [1, Proposition 7.37]. However, the converse need not be true.)

Proposition 4.3. *If $\mathcal{C}$ is an absolute full subcategory of* $\mathbf{Str}\,\Sigma$ *for some signature Σ, then $\mathcal{C}$ supports elementary embeddings.*

Proof. We first prove that $\mathbf{Str}\,\Sigma$ itself supports elementary embeddings. If X is a Σ-structure, then the set $j(X)$ admits operations and relations defined as $\sigma_{j(X)} = j(\sigma_X)$ for every operation symbol σ of Σ, and $\rho_{j(X)} = j(\rho_X)$ for every relation symbol ρ. Thus, $j(X)$ becomes a Σ-structure in such a way that $j \restriction X\colon X \to j(X)$ is a homomorphism of Σ-structures.

Now let $\mathcal{C}$ be an absolute full subcategory of $\mathbf{Str}\,\Sigma$. If X is an object in $\mathcal{C}$ then $j(X)$, viewed as a Σ-structure as in the previous paragraph, is also an object of $\mathcal{C}$ since $\mathcal{C}$ is assumed to be absolute, and the function $j \restriction X$ is automatically a homomorphism of Σ-structures. Since $\mathcal{C}$ is assumed to be full, $j \restriction X$ is a morphism in $\mathcal{C}$. □

Therefore, by [2, Theorem 5.35], accessible categories support elementary embeddings. It is however not true that every absolute category supports elementary embeddings. For example, let $\mathcal{C}$ be the category whose class of objects is the class V of all sets and whose morphisms are defined by $\mathcal{C}(X, Y) = \emptyset$ if $X \neq Y$ and $\mathcal{C}(X, X) = \{\mathrm{id}_X\}$ for all X. Then $\mathcal{C}$ does not support elementary embeddings.

Theorem 4.4. *Let $\mathcal{C}$ be a category equipped with an embedding into sets, and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$. Suppose that $\mathcal{C}$ supports elementary embeddings and both $\mathcal{C}$ and $\mathcal{A}$ can be defined by absolute formulas whose parameters have rank smaller than a supercompact cardinal κ. If $X \in \mathcal{A}$, then there is a subobject of X in $V_\kappa \cap \mathcal{A}$.*

Proof. Let φ be an absolute formula defining $\mathcal{C}$ with parameters $a_1, \dots, a_n$ of rank less than κ, and let ψ be an absolute formula defining $\mathcal{A}$ with parameters $b_1, \dots, b_m$ of rank less than κ. Fix an object $X \in \mathcal{A}$ and let $j\colon V \to M$, with M transitive, be an elementary embedding with critical point κ such that X and the restriction $j \restriction X$ are in M, and $j(\kappa) > \mathrm{rank}(X)$. Notice that $a_1, \dots, a_n$ and $b_1, \dots, b_m$ are also in M, since in fact $j(a_r) = a_r$ for all r and $j(b_s) = b_s$ for all s. Let us write $\vec{a}$ for $a_1, \dots, a_n$ and $\vec{b}$ for $b_1, \dots, b_m$.

Since the category $\mathcal{C}$ supports elementary embeddings, the restriction $j \restriction X\colon X \to j(X)$ underlies a monomorphism F in $\mathcal{C}$. The assumption that

φ and ψ are absolute formulas guarantees that $\varphi(F, \vec{a})$ and $\psi(X, \vec{b})$ hold in M. Hence, in M, $j(X)$ has a subobject (namely X) which satisfies ψ and has rank less than $j(\kappa)$. Therefore the following sentence with the parameters X, $\vec{a}$, $\vec{b}$, κ is true in M:

$$\exists y \, \exists f \, (f \colon y \to j(X) \wedge (f \text{ is injective}) \wedge \varphi(f, \vec{a}) \wedge \psi(y, \vec{b}) \wedge \operatorname{rank}(y) < j(\kappa)).$$

As j is an elementary embedding, the following holds in V:

$$\exists y \, \exists f \, (f \colon y \to X \wedge (f \text{ is injective}) \wedge \varphi(f, \vec{a}) \wedge \psi(y, \vec{b}) \wedge \operatorname{rank}(y) < \kappa).$$

Since morphisms whose underlying function is injective are monomorphisms, this says that X has a subobject in $V_\kappa \cap \mathcal{A}$, which proves the theorem. □

Corollary 4.5. *Let (L, η) be an extremal epireflection on a subcategory $\mathcal{C}$ of sets which supports elementary embeddings. Suppose that both $\mathcal{C}$ and the class of L-local objects can be defined by absolute formulas with parameters whose rank is smaller than a supercompact cardinal κ. Then L is an $\mathcal{F}$-localization for some set $\mathcal{F}$ of morphisms.*

Proof. Let the class of objects of $\mathcal{C}$ that are not L-local play the role of the class $\mathcal{A}$ in Theorem 4.4. Then the conclusion of the theorem is precisely that the set V_κ is transverse to the class of objects of $\mathcal{C}$ that are not L-local. Hence, part (a) of Theorem 2.6 and Remark 2.7 yield the desired result. □

Recall that, if $\mathcal{C}$ is balanced, then every epireflection is extremal. Recall also that every accessible category is absolute and supports elementary embeddings. Hence, the statement of Corollary 4.5 holds for arbitrary epireflections in balanced accessible categories. Moreover, if we assume that $\mathcal{C}$ has coproducts and $\mathcal{C}(X, Y)$ is nonempty for all X and Y, then we may infer, in addition to the conclusion of Corollary 4.5, that L is an f-localization for a single morphism f, which can be chosen to be an epimorphism by Proposition 2.3.

As an application, we give the following result. For any class of groups $\mathcal{A}$, the *reduction* $P_{\mathcal{A}}$ is an epireflection on the category of groups whose local objects are groups G that are $\mathcal{A}$-*reduced*, i.e., for which every homomorphism $A \to G$ is trivial if $A \in \mathcal{A}$. Such an epireflection exists by Proposition 2.2, since the class of $\mathcal{A}$-reduced groups is closed under products and subgroups.

Corollary 4.6. *Let $\mathcal{A}$ be any absolute class of groups. If there is a supercompact cardinal greater than the ranks of the parameters in an absolute formula defining $\mathcal{A}$, then there is a group G such that the class of G-reduced groups coincides with the class of $\mathcal{A}$-reduced groups.*

Proof. The category of groups is balanced and locally presentable. Hence, Corollary 4.5 implies that the reduction functor $P_{\mathcal{A}}$ is an f-localization for some group homomorphism f. As in [12, §3], let G be a universal f-acyclic group, i.e., a group G such that P_G and $P_{\mathcal{A}}$ annihilate the same groups. Then P_G and $P_{\mathcal{A}}$ also have the same class of local objects; that is, the class of G-reduced groups coincides with the class of $\mathcal{A}$-reduced groups. □

For the (non-absolute) class $\mathcal{A}$ of groups of the form $\mathbb{Z}^\kappa / \mathbb{Z}^{<\kappa}$ for all cardinals κ, which was mentioned in the Introduction, the existence of a group G such that the class of G-reduced groups coincides with the class of $\mathcal{A}$-reduced groups is equivalent to the existence of a measurable cardinal; see [3] or [6].

5. A counterexample

We will display an example, indicated to us by Rosický, of an extremal epireflection L on the category **Gra** of graphs which is not an $\mathcal{F}$-localization for any set of maps $\mathcal{F}$. This example is based on [2, Example 6.12] and requires to assume the negation of Vopěnka's principle. As already pointed out in the Introduction, Vopěnka's principle is a stronger set-theoretical assumption than the existence of a supercompact cardinal. Indeed, if Vopěnka's principle holds, then there exists a proper class of supercompact cardinals; see [9, Theorem 20.24 and Lemma 20.25]. Hence, if κ is the least supercompact cardinal and λ is the least inaccessible cardinal greater than κ, then V_λ is a model of ZFC in which κ is supercompact and Vopěnka's principle fails.

Thus, let us assume that Vopěnka's principle does not hold and therefore we may choose a proper class of graphs $\mathcal{A}$ which is *rigid*, that is, such that $\mathbf{Gra}(A,B) = \emptyset$ for all $A \neq B$ in $\mathcal{A}$, and $\mathbf{Gra}(A,A)$ has the identity as its only element for every $A \in \mathcal{A}$. Consider the class $\mathcal{L}$ of graphs that are $\mathcal{A}$-*reduced*, i.e.,

$$\mathcal{L} = \{X \in \mathbf{Gra} \mid \mathbf{Gra}(A,X) = \emptyset \text{ for all } A \in \mathcal{A}\},$$

and note that $\mathcal{A} \cap \mathcal{L} = \emptyset$, while every proper subgraph of a graph in $\mathcal{A}$ is in $\mathcal{L}$. By Proposition 2.2, there is an epireflection L whose class of local objects is precisely $\mathcal{L}$, since $\mathcal{L}$ is closed under products and subobjects in the category of graphs. Moreover, the unit map $\eta_X \colon X \to LX$ is an extremal epimorphism (indeed, surjective on vertices and edges) for all X.

Now suppose that there is a set $\mathcal{F} = \{f_i \colon P_i \to Q_i \mid i \in I\}$ of maps of graphs such that L is an $\mathcal{F}$-localization. Then, if we choose any regular cardinal λ that is bigger than the cardinalities of P_i and Q_i for all $i \in I$, it follows that $\mathcal{L}$ is closed under λ-directed colimits. As in [2, Example 6.12], a contradiction is obtained by choosing a graph $A \in \mathcal{A}$ whose cardinality is bigger than λ, and observing that A is a λ-directed colimit of the diagram of all its proper subgraphs, each of which is in $\mathcal{L}$, while A itself is not in $\mathcal{L}$. This contradicts the fact that $\mathcal{L}$ is closed under λ-directed colimits.

The class $\mathcal{L}$ considered in this example cannot be absolute, since otherwise we would contradict Corollary 4.5 by assuming the existence of a supercompact cardinal above the ranks of the parameters in an absolute formula defining $\mathcal{L}$.

In fact, Theorem 4.4 implies that, if a supercompact cardinal κ exists, then there is no rigid proper class of graphs defined by an absolute formula with parameters of rank smaller than κ. To prove this claim, suppose that such a class $\mathcal{A}$ exists. Then it follows from Theorem 4.4 that each graph in $\mathcal{A}$ has a subgraph in $\mathcal{A}$ of rank less than κ. This contradicts rigidity.

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Joan Bagaria, ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Lògica, Història i Filosofia de la Ciència, Universitat de Barcelona. Montalegre 6, 08001 Barcelona, Spain.
`joan.bagaria@icrea.es`

Carles Casacuberta, Departament d'Àlgebra i Geometria, Universitat de Barcelona. Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain.
`carles.casacuberta@ub.edu`

Adrian R. D. Mathias, ERMIT, Université de la Réunion.
Avenue René Cassin 15, F-97715 Saint Denis de la Réunion, France.
`ardm@dpmms.cam.ac.uk`