Similar Constructing Method for Solving the Boundary Value Problem of the Compound Kummer Equation

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Abstract This article is devoted to analyze the compound Kummer equation. Through the structure of left region and right region similarity kernel function, have similar structure of solution for this boundary value problems. The left field solution of this kind of boundary value problems can be obtained by combing the coefficients of boundary value problems on the left and the left area similar kernel function. The right field solution of this kind of boundary value problems can be expressed by assembling the left, the right area similar kernel function, the left lead solution functions and the coefficients of the convergence conditions. Thus put forward the simple method which was named similar constructive method to solve the boundary value problems of the compound Kummer equation. The put forward of this method provides the convenience for solving this kind of differential equation boundary value problems.

Keywords: boundary value problem, similar structure, similar kernel function, similar constructive method, compound Kummer equation

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1. Introduction

The boundary value problems of differential equations always are needed in the calculation of many physical and engineering [1,2,3,4,5]. So it is not just the initial value problems of differential equation, boundary value problems are also the focus of our study. Compound boundary value problem of Kummer equation is a mathematical problem that closely associated with the practical application. But when in solution of the compound boundary value problem of Kummer equation, we often encounter the complicated solving processe. So it is inconvenience for the practical application. Some boundary value problems of differential equations can be written in the form of solution of continued fractions which was named similar structure type. And this conclusion in 2004 has been proved [6,7,8,9,10]. For example, the boundary value problem of Kummer equation, the boundary value problem of Euler hypergeometric equation. Then the solution of the boundary value problem for composite Kummer equations whether it has the above properties?

This article will study the following compound confluence hypergeometric equation boundary value problem:

$$\begin{cases} xy_1{''} + (\gamma_1 - x)y_1{'} - \alpha_1 y_1 = 0 \,(a < x < c) \\ xy_2{''} + (\gamma_2 - x)y_2{'} - \alpha_2 y_2 = 0 \,(c < x < b) \\ [Ey_1 + (1+EF)y_1{'}]_{x=a} = D \\ y_1 \big|_{x=c} = my_2 \big|_{x=c} \\ y{'}_1 \big|_{x=c} = ny{'}_2 \big|_{x=c} \\ [Gy_2 + Hy{'}_2]_{x=b} = 0 \end{cases} \tag{1}$$

Where $\alpha_1, \alpha_2, \gamma_1, \gamma_2, D, E, R, G, H, a, b, c, m, n$ are all known real constant, and γ_1, γ_2 are not as integers, $D \neq 0, G^2 + H^2 \neq 0, a < c < b$.

2. Preliminary Knowledge

lemma 1 [11] The general solution for $xy{''} + (\gamma - x)y{'} - \alpha y = 0$ is

$$y = AF(\alpha, \gamma, x) + Bx^{1-\gamma}F(\alpha+1-\gamma, 2-\gamma, x) \tag{2}$$

Where γ is not an integer, both A and B are real constants and

$$F(\alpha, \gamma, x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\gamma)_k} x^k$$

lemma 2 Introducing two binary functions

$$\begin{aligned} \phi_{0,0}^i&(x,\xi) \\ &= \xi^{1-\gamma_i}F(\alpha_i, \gamma_i, x)F(\alpha_i+1-\gamma_i, 2-\gamma_i, \xi) \\ &- x^{1-\gamma_i}F(\alpha_i+1-\gamma_i, 2-\gamma_i, x)F(\alpha_i, \gamma_i, \xi) \\ &\quad (i = 1, 2) \end{aligned} \tag{3}$$

then

$$\phi_{1,0}^i(x,\xi) = \frac{\partial \phi_{0,0}^i(x,\xi)}{\partial x}$$

$$= \frac{\alpha_i}{\gamma_i} \xi^{1-\gamma_i} F(\alpha_i+1,\gamma_i+1,x) F(\alpha_i+1-\gamma_i,2-\gamma_i,\xi) \quad (4)$$

$$- (1-\gamma_i) x^{-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,x) F(\alpha_i,\gamma_i,\xi)$$

$$- \frac{\alpha_i+1-\gamma_i}{2-\gamma_i} x^{1-\gamma_i} F(\alpha_i+2-\gamma_i,3-\gamma_i,x) F(\alpha_i,\gamma_i,\xi)$$

$$\phi_{0,1}^i(x,\xi) = \frac{\partial \phi_{0,0}^i(x,\xi)}{\partial \xi}$$

$$= (1-\gamma_i) \xi^{-\gamma_i} F(\alpha_i,\gamma_i,x) F(\alpha_i+1-\gamma_i,2-\gamma_i,\xi)$$

$$+ \frac{\alpha_i+1-\gamma_i}{2-\gamma_i} \xi^{1-\gamma_i} F(\alpha_i,\gamma_i,x) F(\alpha_i+2-\gamma_i,3-\gamma_i,\xi) \quad (5)$$

$$- \frac{\alpha_i}{\gamma_i} x^{1-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,x) F(\alpha_i+1,\gamma_i+1,\xi)$$

$$\phi_{1,1}^i(x,\xi) = \frac{\partial \phi_{0,1}^i(x,\xi)}{\partial x}$$

$$= \frac{\alpha_i}{\gamma_i}(1-\gamma_i) \xi^{-\gamma_i} F(\alpha_i+1,\gamma_i+1,x)$$

$$\times F(\alpha_i+1-\gamma_i,2-\gamma_i,\xi)$$

$$+ \frac{\alpha_i}{\gamma_i} \cdot \frac{\alpha_i+1-\gamma_i}{2-\gamma_i} \xi^{1-\gamma_i} F(\alpha_i+1,\gamma_i+1,x)$$

$$\times F(\alpha_i+2-\gamma_i,3-\gamma_i,\xi) \quad (6)$$

$$- \frac{\alpha_i}{\gamma_i}(1-\gamma_i) x^{-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,x)$$

$$\times F(\alpha_i+1,\gamma_i+1,\xi)$$

$$- \frac{\alpha_i}{\gamma_i} \cdot \frac{\alpha_i+1-\gamma_i}{2-\gamma_i} x^{1-\gamma_i} F(\alpha_i+2-\gamma_i,3-\gamma_i,x)$$

$$\times F(\alpha_i+1,\gamma_i+1,\xi)$$

Proof: Based on the differential properties of the Kummer function we can know that

$$\frac{d}{dx} F(\alpha,\gamma,x) = \frac{\alpha}{\gamma} F(\alpha+1,\gamma+1,x) \quad (7)$$

so

$$\phi_{1,0}^i(x,\xi) = \frac{\partial \phi_{0,0}^i(x,\xi)}{\partial x}$$

$$= \xi^{1-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,\xi) \frac{\partial}{\partial x} F(\alpha_i,\gamma_i,x)$$

$$- (1-\gamma_i) x^{-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,x) F(\alpha_i,\gamma_i,\xi)$$

$$- x^{1-\gamma_i} F(\alpha_i,\gamma_i,\xi) \frac{\partial}{\partial x} F(\alpha_i+1-\gamma_i,2-\gamma_i,x)$$

$$= \frac{\alpha_i}{\gamma_i} \xi^{1-\gamma_i} F(\alpha_i+1,\gamma_i+1,x) F(\alpha_i+1-\gamma_i,2-\gamma_i,\xi)$$

$$- (1-\gamma_i) x^{-\gamma_i} F(\alpha_i+1-\gamma_i,2-\gamma_i,x) F(\alpha_i,\gamma_i,\xi)$$

$$- x^{1-\gamma_i} \frac{\alpha_i+1-\gamma_i}{2-\gamma_i} F(\alpha_i+2-\gamma_i,3-\gamma_i,x) F(\alpha_i,\gamma_i,\xi)$$

Thus: Similarly verifiable (5) and (6).

3. The Main Theorem and Its Proof

Theorem If boundary value problem (1) has a unique solution, in that way the left area ($a < x < c$) is:

$$y_1 = D \cdot \frac{1}{E + \dfrac{1}{R+\varphi(a)}} \cdot \frac{1}{R+\varphi(a)} \cdot \varphi(x) \quad (8)$$

The right area ($c < x < b$) solution is:

$$y_2 = D \cdot \frac{1}{E + \dfrac{1}{R+\varphi(a)}} \cdot \frac{1}{R+\varphi(a)}$$

$$\cdot \frac{\phi_{0,1}^1(c,c)}{m\varphi^*(c)\phi_{1,1}^1(a,c) - n\phi_{1,0}^1(c,b)} \cdot \varphi^*(x) \quad (9)$$

Where $\varphi^*(x)$ is defined as similar kernel function of the right area :

$$\varphi^*(x) = \frac{G\phi_{0,0}^2(x,b) + H\phi_{0,1}^2(x,b)}{G\phi_{1,0}^2(c,b) + H\phi_{1,1}^2(c,b)} (c < x < b) \quad (10)$$

Where $\varphi(x)$ is defined as similar kernel function of the left area :

$$\varphi(x) = \frac{m\varphi^*(c)\ \phi_{0,1}^1(x,c) - n\phi_{0,0}^1(x,c)}{m\varphi^*(c)\ \phi_{1,1}^1(a,c) - n\phi_{1,0}^1(a,c)} (a < x < c) \quad (11)$$

Proof: First of all, Based on the lemma 1 we know that the general solution for

$$xy_1'' + (\gamma_1 - x)y_1' - \alpha_1 y_1 = 0 (a < x < c) \quad (12)$$

$$xy_2'' + (\gamma_2 - x)y_2' - \alpha_2 y_2 = 0 (c < x < b) \quad (13)$$

are as follows

$$y_1 = A_1 F(\alpha_1,\gamma_1,x) + B_1 x^{1-\gamma_1} F(\alpha_1+1-\gamma_1,2-\gamma_1,x) \quad (14)$$

$$y_2 = A_2 F(\alpha_2,\gamma_2,x) + B_2 x^{1-\gamma_2} F(\alpha_2+1-\gamma_2,2-\gamma_2,x) \quad (15)$$

Where A_1, B_1, A_2, B_2 are real constants.

According the left boundary conditions $[Ey_1 + (1+ER)y_1']_{x=a} = D$, we can know that

$$A_1[EF(\alpha_1,\gamma_1,a) + (1+ER)\frac{\alpha_1}{\gamma_1}F(\alpha_1+1,\gamma_1+1,a)]$$

$$+ B_1[Ea^{1-\gamma_1}F(\alpha_1+1-\gamma_1,2-\gamma_1,a)$$

$$+ (1+ER)(1-\gamma_1)a^{-\gamma_1}F(\alpha_1+1-\gamma_1,2-\gamma_1,a) \quad (16)$$

$$+ (1+ER)a^{1-\gamma_1}\frac{\alpha_1+1-\gamma_1}{2-\gamma_1}F(\alpha_1+2-\gamma_1,3-\gamma_1,a)] = D$$

According to the continuity conditions $y_1|_{x=c} = my_2|_{x=c}$, $y_1'|_{x=c} = ny_2'|_{x=c}$ we can get

$$A_1 F(\alpha_1,\gamma_1,c) + B_1 c^{1-\gamma_1} F(\alpha_1+1-\gamma_1,2-\gamma_1,c)$$

$$- mA_2 F(\alpha_2,\gamma_2,c) - mB_2 c^{1-\gamma_2} F(\alpha_2+1-\gamma_2,2-\gamma_2,c) = 0 \quad (17)$$

$$A_1 \frac{\alpha_1}{\gamma_1} F(\alpha_1 + 1, \gamma_1 + 1, c) - nA_2 \frac{\alpha_2}{\gamma_2} F(\alpha_2 + 1, \gamma_2 + 1, c)$$

$$+ B_1 [(1 - \gamma_1) c^{-\gamma_1} F(\alpha_1 + 1 - \gamma_1, 2 - \gamma_1, c)$$

$$+ c^{1-\gamma_1} \frac{\alpha_1 + 1 - \gamma_1}{2 - \gamma_1} F(\alpha_1 + 2 - \gamma_1, 3 - \gamma_1, c)] \qquad (18)$$

$$- nB_2 [(1 - \gamma_2) c^{-\gamma_2} F(\alpha_2 + 1 - \gamma_2, 2 - \gamma_2, c)$$

$$+ c^{1-\gamma_2} \frac{\alpha_2 + 1 - \gamma_2}{2 - \gamma_2} F(\alpha_2 + 2 - \gamma_2, 3 - \gamma_2, c)] = 0$$

By the right boundary conditions of the complex confluence hypergeometric equation boundary value problem (1) that $[Gy_2 + Hy'_2]_{x=b} = 0$ we can get

$$A_2 [GF(\alpha_2, \gamma_2, b) + H \frac{\alpha_2}{\gamma_2} F(\alpha_2 + 1, \gamma_2 + 1, b)]$$

$$、 + B_2 [Gb^{1-\gamma_2} F(\alpha_2 + 1 - \gamma_2, 2 - \gamma_2, b)$$

$$+ H(1 - \gamma_2) b^{-\gamma_2} F(\alpha_2 + 1 - \gamma_2, 2 - \gamma_2, b) \qquad (19)$$

$$+ Hb^{1-\gamma_2} \frac{\alpha_2 + 1 - \gamma_2}{2 - \gamma_2} F(\alpha_2 + 2 - \gamma_2, 3 - \gamma_2, b)] = 0$$

Getting linear equations about of undetermined coefficients A_1, B_1, A_2 and B_2 by combining equation (16), (17), (18) and (19).

If we assume that complex confluence hypergeometric equation boundary value problem (1) has a unique solution, then there will be a determinant of coefficient of linear equations not be 0,mean that $\Delta \neq 0$ .And after calculation using (3), (4), (5), (6) for simplifying we can get

$$\Delta = m[G\phi_{0,0}^2(c,b) + H\phi_{0,1}^2(c,b)]$$

$$\cdot [E\phi_{0,1}^1(a,c) + (1 + ER)\phi_{1,1}^1(a,c)]$$

$$- n[G\phi_{1,0}^2(c,b) + H\phi_{1,1}^2(c,b)] \qquad (20)$$

$$\cdot [E\phi_{0,0}^1(a,c) + (1 + ER)\phi_{1,0}^1(a,c)]$$

Using Kramer rule to solve the equations set which is composed of (14), (15), (16) and (17) equations, we can get

$$A_1 = \frac{D}{\Delta} \{ m \ [G \ \phi_{0,0}^2(c,b) + H \ \phi_{0,1}^2(c,b)]$$

$$\cdot [(1 - \gamma_1) \ c^{-\gamma_1} F(\alpha_1 + 1 - \gamma_1, 2 - \gamma_1, c)$$

$$+ c^{1-\gamma_1} \frac{\alpha_1 + 1 - \gamma_1}{2 - \gamma_1} F(\alpha_1 + 2 - \gamma_1, 3 - \gamma_1, c)] \qquad (21)$$

$$- n \ [G\phi_{1,0}^2(c,b) + H\phi_{1,1}^2(c,b)]$$

$$c^{1-\gamma_1} F(\alpha_1 + 1 - \gamma_1, 2 - \gamma_1, c) \}$$

$$B_1 = -\frac{D}{\Delta} \{ m[G\phi_{0,0}^2(c,b) + H\phi_{0,1}^2(c,b)]$$

$$\cdot \frac{\alpha_1}{\gamma_1} F(\alpha_1 + 1, \gamma_1 + 1, c) \qquad (22)$$

$$- n[G\phi_{1,0}^2(c,b) + H\phi_{1,1}^2(c,b)] F(\alpha_1, \gamma_1, c) \}$$

$$A_2 = \frac{D}{\Delta} [Gb^{1-\gamma_2} F(\alpha_2 + 1 - \gamma_2, 2 - \gamma_2, b)$$

$$+ H(1 - \gamma_2) b^{-\gamma_2} F(\alpha_2 + 1 - \gamma_2, 2 - \gamma_2, b) \qquad (23)$$

$$+ Hb^{1-\gamma_2} \frac{\alpha_2 + 1 - \gamma_2}{2 - \gamma_2}$$

$$\cdot F(\alpha_2 + 2 - \gamma_2, 3 - \gamma_2, b)]\phi_{0,1}^1(c,c)$$

$$B_2 = -\frac{D}{\Delta} \begin{bmatrix} GF(\alpha_2, \gamma_2, b) \\ + H \frac{\alpha_2}{\gamma_2} F(\alpha_2 + 1, \gamma_2 + 1, b) \end{bmatrix} \phi_{0,1}^1(c,c) \quad (24)$$

Then y_1, y_2 can be obtained if we get A_1, B_1, A_2, B_2 substitution of(14),(15). And if we simplify y_1, y_2 using the similar kernel function of the left and the right area, we can get the solution of the left and the right area (8)and(9), respectively.

4. Give an Example

Draw the image of following boundary value problem solution:

$$\begin{cases} xy_1 " + (0.5 - x)y_1 ' - y_1 = 0 (1 < x < 2) \\ xy_2 " + (1.5 - x)y_2 ' - y_2 = 0 (2 < x < 3) \\ [y_1 + 2y_1 ']_{x=1} = 1 \\ y_1 |_{x=2} = y_2 |_{x=2} \\ y'_1 |_{x=2} = y'_2 |_{x=2} \\ [y_2 + y'_2]_{x=3} = 0 \end{cases} \qquad (25)$$

According to the preliminary knowledge we can get the general solution of the target equation as follows.

$$y_1 = A_1 F(1, 0.5, x) + B_1 x^{\frac{1}{2}} \ F(1.5, 1.5, x) \qquad (26)$$

$$y_2 = A_2 F(1, 1.5, x) + B_2 x^{\frac{-1}{2}} \ F(0.5, 0.5, x) \qquad (27)$$

Through the comparison of the toundary value problem (1) and the boundary value problem(25) we found that $\alpha_1 = 1, \alpha_2 = 1, \gamma_1 = 0.5, \gamma_2 = 1.5, D = 1, E = 1, F = 1,$ $G = 1, H = 1, a = 1, b = 3, c = 2, m = 1, n = 1$, using MATHEMATICA language program to draw the solution of the boundary value problem in the specified image definition domain as follows:
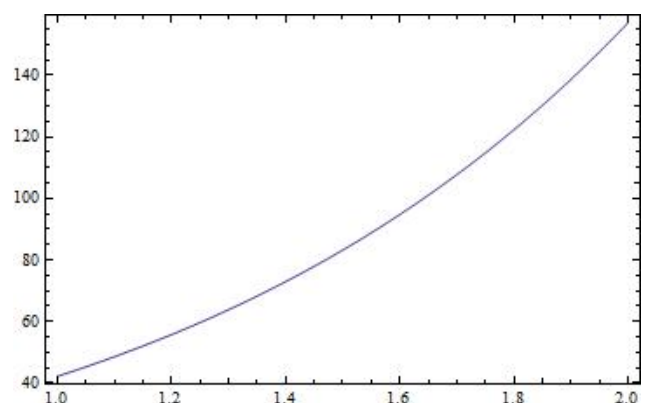

Figure 1. Cuver of solution to boundary value problem(25) (left area)
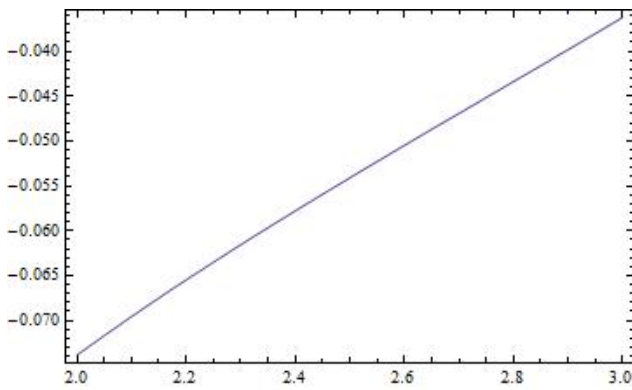
Figure 2. Cuver of solution to boundary value problem(25) (right area)

By changing the parameters of D and G in the condition of observation and analysis of the image change.
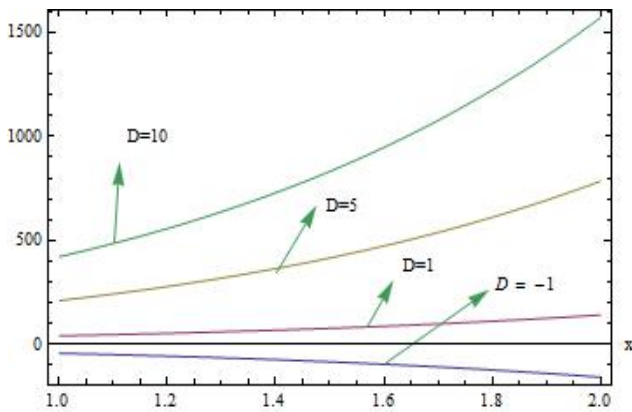

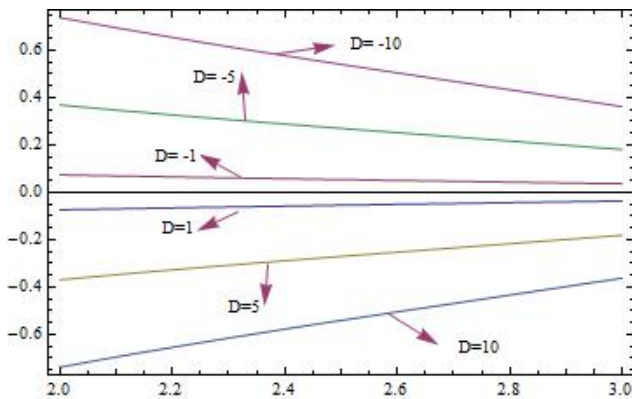**Figure 3.** The influence of parameter D on curve(right area)

Figure 4. The influence of parameter D on curve(right area)
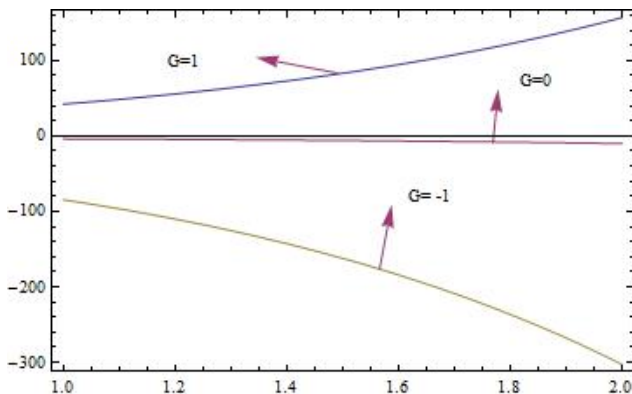

Figure 5. The influence of parameter G on curve(left area)
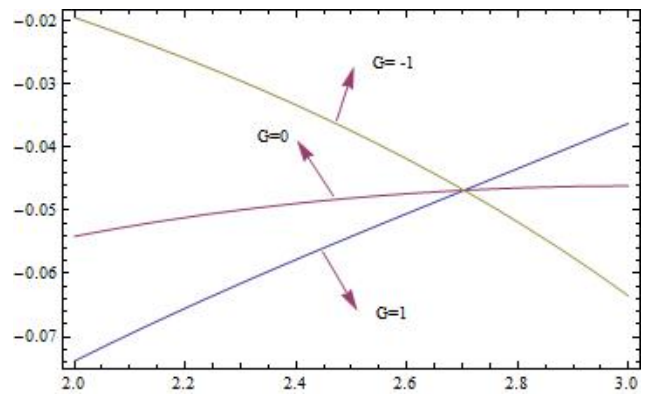

Figure 6. The influence of parameter G on curve(right area)

5. Conclusions and Understanding

1) With the definition of the left, the right area of similar kernel function (10) and (11), the solutions of boundary value problem (1) can be written as the continued fraction structure. In the progress of solving the boundary value problem for composite Kummer equation, we can get the left area and the right area solution directly by constructing similarity kernel function. This simplifies the solving process, improves the solution efficiency, contributes to the researchers compile corresponding analysis software.

2) From the Figure 3 and Figure 4 we can see that the greater absolute value of the parameter D the greater function curve fluctuations in the left area and the right area.

3) From the Figure 4 we can see that the image function on the X axis symmetry when D were obtained from the two opposite number.

4) From the Figure 5 we can see that the greater absolute value of the parameter G the greater function curve fluctuations in the left area.

5) From the Figure 5 we can see that in the right area regardless of how to change the parameters of the G image through a fixed point and the point coordinates for (2.71,-0.04748).

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