ON PERTURBATIONS OF QUASIPERIODIC SCHRÖDINGER OPERATORS

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Abstract. Using relative oscillation theory and the reducibility result of Eliasson, we study perturbations of quasiperiodic Schrödinger operators. In particular, we derive relative oscillation criteria and eigenvalue asymptotics for critical potentials.

1. Introduction

We will be interested in generalizing classical perturbation result of eigenvalues to quasiperiodic operators. We first overview the classical results of interest. Most (if not all) of our results will be parallel to these. For this introduction let H be a self-adjoint realization of

$$H = -\frac{d^2}{dx^2} + q(x) \tag{1.1}$$

on $L^2(1,\infty)$ with $q(x) \to 0$ as $x \to \infty$ and q bounded. A classical result of Weyl now tells us, that the essential spectrum of H, is equal to the one of $-\frac{d^2}{dx^2}$, hence $\sigma_{ess}(H) = [0,\infty)$. We give the generalization of this to quasiperiodic operators in Theorem 3.1.

Kneser answered in [9], the question when 0 is an accumulation point of eigenvalues below 0. One has if

$$\limsup_{x\to\infty} q(x)x^2 < -\frac{1}{4} \tag{1.2}$$

then 0 is an accumulation point of eigenvalues, and if

$$\liminf_{x\to\infty} q(x)x^2 > -\frac{1}{4} \tag{1.3}$$

then 0 is not an accumulation point of eigenvalues. The periodic case was answered by Rofe-Beketov (see here his recent monograph [8]). The generalization to the quasiperiodic case is given in Theorem 3.2.

Once it is known that 0 is an accumulation point of eigenvalues, it is natural to ask how fast do the eigenvalues converge to 0. This question was answered by Kirsch-Simon in [7]. To state their result let $N(\lambda)$ be the number of eigenvalues of $-\frac{d^2}{dx^2} + \frac{\mu}{x^2}$ below λ, then

$$N(\lambda) = \frac{1}{4\pi}\sqrt{\frac{\mu}{\mu_{crit}} - 1}|\ln|\lambda||(1+o(1)), \quad \lambda \uparrow 0, \quad \mu_{crit} = -\frac{1}{4}. \tag{1.4}$$

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For $-\frac{d^2}{dx^2} + \frac{\mu}{x^\gamma}$, $0 < \gamma < 2$, we have

$$N(\lambda) = \frac{1}{\pi}\int_{\{x,\, q(x)<\lambda\}} (\lambda - q(x))^{1/2} dx(1 + o(1)), \qquad \lambda \uparrow 0 \tag{1.5}$$
$$= \frac{\sqrt{\mu/\mu_{crit}}}{\pi} \frac{1}{2-\gamma} \left|\frac{\mu}{\lambda}\right|^{(2-\gamma)/2\gamma} (1 + o(1)), \qquad \lambda \uparrow 0$$

see Theorem XIII.82 in [14]. [1] This result goes back to results in the sixties, see the notes in [14]. The periodic case was answered by Schmidt [15] for $\gamma = 2$. We will answer this question in Theorem 3.7.

Periodic operators have a spectrum made out of the union of finitely or infinitely many bands. That is

$$\sigma_{ess}(-\frac{d^2}{dx^2} + q_0(x)) = [E_0, E_1] \cup [E_2, E_3] \cup \dots, \quad E_j < E_{j+1}, \tag{1.6}$$

for $q_0(x + p) = q_0(x)$, $p > 0$. Since, we now have several boundary points of the spectrum, one can also ask what happens at all, finitely many, ... boundary points of $\sigma_{ess}(H_0)$. Rofe-Beketov gave the following answer to this question: Only finitely many gaps can contain infinitely many eigenvalues for critical perturbations ($q(x) = \mu/x^2$) (see (6.145) in [8]). We will treat this question in Theorem 3.6.

The organization of this paper is as follows. In Section 2, we will state the needed results about quasiperiodic Schrödinger operators. In Section 3, we will state our main results. Most proofs are easy enough to be directly stated. Only the eigenvalue asymptotics requires more work and is stated in the following section. In Section 5, we give an outline of Eliasson's proof and derive some further estimates. In Appendix A, we will review relative oscillation theory, followed by another short appendix on needed methods from the theory of differential equations.

2. Quasiperiodic Operators

We will now recall the basic notations about quasiperiodic Schrödinger operators. Let $\mathbb{T}^d$ be the d-dimensional torus, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Let $Q : \mathbb{T}^d \to \mathbb{R}$ be a real analytic function. We will consider the Schrödinger operator on $L^2(1, \infty)$ given by

$$H_0 = -\frac{d^2}{dx^2} + q_0(x), \quad q_0(x) = Q(\omega x) \tag{2.1}$$

where $\omega \in \mathbb{T}^d$ is fixed. We will assume that ω is a Diophantine number, that is there is some $\tau > d - 1$, $\kappa > 0$, such that

$$DC(\kappa, \tau) : \quad |\langle\omega, n\rangle| \geq \frac{\kappa}{|n|^\tau}, \quad n \in \mathbb{Z}^d \backslash \{0\}, \tag{2.2}$$

holds.

Recall the rotation number $\rho(E)$ from [4]. Denote by $\vartheta(x, E)$ the Prüfer angle of a solution u of $H_0 u = Eu$. That is a continuous function of x such that

$$u(x) = r(x) \sin\vartheta(x, E), \quad u'(x) = r(x) \cos\vartheta(x, E), \quad 0 \leq \vartheta(1, E) < \pi, \tag{2.3}$$

for some continuous function r. The rotation number $\rho(E)$ is now introduced by

$$\rho(E) = \lim_{x\to\infty} \frac{\vartheta(x, E)}{x}. \tag{2.4}$$

[1] We obtain a factor $\frac{1}{2}$ different from [7] in the case $\gamma = 2$, since we are considering half line operators. This factor does not arise for $0 < \gamma < 2$, since the domain of integration also shrinks.

We remark that the integrated density of states $k(E)$ satisfies

$$k(E) = \frac{1}{\pi}\rho(E). \tag{2.5}$$

Johnson and Moser showed

Theorem 2.1. *[[4]] The spectrum $\sigma(H_0)$ is given by*

$$\sigma(H_0) = \{E,\, \rho(E) = \frac{1}{2}\langle\omega, n\rangle, \quad n \in \mathbb{Z}^d\}. \tag{2.6}$$

Furthermore ρ is a continuous function, and constant outside the spectrum.

Now we come to Eliasson's result. Recall that we can rewrite the Schrödinger equation

$$-u''(x) + Q(\omega x)u(x) = Eu(x),$$

as the first order system

$$U'(x) = \begin{pmatrix} 0 & 1 \\ Q(\omega x) - E & 0 \end{pmatrix} U(x) \tag{2.7}$$

where $U(x) = \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}$.

Theorem 2.2. *[[3]] There is an E_0, such that for $E = \frac{1}{2}\langle\omega, m\rangle > E_0$ a boundary point of the spectrum of H_0, there is a function $Y : \mathbb{T}^d \to SL(2,\mathbb{R})$ and $A \in sl(2,\mathbb{R})$ with $A^2 = 0$ such that*

$$X(x) = Y_1 Y(\frac{\omega}{2}x)e^{Ax}, \quad Y_1 = \frac{1}{2\sqrt{E}}\begin{pmatrix} 1 & 1 \\ -\sqrt{E} & \sqrt{E} \end{pmatrix} \tag{2.8}$$

is the fundamental solution of (2.7). Furthermore we have that for $|m| \geq 2$

$$|A| \leq c|m|^{\frac{3}{2}\tau} \tag{2.9}$$

$$|Y| \leq C\log|m|, \quad |\det(Y) - 1| \leq \frac{1}{2}, \tag{2.10}$$

for constants c, C independent of m, and the spectrum of H_0 is purely absolutely continuous above E_0.

We will give an outline of Eliasson's proof in Section 5, and derive the additional estimates there. In fact Eliasson proved that (2.8) holds, when $\rho(E)$ satisfies the next Diophantine condition

$$|\rho - \frac{\langle n, \omega\rangle}{2}| \geq \frac{\tilde{\kappa}}{|n|^\sigma}, \quad n \in \mathbb{Z}^d\backslash\{0\}, \quad \tilde{\kappa} > 0,\, \sigma > 0. \tag{2.11}$$

Eliasson also showed that the spectrum of H_0 will be a Cantor set for generic functions $Q : \mathbb{T}^d \to \mathbb{R}$ in the $|.|_s$ topology given by the norm

$$|Q|_s = \sup_{|\mathrm{Im}(z)|<s} |Q(z)|. \tag{2.12}$$

Furthermore, we could replace $Q(\omega x)$ by $Q(\omega x + \theta)$ for any $\theta \in \mathbb{T}^d$ obtaining the same statement.

3. Main Results

We are interested in decaying perturbations of the quasiperiodic operator H_0. That is for some function Δq consider the operator

$$H_1 = -\frac{d^2}{dx^2} + q_1(x), \quad q_1(x) = q_0(x) + \Delta q(x), \tag{3.1}$$

for $q_0(x) = Q(\omega x)$ as described in Section 2. We then have the next basic stability result of the essential spectrum.

Theorem 3.1. *If $\Delta q \to 0$, then*

$$\sigma_{ess}(H_1) = \sigma_{ess}(H_0) = \mathbb{R}\backslash \bigcup_n G_n, \tag{3.2}$$

for open sets G_n. If Δq is integrable, we have that the spectrum of H_1 is purely absolutely continuous above E_0.

Proof. The first part follows by Weyl's Theorem and Theorem 2.1. For the second part, note that by Theorem 2.2, H_0 has purely absolutely continuous spectrum above E_0, and by Theorem 1.6. of [6] it is invariant under L^1 perturbations. □

It is conjectured in [6], that there is still absolutely continuous spectrum for $\Delta q \in L^2$, but it may not be pure. This was shown for the free case in [1] and for the periodic one in [5]. See also the recent review in [2]. If we write $G_n = (E_n^-, E_n^+)$ for the intervals of the last theorem, and call them gaps. We call E_n^- (resp. E_n^+) a lower (resp. upper) boundary point of the spectrum. The next relative oscillation criterion follows.

Theorem 3.2. *Assume that $\Delta q \to 0$, and let E be a boundary point above E_0 of the essential spectrum of H_0. Then there exists a constant $K = K(E)$ such that E is an accumulation point of eigenvalues of H_1 if*

$$\limsup_{x\to\infty} K \Delta q(x) x^2 < -\frac{1}{4} \tag{3.3}$$

and E is not an accumulation point of eigenvalues if

$$\liminf_{x\to\infty} K \Delta q(x) x^2 > -\frac{1}{4}. \tag{3.4}$$

Furthermore $K > 0$ (resp. $K < 0$) if E is a upper (resp. lower) boundary point.

Proof. Everything follows from Theorem A.6, except for the existence of K. We have from (2.8) that $u_0(t) = U(\frac{\omega}{2}t)$ for a function $U : \mathbb{T}^d \to \mathbb{R}$. We will show

$$\begin{aligned} K &= \liminf_{l\to\infty} \limsup_{x\to\infty} \frac{1}{\ell} \int_x^{x+\ell} u_0(t)^2 dt \\ &= \limsup_{l\to\infty} \liminf_{x\to\infty} \frac{1}{\ell} \int_x^{x+\ell} u_0(t)^2 dt = \int_{\mathbb{T}^d} U(z)^2 dz. \end{aligned}$$

Now note, that (2.2) implies that the system $(\mathbb{T}^d, T_t, \mu)$, where $T_t = \frac{\omega}{2}t$ and μ is the normalized Lebesgue measure is uniquely ergodic. By Birkhoff's ergodic theorem, we have that

$$\lim_{l\to\infty} \frac{1}{\ell} \int_x^{x+\ell} U(\frac{\omega}{2}t)^2 dt = \int_{\mathbb{T}^d} U(z)^2 dz.$$

By unique ergodicity, we know that the limit is uniform in x. Hence, the result follows. □

We even have a whole scale of relative oscillation criteria. To state this, we recall the iterated logarithm $\log_n(x)$ which is defined recursively via

$$\log_0(x) = x, \qquad \log_n(x) = \log(\log_{n-1}(x)).$$

Here we use the convention $\log(x) = \log|x|$ for negative values of x. Then $\log_n(x)$ will be continuous for $x > \mathrm{e}_{n-1}$ and positive for $x > \mathrm{e}_n$, where $\mathrm{e}_{-1} = -\infty$ and $\mathrm{e}_n = \mathrm{e}^{\mathrm{e}_{n-1}}$. Abbreviate further

$$L_n(x) = \frac{1}{\log_{n+1}'(x)} = \prod_{j=0}^{n} \log_j(x), \qquad \tilde{Q}_n(x) = -\frac{1}{4K}\sum_{j=0}^{n-1}\frac{1}{L_j(x)^2}.$$

From Theorem 2.10. of [12].

Theorem 3.3. *Assume the assumptions of the last theorem, and that for some* $n \in \mathbb{N}$

$$\lim_{x\to\infty} L_{n-1}(x)^{-2}(\Delta q(x) - \tilde{Q}_{n-1}(x)) = -\frac{1}{4K}. \tag{3.5}$$

Then E is an accumulation point of eigenvalues of H_1 if

$$\limsup_{x\to\infty} K L_n(x)^2(\Delta q(x) - \tilde{Q}_n(x)) < -\frac{1}{4} \tag{3.6}$$

and E is not an accumulation point of eigenvalues if

$$\liminf_{x\to\infty} K L_n(x)^2(\Delta q(x) - \tilde{Q}_n(x)) > -\frac{1}{4}, \tag{3.7}$$

with the same K as in the last theorem.

The next lemma gives us an estimate on K.

Lemma 3.4. *The constant K of Theorem 3.2, satisfies*

$$|K(E)| \le \frac{C}{|m|^{\tilde{\tau}}\sqrt{E}}, \quad 0 < \tilde{\tau} < \frac{3}{2}\tau \tag{3.8}$$

where $m \in \mathbb{Z}^d$ is such that $E \in \rho^{-1}(\frac{1}{2}\langle\omega, m\rangle)$.

Proof. From Theorem 2.2, we know the existence. We note that $\det(Y_1) = 1$. By (2.9), we have that $|A| \le c|m|^{-\frac{3}{2}\tau}$, where the m is the one such that $\rho(E) = \frac{1}{2}\langle\omega, m\rangle$. Hence, we obtain that $|\beta| \le c|m|^{-\frac{3}{2}\tau}$. Now

$$|K| \le c\frac{\int_{\mathbb{T}} U(z)dz}{|m|^{\frac{3}{2}\tau}\sqrt{E}}.$$

The claim now follows by (2.10). □

Remark 3.5. *One can hope that the estimate (3.10) on $K(E)$ can be improved. It was shown in* [3] *that the matrix A and then β would satisfy the bound $|\beta| \le C|E_+ - E_-|$ for some constant C. Then it was shown in* [13]*, that $|E_+ - E_-| \le ce^{-\gamma|m|}$ for some constants c and γ. Hence one should expect $K(E)$ to decrease exponentially in m. Unfortunately, the estimate of* [13] *depends on further arithmetic properties of m. Hence, it is not clear if it holds at all band edges.*

For simplicity, we will now restrict our attention to perturbations of the form

$$\Delta q(x) = \frac{\mu}{x^\gamma}, \quad \mu \neq 0, \quad \gamma > 0. \tag{3.9}$$

We will denote the operator $H_0 + \frac{\mu}{x^\gamma}$ by H_μ^γ. Now, we come to the question how many gaps above E_0 can contain infinitely many eigenvalues. This question is a bit odder than the one for periodic operators, since there are bounded intervals that contain infinitely many gaps.

Introduce μ_{crit} by

$$\mu_{crit}(E) = -\frac{1}{4K(E)}. \tag{3.10}$$

Then $E > E_0$ is an accumulation point of eigenvalues of H_μ^2 if and only if $\mu/\mu_{crit} > 1$. For H_μ^γ, $0 < \gamma < 2$, this requirement is $\mu/\mu_{crit} > 0$. Now, we come to

Theorem 3.6. *If $\gamma > 2$, then no boundary point of $\sigma(H_0)$ above E_0 is an accumulation point of eigenvalues of $H_\mu^\gamma = H_0 + \frac{\gamma}{x^\alpha}$. If $\gamma < 2$, then if $\mu < 0$ (resp. $\mu > 0$), then all upper (resp. lower) boundary points above E_0 are accumulation points of eigenvalues of H_μ^γ.*

If $\gamma = 2$, we can add infinitely many eigenvalues to each gap by choosing μ large enough. However, for every value of μ only finitely many gaps contain infinitely many eigenvalues of H_μ^γ.

Proof. The first claim follows from Theorem 3.2. The second claim follows from the last lemma and Theorem 3.2. □

Now, we come to the eigenvalue asymptotics. Let E be again a boundary point of the spectrum of H_0. Introduce, if the set $(\tilde{E}, E) \cap \sigma(H_0) = \emptyset$, by $N(\lambda)$ the number

$$N(\lambda) = \operatorname{tr}(P_{(\tilde{E},\lambda)}(H_\mu^\gamma)), \quad \tilde{E} < \lambda < E \tag{3.11}$$

with the obvious modification for $(E, \tilde{E}) \cap \sigma(H_0) = \emptyset$.

Theorem 3.7. *Let E be a boundary point of the spectrum of H_0, which is an accumulation point of eigenvalues of H_μ^γ. Then if $\gamma = 2$*

$$N(\lambda) = \frac{1}{4\pi}\sqrt{\frac{\mu}{\mu_{crit}} - 1} \cdot |\log|E - \lambda|| \cdot (1 + o(1)), \tag{3.12}$$

and if $0 < \gamma < 2$

$$N(\lambda) = \frac{1}{\pi}\frac{1}{2-\gamma}\sqrt{\frac{\mu}{\mu_{crit}}}\left(\frac{|\mu|}{|E-\lambda|}\right)^{(2-\gamma)/2\gamma} \cdot (1 + o(1)). \tag{3.13}$$

where $N(\lambda)$ is the number of eigenvalues near E.

We will give a proof in Section 4. In difference to the proof of [15], our proof only uses the decay of the potential and the behavior of the solution at the boundary point of the spectrum. In fact everything carries over to general elliptic situations. That is, where one has two solutions u_0, u_1 such that $u_0(x)$ and $u_1(x) - xu_0(x)$ are bounded functions.

Remark 3.8. *It was already shown in Corollary 6.6 in* [8] *that $\mu_{crit}(E)$ has to diverge as $E \to \infty$. We also remark that* [8] *develops a different approach to relative oscillation criteria than was used in* [12].

4. Proof of Theorem 3.7

We will now give explicit bounds on the spectral projections.

Lemma 4.1. *Let ψ be a solution of*

$$\psi'(x) = -\Delta q(x)(u_0(x)\cos\psi(x) - v_0(x)\cos\psi(x))^2. \tag{4.1}$$

Then we have that

$$\psi(x) = \left(\frac{1}{2}\sqrt{\frac{\mu}{\mu_{crit}} - 1} + o(1)\right)\log|x| \tag{4.2}$$

if $\Delta q(x) = \mu/x^2$. If $\Delta q(x) = \mu/x^\gamma$, $0 < \gamma < 2$,

$$\psi(x) = \left(\sqrt{\frac{\mu}{\mu_{crit}}}\frac{1}{2-\gamma} + o(1)\right)x^{1-\gamma/2}. \tag{4.3}$$

Proof. Use in (A.5) $\alpha = x$, to obtain if $\gamma = 2$ the next equation

$$\varphi' = \frac{1}{x}(\sin^2\varphi + \cos\varphi\sin\varphi + \mu u_0^2\cos^2\varphi) + O(\frac{1}{x^2}),$$

whose asymptotics can be evaluated with Lemma B.2 and Lemma B.1.

In the case $0 < \gamma < 2$, we choose $\alpha = 1/\sqrt{|\Delta q|K}$, then also the $\sin\varphi\cos\varphi$ term becomes of lower order, hence we obtain by averaging

$$\varphi'(x) = \sqrt{K\Delta q(x)} + O(\Delta q + \frac{\Delta q'}{\Delta q})$$

which implies the claim for Δq of the particular form. □

Then, we have that

Lemma 4.2. *Let the Wronskian $W(u_1(E), u_0(E))$ have n zeros on*

$$\{x, \forall y > x, |\Delta q| \leq |E - \lambda|\}.$$

Then we have that

$$N(\lambda) \leq n + 3. \tag{4.4}$$

Proof. Observe that by the comparison theorem for Wronskians, we have that $W(u_1(\lambda), u_0(\lambda))$ can have at most one zero left of x_n.

Hence, we obtain

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_1) \leq \#_{(1,x_n)}(u_1(\lambda), u_0(\lambda)) + 1$$

Now, by the triangle inequality for Wronskians, we obtain $\#_{(1,x_n)}(u_1(\lambda), u_0(\lambda)) \leq \#_{(1,x_n)}(u_1(\lambda), u_0(E)) + 1$. It, now suffices to note that $\#_{(1,x_n)}(u_1(\lambda), u_0(E))$ is bounded by $\#_{(1,x_n)}(u_1(E), u_0(E)) + 1$ by using the comparison theorem for Wronskians. □

Note, that the last two lemmas imply the next bound on the eigenvalues if $\gamma = 2$

$$N(\lambda) \leq \frac{1}{4\pi}\sqrt{\frac{\mu}{\mu_{crit}} - 1} \cdot |\log|E - \lambda|| \cdot (1 + o(1)), \tag{4.5}$$

and if $0 < \gamma < 2$

$$N(\lambda) \leq \frac{1}{\pi}\frac{1}{2-\gamma}\sqrt{\frac{\mu}{\mu_{crit}}}\left(\frac{|\mu|}{|E-\lambda|}\right)^{(2-\gamma)/2\gamma} \cdot (1 + o(1)). \tag{4.6}$$

The next lemma shows that we have equality. Hence with it Theorem 3.7 is proven.

Lemma 4.3. *Let $0 < \delta < 1$ and $0 < \gamma \leq 2$, then if $\gamma = 2$,*

$$N(\lambda) \geq \frac{1}{4\pi}\sqrt{\frac{\mu}{\mu_{crit}}(1-\delta) - 1} \cdot |\log|E-\lambda|| \cdot (1 + o(1)), \tag{4.7}$$

and if $0 < \gamma < 2$,

$$N(\lambda) \geq \frac{1}{\pi}\frac{1}{2-\gamma}\sqrt{\frac{\mu}{\mu_{crit}}(1-\delta)} \left(\frac{|\mu|}{|E-\lambda|}\right)^{(2-\gamma)/2\gamma} \cdot (1 + o(1)). \tag{4.8}$$

Proof. Let x_{max} be given by

$$x_{max}(\lambda) = \delta \left(\frac{|\mu|}{|E-\lambda|}\right)^{1/\gamma}.$$

Let $\varphi_\lambda(x)$ be a Prüfer angle of $W(u_0(E), u_1(\lambda))$. By the triangle inequality for Wronskians, it is clear that φ_λ is close to the Prüfer angle of $W(u_0(\lambda), u_1(\lambda))$. Now, for $x < x_{max}(\lambda)$, we have that

$$\varphi_\lambda'(x) \geq \frac{\mu(1-\delta)}{x^\gamma}(u_0 \cos\varphi_\lambda(x) - v_0 \sin\varphi_\lambda(x))^2.$$

This is the same equation for all λ. As $x \to \infty$, the solution has the claimed asymptotics by using Lemma 4.1. Hence, the claim follows. □

5. Outline of Eliasson's proof

We now give an outline of Eliasson's proof of reducibility in [3]. The next lemma is an easy computation.

Lemma 5.1. *The equation*

$$X'(x) = \begin{pmatrix} 0 & 1 \\ Q(\omega x) - E & 0 \end{pmatrix} X(x) \tag{5.1}$$

can be transformed by

$$X_1(x) = Y_1^{-1} X(x), \quad Y_1 = \begin{pmatrix} 1 & 1 \\ -\sqrt{E} & \sqrt{E} \end{pmatrix} \tag{5.2}$$

to

$$X_1'(x) = (A_1 + F_1(\omega x, \sqrt{E})) X_1(x), \tag{5.3}$$

where

$$A_1 = \sqrt{E} J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F_1(z, \sqrt{E}) = \frac{Q(z)}{2\sqrt{E}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5.4}$$

Furthermore A_1, F_1 satisfy Hypothesis 5.2.

Hypothesis H.5.2. *Let $A_1 \in sl(2,\mathbb{C})$ and $F_1 : \mathbb{T}^d \to Mat(2,\mathbb{C})$ satisfy*

$$\mathrm{tr}(\hat{F}_1(0)) = 0, \tag{5.5}$$

$$|A_1 - \sqrt{E} J| < 2 \tag{5.6}$$

$$|F_1|_{r_1} < \varepsilon_1, \tag{5.7}$$

for some $\varepsilon_1 > 0$, small.

We have now seen that we can reduce our system to one of the form

$$X_1'(x) = (A_1 + F_1(x))X_1(x), \tag{5.8}$$

where F_1 is small. This system although close to a constant coefficient one cannot be solved explicitly. However, we can reduce it to a system

$$X_2'(x) = (A_2 + F_2(x))X_2(x) \tag{5.9}$$

where F_2 is smaller than F_1, as follows. We will construct A_2, F_2, and a solution Y_1 of the system

$$Y_1'(x) = (A_1 + F_1)Y_1 - Y_1(A_2 + F_2). \tag{5.10}$$

Then for X_2 a solution to (5.9), we have that $Y_1 X_2$ will solve (5.8). Of course, we cannot hope that (5.9) will be explicitly solvable, however we will be able to iterate the above procedure to obtain better and better approximate solutions.

Since, we will require that $F_k \to 0$, and then our final $X_\infty(x) = e^{xA}$. Here $A = \lim_{k\to\infty} A_k$. So the final solution will be

$$\prod_{k=1}^{\infty} Y_k(\frac{\omega}{2}x)e^{Ax} \tag{5.11}$$

We will not attempt to solve (5.10) in this paper, and refer to [3] for the details. However, we will draw further conclusions from Eliasson's method to control our quantities.

Fix $0 < \varepsilon_1 < 1$ sufficiently small. Fix $0 < \sigma < 1$, and let

$$\varepsilon_{j+1} = \varepsilon_j^{1+\sigma} = \varepsilon_1^{(1+\sigma)^j}. \tag{5.12}$$

Furthermore, assume that $r_1 > r_2 > r_3 > \dots$ is a decreasing sequence of positive numbers, satisfying

$$\frac{r_1}{2^{j+1}} \le r_j - r_{j+1}. \tag{5.13}$$

r_j will play the role of the neighborhood, where we suppose to have analyticity. Introduce N_j by

$$N_j = \frac{2\sigma}{r_j - r_{j+1}} \log(\varepsilon_j^{-1}) = \frac{2\sigma(1+\sigma)^j}{r_j - r_{j+1}} \log(\varepsilon_1^{-1}) \le C(2+2\sigma)^j, \quad C > 0. \tag{5.14}$$

Furthermore, one also sees that $N_j \ge \tilde{C}(1+\sigma)^j$ for some other constant $\tilde{C}$. Hence $N_j \to \infty$ as $j \to \infty$. Furthermore, we have

$$\varepsilon_j^\sigma \le \left(\frac{4\sigma}{r_1(1+\sigma)} \log(\varepsilon_1^{-1})(2+2\sigma)^j\right)^{-4\tau} \le N_j^{-4\tau} \tag{5.15}$$

if ε_1 is small enough.

Proposition 5.3. *Assume Hypothesis 5.2 with ε_1 small enough, then there are functions $Y_j : 2\mathbb{T}^d \to GL(2,\mathbb{R})$, $A_j \in sl(2,\mathbb{R})$, and $F_j : \mathbb{T}^d \to Mat(2,\mathbb{R})$, for $j \ge 1$. Furthermore, there are numbers m_j that satisfy*

$$\varepsilon_j^\sigma \le |2\alpha_j - \langle\omega, m_j\rangle| \le 2\varepsilon_j^\sigma, \quad 0 < |m_j| \le N_j, \tag{5.16}$$

or $m_j = 0$ if (5.16) cannot be satisfied. Here α_j is the rotation number of A_k. Furthermore A_j, F_j, and Y_j satisfy

$$\langle Y_{j+1}'(x), \frac{\omega}{2}\rangle = (A_j + F_j(x))Y_{j+1}(x) - Y_{j+1}(x)(A_{j+1} + F_{j+1}(x)), \tag{5.17}$$

$$|\left(Y_{j+1}(.) - \exp\left(\frac{\langle m_k, .\rangle}{\alpha_j} A_j\right)\right)| \le \varepsilon_j^{1/2}, \tag{5.18}$$

$$|\left(A_{j+1} - \left(1 - \frac{\langle \omega, m_j\rangle}{2\alpha_j}\right) A_j\right)| \le \varepsilon_j^{2/3}, \tag{5.19}$$

$$\operatorname{tr}(\hat{F}_{j+1}(0)) = 0, \quad |F_{j+1}|_{r_{j+1}} < \varepsilon_{j+1}, \tag{5.20}$$

$$|A_{j+1}| \le 32|\alpha_{j+1}|N_{j+1}^{\tau}, \qquad \textit{if } |\alpha_{j+1}| \ge \frac{1}{4}N_{j+1}^{-\tau}. \tag{5.21}$$

Proof. This is Lemma 1 and 2 in [3]. □

Remark 5.4. *The requirement of ε_1 being small enough, will in fact determine our lower bound on allowed energies E. Since for $E > E_0$*

$$|F_1|_{r_1} = \frac{C}{\sqrt{E}} < \frac{C}{\sqrt{E_0}} = \varepsilon_1$$

for some constant C. Hence by making E_0 large, we can make ε_1 arbitrarily small.

Lemma 5.5. *Assume that Y_j, A_j and F_j satisfy the conditions given in Proposition 5.3. If for all $j \le k$, $m_j = 0$, then*

$$|A_k - \lambda J| < 3. \tag{5.22}$$

We furthermore obtain, if K is the largest integer less than k such that $m_K \neq 0$, that

$$|A_k| \le C\frac{1}{N_K^{3\tau}} < 3, \quad k \ge K \tag{5.23}$$

where C doesn't depend on K.

Proof. By $m_k = 0$, we have that from (5.16)

$$|2\alpha_k - \langle \omega, n\rangle| \ge \varepsilon_k^{\sigma}, \quad 0 < |n| \le N_j.$$

For $m_j = 0$, $j = 1, \dots, k$, we have by (5.19) that

$$|A_k| \le 2 + \varepsilon_1^{2/3} + \dots + \varepsilon_{k-1}^{3/2} < 3.$$

This shows the first part.

For the second claim, let $l \le k$ be maximal such that the $m_l \neq 0$. Then, we obtain a bound on $|A_j(\lambda)|$ by

$$\begin{aligned} |A_k| &\le \varepsilon_l^{2/3} + \dots + \varepsilon_{k-1}^{3/2} + |\left(1 - \frac{\langle \omega, m_l\rangle}{2\alpha_l}\right) A_l| \\ &\le 2\varepsilon_l^{2/3} + 2\varepsilon_l^{\sigma}|\frac{A_l}{2\alpha_l}| \le 34 N_l^{\tau}\varepsilon_l^{\sigma}, \end{aligned}$$

where we used (5.16) in the middle and (5.21) in the last step. (5.23) follows from (5.15). The last claim is evident. □

Let us now consider $\tilde{\rho} = \frac{1}{2}\sum_{k=1}^{\infty}\langle m_k, \omega\rangle + \alpha$, where $\alpha = \lim_{j\to\infty}\alpha_j$. Furthermore, $\rho_{j+1} = \frac{1}{2}\sum_{k=1}^{j}\langle m_k, \omega\rangle + \alpha_{j+1}$, Furthermore, we know that inside the gap $\alpha = 0$ from [3]. We now obtain

Lemma 5.6. *$\rho_{j+1} \to \tilde{\rho}$ uniformly. If ρ is rational, $m_j = 0$ for j large. Furthemore,*

$$\sum_{k, m_k \neq 0} m_k = m, \tag{5.24}$$

holds.

Proof. The first two parts are Lemma 3 in [3]. The last part follows, since $\alpha \to 0$, and with $\tilde{m} = \sum_{k, m_k \neq 0} m_k$, one has

$$0 = \tilde{\rho} - \frac{1}{2}\langle \omega, m\rangle = \frac{1}{2}\langle \omega, \tilde{m} - m\rangle.$$

Hence $\tilde{m} = m$ by the Diophantine condition. □

Proof of (2.9). We will now show how (5.24) can be used to make the bound from (5.23) only depend on m. By definition $|m_k| \leq N_k$, we have by (5.14)

$$|m| \leq \sum_{k, m_k \neq 0} |m_k| \leq \sum_{k=1}^{K} N_k \leq C(2\sigma + 2)^{K+1}.$$

Hence $K \geq \frac{\log |m|}{\log(2+2\sigma)} - C$ and by (5.14) $N_K \geq C\sqrt{|m|}$ and then (5.23) implies the claim, since it holds for all large k. □

We have that

Lemma 5.7. *If $m_j = 0$ for j large, we have that $\prod Y_j$ converges to some Y uniformly on compact subsets. Furthermore $A_j \to A$ and $F_j \to 0$. Furthermore (2.10) holds.*

Proof. Since r_j is decreasing and positive, it has a limit $r_0 \geq 0$. By (5.20), we have that $|F_j|_{r_j} \to 0$. Since $m_j = 0$ for large j, we have that $|A_{j+1} - A_j| \leq \varepsilon_j^{2/3}$ from (5.19). Hence, $A_j \to A$, since $\sum_{j=N}^{\infty} \varepsilon_j^{2/3} < \infty$.

By (5.18), we have that $|Y_j - \mathbb{I}| \leq \varepsilon_j^{1/2}$, if $m_j = 0$, which implies $\prod Y_j \to Y$ by a similar argument. If $m_j \neq 0$, we have that

$$|Y_{j+1}(\frac{\omega}{2}t) - \mathbb{I}| \leq \varepsilon_j^{1/2} + |\cos(\frac{\langle m_j, t\rangle}{2})\mathbb{I} + \sin(\frac{\langle m_j, t\rangle}{2})\frac{A_j}{\alpha_j} - \mathbb{I}|,$$

where the last term ≤ 3. Since, we can bound the number of these terms by $\log m$, we obtain the claim. By (5.18), we have that $Y_{j+1} - \mathbb{I}$, if $m_j = 0$, resp. $\exp(-\langle m_j, t\rangle A_j/\alpha_j)Y_{j+1} - \mathbb{I}$ are bounded by $\varepsilon_j^{1/2}$. Hence, we can bound $|\det(Y_{j+1}) - 1| \leq \varepsilon_j$, from which the estimate on the determinant follows. □

Appendix A. Relative Oscillation Theory

As introduced in [10], relative oscillation theory is a tool to compute the difference of spectra of two different Schrödinger operators. Let $q_0, q_1 \in L^1_{loc}$ and

$$H_j = -\frac{d^2}{dx^2} + q_j, \quad j = 0, 1 \tag{A.1}$$

be self-adjoint Schrödinger operators on $L^2(1, \infty)$. Introduce $\Delta q = q_1 - q_0$, which we will assume to be sign-definite. Denote by $\#(u_0, u_1)$ the number of zeros of the Wronskian $W(u_0, u_1) = u_0 u_1' - u_0' u_1$ on $(1, \infty)$, for solutions $\tau_j u_j = \lambda_j u_j$. Let $\psi_{j,-}(\lambda)$ be the solution of $\tau_j \psi_{j,-}(\lambda) = \lambda \psi_{j,-}(\lambda)$, which obeys the boundary condition at 1 (e.g. $\psi_{j,-}(\lambda)(1) = 0$). Similarly let $\psi_{j,+}(\lambda)$ be the solution satisfying $\psi_{j,+}(\lambda) \in L^2(1,\infty)$. Then [10] tells us:

Theorem A.1. *Assume that* $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$. *Then, we have that*

$$\mathrm{tr}P_{[\lambda_0,\lambda_1)}(H_1) - \mathrm{tr}P_{(\lambda_0,\lambda_1]}(H_0) = \begin{cases} (\#(\psi_{1,\pm}(\lambda_1), \psi_{0,\mp}(\lambda_1) - \#(\psi_{1,\pm}(\lambda_0), \psi_{0,\mp}(\lambda_0)), & \Delta q < 0 \\ -(\#(\psi_{1,\pm}(\lambda_1), \psi_{0,\mp}(\lambda_1) - \#(\psi_{1,\pm}(\lambda_0), \psi_{0,\mp}(\lambda_0)), & \Delta q > 0 \end{cases} \tag{A.2}$$

Here $\mathrm{tr}P_{[\lambda_0,\lambda_1)}(H_1)$ *denotes the number of eigenvalues of* H_1 *in* $[\lambda_0, \lambda_1)$.

Since one has the next triangle inequality for Wronskians

$$\#(u_0, u_1) + \#(u_1, u_2) - 1 \leq \#(u_0, u_2) \leq \#(u_0, u_1) + \#(u_1, u_2) + 1, \tag{A.3}$$

one can replace $\psi_{j,\pm}(\lambda)$ by any other solution of $\tau_j u = \lambda u$, up to a finite error. We furthermore remark that the next two comparison theorems hold. The first one is found in [11].

Theorem A.2 (Sturm's Comparison theorem)**.** *Let* $q_0 - q_1 > 0$, *and* $H_j u_j = 0$, $j = 0, 1$. *Then between any two zeros of* u_0 *or* $W(u_0, u_1)$, *there is a zero of* u_1.

Similarly, between two zeros of u_1, *which are not at the same time zeros of* u_0, *there is at least one zero of* u_0 *or* $W(u_0, u_1)$.

The next result is found in [10].

Theorem A.3 (Comparison theorem for Wronskians)**.** *Suppose* u_j *satisfies* $\tau_j u_j = \lambda_j u_j$, $j = 0, 1, 2$, *where* $\lambda_0 r - q_0 \leq \lambda_1 r - q_1 \leq \lambda_2 r - q_2$.

If $c < d$ *are two zeros of* $W_x(u_0, u_1)$ *such that* $W_x(u_0, u_1)$ *does not vanish identically, then there is at least one sign flip of* $W_x(u_0, u_2)$ *in* (c, d). *Similarly, if* $c < d$ *are two zeros of* $W_x(u_1, u_2)$ *such that* $W_x(u_1, u_2)$ *does not vanish identically, then there is at least one sign flip of* $W_x(u_0, u_2)$ *in* (c, d).

We call H_1 relatively oscillatory with respect to H_0 at E if for any solutions of $H_j u_j(E) = E u_j(E)$, $j = 0, 1$, we have that $\#(u_0(E), u_1(E))$ is infinite. Otherwise we call it relatively nonoscillatory. Now, we come to relative oscillation criteria.

Lemma A.4. *Let* $\lim_{x\to\infty} \Delta q(x) = 0$. *Then* $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ *and* H_1 *is relatively nonoscillatory with respect to* H_0 *at* $E \in \mathbb{R}\backslash\sigma_{ess}(H_0)$.

By Theorem A.1, this is equivalent if E is a boundary point of the essential spectrum of H_0, to E being an accumulation point of eigenvalues of H_1. In order to state a relative oscillation criterion at a boundary point of the spectrum, some preparations are needed.

Definition A.5. *A boundary point* E *of the essential spectrum of* H_0 *will be called admissible if there is a minimal solution* u_0 *of* $(\tau_0 - E)u_0 = 0$ *and a second linearly independent solution* v_0 *with* $W(u_0, v_0) = 1$ *such that*

$$\begin{pmatrix} u_0 \\ p_0 u_0' \end{pmatrix} = O(\alpha), \quad \begin{pmatrix} v_0 \\ p_0 v_0' \end{pmatrix} - \beta \begin{pmatrix} u_0 \\ p_0 u_0' \end{pmatrix} = o(\alpha\beta)$$

for some weight functions $\alpha > 0$, $\beta \lessgtr 0$, *where* β *is absolutely continuous such that* $\rho = \frac{\beta'}{\beta} > 0$ *satisfies* $\rho(x) = o(1)$ *and* $\frac{1}{\ell}\int_0^\ell |\rho(x+t) - \rho(x)|\, dt = o(\rho(x))$.

It is shown in Lemmas 4.2 and 4.3 of [12], that there exists a Prüfer angle ψ for $W(u_0, u_1)$ such that it obeys

$$\psi'(x) = -\Delta q(x)(u_0(x)\cos(\psi(x)) - v_0(x)\sin(\psi(x)))^2. \tag{A.4}$$

Through the transformation $\cot\psi = \alpha\cot\varphi + \beta$, this can then be transformed to (see Lemma 4.6 of [12])

$$\begin{aligned}\varphi' = {}& \frac{\alpha'}{\alpha}\sin\varphi\cos\varphi + \frac{\beta'}{\alpha}\sin^2\varphi - \Delta q\cdot\alpha u_0^2\cos^2\varphi \\ & + O(\Delta q) + O(\Delta q/\alpha).\end{aligned} \tag{A.5}$$

Through an application of the methods of Appendix B, one comes to the main result of [12].

Theorem A.6. *Suppose E is an admissible boundary point of the essential spectrum of τ_0, with u_0, v_0 and α, β as in Definition A.5. Furthermore, suppose that we have*

$$\Delta q = O\big(\frac{\beta'}{\alpha^2\beta^2}\big). \tag{A.6}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ if

$$\inf_{\ell>0}\limsup_{x\to b}\frac{1}{\ell}\int_x^{x+\ell}\frac{\beta(t)^2}{\beta'(t)}u_0(t)^2\Delta q(t)dt < -\frac{1}{4} \tag{A.7}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ if

$$\sup_{\ell>0}\liminf_{x\to b}\frac{1}{\ell}\int_x^{x+\ell}\frac{\beta(t)^2}{\beta'(t)}u_0(t)^2\Delta q(t)dt > -\frac{1}{4}. \tag{A.8}$$

We finish this section with a closing remark.

Remark A.7. *The requirement made that Δq is of definite sign is not necessary. However, a general theory requires a more difficult definition of $\#(u_0, u_1)$. We refer the interested reader to* [10] *for details.*

Appendix B. Averaging ordinary differential equations

In this section we collect the required results for these ordinary differential equations. Proofs and further references can be found in [12].

Lemma B.1. *Suppose $\rho(x) > 0$ (or $\rho(x) < 0$) is not integrable near b. Then the equation*

$$\varphi'(x) = \rho(x)\Big(A\sin^2\varphi(x) + \cos\varphi(x)\sin\varphi(x) + B\cos^2\varphi(x)\Big) + o(\rho(x)) \tag{B.1}$$

has only unbounded solution if $4AB > 1$ and only bounded solution if $4AB < 1$. In the unbounded case we have

$$\varphi(x) = \left(\frac{\mathrm{sgn}(A)}{2}\sqrt{4AB-1} + o(1)\right)\int^x \rho(t)dt. \tag{B.2}$$

In addition, we also need to look at averages: Let $\ell > 0$, and denote by

$$\overline{g}(x) = \frac{1}{\ell}\int_x^{x+\ell} g(t)dt. \tag{B.3}$$

the average of g over an interval of length ℓ.

Lemma B.2. *Let φ obey the equation*

$$\varphi'(x) = \rho(x)f(x) + o(\rho(x)), \tag{B.4}$$

where $f(x)$ is bounded. If

$$\frac{1}{\ell}\int_0^\ell |\rho(x+t) - \rho(x)|\, dt = o(\rho(x)) \tag{B.5}$$

then

$$\overline{\varphi}'(x) = \rho(x)\overline{f}(x) + o(\rho(x)) \tag{B.6}$$

Moreover, suppose $\rho(x) = o(1)$. If $f(x) = A(x)g(\varphi(x))$, where $A(x)$ is bounded and $g(x)$ is bounded and Lipschitz continuous, then

$$\overline{f}(x) = \overline{A}(x)g(\overline{\varphi}) + o(1). \tag{B.7}$$

Condition (B.5) is a strong version of saying that $\overline{\rho}(x) = \rho(x)(1 + o(1))$ (it is equivalent to the latter if ρ is monotone). It will be typically fulfilled if ρ decreases (or increases) polynomially (but not exponentially). For example, the condition holds if $\sup_{t\in[0,1]} \frac{\rho'(x+t)}{\rho(x)} \to 0$.

Furthermore, note that if $A(x)$ has a limit, $A(x) = A_0 + o(1)$, then $\overline{A}(x)$ can be replaced by the limit A_0 and we have the next result

Corollary B.3. *Let φ obey the equation*

$$\varphi' = \rho\Big(A\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) + B\cos^2(\varphi)\Big) + o(\rho) \tag{B.8}$$

with A, B bounded functions and assume that $\rho = o(1)$ satisfies (B.5). Then the averaged function $\overline{\varphi}$ obeys the equation

$$\overline{\varphi}' = \rho\Big(\overline{A}\sin^2(\overline{\varphi}) + \sin(\overline{\varphi})\cos(\overline{\varphi}) + \overline{B}\cos^2(\overline{\varphi})\Big) + o(\rho). \tag{B.9}$$

Note that in this case φ is bounded (above/below) if and only if $\overline{\varphi}$ is bounded (above/below).

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