

# Analysis of Fractional Splines Interpolation and Optimal Error Bounds

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**Abstract** This paper presents a formulation and a study of three interpolatory fractional splines these are in the class of  $m\alpha$ ,  $m = 2, 4, 6$ ,  $\alpha = 0.5$ . We extend fractional splines function with uniform knots to approximate the solution of fractional equations. The developed of spline method is to analysis convergence fractional order derivatives and estimating error bounds. We propose spline fractional method to solve fractional differentiation equations. Numerical example is given to illustrate the applicability and accuracy of the methods.

**Keywords:** fractional integral and derivative, caputo derivative, error bound

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## 1. Introduction

In the past decades, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [2,10,12,13]. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The fractional spline function of a polynomial form (see [5,6,7,8]) is a new approach to provide an analytical approximation to linear and nonlinear problems, and it is particularly valuable as a tool for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of numerical approximate solutions to both linear and nonlinear differential equations.

In this work we construct a new fractional spline which interpolates the (1/2-th derivative for the first case, 1/2, 3/2 -th derivatives for the second case, and 1/2, 3/2, 5/2-th derivatives for the last case of a given function at the knots and its value at the beginning of the interval considered. We obtain a direct simple formula for the proposed fractional spline. error bounds for the function is derived in the sense of the Hermite interpolation. To illustrate the efficiency and the error analysis two numerical examples are considered.

## 2. Preliminaries

In this section, we recall some relevant definitions. There are many ways to define fractional integral and derivative. In this paper we will use Riemann-Liouville fractional integral and Caputo fractional derivative.

Let  $\alpha$  be a positive real number and  $f(x)$  be a function defined on the right side of  $a$ , then **Definition 1. [1,2,10]** The Riemann-Liouville fractional integral of order  $\alpha > 0$  is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt,$$

$$n-1 < \alpha < n \in \mathbb{N}$$

where  $\Gamma$  is the gamma function.

**Definition 2. [1,2,10]** The Caputo fractional derivative of order  $\alpha > 0$  is defined by

$${}_c D_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, & \text{for } n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x), & \text{for } \alpha = n \in \mathbb{N}. \end{cases}$$

## 3. Description of the Fractional Splines

We construct here a class of interpolating fractional splines of degree  $j\alpha$ , for  $j = 2, 4, 6$  and  $\alpha = 0.5$ . error estimates for these splines are also represented. Since all cases considered are similar, details are given only for the first case of  $2\alpha$ .

Let  $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$  be a uniform partition of  $[0,1]$ . Set the stepsize  $h = x_{i+1} - x_i$  ( $i = 0(1)n$ ) and note that

$$D^{n,\alpha} f = \underbrace{D^\alpha, D^\alpha \dots D^\alpha}_{n\text{-times}}$$

If  $g$  is a real-valued function in  $[0, 1]$ , then  $g_i$  stands for  $g(x_i)$  ( $i = 0(1)n$ ). Since all cases considered are similar, details are given only for the first case. We have the following cases:

### 3.1. Spline of Degree $2\alpha$ Case (Existence and Uniqueness)

We suppose that  $s^{(1/2)}(x) \in C^2[0,1]$  and  $s(x)$  in each subinterval  $[x_i, x_{i+1}]$  has a form:

$$s(x) = a_i(x \dots x_i) + b_i(x \dots x_i)^{1/2} + c_i \tag{1}$$

where  $a_i, b_i$  and  $c_i$  are constants to be determined.

**Theorem 1.** *suppose that  $s^{(1/2)}(x) \in C^2[0,1]$  and  $s(x)$  in each subinterval  $[x_i, x_{i+1}]$  has the form (1). Given the real numbers  $s(1=2) i = f(1=2) i$  ( $i = 0(1)n$ ) and  $f_0$ , there exist a unique  $s(x)$  such that*

$$\begin{aligned} s_i^{(1/2)} &= f_i^{(1/2)} \quad (i = 0(1)n) \\ s_0 &= f_0 \end{aligned} \tag{2}$$

The fractional spline which satisfies (2) in  $[x_i, x_{i+1}]$  is of the form:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h^{1/2} s_i^{(1/2)} A_2(t) \tag{3}$$

where

$$\begin{aligned} A_0(t) &= 1 \dots t^{3/2}, \quad A_1(t) = t^{3/2}, \\ A_2(t) &= \frac{2}{\sqrt{\pi}} (t^{1/2} - t^{3/2}) \end{aligned} \tag{4}$$

and  $x = x_i + th, t \in [0,1]$ , with a similar expression for  $s(x)$  in  $[x_{i-1}, x_i]$ .

The coefficient  $s_i$  in (3) are given by the recurrence formula:

$$s_i = s_{i-1} - \frac{2}{3\sqrt{\pi}} h^2 \left( f_{i-1}^{(1/2)} + 2f_i^{(1/2)} \right), s_0 = f_0 \tag{5}$$

*Proof.* Indeed we can express any  $p(t)$  in  $[0,1]$  in the following form:

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0^{(1/2)} A_2(t)$$

To determine  $A_0, A_1, A_2$ , we write the above equality for  $p(t) = 1, t^{1/2}, t^{3/2}$  we get

$$A_0 + A_1 = 1, \quad A_1 + \frac{\sqrt{\pi}}{2} A_2 = t^{1/2}, \quad A_1 = t^{3/2}$$

Solving this we obtain

$$A_0(t) = 1 - t^{3/2}, \quad A_1(t) = t^{3/2}, \quad A_2(t) = \frac{2}{\sqrt{\pi}} \left( t^{1/2} - t^{3/2} \right)$$

Now for a fixed  $i \in \{0, 1, \dots, n-1\}$ , set  $x = x_i + th, t \in [0,1]$ . In the subinterval  $[x_i, x_{i+1}]$  the fractional spline  $s(x)$  satisfying (2) is:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h^{1/2} s_i^{(1/2)} A_2(t)$$

We have a similar expression for  $s(x)$  in  $[x_{i-1}, x_i]$ .

From the continuity condition of  $s^{(1/2)}(x_i^-) = s^{(1/2)}(x_i^+)$  we arrive the above recurrence formula (5). This completes the proof.

### 3.2. Error Bounds for the Fractional Spline of Degree $2\alpha$ Case

In this section, the  $L_\infty$  error estimates are presented for the above interpolating fractional spline in  $[0,1]$  using one of the best theorem of the Hermite interpolation (Theorem 2). Note that  $\|\cdot\|$  denotes the  $L_\infty$  norm.

**Theorem 2.** [4,8,9] *Let  $g \in C^{2m}[0, h]$  be given. Let  $p_{2m-1}$  be the unique Hermite interpolation polynomial of degree  $2m-1$  that matches  $g$  and its first  $m-1$  derivatives  $g^{(r)}$  at 0 and  $h$ . Then*

$$\left| e^{(r)}(x) \right| \leq \frac{h^r [x(h-x)]^{m-r} G}{r!(2m-1)!}, \quad r = 0(1)m, 0 \leq x \leq h, \tag{6}$$

where

$$\begin{aligned} \left| e^{(r)}(x) \right| &= \left| g^{(r)}(x) - p_{2m-1}^{(r)}(x) \right| \text{ and} \\ G &= \max_{0 \leq x \leq h} \max_{0 \leq x \leq h} \left| g^{(2m)}(x) \right| \end{aligned} \tag{7}$$

The bounds in (6) are best possible for  $r = 0$  only.

**Theorem 3.** *Suppose that  $s(x)$  be the fractional spline defined in section 3.1,  $f^{(1/2)}(x) \in C^2[0,1]$  and that  $f'(0) = f'(1) = 0$ , then for any  $x \in [0,1]$  we have*

$$\left| s(x) - f(x) \right| \leq \frac{h^2}{4\sqrt{\pi}} f^{(5/2)} \tag{8}$$

*Proof.* Since we have  $s^{(1/2)}(x)$  is the Hermite interpolation polynomial of degree 1 matching  $f^{(1/2)}$  at  $x = x_i, x_{i+1}$ . So for any  $x \in [x_i, x_{i+1}]$  we have using (6) with

$$\begin{aligned} m &= 1, \quad g = f^{(1/2)}, \text{ and } p_1 = s^{(1/2)} \\ \left| s^{(1/2)}(x) - f^{(1/2)}(x) \right| &\leq \frac{h^2}{4 \cdot 2!} D^2 D^{1/2} \end{aligned}$$

From which, we get

$$\left| I_{0,x}^{1/2} \left( s^{(1/2)}(x) - f^{(1/2)}(x) \right) \right| \leq I_{0,x}^{1/2} \left( \frac{h^2}{4 \cdot 2!} D^2 D^{1/2} \right)$$

This gives

$$|s(x) - s(0) - f(x) + f(0)| \leq \frac{2}{\sqrt{\pi}} x^2 \left( \frac{h^2}{4 \cdot 2!} D^2 D^{\frac{1}{2}} \right)$$

since,  $s(0) = f(0)$  and  $x \in [0,1]$  then the last equation becomes

$$|s(x) - f(x)| \leq \frac{h^2}{4\sqrt{\pi}} D^2 D^{\frac{1}{2}}$$

and since  $f'(0) = f''(0) = 0$ , following [11], p. 20, we have

$$D^2 D^{\frac{1}{2}} f = D^{\frac{5}{2}} f = f^{(\frac{5}{2})}$$

which leads to

$$|s(x) - f(x)| \leq \frac{h^2}{4\sqrt{\pi}} f^{(\frac{5}{2})}$$

Thus we have proved the theorem.

### 3.3. Spline of Degree $4\alpha$ Case (Existence and Uniqueness)

We suppose that  $s^{(\frac{1}{2})}(x) \in C^4[0,1]$  and  $s(x)$  in each subinterval  $[x_i, x_{i+1}]$  has a form:

$$s(x) = a_i(x - x_i)^2 + b_i(x - x_i)^{\frac{3}{2}} + c_i(x - x_i) + d_i(x - x_i)^{\frac{1}{2}} + e_i \tag{9}$$

From which the following theorem can be obtained:

**Theorem 4.** Suppose that  $s^{(\frac{1}{2})}(x) \in C^2[0,1]$  and  $s(x)$  in each subinterval  $[x_i, x_{i+1}]$  has a form (1). Given the real numbers  $s_i^{(\frac{1}{2})} = f_i^{(\frac{1}{2})}$ ,  $s_i^{(\frac{3}{2})} = f_i^{(\frac{3}{2})}$  ( $i=0(1)n$ ) and  $f_0$ , there exist a unique  $s(x)$  such that

$$s_i^{(\frac{1}{2})} = f_i^{(\frac{1}{2})}, s_i^{(\frac{3}{2})} = f_i^{(\frac{3}{2})}, (i=0(1)n), s_0 = f_0 \tag{10}$$

The fractional spline which satisfies (10) in  $[x_i, x_{i+1}]$  is of the form:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + \frac{1}{h^2} \left[ f_i^{(\frac{1}{2})} A_2(t) + f_{i+1}^{(\frac{1}{2})} A_3(t) \right] + h^{\frac{3}{2}} f_i^{(\frac{3}{2})} A_4(t) \tag{11}$$

where

$$A_0(t) = \frac{1}{5} \left( 2t^{\frac{7}{2}} - 7t^{\frac{5}{2}} \right) + 1, A_1(t) = -\frac{1}{5} t^{\frac{5}{2}} (2t - 7),$$

$$A_2(t) = \frac{2}{75\sqrt{\pi}} t^{\frac{1}{2}} (14t^3 - 89t^2 + 75), \tag{12}$$

$$A_3(t) = \frac{32}{75\sqrt{\pi}} t^{\frac{5}{2}} (t - 1), A_4(t) = \frac{4}{75\sqrt{\pi}} t^{\frac{3}{2}} (2t - 5)(t - 1)$$

and  $x = x_i + th, t \in [0,1]$ , with a similar expression for  $s(x)$  in  $[x_{i-1}, x_i]$ .

The coefficient  $s_i$  in (11) are given by the recurrence formula:

$$\frac{21}{16} \sqrt{\pi} (s_i - s_{i-1}) = h^{\frac{1}{2}} \left[ \frac{129}{40} f_{i-1}^{(\frac{1}{2})} + \frac{3}{5} f_i^{(\frac{1}{2})} \right] + h^{\frac{3}{2}} \left[ f_i^{(\frac{3}{2})} - \frac{27}{20} f_{i-1}^{(\frac{3}{2})} \right], \tag{13}$$

$$s_0 = f_0$$

*Proof.* In this case we can express any  $p(t)$  in  $[0,1]$  in the following form:

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0^{(\frac{1}{2})} A_2(t) + p_1^{(\frac{1}{2})} A_3(t) + p_0^{(\frac{3}{2})} A_4(t)$$

and to determine the coefficients  $A_j, j=0,1,2,3,4$ , we write the above equality for  $p(t) = 1, t^2, t^{\frac{3}{2}}, t^{\frac{5}{2}}, t^{\frac{7}{2}}$ .

By the same technique of theorem 1 we obtain the desired result and consequently the proof is completed.

### 3.4. Error Bounds for the Fractional Spline of Degree $4\alpha$ Case

Here we will derive the  $L_\infty$  error estimates are presented for the fractional spline that we have mentioned in Section 3.3, the error bounds have shown in the below theorem and its proof is similar subsequence of theorem 3.

**Theorem 5.** Suppose that  $s(x)$  be the fractional spline defined in section 3.3,  $f^{(\frac{1}{2})}(x) \in C^4[0,1]$  and that  $f^{(p)}(0) = 0, p = 1, 2, 3, 4$ , then for any  $x \in [0,1]$  we have

$$|s(x) - f(x)| \leq \frac{h^4}{(8)(4!)\sqrt{\pi}} f^{(\frac{9}{2})}. \tag{14}$$

*Proof.* Because  $s^{(\frac{1}{2})}(x)$  is the Hermite interpolation

polynomial of degree 2 matching  $f^{(\frac{1}{2})}, f^{(\frac{3}{2})}$  at  $x = x_i, x_{i+1}$ . So for any  $x \in [x_i, x_{i+1}]$  we have using (6) with

$$m = 2, g = f^{(\frac{1}{2})}, \text{ and } p_2 = s^{(\frac{1}{2})},$$

$$|s(x) - f(x)| \leq \frac{h^4}{(8)(4)\sqrt{\pi}} D^4 D^{\frac{1}{2}} f$$

and following [11], we have

$$D^4 D^{\frac{1}{2}} f = D^{\frac{9}{2}} f = f^{(\frac{9}{2})},$$

which leads to

$$|s(x) - f(x)| \leq \frac{h^4}{(8)(4)\sqrt{\pi}} f^{(9)}\left(\frac{1}{2}\right)$$

Which proves the theorem.

### 3.5. Spline of Degree $6\alpha$ Case (Existence and Uniqueness)

We suppose that  $s^{(1)}\left(\frac{1}{2}\right)(x) \in C^6[0,1]$  and  $s(x)$  in each subinterval  $[x_i, x_{i+1}]$  has a form:

$$s(x) = a_i(x-x_i)^3 + b_i(x-x_i)^{\frac{5}{2}} + c_i(x-x_i)^2 + d_i(x-x_i) \tag{15}$$

Which deduces the following theorem:

$$\frac{3}{2} + e_i(x-x_i) + f_i$$

**Theorem 6.** Let  $s(x)$  be the fractional spline defined in Section 3.5. Given the real numbers  $s_i^{(1)}\left(\frac{1}{2}\right) = f_i^{(1)}\left(\frac{1}{2}\right), s_i^{(3)}\left(\frac{3}{2}\right) = f_i^{(3)}\left(\frac{3}{2}\right), s_i^{(5)}\left(\frac{5}{2}\right) = f_i^{(5)}\left(\frac{5}{2}\right) (i=0(1)n)$  and  $f_0$ , there exist a unique  $s(x)$  such that

$$s_i^{(1)}\left(\frac{1}{2}\right) = f_i^{(1)}\left(\frac{1}{2}\right), s_i^{(3)}\left(\frac{3}{2}\right) = f_i^{(3)}\left(\frac{3}{2}\right), s_i^{(5)}\left(\frac{5}{2}\right) = f_i^{(5)}\left(\frac{5}{2}\right), (i=0(1)n) \tag{16}$$

$$s_0 = f_0$$

The fractional spline which satisfies (16) in  $[x_i, x_{i+1}]$  is of the form:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h^{\frac{1}{2}} \left[ f_i^{(1)}\left(\frac{1}{2}\right) A_2(t) + f_{i+1}^{(1)}\left(\frac{1}{2}\right) A_3(t) \right] + h^{\frac{3}{2}} \left[ f_i^{(3)}\left(\frac{3}{2}\right) A_4(t) + f_{i+1}^{(3)}\left(\frac{3}{2}\right) A_5(t) \right] + h^{\frac{5}{2}} f_i^{(5)}\left(\frac{5}{2}\right) A_6(t) \tag{17}$$

where

$$A_0(t) = \frac{1}{3} \left( 176t^{\frac{9}{2}} - 99t^{\frac{7}{2}} - 80t^{\frac{11}{2}} + 3 \right) + 1,$$

$$A_1(t) = \frac{1}{3} t^{\frac{7}{2}} (80t^2 - 176t + 99),$$

$$A_2(t) = -\frac{2}{63\sqrt{\pi}} t^{\frac{1}{2}} (656t^5 - 1520t^4 + 927t^3 - 63),$$

$$A_3(t) = -\frac{256}{63\sqrt{\pi}} t^{\frac{7}{2}} (8t-9)(t-1), \tag{18}$$

$$A_4(t) = -\frac{4}{945\sqrt{\pi}} t^{\frac{3}{2}} (1360t^4 - 3376t^3 + 2331t^2 - 315),$$

$$A_5(t) = \frac{256}{945\sqrt{\pi}} t^{\frac{7}{2}} (10t-9)(t-1),$$

$$A_6(t) = -\frac{8}{945\sqrt{\pi}} t^{\frac{5}{2}} (4t-3)(20t-21)(t-1)$$

and  $x = x_i + th, t \in [0,1]$ , with a similar expression for  $s(x)$  in  $[x_{i-1}, x_i]$ .

The coefficient  $s_i$  in (17) are given by the recurrence formula:

$$\frac{1155}{16} \sqrt{\pi} (s_i - s_{i-1}) = h^{\frac{1}{2}} \left[ \frac{355}{8} f_{i-1}^{(1)}\left(\frac{1}{2}\right) + 100 f_i^{(1)}\left(\frac{1}{2}\right) \right] + h^{\frac{3}{2}} \left[ \frac{115}{12} f_i^{(3)}\left(\frac{3}{2}\right) - \frac{40}{3} f_{i-1}^{(3)}\left(\frac{3}{2}\right) \right] + h^{\frac{5}{2}} \left[ f_{i-1}^{(5)}\left(\frac{5}{2}\right) + \frac{59}{12} f_i^{(5)}\left(\frac{5}{2}\right) \right] \tag{19}$$

$$s_0 = f_0$$

*Proof.* In this case we can express any  $p(t)$  in  $[0,1]$  in the following form:

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0^{(1)}\left(\frac{1}{2}\right) A_2(t) + p_1^{(1)}\left(\frac{1}{2}\right) A_3(t) + p_0^{(3)}\left(\frac{3}{2}\right) A_4(t) + p_1^{(3)}\left(\frac{3}{2}\right) A_5(t) + p_0^{(5)}\left(\frac{5}{2}\right) A_6(t)$$

and to determine the coefficients  $A_j, j=0,1,\dots,6$ , we

write the above equality for  $p(t) = 1, t^{\frac{1}{2}}, t^{\frac{3}{2}}, t^{\frac{5}{2}}, t^{\frac{7}{2}}, t^{\frac{9}{2}}, t^{\frac{11}{2}}$ . By the same technique of theorem 1 we obtain the desired result and consequently the proof is completed.

### 3.6. Error Bounds for the Fractional Spline of Degree $46$ Case

Error estimates for the fractional spline that we have mentioned in Section 3.5 are explained by the following theorem:

**Theorem 5.** Suppose that  $s(x)$  be the fractional spline defined in section 3.3,  $f^{(1)}\left(\frac{1}{2}\right)(x) \in C^6[0,1]$  and that  $f^{(p)}(0) = 0, p = 1, 2, \dots, 6$ , then for any  $x \in [0,1]$  we have

$$|s(x) - f(x)| \leq \frac{h^6}{(32)(6!)\sqrt{\pi}} f^{(13)}\left(\frac{13}{2}\right) \tag{20}$$

*Proof.* Because  $s^{(1)}\left(\frac{1}{2}\right)(x)$  is the Hermite interpolation polynomial of degree 3 matching  $f^{(1)}\left(\frac{1}{2}\right), f^{(3)}\left(\frac{3}{2}\right), f^{(5)}\left(\frac{5}{2}\right)$  at  $x = x_i, x_{i+1}$ . So for any  $x \in [x_i, x_{i+1}]$  we have using (6) with

$$m = 3, g = f^{(1)}\left(\frac{1}{2}\right), \text{ and } p_3 = s^{(1)}\left(\frac{1}{2}\right)$$

$$|s(x) - f(x)| \leq \frac{h^6}{(32)(6!)\sqrt{\pi}} D^6 D^{\frac{1}{2}} f$$

and following [11], we have

$$D^6 D^{\frac{1}{2}} f = D^{\frac{13}{2}} f = f^{(13)}\left(\frac{13}{2}\right)$$

This gives

$$|s(x) - f(x)| \leq \frac{h^6}{(32)(6!)\sqrt{\pi}} f^{(13)}\left(\frac{13}{2}\right)$$

Which proves the theorem.

### 4. Algorithms

The following remarks are needed in solving a problem:

1. Note that the above formulation and analysis was done in  $[0,1]$ . However, this does not constitute a serious restriction since the same techniques can be carried out for the general interval  $[a,b]$ . This is achieved using the linear transformation

$$x = \frac{1}{b-a}t - \frac{a}{b-a} \tag{21}$$

Form  $[a,b]$  to  $[0,1]$ .

2. Use the equations (5), (13) and (19) to compute  $s_i, (i=1(1)n)$ , respectively, in each cases.
3. Use the equations (3), (11) and (17) to compute  $s(x)$  at  $n$  equally spaced points in each subinterval  $[x_i, x_{i+1}] (i=1(1)n-1)$  and in each cases.

### 5. Illustrations Results

To illustrate our methods as error estimates has been found in theorems (3, 5 and 7) and to compare each of them with the other one, we have solved two examples of fractional equation. We have implemented all of problems' calculations with MATLAB 12b.

**Example 1.** Consider the following fractional differential equation

$$f^{(1/2)}(x) - \sqrt{\pi}x^{15} = 0, x \in [0,1], \tag{22}$$

with  $f(0) = 0$

For which, all actual error bounds for each cases are presented in Table 1,

**Table 1. The observed maximum errors**

h	Fractional Splines		
	Degree $2\alpha$	Degree $4\alpha$	Degree $6\alpha$
1/10	1.2187E-01	6.2841E-04	4.5822E-07
1/20	3.0468E-02	3.9276E-05	7.1597E-09
1/30	7.6171E-03	2.4547E-06	1.1187E-10
1/40	1.9042E-03	1.5342E-07	1.7479E-12

**Example 2.** Let

$$f(x) = x^{13} = 0, x \in [0,1] \tag{23}$$

For which, all actual error bounds for each cases are presented in Table 2.

**Table 2. The observed maximum errors**

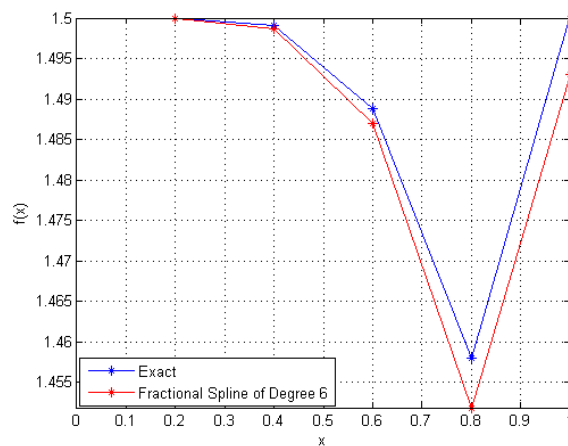
h	Fractional Splines		
	Degree $2\alpha$	Degree $4\alpha$	Degree $6\alpha$
1/10	3.4549E-01	8.6372E-04	1.4395E-07
1/20	8.6372E-02	5.3982E-05	2.2492E-09
1/30	2.1593E-02	3.3739E-06	3.5145E-11
1/40	5.3982E-03	2.1087E-07	5.4914E-13

**Example 3.** Consider the following fractional differential equation

$$f^{(1/2)}(x) - \frac{40320}{\Gamma(8.5)}x^{15} + \frac{5040}{\Gamma(7.5)}x^{13} = 0, \tag{24}$$

$$f(0) = 1.5, x \in [0,1],$$

Numerical and exact solutions are presented in Table 3. we give here the fractional spline of degree  $6\alpha$  for  $h = 0.1$ . Also, the exact and numerical solutions are demonstrated in Figure 1. for  $h = 0.2$ .



**Figure 1.** Exact and approximate solutions of Example 3 with  $h = 0.2$ .

**Table 3. Exact, approximate and absolute error**

x	Exact	Approximate	Absolute Error
0.0	1.5000	1.50000	0
0.1	1.4999	1.49999	2.06501E-08
0.2	1.49998	1.49999	8.21765E-06
0.3	1.49984	1.49997	1.28093E-04
0.4	1.49901	1.49984	8.26040E-04
0.5	1.49609	1.49935	3.26386E-03
0.6	1.48880	1.49806	9.25965E-03
0.7	1.47529	1.49542	2.01315E-02
0.8	1.45805	1.49144	3.33924E-02
0.9	1.45217	1.48816	3.59953E-02
1.0	1.50000	1.49257	7.42376E-03

### 6. Conclusions

In this paper, the existence and uniqueness of three fractional splines of degree  $m\alpha, m = 2, 4, 6, \alpha = 0.5$  are derived and in each case we have obtained direct simple formulas. These formulas are agree with those obtained for degree of integer, such as in [3], where a different approach was used. Also, Error estimates are derived which, together with the numerical results, showed the method to be efficient.

### References

- [1] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives-Theory and Applications, Gordon and Breach Science, Amsterdam, 1993.
- [2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
- [3] J. E. Phythian and R. Williams, Direct cubic spline approximation to integrals, Int. j. numer. methodseng., 23 (1986), 305-315.
- [4] P. G. Clarfelt, M. H. Schultz and R. S. Varga, Numerical methods of high order accuracy, Numer. Math., 9 (1967), 394-430.

- [5] Zahra W. K. and Elkholy S. M., Quadratic spline solution for boundary value problem of fractional order, *Numer Algor*, 59 (2012), 373-391.
- [6] G. Micula, T. Fawzy, and Z. Ramadan, A polynomial spline approximation method for solving system of ordinary differential equations, *Babes-Bolyai Cluj-Napoca. Mathematica*, vol. 32, no. 4 (1987) 55-60.
- [7] M. A. Ramadan, Spline solutions of first order delay differential equations, *Journal of the Egyptian Mathematical Society*, vol. 13, no. 1 (2005) 7-18.
- [8] G. Birkhoff and A. Priver, Hermite interpolation errors for derivatives, *J. Math. Phys.*, 46 (1967) 440-447.
- [9] A. K. Varma and G. Howell, Best error bounds for derivatives in two point Birkhoff interpolation problems, *J. Approx. Theory*, 38 (1983) 258-268.
- [10] Richard Herrmann, *Fractional calculus: an introduction for physicists*, Giga Hedron, Germany, 2nd edition, 2014.
- [11] M. Ishteva, Properties and applications of the Caputo fractional operator, Msc. Thesis, Dept. of Math., Universit ă Karlsruhe (TH), Sofia, Bulgaria, 2005.
- [12] A. K. Varma and G. Howell, Cantor-type cylindrical coordinate method for differential equations with local fractional derivatives, *Physics Letters A*, 377 (28) (2013) 1696-1700.
- [13] A. K. Varma and G. Howell, Thermal impedance estimations by semi-derivatives and semi-integrals: 1-D semiinfinite cases, *Thermal Science*, 17 (2) (2013) 581-589.
- [14] Faraidun K. Hamasalh and Pshtiwan O. Muhammad, Generalized Quartic Fractional Spline Interpolation with Applications, *Int. J. Open Problems Compt. Math*, Vol. 8, No. 1 (2015), 67-80.