INDEPENDENCE IN ABSTRACT ELEMENTARY CLASSES

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Abstract. We study general methods to build forking-like notions in the framework of tame abstract elementary classes (AECs) with amalgamation. We show that whenever such classes are categorical in a high-enough cardinal, they admit a good frame: a forking-like notion for types of singleton elements.

Theorem 0.1 (Superstability from categoricity)**.** Let K be a $(<\kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > \text{LS}(K)$ and K is categorical in a $\lambda > \kappa$, then:

- K is stable in all cardinals $\geq \kappa$.
- K is categorical in κ.
- There is a type-full good λ-frame with underlying class K_λ.

Under more locality conditions, we prove that the frame extends to a global independence notion (for types of arbitrary length).

Theorem 0.2 (A global independence notion from categoricity)**.** Let K be a densely type-local, fully tame and type short AEC with amalgamation. If K is categorical in unboundedly many cardinals, then there exists $\lambda \geq \text{LS}(K)$ such that $K_{\geq \lambda}$ admits a global independence relation with the properties of forking in a superstable first-order theory.

Modulo an unproven claim of Shelah, we deduce that Shelah's categoricity conjecture follows from the weak generalized continuum hypothesis and the existence of unboundedly many strongly compact cardinals.

Contents

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1. Introduction

Independence (or forking) is one of the central notions of modern classification theory. In the first-order setup, it was introduced by Shelah [She78] and is one of the main device of his book. One can ask whether there is such a notion in the nonelementary context. In homogeneous model theory, this was investigated in [HL02] for the superstable case and [BL03] for the simple and stable cases. Some of their results were later generalized by Hyttinen and Kesälä [HK06] to tame and $\aleph_0$-stable finitary abstract elementary classes[1]. What about general[2] abstract elementary classes? There we believe the answer is still a work in progress.

Already in [She99, Remark 4.9.1] it was asked by Shelah whether there is such a notion as forking in AECs. In his book on AECs [She09], the central concept is that of a good frame (a local independence notion for types of singletons) and some conditions are given for their existence. Shelah's main construction (see [She09, Theorem II.3.7]) uses categoricity in two successive cardinals and non-ZFC principles like the

[1]Note that by [HK06, Theorem 4.11], such classes are actually fully $(< \aleph_0)$-tame and short.

[2]For a discussion of how the framework of tame AECs compare to other non first-order frameworks, see the introduction of [Vasb].

weak diamond[3]. We argue that replacing Shelah's strong local model-theoretic hypotheses by the global hypotheses of amalgamation and tameness (a locality property for types introduced by Grossberg and VanDieren [GV06b]) makes life easier: the theorems are in ZFC and their proofs are usually much simpler. Furthermore, one can argue that any "reasonable" AEC should be tame and have amalgamation, see for example the discussion in Section 5 of [BG], and the introductions of [Bon14b] or [GV06b]. Examples of the use of tameness and amalgamation include [GV06c] (an upward categoricity transfer theorem which Boney [Bon14b] used to prove that Shelah's categoricity conjecture for a successor follows from unboundedly many strongly compact) and [Bon14a, BVb, Jarb], showing that good frames behave well in tame classes.

In [Vasa], we managed to build good frames in ZFC using global model-theoretic hypotheses: tameness, amalgamation, and categoricity in a cardinal of high-enough cofinality. However we were unable to remove the assumption on the cofinality of the cardinal or to show that the frame was *successful*, a key technical property of frames. Both in Shelah's book and in [Vasa], the question of whether there exists a *global* independence notion (for longer types) was also left open. In this paper, we continue working in ZFC with tameness and amalgamation, and make progress toward these problems. Regarding the cofinality of the categoricity cardinal, we show (Theorem 10.14):

Theorem 1.1. Let K be a $(< \kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > \text{LS}(K)$ and K is categorical in a $\lambda > \kappa$, then there is a type-full good λ-frame with underlying class K_λ.

As a consequence, the class K above has many superstable-like properties: for all $\lambda' \geq \lambda$, K is stable[4] in λ' (this is also part of Theorem 10.14) and has a unique limit model of cardinality λ' (by e.g. [BVb, Theorem 1.1] and [She09, Lemma II.4.8], see [Bon14a, Theorem 9.2] for

[3]Shelah claims to construct a good frame in ZFC in [She09, Theorem IV.4.10] but he has to change the class and still uses the weak diamond to show his frame is successful.

[4]A downward stability transfer from categoricity is well known (see e.g [She99, Claim 1.7]), but the upward transfer is new and improves on [Vasa, Theorem 7.5]. In fact, the proof is new even when K is the class of models of a first-order (uncountable) theory.

a detailed proof). The proof of Theorem 1.1 also yields a downward categoricity transfer[5]:

Theorem 1.2. Let K be a $(< \kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > \text{LS}(K)$ and K is categorical in a $\lambda > \kappa$, then K is categorical in κ.

The construction of the good frame in the proof of Theorem 1.1 is similar to that in [Vasa] but uses local character of coheir (or $(< \kappa)$-satisfiability) rather than splitting. A milestone study of coheir in the nonelementary context is [MS90], working in classes of models of an $L_{\kappa,\omega}$-sentence, κ a strongly compact cardinal. Makkai and Shelah's work was generalized to fully tame and short AECs in [BG], and some results were improved in [Vasb]. Building on these works, we are able to show that under the assumptions above, coheir has enough superstability-like properties to apply the methods of [Vasa], and obtain that coheir restricted to types of length one in fact induces a good frame.

Note that coheir is a candidate for a global independence relation. In fact, one of the main result of [BGKV] is that it is canonical: if there is a global forking-like notion, it must be coheir. Unfortunately, the paper assumes that coheir has the extension property, and it is not clear that it is a reasonable assumption. Here, we prove that coheir is canonical without this assumption (Theorem 9.3). We also obtain results on the canonicity of good frames. For example, any two type-full good λ-frames with the same *categorical* underlying AEC must be the same (Theorem 9.7). This answers several questions asked in [BGKV].

Using that coheir is global and (under categoricity) induces a good frame, we can use more locality assumptions to get that the good frame is successful (Theorem 15.6):

Theorem 1.3. Let K be a fully $(< \kappa)$-tame and short AEC. If $\text{LS}(K) < \kappa = \beth_\kappa < \lambda = \beth_\lambda$ and K is categorical in a $\lambda' \geq \lambda$, then there exists an *ω-successful* type-full good λ-frame with underlying class K_λ.

We believe that the locality hypotheses in Theorem 1.3 are reasonable: they follow from large cardinals [Bon14b] and slightly weaker assumptions can be derived from the existence of a global forking-like notion, see the discussion in Section 15.

[5]Recall that [MS90, Conclusion 5.1] proved a similar conclusion with much stronger assumptions (namely that K is the class of models of an $L_{\kappa,\omega}$ sentence, κ a strongly compact cardinal).

Theorem 1.3 can be used to build a global independence notion (Theorem 15.1 formalizes Theorem 0.2 from the abstract). Unfortunately we assume one more locality hypothesis (dense type-locality) there, but we suspect it can be removed, see the discussion in Section 15. Without dense type-locality, one still obtains an independence relation for types of length less than or equal to λ.

These results bring us closer to solving one of the main test questions in the classification theory of abstract elementary classes (a version of which already appears as [She90, Open problem D.3(a)], see [Gro02] for history and motivation):

Conjecture 1.4 (Shelah's categoricity conjecture)**.** If K is an AEC that is categorical in unboundedly many cardinals, then K is categorical on a tail of cardinals.

Even without ω-successful good frames, Theorem 1.2 gives[6]:

Theorem 1.5. Let K be an $\text{LS}(K)$-tame AEC with amalgamation. If K is categorical in unboudedly many cardinals, then K is categorical on a closed unbounded set of cardinals (the set of $\kappa > \text{LS}(K)$ such that $\kappa = \beth_\kappa$).

Now the power of ω-successful frames comes from Shelah's analysis in Chapter III of his book. Unfortunately, Shelah could not quite prove the stronger results he had hoped for. In [She09, Discussion III.12.40], Shelah claims the following (a proof should appear in [She]):

Claim 1.6. Assume the weak generalized continuum hypothesis[7] (WGCH). Let K be an AEC such that there is an ω-successful good λ-frame with underlying class K_λ. Then K is categorical in some[8] $\mu > \lambda^{+\omega}$ if and only if K is categorical in all $\mu > \lambda^{+\omega}$.

Modulo this claim, we obtain the consistency of Shelah's categoricity conjecture from large cardinals:

Theorem 1.7. Assume Claim 1.6 and WGCH.

(1) Shelah's categoricity conjecture holds for fully tame and short AECs with amalgamation.

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[6] In [She09, Section IV.3], Shelah proves a slightly weaker conclusion from fewer assumptions.

[7] Namely, $2^\lambda < 2^{\lambda^+}$ for all cardinals λ.

[8] Note that (in contrast to previous works of Shelah, e.g. [She99]) μ is *not* assumed to be a successor cardinal.

(2) If there exists unboundedly many strongly compact cardinals, then Shelah's categoricity conjecture holds.

Proof. Let K be an AEC which is categorical in unboundedly many cardinals.

(1) Assume K is fully $\text{LS}(K)$-tame and short and has amalgamation. Let $\text{LS}(K) < \kappa = \beth_\kappa < \lambda = \beth_\lambda$. By Theorem 1.3, there is an ω-successful good λ-frame on K_λ. By Claim 1.6, K is categorical in all $\mu > \lambda^{+\omega}$.
(2) Let $\kappa > \text{LS}(K)$ be strongly compact. By [Bon14b], K is fully $(< \kappa)$-tame and short. By the methods of [MS90, Proposition 1.13], $K_{\geq \kappa}$ has amalgamation. Now apply the previous part to $K_{\geq \kappa}$.

□

Note that [She09, Theorem IV.12] is stronger than Theorem 1.7 (since Shelah assumes only amalgamation) but we haven't checked Shelah's argument.

This paper is organized as follows. In Section 2 we give review some background. In Sections 3-4, we introduce the framework with which we will study independence. In Sections 5-8, we introduce the definition of a *generator* for an independence relation and show how to use it to build good frames. In Section 9, we use the theory of generators to prove results on the canonicity of coheir and good frames. In Section 10, we use generators to give a general definition of superstability (closely related to those implicit in [GVV, Vasa]). We use it to build good frames and show it follows from categoricity. In Section 11, we show how to prove a good frame is ω-successful provided it is induced by coheir. In Sections 12-14, we show how to extend such a frame to a global independence relation. Finally, in Section 15, we prove some of the main theorems of this paper and conclude.

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2. Preliminaries

We review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed. We refer the reader to the preliminaries of [Vasb] for more motivation on some of the definitions below.

2.1. Set theoretic terminology.

Notation 2.1. When we say that $\mathcal{F}$ *is an interval of cardinals*, we mean that $\mathcal{F} = [\lambda, \theta)$, the set of cardinals μ such that $\lambda \leq \mu < \theta$. Here, $\lambda \leq \theta$ are (possibly finite) cardinals except we also allow $\theta = \infty$.

We will often use the following function:

Definition 2.2 (Hanf function)**.** For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$.

Note that for λ infinite, $\lambda = \beth_\lambda$ if and only if for all $\mu < \lambda$, $h(\mu) < \lambda$.

Definition 2.3. For κ an infinite cardinal, let κ_r be the least regular cardinal $\geq \kappa$. That is, κ_r is κ^+ if κ is singular and κ otherwise.

2.2. Abstract classes.

Recall [Vasb, Definition 2.6] that an abstract class is a pair $(K, \leq)$, where K is a class of structures of the same (possibly infinitary) language and $\leq$ is an ordering on K extending substructure and respecting isomorphisms. We will use the same notation as in [Vasb]; for example $M < N$ means $M \leq N$ and $M \neq N$.

Definition 2.4. Let K be an abstract class and let R be a binary relation on K. A sequence $\langle M_i : i < \delta \rangle$ of elements of K is *R-increasing* if for all $i < j < \delta$, $M_i R M_j$. When $R = \leq$, we omit it. *Strictly increasing* means $<$-increasing. $\langle M_i : i < \delta \rangle$ is *continuous* if for all limit $i < \delta$, $M_i = \bigcup_{j<i} M_j$.

Notation 2.5. For K an abstract class, $\mathcal{F}$ an interval of cardinals, we write $K_{\mathcal{F}} := \{M \in K \mid \|M\| \in \mathcal{F}\}$. When $\mathcal{F} = \{\lambda\}$, we write K_λ for $K_{\{\lambda\}}$. We also use notation like $K_{\geq \lambda}$, $K_{<\lambda}$, etc.

Definition 2.6. An abstract class K is *in* $\mathcal{F}$ if $K_{\mathcal{F}} = K$.

We now recall the definition of an abstract elementary class (AEC) in $\mathcal{F}$, for $\mathcal{F}$ an interval of cardinal. Localizing to an interval is convenient when dealing with good frames and appears already (for $\mathcal{F} = \{\lambda\}$) in [JS13, Definition 1.0.3.2]. Confusingly, Shelah earlier on called an AEC in λ a λ-AEC (in [She09, Definition II.1.18]).

Definition 2.7. For $\mathcal{F} = [\lambda, \theta)$ an interval of cardinals, we say an abstract class K in $\mathcal{F}$ is an *abstract elementary class* (AEC for short) *in* $\mathcal{F}$ if it satisfies:

(1) Coherence: If M_0, M_1, M_2 are in K, $M_0 \leq M_2$, $M_1 \leq M_2$, and $|M_0| \subseteq |M_1|$, then $M_0 \leq M_1$.
(2) $L(K)$ is finitary.
(3) Tarski-Vaught axioms: If $\langle M_i : i < \delta \rangle$ is an increasing chain in K and $\delta < \theta$, then $M_\delta := \bigcup_{i<\delta} M_i$ is such that:
  (a) $M_\delta \in K$.
  (b) $M_0 \leq M_\delta$.
  (c) If $M_i \leq N$ for all $i < \delta$, then $M_\delta \leq N$.
(4) Löwenheim-Skolem axiom: There exists a cardinal $\lambda \geq |L(K)| + \aleph_0$ such that for any $M \in K$ and any $A \subseteq |M|$, there exists $M_0 \leq M$ containing A with $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(K)$ (the *Löwenheim-Skolem number of* K) for the least such cardinal.

When $\mathcal{F} = [0, \infty)$, we omit it. We say K is an *AEC in* λ if it is an AEC in $\{\lambda\}$.

Recall that an AEC in $\mathcal{F}$ can be made into an AEC:

Fact 2.8 (Lemma II.1.23 in [She09])**.** If K is an AEC in $\lambda := \mathrm{LS}(K)$, then there exists a unique AEC K' such that $(K')_\lambda = K$ and $\mathrm{LS}(K') = \lambda$. The same holds if K is an AEC in $\mathcal{F}$, $\mathcal{F} = [\lambda, \theta)$ (apply the previous sentence to K_λ).

Notation 2.9. Let K be an AEC in $\mathcal{F}$ with $\mathcal{F} = [\lambda, \theta)$, $\lambda = \mathrm{LS}(K)$. Write K^{up} for the unique AEC K' described by Fact 2.8.

When studying independence, the following definition will be useful:

Definition 2.10. An ∞-*AEC* (or AC with coherence) in $\mathcal{F}$ is an abstract class in $\mathcal{F}$ satisfying the coherence property (see Definition 2.7).

We also define the following weakening of the existence of a Löwenheim-Skolem number:

Definition 2.11. An abstract class K is $(< \lambda)$-*closed* if for any $M \in K$ and $A \subseteq |M|$ with $|A| < \lambda$, there exists $M_0 \leq M$ which contains A and has size less than λ. λ-closed means $(< \lambda^+)$-closed.

Remark 2.12. An AEC K is $(< \lambda)$-closed in every $\lambda > \mathrm{LS}(K)$.

We will sometimes use the following consequence of Shelah's presentation theorem:

Fact 2.13 (Conclusion I.1.11 in [She09])**.** Let K be an AEC. If $K_{\geq\lambda} \neq \emptyset$ for every $\lambda < h(\text{LS}(K))$, then K has arbitrarily large models.

As in the preliminaries of [Vasb], we can define a notion of embedding for abstract classes and go on to define amalgamation, joint embedding, no maximal models, Galois types, tameness, and type-shortness (that we will just call shortness).

The following fact tells us that an AEC with amalgamation is a union of AECs with amalgamation and joint embedding:

Fact 2.14 (Lemma 16.14 in [Bal09])**.** Let K be an AEC with amalgamation. Then we can write $K = \bigcup_{i\in I} K^i$ where the K^i's are disjoint AECs with $\text{LS}(K^i) = \text{LS}(K)$ and each K^i has joint embedding and amalgamation.

Using Galois types, a natural notion of saturation can be defined (see [Vasb, Definition 2.21] for more explanation on the definition):

Definition 2.15. Let K be an abstract class and μ be an infinite cardinal.

(1) For $N \in K$, $A \subseteq |N|$ is *μ-saturated in N* if for any $A_0 \subseteq A$ of size less than μ, any $p \in \text{gS}^{<\mu}(A_0; N)$ is realized inside A.
(2) A model $M \in K$ is *μ-saturated* if it is μ-saturated in N for all $N \geq M$. When $\mu = \|M\|$, we omit it.
(3) We write $K^{\mu\text{-sat}}$ for the class of μ-saturated models of $K_{\geq\mu}$ (ordered by the ordering of K).

Remark 2.16. By [She09, Lemma II.1.14], if K is an AEC with amalgamation and $\mu > \text{LS}(K)$, $M \in K$ is μ-saturated if and only if for all $N \geq M$ and all $A_0 \subseteq |M|$ with $|A_0| < \mu$, any $p \in \text{gS}(A_0; N)$ is realized in M. That is, it is enough to consider types of length 1 in the definition. We will use this fact freely.

Finally, we recall there is a natural notion of stability in this context. Our definition follows [Vasb, Definition 2.19].

Definition 2.17 (Stability)**.** Let α be a cardinal, μ be a cardinal. A model $N \in K$ is *$(<\alpha)$-stable in μ* if for all $A \subseteq |N|$ of size $\leq \mu$, $|\text{gS}^{<\alpha}(A; N)| \leq \mu$. Here and below, α-stable means $(<(\alpha^+))$-stable. We say "stable" instead of "1-stable".

K is *$(<\alpha)$-stable in μ* if every $N \in K$ is $(<\alpha)$-stable in μ. K is *$(<\alpha)$-stable* if it is $(<\alpha)$-stable in unboundedly many cardinals.

A corresponding definition of the order property in AECs appears in [She99, Definition 4.3]. For simplicity, we have removed one parameter from the definition.

Definition 2.18. Let α and μ be cardinals and let K be an abstract class. A model $M \in K$ has the *α-order property of length μ* if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside M with $\ell(\bar{a}_i) = \alpha$ for all $i < \mu$, such that for any $i_0 < j_0 < \mu$ and $i_1 < j_1 < \mu$, $\text{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \text{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N)$.

M has the *$(<\alpha)$-order property of length μ* if it has the β-order property of length μ for some $\beta < \alpha$. M has the *order property of length μ* if it has the α-order property of length μ for some α.

K has the α-order of length μ if some $M \in K$ has it. *K has the order property* if it has the order property for every length.

For completeness, we also recall the definition of the following semantic variation of the $(<\kappa)$-order property of length κ that appears in [BG, Definition 4.2] (but is adapted from a previous definition of Shelah, see there for more background):

Definition 2.19. Let K be an AEC. For $\kappa > \text{LS}(K)$, K has the *weak κ-order property* if there are $\alpha, \beta < \kappa$, $M \in K_{<\kappa}$, $N \geq M$, types $p \neq q \in \text{gS}^{\alpha+\beta}(M)$, and sequences $\langle \bar{a}_i : i < \kappa \rangle$, $\langle \bar{b}_i : i < \kappa \rangle$ from N so that for all $i, j < \kappa$:

(1) $i \leq j$ implies $\text{gtp}(\bar{a}_i\bar{b}_j/M; N) = p$.
(2) $i > j$ implies $\text{gtp}(\bar{a}_i\bar{b}_j/M; N) = q$.

The following sums up all the results we will use about stability and the order property:

Fact 2.20. Let K be an AEC.

(1) [Vasb, Lemma 4.8] Let $\kappa = \beth_\kappa > \text{LS}(K)$. The following are equivalent:
  (a) K has the weak κ-order property.
  (b) K has the $(<\kappa)$-order property of length κ.
  (c) K has the $(<\kappa)$-order property.
(2) [Vasb, Theorem 4.13] Assume K is $(<\kappa)$-tame and has amalgamation. The following are equivalent:
  (a) K is stable in some $\lambda \geq \kappa + \text{LS}(K)$.
  (b) There exists $\mu \leq \lambda_0 < h(\kappa + \text{LS}(K))$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda = \lambda^{<\mu}$.
  (c) K does not have the order property.
  (d) K does not have the $(<\kappa)$-order property.

(3) [BKV06, Theorem 4.5] If K is $\mathrm{LS}(K)$-tame, has amalgamation, and is stable in $\mathrm{LS}(K)$, then it is stable in $\mathrm{LS}(K)^+$.

2.3. Universal and limit extensions.

Definition 2.21. Let K be an abstract class, λ be a cardinal.

(1) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if whenever we have $N' \geq N$ such that $\|N'\| = \|N\|$ and $M \leq M' \leq N'$, then there exists $f : M' \xrightarrow[M]{} N$. Say $M \leq_{\text{univ}} N$ if and only if $M = N$ or $M <_{\text{univ}} N$.
(2) For $M, N \in K$, λ a cardinal and $\delta \leq \lambda^+$, say $M <_{\lambda,\delta} N$ (N is *(λ, δ)-limit over* M) if and only if $M \in K_\lambda$, $N \in K_{\lambda+|\delta|}$, $M < N$, and there exists $\langle M_i : i \leq \delta \rangle$ increasing continuous such that $M_0 = M$, $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$, and $M_\delta = N$ if $\delta > 0$. Say $M \leq_{\lambda,\delta}$ if $M = N$ or $M <_{\lambda,\delta} N$. We say $N \in K$ is a (λ, δ)-*limit model* if $M <_{\lambda,\delta} N$ for some M. We say M is λ-*limit* if it is (λ, δ)-limit for some limit $\delta < \lambda^+$. When λ is clear from context, we omit it.

Remark 2.22. So for $M, N \in K_\lambda$, $M <_{\lambda,0} N$ if and only if $M < N$, while $M <_{\lambda,1}$ if and only if $M <_{\text{univ}} N$.

Remark 2.23. Variations on $<_{\lambda,\delta}$ already appear as [She99, Definition 2.1]. Our definition of being universal is different from the usual one (see e.g. [Van06, Definition I.2.1.2]): first, we only work locally as usual (but if amalgamation holds this does not matter), and second we ask only for $\|M\| \leq \|M'\| \leq \|N\|$ rather than $\|M'\| = \|M\|$.

Fact 2.24. Let K be an AC, λ be an infinite cardinal, and $\delta \leq \lambda^+$. Then:

(1) $M_0 <_{\text{univ}} M_1 \leq M_2$ implies $M_0 <_{\text{univ}} M_2$.
(2) If K has amalgamation, then $M_0 \leq M_1 <_{\text{univ}} M_2$ implies $M_0 <_{\text{univ}} M_2$.
(3) If K_λ has amalgamation and $M_0 \in K_\lambda$, then $M_0 \leq M_1 <_{\lambda,\delta} M_2$ implies $M_0 <_{\lambda,\delta} M_2$.
(4) If $\delta < \lambda^+$, K_λ is an AEC in $\lambda \geq \mathrm{LS}(K)$ with amalgamation, no maximal models, and stability in λ, then for any $M_0 \in K_\lambda$ there exists M_0' such that $M_0 <_{\lambda,\delta} M_0'$.

Proof. All are straightforward, except perhaps the last which is due to Shelah. For proofs and references see [Vasa, Proposition 2.12]. ☐

By a routine back and forth argument, we have:

Fact 2.25 (Fact 1.3.6 in [SV99])**.** Let K be an AEC in $\lambda := \text{LS}(K)$ with amalgamation. Let $\delta \leq \lambda^+$ be a limit ordinal and for $\ell = 1, 2$, let $\langle M_i^\ell : i \leq \delta \rangle$ be increasing continuous with $M_0 := M_0^1 = M_0^2$ and $M_i^\ell <_{\text{univ}} M_{i+1}^\ell$ for all $i < \delta$ (so they witness $M_0^\ell <_{\lambda,\delta} M_\delta^\ell$).

Then there exists $f : M_\delta^1 \cong_{M_0} M_\delta^2$ such that for all $i < \delta$, there exists $j < \delta$ such that $f[M_i^1] \leq M_j^2$ and $f^{-1}[M_i^2] \leq M_j^1$.

Remark 2.26. Uniqueness of limit models that are *not* of the same length (i.e. the statement $M_0 <_{\lambda,\delta} M_1$, $M_0 <_{\lambda,\delta'} M_2$ implies $M_1 \cong_{M_0} M_2$ for any limit $\delta, \delta' < \lambda^+$) has been argued to be an important dividing line, akin to superstability in the first-order theory. See for example [SV99, Van06, GVV].

We couldn't find a proof of the next result in the literature, so we included one here.

Lemma 2.27. Let K be an AEC with amalgamation. Let δ be a (not necessarily limit) ordinal and assume $(M_i)_{i \leq \delta}$ is increasing continuous with $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$. Then $M_i <_{\text{univ}} M_\delta$ for all $i < \delta$.

Proof. By induction on δ. If $\delta = 0$, there is nothing to do. If $\delta = \alpha + 1$ is a successor, let $i < \delta$. We know $M_i \leq M_\alpha$. By hypothesis, $M_\alpha <_{\text{univ}} M_\delta$. By Fact 2.24, $M_i <_{\text{univ}} M_\delta$. Assume now δ is limit. In that case it is enough to show $M_0 <_{\text{univ}} M_\delta$. By the induction hypothesis, we can further assume that $\delta = \text{cf}(\delta)$. Let $N \geq M_0$ be given such that $\mu := \|N\| \leq \|M_\delta\|$, and N, M_δ are inside a common model $\widehat{N}$. If $\mu < \|M_\delta\|$, then there exists $i < \delta$ such that $\mu \leq \|M_i\|$, and we can use the induction hypothesis, so assume $\mu = \|M_\delta\|$. We can further assume $\mu > \|M_0\|$, for otherwise N directly embeds into M_1 over M_0. The M_is show that $\gamma := \text{cf}(\mu) \leq \delta$. Let $\langle N_i : i \leq \gamma \rangle$ be increasing continuous such that for all $i < \gamma$.

(1) $N_0 = M_0$.
(2) $N_\gamma = N$.
(3) $\|N_i\| < \mu$.

This exists since $\gamma = \text{cf}(\mu)$.

Build $\langle f_i : i \leq \gamma \rangle$, increasing continuous such that for all $i < \gamma$, $f_i : N_i \xrightarrow[M_0]{} M_{k_i}$ for some $k_i < \delta$. This is enough, since then f_γ will be the desired embedding. This is possible: For $i = 0$, take $f_0 := \text{id}_{M_0}$. At limits, take unions: since δ is regular and $\gamma \leq \delta$, $k_j < \delta$ for all $j < i < \gamma$ implies $k_i := \sup_{j<i} k_j < \delta$.

Now given $i = j + 1$, first pick $k = k_j < \delta$ such that $f_j[N_j] \leq M_k$. Such a k exists by the induction hypothesis. Find $k' > k$ such that $\|N_i\| \leq \|M_{k'}\|$. This exists since $\|N_i\| < \mu = \|M_\delta\|$. Now by the induction hypothesis, $M_k <_{\text{univ}} M_{k'}$, so by Fact 2.24, $f_j[N_j] <_{\text{univ}} M_{k'}$. Hence by some renaming, we can extend f_i as desired. □

Remark 2.28. $(K, \leq_{\text{univ}})$ is in general not an AEC as it may fail the Löwenheim-Skolem axiom, the coherence axiom, and the last of the Tarski-Vaught axioms.

3. Independence relations

Since this section mostly lists definitions, the reader already familiar with independence (e.g. in the first-order context) may want to skip it and refer to it as needed. We would like a general framework in which to study independence. One such framework is Shelah's good λ-frames [She09, Section II.6]. Another is given by the definition of independence relation in [BGKV, Definition 3.1] (itself adapted from [BG, Definition 3.3]). Both definitions describe a relation "p does not fork over M" for p a Galois type over N and $M \leq N$.

In [BGKV], it is also shown how to "close" such a relation to obtain a relation "p does not fork over M" when p is a type over an arbitrary set. We find that starting with such a relation makes the statement of symmetry transparent, and hence makes many proofs easier. Perhaps even more importantly, we can be very precise[9] when dealing with chain local character properties (see Definition 3.14). We also give a more general definition than [BGKV], as we do not assume that everything happens in a big homogeneous monster model, and we allow working inside ∞-abstract elementary classes[10] rather than only abstract elementary classes. The later feature is convenient when working with classes of saturated models.

Because we quote extensively from [She09], which deals with frames, and also because it is sometimes convenient to "forget" the extension

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[9]Assume for example that $\mathfrak{s}$ is a good-frame on a class of saturated models of an AEC K. Let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models. Let $M_\delta := \bigcup_{i<\delta} M_i$ and let $p \in \text{gS}(M_\delta)$. We would like to say that there is $i < \delta$ such that p does not fork over M_i but we may not know that M_δ is saturated, so maybe forking is not even defined for types over M_δ. However if the nonforking relation were defined for types over sets, there would be no problem.

[10]Recall (Definition 2.7 and the remark following it) that ∞-AECs are just ACs satisfying the coherence axiom.

of the relation to arbitrary sets, we will still define frames and recall their relationship with independence relations over sets.

3.1. **Frames.** Shelah's definition of a pre-frame appears in [She09, Definition III.0.2.1] and is meant to axiomatize the bare minimum of properties a relation must satisfy in order to be a meaningful independence notions.

We make several changes: we do not mention basic types (we have no use for them), so in Shelah's terminology our pre-frames will be *type-full*. In fact, it is notationally convenient for us to define our frame on every type, not just the nonalgebraic ones. The *disjointness* property (see Definition 3.10) tells us that the frame behaves trivially on the algebraic types. We do not require it (as it is not required in [BGKV, Definition 3.1]) but it will hold of all frames we consider.

We require that the class on which the independence relation operates has amalgamation[11], and we do not require that the base monotonicity property holds (this is to preserve the symmetry between right and left properties in the definition. All the frames we consider will have base monotonicity). Finally, we allow the size of the models to lie in an interval rather than just be restricted to a single cardinal as Shelah does. We also parametrize on the length of the types. This allows more flexibility and was already the approach favored in [Vasa, BVb].

Definition 3.1. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\aleph_0 \leq \lambda < \theta$, $\alpha \leq \theta$ be a cardinal or ∞.

A *type-full pre-*$(<\alpha, \mathcal{F})$*-frame* is a pair $\mathfrak{s} = (K, \mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}})$, where:

(1) K is an ∞-abstract elementary class in $\mathcal{F}$ (see Definition 2.7) with amalgamation.
(2) $\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}$ is a relation on quadruples of the form (M_0, A, M, N), where $M_0 \leq M \leq N$ are all in K, $A \subseteq |N|$ is such that $|A \backslash |M_0|| < \alpha$. We write $\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}(M_0, A, M, N)$ or $A \overset{N}{\underset{M_0}{\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}}} M$ instead of $(M_0, A, M, N) \in \mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}$.
(3) The following properties hold:
    (a) <u>Invariance</u>: If $f : N \cong N'$ and $A \overset{N}{\underset{M_0}{\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}}} M$, then $f[A] \overset{N'}{\underset{f[M_0]}{\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}}} f[M]$.
    (b) <u>Monotonicity</u>: Assume $A \overset{N}{\underset{M_0}{\mathop{\smile\kern-0.9em\raise0.6ex\hbox{$|$}}}} M$. Then:

[11] This is a required axiom in good frames, but not in pre-frames.

(i) Ambient monotonicity: If $N' \geq N$, then $A \overset{N'}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$. If $M \leq N_0 \leq N$ and $A \subseteq |N_0|$, then $A \overset{N_0}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$.

(ii) Left and right monotonicity: If $A_0 \subseteq A$, $M_0 \leq M' \leq M$, then $A_0 \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M'$.

(c) Left normality: If $A \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$, then[12] $AM_0 \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$.

When α or $\mathcal{F}$ are clear from context or irrelevant, we omit them and just say that $\mathfrak{s}$ is a pre-frame (or just a frame). We may omit the "type-full". A $(\leq \alpha)$-frame is just a $(< (\alpha+1))$-frame. We might omit α when $\alpha = 2$ (i.e. $\mathfrak{s}$ is a (≤ 1)-frame) and we might talk of a λ-frame or a $(\geq \lambda)$-frame instead of a $\{\lambda\}$-frame or a $[\lambda, \infty)$-frame.

Notation 3.2. For $\mathfrak{s} = (K, \mathop{\smile}\limits^{\mid})$ a pre-$(< \alpha, \mathcal{F})$-frame with $\mathcal{F} = [\lambda, \theta)$, write[13] $K_{\mathfrak{s}} := K$, $\underset{\mathfrak{s}}{\mathop{\smile}\limits^{\mid}} := \mathop{\smile}\limits^{\mid}$, $\alpha_{\mathfrak{s}} := \alpha$, $\mathcal{F}_{\mathfrak{s}} = \mathcal{F}$, $\lambda_{\mathfrak{s}} := \lambda$, $\theta_{\mathfrak{s}} := \theta$.

Notation 3.3. For $\mathfrak{s} = (K, \mathop{\smile}\limits^{\mid})$ a pre-frame, we write $\mathop{\smile}\limits^{\mid}(M_0, \bar{a}, M, N)$ or $\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$ for $\text{ran}(\bar{a}) \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$ (similarly when other parameters are sequences). When $p \in \text{gS}^{<\infty}(M)$, we say p *does not* $\mathfrak{s}$*-fork over* M_0 (or just *does not fork over* M_0 if $\mathfrak{s}$ is clear from context) if $\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$ whenever $p = \text{gtp}(\bar{a}/M; N)$ (using monotonicity and invariance, it is easy to check that this does not depend on the choice of representatives).

Remark 3.4. In the definition of a pre-frame given in [BVb, Definition 3.1], the left hand side of the relation $\mathop{\smile}\limits^{\mid}$ is a sequence, not just a set. Here, we simply assume outright that the relation is defined so that order does not matter.

Remark 3.5. We can go back and forth from our definition of pre-frame to Shelah's. We sketch how. From a pre-frame $\mathfrak{s}$ in our sense (with $K_{\mathfrak{s}}$ an AEC), we can let $S^{\text{bs}}(M)$ be the set of nonalgebraic $p \in \text{gS}(M)$ that do not $\mathfrak{s}$-fork over M. Then restricting $\underset{\mathfrak{s}}{\mathop{\smile}\limits^{\mid}}$ to the basic types we obtain (assuming that $\mathfrak{s}$ has base monotonicity, see Definition 3.10) a pre-frame in Shelah's sense. From a pre-frame $(K, \mathop{\smile}\limits^{\mid}, S^{\text{bs}})$ in Shelah's sense (where K has amalgamation), we can extend $\mathop{\smile}\limits^{\mid}$ by specifying

[12]For sets A and B, we sometimes write AB instead of $A \cup B$.

[13]Really, α, $\mathcal{F}$, and θ should be part of the data of the frame but we usually ignore this detail.

that algebraic and basic types do not fork over their domains. We then get a pre-frame $\mathfrak{s}$ in our sense with base monotonicity and disjointness.

3.2. **Independence relations.** We now give a definition for an independence notion that also takes sets on the right hand side.

Definition 3.6 (Independence relation)**.** Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\aleph_0 \leq \lambda < \theta$, $\alpha, \beta \leq \theta$ be cardinals or ∞. A $(<\alpha, \mathcal{F}, <\beta)$-*independence relation* is a pair $\mathfrak{i} = (K, \perp)$, where:

(1) K is an ∞-abstract elementary class in $\mathcal{F}$ with amalgamation.
(2) $\perp$ is a relation on quadruples of the form (M, A, B, N), where $M \leq N$ are all in K, $A \subseteq |N|$ is such that $|A \backslash |M|| < \alpha$ and $B \subseteq |N|$ is such that $|B \backslash |M|| < \beta$. We write $\perp(M, A, B, N)$ or $A \overset{N}{\underset{M}{\perp}} B$ instead of $(M, A, B, N) \in \perp$.
(3) The following properties hold:
   (a) Invariance: If $f : N \cong N'$ and $A \overset{N}{\underset{M}{\perp}} B$, then $f[A] \overset{N'}{\underset{f[M]}{\perp}} f[B]$.
   (b) Monotonicity: Assume $A \overset{N}{\underset{M}{\perp}} B$. Then:
      (i) Ambient monotonicity: If $N' \geq N$, then $A \overset{N'}{\underset{M}{\perp}} B$. If $M \leq N_0 \leq N$ and $A \cup B \subseteq |N_0|$, then $A \overset{N_0}{\underset{M}{\perp}} B$.
      (ii) Left and right monotonicity: If $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \overset{N}{\underset{M}{\perp}} B_0$.
   (c) Left and right normality: If $A \overset{N}{\underset{M}{\perp}} B$, then $AM \overset{N}{\underset{M}{\perp}} BM$.

When $\beta = \theta$, we omit it. More generally, when α, β are clear from context or irrelevant, we omit them and just say that $\mathfrak{i}$ is an independence relation.

We adopt the same notational conventions as for pre-frames.

Remark 3.7. It seems that in every case of interest $\beta = \theta$. We did not make it part of the definition to avoid breaking the symmetry between α and β. Note also that the case $\alpha > \lambda$ is of particular interest in Section 14.

Before listing the properties independence relations and frames could have, we discuss how to go from one to the other. The cl operation is called the *minimal closure* in [BGKV, Definition 3.4].

Definition 3.8.

(1) Given a pre-frame $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\mid})$, let $\text{cl}(\mathfrak{s}) := (K, \overset{\text{cl}}{\mathop{\smile\!\!\!\!\!\mid}})$, where $\overset{\text{cl}}{\mathop{\smile\!\!\!\!\!\mid}}(M, A, B, N)$ if and only if $M \leq N$, $|B| < \theta_{\mathfrak{s}}$, and there exists $N' \geq N$, $M' \geq M$ containing B such that $\mathop{\smile\!\!\!\!\!\mid}(M, A, M', N')$.
(2) Given a $(< \alpha, \mathcal{F})$-independence relation $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\mid})$ let $\text{pre}(\mathfrak{i}) := (K, \overset{\text{pre}}{\mathop{\smile\!\!\!\!\!\mid}})$, where $\overset{\text{pre}}{\mathop{\smile\!\!\!\!\!\mid}}(M, A, M', N)$ if and only if $M \leq M' \leq N$ and $\mathop{\smile\!\!\!\!\!\mid}(M, A, M', N)$.

Remark 3.9.

(1) If $\mathfrak{i}$ is a $(< \alpha, \mathcal{F})$-independence relation, then $\text{pre}(\mathfrak{i})$ is a pre-$(< \alpha, \mathcal{F})$-frame.
(2) If $\mathfrak{s}$ is a pre-$(< \alpha, \mathcal{F})$-frame, then $\text{cl}(\mathfrak{s})$ is a $(< \alpha, \mathcal{F})$-independence relation and $\text{pre}(\text{cl}(\mathfrak{s})) = \mathfrak{s}$.

Other properties of cl and pre are given by Proposition 4.1.

Next, we give a long list of properties that an independence relation may or may not have. We also give the corresponding property for frames. Most are classical and already appear for example in [BGKV]. We give them here again both for the convenience of the reader and because their definition is sometimes slightly modified compared to [BGKV] (for example, symmetry there is called right full symmetry here, and some properties like uniqueness and extensions are complicated by the fact we do not work in a monster model). They will be used throughout this paper (for example, Section 4 discusses implications between the properties).

Definition 3.10 (Properties of independence relations)**.** Let $\mathfrak{i} := (K, \mathop{\smile\!\!\!\!\!\mid})$ be a $(< \alpha, \mathcal{F}, < \beta)$-independence relation.

(1) $\mathfrak{i}$ has *disjointness* if $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\mid}}} B$ implies $A \cap B \subseteq |M|$.
(2) $\mathfrak{i}$ has *symmetry* if $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\mid}}} B$ implies that for all[14] $B_0 \subseteq B$ of size less than α and all $A_0 \subseteq A$ of size less than β, $B_0 \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\mid}}} A_0$.

[14]Why not just take $B_0 = B$? Because the definition of $\mathop{\smile\!\!\!\!\!\mid}$ requires that the left hand side has size less than α. Similarly for right full symmetry.

(3) $\mathfrak{i}$ has *right full symmetry* if $A \overset{N}{\underset{M}{\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.2ex\hbox{$\mid$}}}} B$ implies that for all $B_0 \subseteq B$ of size less than α and all $A_0 \subseteq A$ of size less than β, there exists $N' \geq N$, $M' \geq M$ containing A_0 such that $B_0 \overset{N'}{\underset{M}{\downarrow}} M'$.

(4) $\mathfrak{i}$ has *right base monotonicity* if $A \overset{N}{\underset{M}{\downarrow}} B$ and $M \leq M' \leq N$, $|M'| \subseteq B \cup |M|$ implies $A \overset{N}{\underset{M'}{\downarrow}} B$.

(5) $\mathfrak{i}$ has *right existence* if $A \overset{N}{\underset{M}{\downarrow}} M$ for any $A \subseteq |N|$ with $|A| < \alpha$.

(6) $\mathfrak{i}$ has *right uniqueness* if whenever $M_0 \leq M \leq N_\ell$, $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$, $q_\ell \in \mathrm{gS}^{<\alpha}(B; N_\ell)$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$, and q_ℓ does not fork over M_0, then $q_1 = q_2$.

(7) $\mathfrak{i}$ has *right extension* if whenever $p \in \mathrm{gS}^{<\alpha}(MB; N)$ does not fork over M and $B \subseteq C \subseteq |N|$ with $|C| < \beta$, there exists $N' \geq N$ and $q \in \mathrm{gS}^{<\alpha}(MC; N')$ extending p such that q does not fork over M.

(8) $\mathfrak{i}$ has *right independent amalgamation* if $\alpha > \lambda$, $\beta = \theta$, and[15] whenever $M_0 \leq M_\ell$ are in K, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$ such that $f_1[M_1] \overset{N}{\underset{M_0}{\downarrow}} f_2[M_2]$.

(9) $\mathfrak{i}$ has the *right* $(< \kappa)$*-model-witness property* if whenever $M \leq M' \leq N$, $\|M'|\backslash|M\| < \beta$, $A \subseteq |N|$, and $A \overset{N}{\underset{M}{\downarrow}} B_0$ for all $B_0 \subseteq |M'|$ of size less than κ, then $A \overset{N}{\underset{M}{\downarrow}} M'$. $\mathfrak{i}$ has the *right* $(< \kappa)$*-witness property* if this is true when M' is allowed to be an arbitrary set. The λ*-[model-]witness property* is the $(< \lambda^+)$-witness property.

(10) $\mathfrak{i}$ has *right transitivity* if whenever $M_0 \leq M_1 \leq N$, $A \overset{N}{\underset{M_0}{\downarrow}} M_1$ and $A \overset{N}{\underset{M_1}{\downarrow}} B$ implies $A \overset{N}{\underset{M_0}{\downarrow}} B$. *Strong right transitivity* is the same property when we do not require $M_0 \leq M_1$.

(11) $\mathfrak{i}$ has *right full model-continuity* if K is an AEC in $\mathcal{F}$, $\alpha > \lambda$, $\beta = \theta$, and whenever $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous

[15]Note that even though the next condition is symmetric, the condition on α and β make the left version of the property different from the right.

with δ limit, $\ell \leq 3$, for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$, $\ell = 1, 2$, $\|M_\delta^1\| < \alpha$, and $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\mathop{\smile}\limits^{\mid}}} M_i^2$ for all $i < \delta$, then $M_\delta^1 \overset{M_\delta^3}{\underset{M_\delta^0}{\mathop{\smile}\limits^{\mid}}} M_\delta^2$.

(12) *Weak chain local character* a technical property used to generate weakly good independence relations, see Definition 6.6.

Whenever this makes sense, we similarly define the same properties for pre-frames.

Note that we have defined the right version of the asymmetric properties. One can define a left version by looking at the *dual independence relation.*

Definition 3.11. Let $\mathfrak{i} := (K, \mathop{\smile}\limits^{\mid})$ be a $(< \alpha, \mathcal{F}, < \beta)$-independence relation. Define the *dual independence relation* $\mathfrak{i}^d := (K, \overset{d}{\mathop{\smile}\limits^{\mid}})$ by $\overset{d}{\mathop{\smile}\limits^{\mid}}(M, A, B, N)$ if and only if $\mathop{\smile}\limits^{\mid}(M, B, A, N)$.

Remark 3.12.

(1) If $\mathfrak{i}$ is a $(< \alpha, \mathcal{F}, < \beta)$-independence relation, then $\mathfrak{i}^d$ is a $(< \beta, \mathcal{F}, < \alpha)$-independence relation and $\left(\mathfrak{i}^d\right)^d = \mathfrak{i}$.
(2) Let $\mathfrak{i}$ be a $(< \alpha, \mathcal{F}, < \alpha)$-independence relation. Then $\mathfrak{i}$ has symmetry if and only if $\mathfrak{i} = \mathfrak{i}^d$.

Definition 3.13. For P a property, we will say $\mathfrak{i}$ *has left* P if $\mathfrak{i}^d$ has right P, similarly if we swap left and right. When we omit left/right, we mean the right version of the property.

Definition 3.14 (Locality cardinals)**.** Let $\mathfrak{i} = (K, \mathop{\smile}\limits^{\mid})$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$. Let $\alpha_0 < \alpha$.

(1) Let $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \lambda^+$ such that for any $M \leq N$ in K, any $A \subseteq |N|$ with $|A| \leq \alpha_0$, there exists $M_0 \leq M$ in $K_{<\mu}$ with $A \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$. When μ does not exist or $\mu = \theta$, we set $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \infty$.
(2) For R a binary relation on K, Let $\kappa_{\alpha_0}(\mathfrak{i}, R)$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \aleph_0$ such that for any regular $\delta \geq \mu$, any R-increasing (recall Definition 2.4) $\langle M_i : i < \delta \rangle$ in K, any N extending all the M_i's, and any $A \subseteq |N|$ of size $\leq \alpha_0$, there exists $i < \delta$ such that $A \overset{N}{\underset{M_i}{\mathop{\smile}\limits^{\mid}}} M_\delta$. Here, we have set[16] $M_\delta :=$

[16]Recall that K is only an ∞-abstract elementary class, so may not be closed under unions of chains of length δ.

$\bigcup_{i<\delta} M_\delta$. When $R = \leq$, we omit it. When μ does not exist or $\mu = \theta$, we set $\kappa_{\alpha_0}(\mathfrak{i}) = \infty$.

When K is clear from context, we may write $\bar{\kappa}_{\alpha_0}(\mathop{\smile\!\!\!\!\mid})$. For $\alpha_0 \leq \alpha$, we also let $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) := \sup_{\alpha_0' < \alpha_0} \kappa_{\alpha_0'}(\mathfrak{i})$. Similarly for $\kappa_{<\alpha_0}$.

We similarly define $\bar{\kappa}_{\alpha_0}(\mathfrak{s})$ and $\kappa_{\alpha_0}(\mathfrak{s})$ for $\mathfrak{s}$ a pre-frame (in the definition of $\kappa_{\alpha_0}(\mathfrak{s})$, we require in addition that M_δ be member of K).

We will use the following notation to restrict independence relations to smaller domains:

Notation 3.15. Let $\mathfrak{i}$ be a $(<\alpha, \mathcal{F}, <\beta)$-independence relation.

(1) For $\alpha_0 \leq \alpha$, $\beta_0 \leq \beta$, let $\mathfrak{i}^{<\alpha_0, <\beta_0}$ denotes the $(<\alpha_0, \mathcal{F}, <\beta_0)$ obtained by restricting the types to have length α_0 and the right hand side to have size less than β_0 (in the natural way). When $\beta_0 = \beta$, we omit it.
(2) When $\mathfrak{i}$ is a $(<\alpha, \mathcal{F})$-independence relation and $\mathcal{F}_0 \subseteq \mathcal{F}$ is an interval of cardinals, we let $\mathfrak{i}_{\mathcal{F}_0}$ be the restriction of $\mathfrak{i}$ to models of size in $\mathcal{F}_0$, and types of appropriate length (that is, if $\mathcal{F}_0 = [\lambda, \theta_0)$, $\mathfrak{i}_{\mathcal{F}_0} = \left(\mathfrak{i}^{<\min(\alpha, \theta_0)}\right)_{\mathcal{F}_0}$).
(3) For K' a sub-∞-AEC of $K_{\mathfrak{i}}$, let $\mathfrak{i} \upharpoonright K'$ be the $(<\alpha, \mathcal{F}, <\beta)$-independence relation obtained by restricting the underlying class to K'.

We end this section with two examples of independence relations. The first is coheir. In first-order logic, coheir was first defined in [LP79]. A definition of coheir for classes of models in $L_{\kappa,\omega}$ appears in [MS90] and was later adapted to general AECs in [BG]. In [Vasb], we gave a more conceptual (but equivalent) definition and improved some of the results of Boney and Grossberg. Here, we use Boney and Grossberg's definition but rely on [Vasb].

Definition 3.16. Let K be an AEC with amalgamation and let $\kappa > \mathrm{LS}(K)$.

Define $\mathfrak{i}_{\kappa\text{-ch}}(K) := (K^{\kappa\text{-sat}}, \mathop{\smile\!\!\!\!\mid})$ by $\mathop{\smile\!\!\!\!\mid}(M, A, B, N)$ if and only if $M \leq N$ are in $K^{\kappa\text{-sat}}$, $A \cup B \subseteq |N|$, and for any $\bar{a} \in {}^{<\kappa}A$ and $B_0 \subseteq |M| \cup B$ of size less than κ, there exists $\bar{a}' \in {}^{<\kappa}|M|$ such that $\mathrm{gtp}(\bar{a}/B_0; N) = \mathrm{gtp}(\bar{a}'/B_0; M)$.

Fact 3.17 (Theorem 5.13 in [Vasb])**.** Let K be an AEC with amalgamation and let $\kappa > \mathrm{LS}(K)$. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)$.

(1) $\mathfrak{i}$ is a $(<\infty, [\kappa, \infty))$-independence relation with disjointness, base monotonicity, left and right existence, left and right $(<\kappa)$-witness property, and strong left transitivity.
(2) If K does not have the $(<\kappa)$-order property of length κ, then:
   (a) $\mathfrak{i}$ has symmetry and strong right transitivity.
   (b) For all α, $\bar{\kappa}_\alpha(\mathfrak{i}) \leq \left((\alpha + \text{LS}(K))^{<\kappa}\right)^+$.
   (c) If $M_0 \leq M \leq N_\ell$ for $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$. $q_\ell \in \text{gS}^{<\infty}(B; N_\ell)$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$, q_ℓ does not $\mathfrak{i}$-fork over M_0 for $\ell = 1, 2$, and K is $(<\kappa)$-tame and short for $\{q_1, q_2\}$, then $q_1 = q_2$.
   (d) If K is $(<\kappa)$-tame and short for types of length less than α, then $\text{pre}(\mathfrak{i}^{<\alpha})$ has uniqueness. Moreover[17] $\mathfrak{i}^{<\alpha}_{[\kappa,\alpha)}$ has uniqueness.

Remark 3.18. The extension property[18] seems to be more problematic. In [BG], Boney and Grossberg simply assumed it (they also showed that it followed from κ being strongly compact [BG, Theorem 8.2.1]). From superstability-like hypotheses, we will obtain more results on it (see Theorem 10.14, Theorem 15.1, and Theorem 15.6).

We now consider another independence notion: splitting. This was first defined for AECs in [She99, Definition 3.2]. Here we define the negative property (nonsplitting), as it is the one we use the most.

Definition 3.19 (λ-nonsplitting)**.** Let K be an ∞-AEC with amalgamation.

(1) For λ an infinite cardinal, define $\mathfrak{s}_{\lambda\text{-ns}}(K) := (K, \perp)$ by $\bar{a} \overset{N}{\underset{M_0}{\perp}} M$ if and only if $M_0 \leq M \leq N$, $A \subseteq |N|$, and whenever $M_0 \leq N_\ell \leq M$, $N_\ell \in K_{\leq\lambda}$, $\ell = 1, 2$, and $f : N_1 \cong_{M_0} N_2$, then $f(\text{gtp}(\bar{a}/N_1; N)) = \text{gtp}(\bar{a}/N_2; N)$.
(2) Define $\mathfrak{s}_{\text{ns}}(K)$ to have underlying AEC K and nonforking relation defined such that $p \in \text{gS}^{<\infty}(M)$ does not $\mathfrak{s}_{\text{ns}}(K)$-fork over $M_0 \leq M$ if and only if p does not $\mathfrak{s}_{\lambda\text{-ns}}(K)$-fork over M_0 for all infinite λ.
(3) Let $\mathfrak{i}_{\lambda\text{-ns}}(K) := \text{cl}(\mathfrak{s}_{\lambda\text{-ns}}(K))$, $\mathfrak{i}_{\text{ns}}(K) := \text{cl}(\mathfrak{s}_{\text{ns}}(K))$.

Fact 3.20. Assume K is an ∞-AEC in $\mathcal{F} = [\lambda, \theta)$ with amalgamation. Let $\mathfrak{s} := \mathfrak{s}_{\text{ns}}(K)$, $\mathfrak{s}' := \mathfrak{s}_{\lambda\text{-ns}}(K)$.

[17] Of course, this is only interesting if $\alpha \leq \kappa$.

[18] A word of caution: In [HL02, Section 4], the authors give an example of an ω-stable class that does not have extension. However, the extension property they consider is *over all sets*, not only over models.

(1) $\mathfrak{s}$ and $\mathfrak{s}'$ are pre-$(<\infty,\mathcal{F})$-frame with base monotonicity, left and right existence. If K is λ-closed, $\mathfrak{s}'$ has the right λ-model-witness property.
(2) If K is an AEC in $\mathcal{F}$ and is stable in λ, then $\bar{\kappa}_{<\omega}(\mathfrak{s}') = \lambda^+$.
(3) If $\mathfrak{t}$ is a pre-$(<\infty,\mathcal{F})$-frame with uniqueness and $K_{\mathfrak{t}} = K$, then $\underset{\mathfrak{t}}{\downarrow} \subseteq \underset{\mathfrak{s}}{\downarrow}$.
(4) Always, $\underset{\mathfrak{s}}{\downarrow} \subseteq \underset{\mathfrak{s}'}{\downarrow}$. Moreover if K is λ-tame for types of length less than α, then $\mathfrak{s}^{<\alpha} = (\mathfrak{s}')^{<\alpha}$.
(5) Let $M_0 <_{\text{univ}} M \leq N$ with $\|M\| = \|N\|$.
   (a) Weak uniqueness: If $p_\ell \in \text{gS}^\alpha(N)$, $\ell = 1, 2$, do not $\mathfrak{s}$-fork over M_0 and $p_1 \upharpoonright M = p_2 \upharpoonright M$, then $p_1 = p_2$.
   (b) Weak extension: If $p \in \text{gS}^{<\infty}(M)$ does not $\mathfrak{s}$-fork over M_0 and $f : N \underset{M_0}{\longrightarrow} M$, then $q := f^{-1}(p)$ is an extension of p to $\text{gS}^{<\infty}(N)$ that does not $\mathfrak{s}$-fork over M_0. Moreover q is algebraic if and only if p is algebraic.

Proof.

(1) Easy.
(2) By [She99, Claim 3.3.1] (see also [GV06b, Fact 4.6]).
(3) By [BGKV, Lemma 4.2].
(4) By [BGKV, Proposition 3.12]).
(5) By [Van06, Theorem I.4.10, Theorem I.4.12] (the moreover part is easy to see from the definition of q).

□

Remark 3.21. Fact 3.20.(3) tells us that any reasonable independence relation will be extended by nonsplitting. In this sense, nonsplitting is a *maximal* candidate for an independence relation[19].

[19]Moreover, $(<\kappa)$-coheir is a *minimal* candidate in the following sense: Let us say an independence relation $\mathfrak{i} = (K, \downarrow)$ has the *strong* $(<\kappa)$*-witness property* if whenever $A \overset{N}{\underset{M}{\not\downarrow}} B$, there exists $\bar{a}_0 \in {}^{<\kappa}A$ and $B_0 \subseteq |M| \cup B$ of size less than κ such that $\text{gtp}(\bar{a}_0'/B_0; N) = \text{gtp}(\bar{a}_0/B_0; N)$ implies $\bar{a}_0 \overset{N}{\underset{M}{\not\downarrow}} B$. Intuitively, this says that forking is witnessed by a formula (and this could be made precise using the methods of [Vasb]). It is easy to check that $(<\kappa)$-coheir has this property, and any independence relation with strong $(<\kappa)$-witness and left existence must extend $(<\kappa)$-coheir.

4. Some independence calculus

We investigate relationship between properties and how to go from a frame to an independence relation. Most of it appears already in [BGKV] and has a much longer history, described there. The following are new: Lemma 4.5 gives a way to get the witness properties from tameness, partially answering [BGKV, question 5.5]. Lemmas 4.7 and 4.8 are technical results used in the last sections.

The following proposition investigates what properties are preserved by the operations cl and pre (recall Definition 3.8). This was done already in [BGKV, Section 5.1], so we mostly cite from there.

Proposition 4.1. Let $\mathfrak{s}$ be a $(< \alpha, \mathcal{F})$-pre-frame and let $\mathfrak{i}$ be a $(< \alpha, \mathcal{F})$-independence relation.

(1) For P in the list of properties of Definition 3.10, if $\mathfrak{i}$ has P, then $\text{pre}(\mathfrak{i})$ has P.
(2) For P a property in the following list, $\mathfrak{i}$ has P if (and only if) $\text{pre}(\mathfrak{i})$ has P: existence, independent amalgamation, full model-continuity.
(3) For P a property in the following list, $\text{cl}(\mathfrak{s})$ has P if (and only if) $\mathfrak{s}$ has P: disjointness, full symmetry, base monotonicity, extension, transitivity.
(4) If $\text{pre}(\mathfrak{i})$ has extension, then $\text{cl}(\text{pre}(\mathfrak{i})) = \mathfrak{i}$ if and only if $\mathfrak{i}$ has extension.
(5) The following are equivalent:
   (a) $\mathfrak{s}$ has full symmetry.
   (b) $\text{cl}(\mathfrak{s})$ has symmetry.
   (c) $\text{cl}(\mathfrak{s})$ has full symmetry.
(6) If $\text{pre}(\mathfrak{i})$ has uniqueness and $\mathfrak{i}$ has extension, then $\mathfrak{i}$ has uniqueness.
(7) If $\text{pre}(\mathfrak{i})$ has extension and $\mathfrak{i}$ has uniqueness, then $\mathfrak{i}$ has extension.
(8) For all $\alpha_0 < \alpha$, $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \bar{\kappa}_{\alpha_0}(\text{pre}(\mathfrak{i}))$.
(9) For all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}) \leq \kappa_{\alpha_0}(\text{pre}(\mathfrak{i}))$. If $K_{\mathfrak{i}}$ is an AEC, then this is an equality.

Proof. All are straightforward. See [BGKV, Lemma 5.3, Lemma 5.4]. □

To what extent is an independence relation determined by its corresponding frame? There is an easy answer:

Lemma 4.2. Let $\mathfrak{i}$ and $\mathfrak{i}'$ be independence relations with $\text{pre}(\mathfrak{i}) = \text{pre}(\mathfrak{i}')$. If $\mathfrak{i}$ and $\mathfrak{i}'$ both have extension, then $\mathfrak{i} = \mathfrak{i}'$.

Proof. By Proposition 4.1.(4), $\mathfrak{i} = \text{cl}(\text{pre}(\mathfrak{i}))$ and $\mathfrak{i}' = \text{cl}(\text{pre}(\mathfrak{i}')) = \text{cl}(\text{pre}(\mathfrak{i})) = \mathfrak{i}$. □

The next proposition gives relationships between the properties. We state most results for frames, but they usually have an analog for independence relations that can be obtained using Proposition 4.1.

Proposition 4.3. Let $\mathfrak{i}$ be a $(<\alpha, \mathcal{F})$-independence relation with base monotonicity. Let $\mathfrak{s}$ be a $(<\alpha, \mathcal{F})$-pre-frame with base monotonicity.

(1) If $\mathfrak{i}$ has full symmetry, then it has symmetry. If $\mathfrak{i}$ has the $(<\kappa)$-witness property, then it has the $(<\kappa)$-model-witness property. If $\mathfrak{i}$ [$\mathfrak{s}$] has strong transitivity, then it has transitivity.
(2) If $\mathfrak{s}$ has uniqueness and extension, then it has transitivity.
(3) For $\alpha > \lambda$, if $\mathfrak{s}$ has extension and existence, then $\mathfrak{s}$ has independent amalgamation. Conversely, if $\mathfrak{s}$ has transitivity and independent amalgamation, then $\mathfrak{s}$ has extension and existence. Moreover if $\mathfrak{s}$ has uniqueness and independent amalgamation, then it has transitivity.
(4) If[20] $\min(\kappa_{<\alpha}(\mathfrak{s}), \bar{\kappa}_{<\alpha}(\mathfrak{s})) < \infty$, then $\mathfrak{s}$ has existence.
(5) If $K_{\mathfrak{s}}$ is an AEC in $\mathcal{F}$, then $\kappa_{<\alpha}(\mathfrak{i}) \leq \bar{\kappa}_{<\alpha}(\mathfrak{i})$.
(6) If $K_{\mathfrak{s}}$ is $\lambda_{\mathfrak{s}}$-closed, $\bar{\kappa}_{<\alpha}(\mathfrak{s}) = \lambda_{\mathfrak{s}}^+$ and $\mathfrak{s}$ has transitivity, then $\mathfrak{s}$ has the right λ-model-witness property.
(7) If $K_{\mathfrak{s}}$ does not have the order property (Definition 2.18), any chain in $K_{\mathfrak{s}}$ has an upper bound, $\theta = \infty$, and $\mathfrak{s}$ has uniqueness, existence, and extension, then $\mathfrak{s}$ has full symmetry.

Proof.

(1) Easy.
(2) As in the proof of [She09, Claim II.2.18].
(3) The first sentence is easy, since independent amalgamation is a particular case of extension and existence. Moreover to show existence it is enough by monotonicity to show it for types of models. The proof of transitivity from uniqueness and independent amalgamation is as in (2).
(4) By definition of the local character cardinals.

[20]Note that maybe $\alpha = \infty$. However we can always apply the proposition to $\mathfrak{s}^{<\alpha_0}$ for an appropriate $\alpha_0 \leq \alpha$.

(5) Let $\delta = \text{cf}(\delta) \geq \bar{\kappa}_{<\alpha}(\mathfrak{i})$ and $\langle M_i : i < \delta \rangle$ be increasing in K, $N \geq M_i$ for all $i < \delta$ and $A \subseteq |N|$ with $|A| < \alpha$ then $M_\delta := \bigcup_{i<\delta} M_i$ is in K since K is an AEC. By definition of $\bar{\kappa}_{<\alpha}$ there exists $N \leq M_\delta$ of size less than $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$ such that p does not fork over N. Now use regularity of δ to find $i < \delta$ with $N \leq M_i$.

(6) Let $\lambda := \lambda_{\mathfrak{s}}$, say $\mathfrak{s} = (K, \perp)$. Let $M_0 \leq M \leq N$ and assume $A \overset{N}{\underset{M_0}{\perp}} B$ for all $B \subseteq |M|$ with $|B| \leq \lambda$. By definition of $\bar{\kappa}_{<\alpha}(\mathfrak{s})$, there exists $M_0' \leq M$ of size λ such that $A \overset{N}{\underset{M_0'}{\perp}} M$. By λ-closure and base monotonicity, we can assume without loss of generality that $M_0 \leq M_0'$. By assumption, $A \overset{N}{\underset{M_0}{\perp}} M_0'$, so by transitivity, $A \overset{N}{\underset{M_0}{\perp}} M$.

(7) As in [BGKV, Corollary 5.15].

□

Remark 4.4. The precise statement of [BGKV, Corollary 5.15] shows that Proposition 4.3.(7) is local in the sense that to prove symmetry over the base model M, it is enough to require uniqueness and extension over this base model (i.e. any two types that do not fork over M, have the same domain, and are equal over M are equal over their domain, and any type over M can be extended to an arbitrary domain so that it does not fork over M).

Lemma 4.5. Let $\mathfrak{i} = (K, \perp)$ be a $(< \alpha, \mathcal{F})$-independence relation. If $\mathfrak{i}$ has extension and uniqueness, then:

(1) If K is $(< \kappa)$-tame for types of length less than α, then K has the right $(< \kappa)$-model-witness property.

(2) If K is $(< \kappa)$-tame and short for types of length less than $\theta_{\mathfrak{i}}$, then K has the right $(< \kappa)$-witness property.

(3) If K is $(< \kappa)$-tame and short for types of length less than $\kappa + \alpha$ and $\mathfrak{i}$ has symmetry, then K has the left $(< \kappa)$-witness property.

Proof.

(1) Let $M \leq M' \leq N$ be in K, $A \subseteq |N|$ have size less than α. Assume $A \overset{N}{\underset{M}{\perp}} B_0$ for all $B_0 \subseteq |M'|$ of size less than κ. We

want to show that $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\;\mid}}} M'$. Let $\bar{a}$ be an enumeration of A, $p := \text{gtp}(\bar{a}/M; N)$. Note that (taking $B_0 = \emptyset$ above) normality implies p does not fork over M. By extension, let $q \in \text{gS}^{<\alpha}(M')$ be an extension of p that does not fork over M. Using amalgamation and some renaming, we can assume without loss of generality that q is realized in N. Let $p' := \text{gtp}(\bar{a}/M'; N)$. We claim that $p' = q$, which is enough by invariance. By the tameness assumption, it is enough to check that $p' \upharpoonright B_0 = q \upharpoonright B_0$ for all $B_0 \subseteq |M'|$ of size less than κ. Fix such a B_0. By assumption, $p' \upharpoonright B_0$ does not fork over M. By monotonicity, $q \upharpoonright B_0$ does not fork over M. By the previous part, $\mathfrak{i}$ has uniqueness so $p' \upharpoonright B_0 = q \upharpoonright B_0$, as desired.

(2) Similar to before, noting that for $M \leq N$, $\text{gtp}(\bar{a}/M\bar{b}; N) = \text{gtp}(\bar{a}'/M\bar{b}; N)$ if and only if $\text{gtp}(\bar{a}\bar{b}/M; N) = \text{gtp}(\bar{a}'\bar{b}/M; N)$.

(3) Observe that in the proof of the previous part, if the set on the right hand side has size less than κ, it is enough to require $(< \kappa)$-tameness and shortness for types of length less than $(\alpha + \kappa)$. Now use symmetry.

□

Having a nice independence relation makes the class nice. The results below are folklore:

Proposition 4.6. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!\;\mid})$ be a $(< \alpha, \mathcal{F})$-independence relation with base monotonicity. Assume K is an AEC with $\text{LS}(K) = \lambda_{\mathfrak{i}}$.

(1) If $\mathfrak{i}$ has uniqueness, and $\kappa := \bar{\kappa}_{<\alpha}(\mathfrak{i}) < \infty$, then K is $(< \kappa)$-tame for types of length less than α.
(2) If $\mathfrak{i}$ has uniqueness and $\kappa := \bar{\kappa}_{<\alpha}(\mathfrak{i}) < \infty$, then K is $(< \alpha)$-stable in any infinite μ such that $\mu = \mu^{<\kappa}$.
(3) If $\mathfrak{i}$ has uniqueness, $\mu > \text{LS}(K)$, K is $(< \alpha)$-stable in unboundedly many $\mu_0 < \mu$, and $\text{cf}(\mu) \geq \kappa_{<\alpha}(\mathfrak{i})$, then K is $(< \alpha)$-stable in μ.

Proof.

(1) As in the proof of [Bon14a, Theorem 3.2].
(2) Let $\mu = \mu^{<\kappa}$ be infinite. Let $M \in K_{\leq \mu}$, $\langle p_i : i < \mu^+ \rangle$ be elements in $\text{gS}^{<\alpha}(M)$. It is enough to show that for some $i < j$, $p_i = p_j$. For each $i < \lambda^+$, there exists $M_i \leq M$ in $K_{<\kappa}$ such that p_i does not fork over M_i. Since $\mu = \mu^{<\kappa}$, we can assume without loss of generality that $M_i = M_0$ for all $i < \mu^+$. Also,

$|\text{gS}^{<\alpha}(M_0)| \leq 2^{<\kappa} \leq \mu^{<\kappa} = \mu$ so there exists $i < j < \lambda^+$ such that $p_i \restriction M_0 = p_j \restriction M_0$. By uniqueness, $p_i = p_j$, as needed.

(3) As in the proof of [Vasa, Lemma 5.5].

□

The next lemma clarifies the relationship between full model continuity and local character. Essentially, it says that local character for types up to a certain length plus full model-continuity implies local character for all lengths. It will be used in Section 14.

Lemma 4.7. Let $\mathfrak{i} = (K, \perp)$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$ and $\lambda < \alpha \leq \theta$. Assume:

(1) K is an AEC with $\text{LS}(K) = \lambda$.
(2) $\mathfrak{i}$ has base monotonicity, transitivity, and left extension.
(3) $\mathfrak{i}$ has full model continuity.
(4) For all cardinals $\mu < \lambda^+$, $\bar{\kappa}_\mu(\mathfrak{i}) = \lambda^+$.

Then for all cardinals $\mu < \alpha$, $\bar{\kappa}_\mu(\mathfrak{i}) = \lambda^+ + \mu^+$.

Proof. By induction on μ. If $\lambda \geq \mu$, this holds by hypothesis, so assume $\lambda < \mu$. Let $\delta := \text{cf}(\mu)$.

Let $M \leq N$ be in K and let $A \subseteq |N|$ have size μ. We want to find $M^0 \leq M$ such that $A \underset{M^0}{\overset{N}{\perp}} M$ and $\|M_0\| \leq \mu$. Let $\langle A_i : i \leq \delta \rangle$ be increasing continuous such that $A = A_\delta$ and $|A_i| < \mu$ for all $i < \delta$. Build increasing continuous $\langle M_i^0 : i \leq \delta \rangle$, $\langle N_i^0 : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ such that for all $i < \delta$:

(1) $M_i \leq M$, $N \leq N_i$.
(2) $N_i^0 \leq N_i$.
(3) $M_i \leq N_i^0$.
(4) $A_i \subseteq |N_{i+1}^0|$.
(5) $\|N_i^0\| < \mu$.
(6) For $i > 0$, $N_i^0 \underset{M_i}{\overset{N_i}{\perp}} M$.

<u>This is possible</u>. By induction on $i < \mu$. If i is limit, take unions. If $i = 0$, take $N_0 := N$, any $M_0 \leq M$ in $K_{<\mu}$, and any $N_0^0 \leq N$ in $K_{<\mu}$ with $M_0 \leq N_0^0$.

Assume now that $i = j + 1$ and M_j, N_j, N_j^0 have been defined. Since $\|N_j^0\| < \mu$, the induction hypothesis implies there exists $M_i \leq M$ in

$K_{<\mu}$ such that $A_j N_j^0 \mathop{\perp}\limits_{M_i}^{N_j} M$, and (by base monotonicity) without loss of generality $M_j \leq M_i$. By left set-extension, we can find $N_i \geq N_j$ and $N_i^0 \leq N_i$ in $K_{<\mu}$ such that $A_j \subseteq |N_i^0|$, $M_i \leq N_i^0$, $N_j^0 \leq N_i^0$, and $N_i^0 \mathop{\perp}\limits_{M_i}^{N_i} M$.

<u>This is enough</u>. $A = A_\delta \subseteq |N_\delta^0|$ and by full model continuity, $N_\delta^0 \mathop{\perp}\limits_{M_\delta}^{N_\delta} M$. Also, $M_\delta \leq N_\delta^0$ which is in $K_{\leq\mu}$, and by monotonicity, $A \mathop{\perp}\limits_{M_\delta}^{N} M$, so $M^0 := M_\delta$ is as desired. $\square$

The following technical result is also used in the last sections. Roughly, it gives conditions under which we can take the base model given by local character to be contained in both the left and right hand side.

Lemma 4.8. Let $\mathfrak{i} = (K, \perp)$ be a $(<\alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$, with $\alpha > \lambda$. Assume:

(1) K is an AEC with $\text{LS}(K) = \lambda$.
(2) $\mathfrak{i}$ has base monotonicity and transitivity.
(3) μ is a cardinal, $\lambda \leq \mu < \theta$.
(4) $\mathfrak{i}$ has the left $(<\kappa)$-model-witness property for some regular $\kappa \leq \mu$.
(5) $\bar{\kappa}_\mu(\mathfrak{i}) = \mu^+$.

Let $M^0 \leq M^\ell \leq N$ be in K, $\ell = 1, 2$ and assume $M^1 \mathop{\perp}\limits_{M^0}^{N} M^2$. Let $A \subseteq |M^1|$, be such that $|A| \leq \mu$. Then there exists $N^1 \leq M^1$ and $N^0 \leq M^0$ such that:

(1) $A \subseteq |N^1|$, $A \cap |M^0| \subseteq |N^0|$.
(2) $N^0 \leq N^1$ are in $K_{\leq\mu}$.
(3) $N^1 \mathop{\perp}\limits_{N^0}^{N} M^2$.

Proof. For $\ell = 0, 1$, we build $\langle N_i^\ell : i \leq \kappa \rangle$ increasing continuous in $K_{\leq\mu}$ such that for all $i < \kappa$ and $\ell = 0, 1$:

(1) $A \subseteq |N_0^1|$, $A \cap |M^0| \subseteq |N_0^0|$.
(2) $N_i^\ell \leq M^\ell$.
(3) $N_i^0 \leq N_i^1$.

(4) $N_i^1 \overset{N}{\underset{N_{i+1}^0}{\downarrow}} M^2$.

This is possible. Pick any $N_0^0 \leq M^0$ in $K_{\leq \mu}$ containing $A \cap |M^0|$. Now fix $i < \kappa$ and assume inductively that $\langle N_j^0 : j \leq i \rangle$, $\langle N_j^1 : j < i \rangle$ have been built. If i is a limit, we take unions. Otherwise, pick any $N_i^1 \leq M^1$ in $K_{\leq \mu}$ that contains A, N_j^1 for all $j < i$ and N_i^0. Now use right transitivity and $\bar{\kappa}_\mu(\mathfrak{i}) = \mu^+$ to find $N_{i+1}^0 \leq M^0$ such that $N_i^1 \overset{N}{\underset{N_{i+1}^0}{\downarrow}} M^2$. By base monotonicity, we can assume without loss of generality that $N_i^0 \leq N_{i+1}^0$.

This is enough. We claim that $N^\ell := N_\kappa^\ell$ are as required. By coherence, $N^0 \leq N^1$ and since $\kappa \leq \mu$ they are in $K_{\leq \mu}$. Since $A \subseteq |N_0^1|$, $A \subseteq |N^1|$. It remains to see $N^1 \overset{N}{\underset{N^0}{\downarrow}} M^2$. By the left witness property[21], it is enough to check it for every $B \subseteq |N^1|$ of size less than κ. Fix such a B. Since κ is regular, there exists $i < \kappa$ such that $B \subseteq |N_i^1|$. By assumption and monotonicity, $B \overset{N}{\underset{N_{i+1}^0}{\downarrow}} M^2$. By base monotonicity, $B \overset{N}{\underset{N_\kappa^0}{\downarrow}} M^2$, as needed. □

5. Skeletons

We define what it means for an abstract class K' to be a *skeleton* of an abstract class K. The main examples are classes of saturated models with the usual ordering (or even universal or limit extension). Except perhaps for Lemma 5.7, the results of this section are either easy or well known, we simply put them in our general language.

We will use skeletons to generalize various statements of chain local character (for example in [GVV, Vasa]) that only ask that if $\langle M_i : i < \delta \rangle$ is an increasing chain with respect to *some restriction of the* ordering of K (usually being universal over) and the M_is are inside some subclass of K (usually some class of saturated models), then any $p \in \text{gS}(\bigcup_{i<\delta} M_i)$ does not fork over some M_i. Lemma 6.8, is they key upward transfer of that property. Note that Lemma 6.7 shows that one can actually assume that skeletons have a particular form. However the generality is

[21]Note that we do *not* need to use full model continuity, as we only care about chains of cofinality $\geq \kappa$.

still useful when one wants to actually prove the actual local character statement. Also, it seems that many key subclasses appearing in the theory of AECs are skeletons, see the examples below.

Definition 5.1. For $(K, \leq)$ an abstract class, we say $(K', \trianglelefteq)$ is a sub-AC of K if $K' \subseteq K$, $(K', \trianglelefteq)$ is an AC, and $M \trianglelefteq N$ implies $M \leq N$. We similarly define sub-AEC, etc. When $\trianglelefteq = \leq \restriction K'$, we omit it (or may abuse notation and write $(K', \leq)$).

Definition 5.2. For $(K, \leq)$ an abstract class, we say a set $S \subseteq K$ is *dense* in $(K, \leq)$ if for any $M \in K$ there exists $M' \in S$ with $M \leq M'$.

Definition 5.3. An abstract class $(K', \trianglelefteq)$ is a *skeleton* of $(K, \leq)$ if:

(1) $(K', \trianglelefteq)$ is a sub-AC of $(K, \leq)$.
(2) K' is dense in $(K, \leq)$.
(3) If $\langle M_i : i < \alpha \rangle$ is a $\trianglelefteq$-increasing chain in K' (α not necessarily limit) and there exists $N \in K'$ such that $M_i < N$ for all $i < \alpha$, then we can choose such an N with $M_i \triangleleft N$ for all $i < \alpha$.

Remark 5.4. The term "skeleton" is inspired from the term "skeletal" in [Vasa], although there "skeletal" is applied to frames. The intended philosophical meaning is the same: K' has enough information about K so that for many purposes we can work with K' rather than K.

Remark 5.5. Let $(K, \leq)$ be an abstract class. Assume $(K', \trianglelefteq)$ is a dense sub-AC of $(K, \leq)$ with no maximal models satisfying in addition: If $M_0 \leq M_1 \triangleleft M_2$ are in K', then $M_0 \triangleleft M_2$. Then $(K', \trianglelefteq)$ is a skeleton of $(K, \leq)$. This property of the ordering already appears in the definition of an abstract universal ordering in [Vasa, Definition 2.13]. In the terminology there, if $(K, \leq)$ is an AEC and $\triangleleft$ is an (invariant) universal ordering on K_λ, then $(K_\lambda, \trianglelefteq)$ is a skeleton of $(K_\lambda, \leq)$.

Example 5.6. Let K be an AEC.

(1) Assume K has no maximal models. Let K' be the class of Ehrenfeucht-Mostowski models of K (for some fixed diagram, see for example [Bal09, Theorem 8.18]). Then K' is dense in K (because for a fixed $M \in K$, the AEC K_M of models in K containing a copy of M has EM models), so $(K', \leq)$ is a skeleton of $(K, \leq)$.
(2) Let $\mathfrak{s}$ be a weakly successful good λ-frame (see e.g. Definition 11.4 or [She09, Chapter II]) with $K_{\mathfrak{s}} = K_\lambda$. Then $(K_{\lambda^+}^{\lambda^+\text{-sat}}, \leq_{\lambda^+}^{\text{NF}})$ (see [JS13, Definition 6.1.4] or [She09, Definition II.7.2], where $\leq_{\lambda^+}^{\text{NF}}$ is called $\leq_{\lambda^+}^{*}$) is a skeleton of $(K_{\lambda^+}, \leq)$ (use [She09, Claim II.7.4.1, II.7.7.3]).

(3) Let $\lambda \geq \text{LS}(K)$. Assume K_λ has amalgamation, no maximal models and is stable in λ. Let K' be dense in K_λ and let $\delta < \lambda^+$. Then $(K', \leq_{\lambda,\delta})$ (recall Definition 2.21) is a skeleton of $(K_\lambda, \leq)$ (use Fact 2.24 and Remark 5.5).

We will only use Example 5.6.(3). However the above suggests that other types of skeletons are also relevant. For example *solvability*, Shelah's definition of superstability [She09, Definition IV.1.4], uses EM models (see the discussion at the beginning of section 10).

The next lemma is a useful tool to find extensions in the skeleton of an AEC with amalgamation:

Lemma 5.7. Let $(K', \trianglelefteq)$ be a skeleton of $(K, \leq)$. Assume K is an AEC in $\mathcal{F} := [\lambda, \theta)$ with amalgamation. If $M \leq N$ are K', then there exists $N' \in K'$ such that $M \trianglelefteq N'$ and $N \trianglelefteq N'$.

Proof. If N is not maximal (with respect to either of the orderings, it does not matter by definition of a skeleton), then by definition of a skeleton we can find $N' \in K'$ such that $N \triangleleft N'$ and $M \triangleleft N'$, as needed.

Now assume N is maximal. We claim that $M \trianglelefteq N$, so $N' := N$ is as desired. Suppose not. Let $\mu := \|N\|$.

We build $\langle M_i : i < \mu^+ \rangle$ and $\langle f_i : M_i \xrightarrow[M]{} N : i < \mu^+ \rangle$ such that:

(1) $\langle M_i : i < \mu^+ \rangle$ is a strictly increasing chain in $(K', \trianglelefteq)$ with $M_0 = M$.
(2) $\langle f_i : i < \mu^+ \rangle$ is a strictly increasing chain of K-embeddings.

<u>This is enough</u>. Let $B_{\mu^+} := \bigcup_{i<\mu^+} |M_i|$ and $f_{\mu^+} := \bigcup_{i<\mu^+} f_i$ (Note that it could be that $\mu^+ = \theta$, so B_{μ^+} is just a set and we do not claim that f_{μ^+} is a K-embedding). Then f_{μ^+} is an injection from B_{μ^+} into $|N|$. This is impossible because $|B_{\mu^+}| \geq \mu^+ > \mu = \|N\|$.

<u>This is possible</u>. Set $M_0 := M$, $f_0 := \text{id}_M$. If $i < \mu^+$ is limit, let $M_i' := \bigcup_{j<i} M_j \in K$. By density, find $M_i'' \in K'$ such that $M_i' \leq M_i''$. We have that $M_j < M_i''$ for all $j < i$. By definition of a skeleton, this means we can find $M_i \in K'$ with $M_j \triangleleft M_i$ for all $j < i$. Let $f_i' := \bigcup_{j<i} f_j$. Using amalgamation and the fact that N is maximal, we can extend it to $f_i : M_i \xrightarrow[M]{} N$. If $i = j + 1$ is successor, we consider two cases:

- If M_j is not maximal, let $M_i \in K'$ be a $\lhd$-extension of M_j. Using amalgamation and the fact N is maximal, pick $f_i : M_i \underset{M}{\longrightarrow} N$ an extension of f_j.
- If M_j is maximal, then by amalgamation and the fact both N and M_j are maximal, we must have $N \cong_M M_j$. However by assumption $M_0 \unlhd M_j$ so $M = M_0 \unlhd N$, a contradiction.

□

Thus we get that many properties of a class transfer to its skeletons.

Proposition 5.8. Let $(K, \leq)$ be an AEC in $\mathcal{F}$ and let $(K', \unlhd)$ be a skeleton of K.

(1) $(K, \leq)$ has no maximal models if and only if $(K', \unlhd)$ has no maximal models.
(2) If $(K, \leq)$ has amalgamation, then:
  (a) $(K', \unlhd)$ has amalgamation.
  (b) $(K, \leq)$ has joint embedding if and only if $(K', \unlhd)$ has joint embedding.
  (c) Galois types are the same in $(K, \leq)$ and $(K', \unlhd)$: For any $N \in K'$, $A \subseteq |N|$, $\bar{b}, \bar{c} \in {}^{\alpha}|N|$, $\text{gtp}_K(\bar{b}/A; N) = \text{gtp}_K(\bar{c}/A; N)$ if and only if $\text{gtp}_{K'}(\bar{b}/A; N) = \text{gtp}_{K'}(\bar{c}/A; N)$. Here, by gtp_K we denote the Galois type computed in $(K, \leq)$ and by $\text{gtp}_{K'}$ the Galois type computed in $(K', \unlhd)$.
  (d) $(K, \leq)$ is α-stable in λ if and only if $(K', \unlhd)$ is α-stable in λ.

Proof.

(1) Directly from the definition.
(2) (a) Let $M_0 \unlhd M_\ell$ be in K', $\ell = 1, 2$. By density, find $N \in K'$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$ K-embeddings. By Lemma 5.7, there exists $N_1 \in K'$ such that $N \unlhd N_1$, $f_1[M_1] \unlhd N_1$. By Lemma 5.7 again, there exists $N_2 \in K'$ such that $N_1 \unlhd N_2$, $f_2[M_2] \unlhd N_2$. Thus we also have $f_1[M_1] \unlhd N_2$. It follows that $f_\ell : M_\ell \underset{M}{\longrightarrow} N_2$ is a $\unlhd$-embedding.
  (b) If $(K', \unlhd)$ has joint embedding, then by density $(K, \leq)$ has joint embedding. The converse is similar to the proof of amalgamation above.
  (c) Note that by density any Galois type (in K) is realized in an element of K'. Since $(K', \unlhd)$ is a sub-AC of $(K, \leq)$, equality of the types in K' implies equality in K (this

doesn't use amalgamation). Conversely, assume $\mathrm{gtp}_K(\bar{b}/A; N) = \mathrm{gtp}_K(\bar{c}/A; N)$. Fix $N' \geq N$ in K and a K-embedding $f : N \xrightarrow[A]{} N'$ such that $f(\bar{b}) = \bar{c}$. By density, we can assume without loss of generality that $N' \in K'$. By Lemma 5.7, find $N'' \in K'$ such that $N \trianglelefteq N''$, $N' \trianglelefteq N''$. By Lemma 5.7 again, find $N''' \in K'$ such that $f[N] \trianglelefteq N'''$, $N'' \trianglelefteq N'''$. By transitivity, $N \trianglelefteq N'''$ and $f : N \xrightarrow[A]{} N'''$ witnesses equality of the Galois types in $(K', \trianglelefteq)$.

(d) Because Galois types are the same in K and K'.

□

We end with an observation concerning universal extensions that will be used in the proof of Lemma 6.7.

Lemma 5.9. Let K be an AEC in $\lambda := \mathrm{LS}(K)$. Assume K has amalgamation, no maximal models, and is stable in λ. Let $(K', \trianglelefteq)$ be a skeleton of K. For any $M \in K'$, there exists $N \in K'$ such that both $M \triangleleft N$ and $M <_{\mathrm{univ}} N$. Thus $(K', \trianglelefteq \cap \leq_{\mathrm{univ}})$ is a skeleton of K.

Proof. For the last sentence, let $\trianglelefteq' := \trianglelefteq \cap \leq_{\mathrm{univ}}$. Note that if $\langle M_i : i < \alpha \rangle$ is a $\trianglelefteq'$-increasing chain in K' and $M \in K'$ is such that $M_i < M$ for all $i < \alpha$, then by definition of a skeleton we can take M so that $M_i \triangleleft M$ for all $i < \alpha$. If we know that there exists $N \in K'$ with $M \triangleleft N$ and $M <_{\mathrm{univ}} N$, then for all $i < \alpha$, $M_i \triangleleft N$ by transitivity, and $M_i <_{\mathrm{univ}} N$ by Lemma 2.27.

Now let $M \in K'$. By Lemma 5.6, there exists $N \in K$ with $M <_{\mathrm{univ}} N$. By density (note that if $N' \geq N$ is in K, then $M <_{\mathrm{univ}} N'$) we can take $N \in K'$. By Lemma 5.7, there exists $N' \in K'$ such that $M \trianglelefteq N'$ and $N \trianglelefteq N'$. Thus $M <_{\mathrm{univ}} N'$, as desired. □

6. Generating an independence relation

In [She09, Section II.2], Shelah showed how to extend a good λ-frame to a pre-$(\geq \lambda)$-frame. Later, [Bon14a] (with improvements in [BVb]) gave conditions under which all the properties transferred. Similar ideas are used in [Vasa] to directly build a good frame. In this section we adapt Shelah's definition to our more general setup. It is useful to think of the initial λ-frame as a *generator*[22] for a $(\geq \lambda)$-frame, since in case the frame is not good we usually can only get a nice independence

[22] In [Vasa], we called a generator a skeletal frame (and in earlier version a poor man's frame).

relation on λ^+-saturated models. Moreover, it is often useful to work with the independence relation being only defined on a dense sub-AC of the original AECs.

Definition 6.1. $(K, \mathfrak{i})$ is a *λ-generator for a $(< \alpha)$-independence relation* if:

(1) α is a cardinal with $2 \leq \alpha \leq \lambda^+$. λ is an infinite cardinal.
(2) K is an AEC in $\lambda := \text{LS}(K)$
(3) $\mathfrak{i} = (K', \perp)$, where $(K', \leq)$ is a dense sub-AC (recall Definitions 5.1, 5.2) of[23] $(K, \leq)$.
(4) $\mathfrak{i}$ is a $(< \alpha, \lambda)$-independence relation.
(5) K^{up} (recall Definition 2.9) has amalgamation.

Remark 6.2. We could similarly define a λ-generator for a $(< \alpha)$-independence relation below θ, where we require $\theta \geq \lambda^{++}$ and only $K^{\text{up}}_{\mathcal{F}}$ has amalgamation (so when $\theta = \infty$ we recover the above definition). We will not adopt this approach as we have no use for the extra generality and do not want to complicate the notation further. We could also have required less than "K is an AEC in λ" but again we have no use for it.

Definition 6.3. Let $(K, \mathfrak{i})$ be a λ-generator for a $(< \alpha)$-independence relation. Define $(K, \mathfrak{i})^{\text{up}} := (K^{\text{up}}, \perp^{\text{up}})$ by $\perp^{\text{up}}(M, A, B, N)$ if and only if $M \leq N$ are in K^{up} and there exists $M_0 \leq M$ in K' such that for all $B_0 \subseteq B$ with $|B_0| \leq \lambda$ and all $N_0 \leq N$ in K' with $A \cup B_0 \subseteq |N_0|$, $M_0 \leq N_0$, we have $\underset{\mathfrak{i}}{\perp}(M_0, A, B_0, N_0)$.

When $K = K_{\mathfrak{i}}$, we write $\mathfrak{i}^{\text{up}}$ for $(K, \mathfrak{i})^{\text{up}}$.

Remark 6.4. In general, we do not claim that $(K, \mathfrak{i})^{\text{up}}$ is even an independence relation (the problem is that given $A \subseteq |N|$ with $N \in K^{\text{up}}$ and $|A| \leq \lambda$, there might not be any $M \in K'$ with $M \leq N$ and $A \subseteq |M|$ so the monotonicity properties can fail). Nevertheless, we will abuse notation and use the restriction operations on it.

Lemma 6.5. Let $(K, \mathfrak{i})$ be a λ-generator for a $(< \alpha)$-independence relation. Then:

(1) If $K = K_{\mathfrak{i}}$, then $\mathfrak{i}^{\text{up}} := (K, \mathfrak{i})^{\text{up}}$ is an independence relation.
(2) $(K, \mathfrak{i})^{\text{up}} \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$ is an independence relation.

[23]Why not be more general and require only $(K', \trianglelefteq)$ to be a skeleton of K? Because our main examples of skeletons do not satisfy the coherence axiom which appears in the definition of an independence relation.

Proof. As in [She09, Claim II.2.11], using density and homogeneity in the second case. □

The case (1) of Lemma 6.5 has been well studied (at least for the case when $\alpha = 2$): see [She09, Section II.2] and [Bon14a, Bon14b]. We will further look at it in the last sections. We will focus on case (2) for now. It has been studied (implicitly) in [Vasa] when $\mathfrak{i}$ is nonsplitting and satisfies some superstability-like assumptions. We will use the same methods to obtain more general results. The generality will be used, since for example we also care about what happens when $\mathfrak{i}$ is coheir.

The following property of a generator will be very useful in the next section. The point is that $\bigcup_{i<\lambda^+} M_i$ below is usually not a member of $K_{\mathfrak{i}}$ so nonforking is not defined on it.

Definition 6.6. Let $(K, \mathfrak{i})$ be a λ-generator for a $(< \alpha)$-independence relation.

$(K, \mathfrak{i})$ has *weak chain local character* if there exists $\trianglelefteq$ such that $(K_{\mathfrak{i}}, \trianglelefteq)$ is a skeleton of K and whenever $\langle M_i : i < \lambda^+ \rangle$ is $\triangleleft$-increasing in $K_{\mathfrak{i}}$ and $p \in \text{gS}^{<\alpha}(\bigcup_{i<\lambda^+} M_i)$, there exists $i < \lambda^+$ such that $p \upharpoonright M_{i+1}$ does not fork over M_i.

The following technical lemma shows that local character in a skeleton implies local character on a bigger class with the universal ordering:

Lemma 6.7. Let $(K, \mathfrak{i})$ be a λ-generator for a $(< \alpha)$-independence relation.

Assume that K has amalgamation, no maximal models, and is stable in λ. Assume $\mathfrak{i}$ has base monotonicity and let $K' := K_{\mathfrak{i}}$. Let $(K'', \trianglelefteq)$ be a skeleton of $(K', \leq)$ and let $\mathfrak{i}' := \mathfrak{i} \upharpoonright (K'', \leq)$. Then:

1. $\kappa_{<\alpha}(\mathfrak{i}, \leq_{\text{univ}}) \leq \kappa_{<\alpha}(\mathfrak{i}', \trianglelefteq)$.
2. If $(K, \mathfrak{i}')$ has weak chain local character, then $(K, \mathfrak{i})$ has it and it is witnessed by $<_{\text{univ}}$.

Proof.

1. By Lemma 5.9, we can (replacing $\trianglelefteq$ by $\trianglelefteq \cap \leq_{\text{univ}}$) assume without loss of generality that $\trianglelefteq$ is extended by $\leq_{\text{univ}}$. Let $\langle M_i : i < \delta \rangle$ be $\leq_{\text{univ}}$-increasing in K', $\delta = \text{cf}(\delta) \geq \kappa_{<\alpha}(\mathfrak{i}, \leq_{\text{univ}})$, $\delta < \lambda^+$. Without loss of generality, $\langle M_i : i < \delta \rangle$ is $<_{\text{univ}}$-increasing. Let $M_\delta := \bigcup_{i<\delta} M_i$ and let $p \in \text{gS}^{<\alpha}(M_\delta)$.

   By density, pick $M_0' \in K''$ such that $M_0 <_{\text{univ}} M_0'$. Now build $\langle M_i' : i < \delta \rangle$ $\triangleleft$-increasing in K''. Let $M_\delta' := \bigcup_{i<\delta} M_i'$. By Fact

2.25, there exists $f : M'_\delta \cong_{M_0} M_\delta$ such that for every $i < \delta$ there exists $j < \delta$ with $f[M'_i] \le M_j$, $f^{-1}[M_i] \le M'_j$. By definition of $\kappa_{<\alpha}(\mathfrak{i}', \trianglelefteq)$, there exists $i < \delta$ such that $f^{-1}(p)$ does not $\mathfrak{i}'$-fork over M'_i. Let $j < \delta$ be such that $f[M'_i] \le M_j$. By invariance, p does not $\mathfrak{i}'$-fork over $f[M_i]$, so does not $\mathfrak{i}$-fork over $f[M_i]$. By base monotonicity, p does not $\mathfrak{i}$-fork over M_j, as desired.

(2) Similar.

□

The last lemma of this section investigates what properties directly transfer up.

Lemma 6.8. Let $(K, \mathfrak{i})$ be a λ-generator for a $(< \alpha)$-independence relation. Let $\mathfrak{i}' := (K, \mathfrak{i})^{\text{up}} \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$.

(1) If $\mathfrak{i}$ has base monotonicity, then $\mathfrak{i}'$ has base monotonicity.
(2) Assume $\mathfrak{i}$ has base monotonicity and $(K, \mathfrak{i})$ has weak chain local character. Then:
   (a) $\bar{\kappa}_{<\alpha}(\mathfrak{i}') = \lambda^{++}$.
   (b) If $\trianglelefteq$ is an ordering such that $(K_{\mathfrak{i}}, \trianglelefteq)$ is a skeleton of K, then for any $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}') \le \kappa_{\alpha_0}(\mathfrak{i}, \trianglelefteq)$.

Proof.

(1) As in [She09, Claim II.2.11]
(2) This is a generalization of the proof of [Vasa, Lemma 4.8] but we have to say slightly more so we give the details. We first prove (2b). Fix $\alpha_0 < \alpha$, and assume $\kappa_{\alpha_0}(\mathfrak{i}, \trianglelefteq) < \infty$. Then by definition $\kappa_{\alpha_0}(\mathfrak{i}, \trianglelefteq) \le \lambda$. Let $\delta = \text{cf}(\delta) \ge \kappa_{\alpha_0}(\mathfrak{i}, \trianglelefteq)$.

   Let $\langle M_i : i < \delta \rangle$ be increasing in $K^{\lambda^+\text{-sat}}$ and write $M_\delta := \bigcup_{i<\delta} M_i$ (note that we do not claim $M_\delta \in K^{\lambda^+\text{-sat}}$. However, $M_\delta \in K_{\ge\lambda}$). Let $p \in \text{gS}^{\alpha_0}(M_\delta)$. We want to find $i < \delta$ such that p does not fork over M_i. There are two cases:
   - Case 1: $\delta < \lambda^+$:
     We imitate the proof of [She09, Claim II.2.11.5]. Assume the conclusion fails. Build $\langle N_i : i < \delta \rangle$ $\trianglelefteq$-increasing in K', $\langle N'_i : i < \delta \rangle$ $\le$-increasing in K' such that for all $i < \delta$:
       (a) $N_i \le M_i$.
       (b) $N_i \le N'_i \le M_\delta$.
       (c) $p \upharpoonright N'_i$ $\mathfrak{i}$-forks over N_i.
       (d) $\bigcup_{j<i}(|N'_j| \cap |M_j|) \subseteq |N_i|$.

This is possible. Assume N_j and N_j' have been constructed for $j < i$. Choose $N_i \leq M_i$ satisfying (2d) so that $N_j \trianglelefteq N_i$ for all $j < i$ (This is possible: use that M_i is λ^+-model-homogeneous and that in skeletons of AECs, chains have upper bounds). By assumption, p $\mathfrak{i}'$-forks over M_i, and so by definition of forking there exists $N_i' \leq M_\delta$ in K' such that $p \upharpoonright N_i'$ forks over N_i. By monotonicity, we can of course assume $N_i' \geq N_i$, $N_i' \geq N_j'$ for all $j < i$.
This is enough. Let $N_\delta := \bigcup_{i<\delta} N_i$, $N_\delta' := \bigcup_{i<\delta} N_i'$. By local character for $\mathfrak{i}$, there is $i < \delta$ such that $p \upharpoonright N_\delta$ does not fork over N_i. By (2b) and (2d), $N_\delta' \leq N_\delta$. Thus by monotonicity $p \upharpoonright N_i'$ does not $\mathfrak{i}$-fork over N_i, contradicting (2c).

- Case 2: $\delta \geq \lambda^+$: Assume the conclusion fails. As in the previous case (in fact it is easier), we can build $\langle N_i : i < \lambda^+ \rangle$ $\trianglelefteq^0$-increasing in K' such that $N_i \leq M_\delta$ and $p \upharpoonright N_{i+1}$ $\mathfrak{i}$-forks over N_i. Since $\mathfrak{i}$ has weak chain local character, there exists $i < \lambda^+$ such that $p \upharpoonright N_{i+1}$ does not $\mathfrak{i}$-fork over N_i, contradiction.

For (2a), assume not: then there exists $M \in K^{\lambda^+\text{-sat}}$ and $p \in \text{gS}^{<\alpha}(M)$ such that for all $M_0 \leq M$ in $K^{\lambda^+\text{-sat}}_{\lambda^+}$, p $\mathfrak{i}'$-forks over M_0. By stability, for any $A \subseteq |M|$ with $|A| \leq \lambda$, there exists $M_0 \leq M$ containing A which is λ^+-saturated of size λ^+. As in case 2 above, we build $\langle N_i : i < \lambda^+ \rangle$ $\trianglelefteq^0$-increasing in K' such that $N_i \leq M$ and $p \upharpoonright N_{i+1}$ $\mathfrak{i}$-fork over N_i. This is possible (for the successor step, given N_i, take any $M_0 \leq M$ saturated of size λ^+ containing N_i. By definition of $\mathfrak{i}'$ and the fact p $\mathfrak{i}'$-forks over M_0, there exists $N_{i+1}' \leq M$ in K' witnessing the forking. This can further extended to N_{i+1} which is as desired). This is enough: we get a contradiction to weak chain local character.

□

7. Weakly good independence relations

Interestingly, nonsplitting and $(< \kappa)$-coheir (for a suitable choice of κ) are already well-behaved if the AEC is stable. This raises the question of whether there is an object playing the role of a good frame (see the next section) in strictly stable classes. Note that [BGKV] proves the canonicity of independence relations that satisfy much less than the full properties of good frames, so it is reasonable to expect existence of such a relation. The next definition comes from extracting all the properties we are able to prove from the construction of a good frame in [Vasa] assuming only stability.

Definition 7.1. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!|}\,)$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$. $\mathfrak{i}$ is *weakly good* if:

(1) $K \neq \emptyset$, is λ-closed (Recall Definition 2.11), and every chain in K of ordinal length less than θ has an upper bound.
(2) K is stable in λ.
(3) $\mathfrak{i}$ has base monotonicity, disjointness, existence, uniqueness, and transitivity.
(4) $\mathfrak{i}$ has the left λ-witness property and the right λ-model-witness property.
(5) Local character: For all $\alpha_0 < \min(\lambda^+, \alpha)$, $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \lambda^+$.
(6) Local extension: $\mathfrak{i}_\lambda^{<\lambda^+}$ has extension.

We say a pre-$(< \alpha, \mathcal{F})$-frame $\mathfrak{s}$ is *weakly good* if $\text{cl}(\mathfrak{s})$ is weakly good. $\mathfrak{i}$ is *pre-weakly good* if $\text{pre}(\mathfrak{i})$ is weakly good.

Remark 7.2. By Proposition 4.3.(6), 4.3.(4), existence and the right λ-witness property follow from the others.

Our main tool to build weakly good independence relations will be to start from a λ-generator (see Definition 6.1) which satisfies some additional properties:

Definition 7.3. $(K, \mathfrak{i})$ is a *λ-generator for a weakly good $(< \alpha)$-independence relation* if:

(1) $(K, \mathfrak{i})$ is a λ-generator for a $(< \alpha)$-independence relation.
(2) K is nonempty, has no maximal models, and is stable in λ.
(3) $(K^{\text{up}})^{\lambda^+\text{-sat}}$ is λ-tame for types of length less than α.
(4) $\mathfrak{i}$ has base monotonicity, existence, and is extended by λ-nonsplitting: whenever $p \in \text{gS}^{<\alpha}(M)$ does not $\mathfrak{i}$-fork over $M_0 \leq M$, then p does not $\mathfrak{s}_{\lambda\text{-ns}}(K_{\mathfrak{i}})$-fork over M_0.
(5) $(K, \mathfrak{i})$ has weak chain local character.

Both coheir and λ-nonsplitting induce a generator for a weakly good independence relation:

Proposition 7.4. Let K be an AEC with amalgamation and let $\lambda \geq \text{LS}(K)$ be such that K_λ is nonempty, has no maximal models, and K is stable in λ. Let $2 \leq \alpha \leq \lambda^+$.

(1) Let $\text{LS}(K) < \kappa \leq \lambda$. Assume that K is $(< \kappa)$-tame and short for types of length less than α. Let $\mathfrak{i} := (\mathfrak{i}_{\kappa\text{-ch}}(K))^{<\alpha}$.

   (a) If K does not have the $(< \kappa)$-order property of length κ, $\kappa_{<\alpha}(\mathfrak{i}) \leq \lambda^+$, and $K_\lambda^{\kappa\text{-sat}}$ is dense in K_λ, then $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(< \alpha)$-independence relation.
   (b) If $\kappa = \beth_\kappa$, $(\alpha_0 + \mathrm{LS}(K))^{<\kappa} \leq \lambda$ for all $\alpha_0 < \alpha$, and $\lambda \geq \kappa_r$ (recall Definition 2.3), then $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(< \alpha)$-independence relation.
(2) Assume $\alpha \leq \omega$ and $K^{\lambda^+\text{-sat}}$ is λ-tame for types of length less than α. Then $\left(K_\lambda, (\mathfrak{i}_{\lambda\text{-ns}}(K_\lambda))^{<\alpha}\right)$ is a λ-generator for a weakly good $(< \alpha)$-independence relation.
(3) Let K' be a dense sub-AC of K such that $K^{\lambda^+\text{-sat}} \subseteq K'$ and let $\mathfrak{i}$ be a $(< \alpha, \geq \lambda)$-independence relation with $K_{\mathfrak{i}} = K'$, such that $\mathrm{pre}(\mathfrak{i})$ has uniqueness, $\mathfrak{i}$ has base monotonicity, and $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$. If K'_λ is dense in K_λ, then $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(< \alpha)$-independence relation.

Proof.

(1) (a) By Fact 3.17, $\mathfrak{i}$ has base monotonicity, existence, and uniqueness. By Fact 3.20.(3), coheir is extended by λ-nonsplitting. The other properties are easy. For example, weak chain local character follows from $\kappa_{<\alpha}(\mathfrak{i}) \leq \lambda^+$ and monotonicity.
   (b) We check that K and $\mathfrak{i}$ satisfy all the conditions of the previous part. By Fact 2.20, K does not have the $(< \kappa)$-order property of length κ. By (the proof of) Proposition 4.3.(5) and Fact 3.17:

$$\kappa_{<\alpha}(\mathfrak{i}) \leq \bar{\kappa}_{<\alpha}(\mathfrak{i}) \leq \sup_{\alpha_0 < \alpha} \left((\alpha_0 + \mathrm{LS}(K))^{<\kappa}\right)^+ \leq \lambda^+$$

   Since K is stable in λ, if $\kappa < \lambda$ then $K_\lambda^{\kappa\text{-sat}}$ is dense in K_λ. If $\kappa = \lambda$, then κ is regular, hence inaccessible so $\kappa = \kappa^{<\kappa}$ so again it is easy to check that $K_\lambda^{\kappa\text{-sat}}$ is dense in K_λ.
(2) Let $\mathfrak{i} := (\mathfrak{s}_{\lambda\text{-ns}}(K))^{<\alpha}$. By Fact 3.20.(2) and Proposition 4.3.(5), $\kappa_{<\alpha}(\mathfrak{i}) = \lambda^+$. By monotonicity, weak chain local character follows. The other properties are easy to check.
(3) By Fact 3.20.(3), $\mathfrak{i}$ is extended by λ-nonsplitting. Weak chain character follows from $\kappa_{<\alpha}(\mathfrak{i}) = \lambda^+$. By (the proof of) Proposition 4.6, $K^{\lambda^+\text{-sat}}$ is λ-tame for types of length less than α. The other properties are easy to check.

□

The next result is that a generator for a weakly good independence relation indeed induces a weakly good independence relation.

Theorem 7.5. Let $(K, \mathfrak{i})$ be a λ-generator for a weakly good $(<\alpha)$-independence relation. Then $(K, \mathfrak{i})^{\text{up}} \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$ is a pre-weakly good $(<\alpha, \geq \lambda^+)$-independence relation.

Proof. This essentially follows from the methods of [Vasa], but we give some details. Let $\mathfrak{i}' := (K, \mathfrak{i})^{\text{up}} \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$. Let $\mathop{\smile}\limits^{\vert} := \mathop{\smile}\limits^{\vert}_{\mathfrak{i}'}$, $K' := K_{\mathfrak{i}}$, $\mathfrak{s}' := \text{pre}(\mathfrak{i}')$. We check the conditions in the definition of a weakly good independence relation. Note that by Remark 7.2 we do not need to check existence or the right λ^+-witness property.

- $\mathfrak{i}'$ is a $(<\alpha, \geq \lambda^+)$-independence relation: By Lemma 6.5.
- $K_{\mathfrak{i}'}$ is stable in λ^+: By Fact 2.20, K^{up} is stable in λ^+. By stability, $K_{\mathfrak{i}'}$ is dense in K so by Proposition 5.8, $K_{\mathfrak{i}'}$ is stable in λ^+.
- $K_{\mathfrak{i}'} \neq \emptyset$ since it is dense in $K^{\text{up}}_{\lambda^+}$ and $K^{\text{up}}_{\lambda} = K$ is nonempty and has no maximal models. Every chain $\langle M_i : i < \delta \rangle$ in $K_{\mathfrak{i}'}$ has an upper bound: we have $M_\delta := \bigcup_{i<\delta} M_i \in K$, and by density there exists $M \geq M_\delta$ in $K_{\mathfrak{i}'}$. $K_{\mathfrak{i}'}$ is λ^+-closed by an easy increasing chain argument, using stability in λ^+.
- Local character: $\bar{\kappa}_{<\alpha}(\mathfrak{i}') = \lambda^{++}$ by Lemma 6.8.
- $\mathfrak{s}'$ has:
  - Base monotonicity: By Lemma 6.8.
  - Uniqueness: First observe that using local character, base monotonicity, λ^+-closure, and the fact that $K_{\mathfrak{i}'}$ is λ^+-tame for types of length less than α, it is enough to show uniqueness for $(\mathfrak{s}')_{\lambda^+}$. For this imitate the proof of [Vasa, Lemma 5.3] (the key is weak uniqueness: Fact 3.20.(5)).
  - Local extension: Let $p \in \text{gS}^{<\alpha}(M)$, $M_0 \leq M \leq N$ be in $(K_{\mathfrak{i}'})_{\lambda^+}$ such that p does not fork over M_0. Let $M_0' \leq M_0$ be in K' and witness it. By homogeneity, $M_0' <_{\text{univ}} M$ so there exists $f : N \xrightarrow[M_0']{M}$. Extend f to an isomorphism $\widehat{f} : \widehat{N} \cong_{M_0'} M$. Let $q := f^{-1}(p) \upharpoonright N$. By invariance, q does not fork over M_0 (as witnessed by M_0'). Since λ-nonsplitting extends nonforking, p does not $\mathfrak{s}_{\lambda\text{-ns}}$-fork over M_0'. By Fact 3.20.(4), p does not $\mathfrak{s}_{\text{ns}}$-fork over M_0'. By weak extension (Fact 3.20.(5), q extends p and is algebraic if and only if q is.
  - Transitivity: Imitate the proof of [Vasa, Lemma 4.7].

– Disjointness: It is enough to prove it for types of length 1 so assume $\alpha = 2$. Assume $a \overset{N}{\underset{M_0}{\downarrow}} M$ (with $M_0 \leq M \leq N$ in $K^{\lambda^+\text{-sat}}$) and $a \in M$. We show $a \in M_0$. Using local character, we can assume without loss of generality that $\|M_0\| = \lambda^+$ and (by taking a submodel of M containing a of size λ^+) that also $\|M\| = \lambda^+$. Find $M_0' \leq M_0$ in K' witnessing the nonforking. By the proof of local extension, we can find $p \in \text{gS}(M)$ extending $p_0 := \text{gtp}(a/M_0; N)$ such that p_0 is algebraic if and only if p is. Since $a \in N$, we must have by uniqueness that p is algebraic so p_0 is algebraic, i.e. $a \in M_0$.

Now by Proposition 4.1, $\text{cl}(\mathfrak{s}')$ has the above properties.

- $\text{cl}(\mathfrak{s}')$ has the left λ-witness property: Because $\alpha \leq \lambda^+$.

□

Interestingly, the generator can always be taken to have a particular form:

Lemma 7.6. Let $(K, \mathfrak{i})$ be a λ-generator for a weakly good $(< \alpha)$-independence relation. Let $\mathfrak{i}' := \mathfrak{i}_{\lambda\text{-ns}}(K)^{<\alpha}$. Then:

(1) $(K, \mathfrak{i}')$ is a λ-generator for a weakly good $(< \alpha)$-independence relation and $<_{\text{univ}}$ is the ordering witnessing weak chain local character.
(2) $\text{pre}((K, \mathfrak{i})^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}} = \text{pre}((K, \mathfrak{i}')^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$.

Proof.

(1) By Lemma 6.7 (with K, $\mathfrak{i}'$, $K_{\mathfrak{i}}$ here standing for $K, \mathfrak{i}$, K'' there), $(K, \mathfrak{i}')$ has weak chain local character (witnessed by $<_{\text{univ}}$) and the other properties are easy to check.
(2) Let $\mathfrak{s} := \text{pre}((K, \mathfrak{i})^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$, $\mathfrak{s}' := \text{pre}((K, \mathfrak{i}')^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$. We want to see that $\underset{\mathfrak{s}}{\downarrow} = \underset{\mathfrak{s}'}{\downarrow}$. Since $\text{pre}(\mathfrak{i})$ is extended by λ-nonsplitting, it is easy to check that $\underset{\mathfrak{s}}{\downarrow} \subseteq \underset{\mathfrak{s}'}{\downarrow}$. By the proof of [BGKV, Lemma 4.1], $\underset{\mathfrak{s}_{\lambda^+}}{\downarrow} = \underset{\mathfrak{s}'_{\lambda^+}}{\downarrow}$. By the right λ-model-witness property, $\underset{\mathfrak{s}}{\downarrow} = \underset{\mathfrak{s}'}{\downarrow}$.

□

Note that if the independence relation of the generator is coheir, then the weakly good independence relation obtained from it is also coheir. We first prove a slightly more abstract lemma:

Lemma 7.7. Let K be an AEC, $\lambda \geq \text{LS}(K)$. Let K' be a dense sub-AC of K such that $K^{\lambda^+\text{-sat}} \subseteq K'$ and K'_λ is dense in K_λ. Let $\mathfrak{i}$ be a $(<\alpha, \geq \lambda)$-independence relation with base monotonicity and $K_{\mathfrak{i}} = K'$, $2 \leq \alpha \leq \lambda^+$. Assume that $\mathfrak{i}$ has base monotonicity and the right λ-model-witness property.

Assume $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$ and $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(<\alpha)$-independence relation. Let $\mathfrak{i}' := (K_\lambda, \mathfrak{i}_\lambda)_{\geq \lambda} \upharpoonright K^{\lambda^+\text{-sat}}$. Then $\text{pre}(\mathfrak{i}') = \text{pre}(\mathfrak{i}) \upharpoonright K^{\lambda^+\text{-sat}}$.

Moreover if $\mathfrak{i}$ has the right λ-witness property, then $\mathfrak{i}' = \mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}}$.

Proof. We prove the moreover part and it will be clear how to change the proof to prove the weaker statement (just replace the use of the witness property by the model-witness property).

Let $M \leq N$ be in $K^{\lambda^+\text{-sat}}$, $p \in \text{gS}^{<\alpha}(B; N)$. We want to show that p does not $\mathfrak{i}$-fork over M if and only if there exists $M_0 \leq M$ in K'_λ so that for all $B_0 \subseteq B$ of size $\leq \lambda$, $p \upharpoonright B_0$ does not $\mathfrak{i}$-fork over M_0. Assume first that p does not $\mathfrak{i}$-fork over M. Since $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$, there exists $M_0 \leq M$ in K_λ such that p does not $\mathfrak{i}$-fork over M_0. By base monotonicity and homogeneity, we can assume that $M_0 \in K'_\lambda$. I particular $p \upharpoonright B_0$ does not $\mathfrak{i}$-fork over B_0 for all $B \subseteq B$ of size $\leq \lambda$.

Conversely, assume p does not $\mathfrak{i}'$-fork over M, and let $M_0 \leq M$ in K'_λ witness it. Then by the right λ-witness property, p does not $\mathfrak{i}$-fork over M_0, so over M, as desired.

□

Lemma 7.8. Let K be an AEC, $\text{LS}(K) < \kappa \leq \kappa' \leq \lambda$. Let $2 \leq \alpha \leq \lambda^+$. Let $\mathfrak{i} := (\mathfrak{i}_{\kappa\text{-ch}}(K))^{<\alpha}_{\geq \lambda} \upharpoonright K^{\kappa'\text{-sat}}$.

Assume $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$ and $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(<\alpha)$-independence relation. Let $\mathfrak{i}' := (K_\lambda, \mathfrak{i}_\lambda)_{\geq \lambda} \upharpoonright K^{\lambda^+\text{-sat}}$. Then $\mathfrak{i}' = \mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}}$.

Proof. By Lemma 7.7 applied with $K' = K^{\kappa'\text{-sat}}$. □

We end this section by showing how to build a weakly good independence relation in any stable fully tame and short AEC (with amalgamation and no maximal models).

Theorem 7.9. Let K be a $\text{LS}(K)$-tame AEC with amalgamation and no maximal models. Let $\kappa = \beth_\kappa > \text{LS}(K)$. Assume K is stable and $(< \kappa)$-tame and short for types of length less than α, $\alpha \geq 2$.

If $K_\kappa \neq \emptyset$, then $\mathfrak{i}_{\kappa\text{-ch}}(K)^{<\alpha} \upharpoonright K^{(\kappa^{<\kappa})^+\text{-sat}}$ is a pre-weakly good $(< \alpha, \geq (\kappa^{<\kappa})^+)$-independence relation. Moreover if $\alpha = \infty$, then it is weakly good.

Proof. Let $\lambda := \kappa^{<\kappa}$. By Fact 3.17, $\mathfrak{i}_{\kappa\text{-ch}}(K)^{<\alpha} \upharpoonright K^{\lambda^+\text{-sat}}$ already has many properties of a weakly good independence relation, and in particular has the left λ-witness property so it is enough to check that $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)^{<(\min(\alpha,\lambda^+))} \upharpoonright K^{\lambda^+\text{-sat}}$ is weakly good, so assume now without loss of generality that $\alpha \leq \lambda^+$. Note that by Fact 3.17, $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = (\lambda^{<\kappa})^+ = \lambda^+$. By Lemma 7.8 it is enough to check that $(K_\lambda, \mathfrak{i}_\lambda)$ is a λ-generator for a weakly good $(< \alpha)$-independence relation. From Fact 2.20, we get that K is stable in λ. Note also (Fact 2.13) that K has arbitrarily large models, so $K_\lambda \neq \emptyset$. Finally, $\lambda \geq \kappa_r$ (as if $\lambda = \kappa$ then $\kappa = \kappa^{<\kappa}$ so is regular). Now apply Proposition 7.4.

If $\alpha = \infty$, then by Fact 3.17, $\mathfrak{i}$ has uniqueness. Since $\mathfrak{i}$ is pre-weakly good, $\text{pre}(\mathfrak{i}_\lambda)$ has extension, so by Proposition 4.1.(7), $\mathfrak{i}_\lambda$ also has extension. The other properties of a weakly good independence relation follow from Fact 3.17. □

8. Good independence relations

Good frames were introduced by Shelah [She09, Definition II.2.1] as a "bare bone" definition of superstability in AECs. Here we recall the definition of a good independence relation. We also define a variation, being *fully* good. This is only relevant when the types are allowed to have length $\geq \lambda$, and asks for more continuity (like in [BVb], but the continuity property asked for is different). This is used to enlarge a good frame in the last sections.

Definition 8.1.

(1) A *good* $(< \alpha, \mathcal{F})$*-independence relation* $\mathfrak{i} = (K, \perp)$ is a $(< \alpha, \mathcal{F})$-independence relation satisfying:
  (a) K is an AEC in $\mathcal{F}$, $K \neq \emptyset$, $\text{LS}(K) = \lambda_\mathfrak{i}$, K has no maximal models and joint embedding, K is stable in all cardinals in $\mathcal{F}$.
  (b) $\mathfrak{i}$ has base monotonicity, disjointness, symmetry, uniqueness, extension, the left $\lambda_\mathfrak{i}$-witness property, and for all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}) = |\alpha_0|^+ + \aleph_0$ and $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = |\alpha_0|^+ + \lambda_\mathfrak{i}^+$.

(2) A *type-full good* $(<\alpha, \mathcal{F})$*-frame* $\mathfrak{s}$ is a pre-$(<\alpha, \mathcal{F})$-frame so that $\text{cl}(\mathfrak{s})$ is good.
(3) $\mathfrak{i}$ is *pre-good* if $\text{pre}(\mathfrak{i})$ is good.

When we add "fully", we require in addition that the frame/independence relation satisfies full model-continuity.

Remark 8.2. Our definition is equivalent to that of Shelah [She09, Definition II.2.1] if we remove the requirement there on the existence of a superlimit (as was done in almost all subsequent papers, for example in [JS13]) and assume the frame is type-full (i.e. the basic types are all the nonalgebraic types). For example, the continuity property Shelah requires follows from $\kappa_1(\mathfrak{s}) = \aleph_0$ ([She09, Claim II.2.17.3]).

Remark 8.3. If $\mathfrak{i}$ is a good $(<\alpha, \mathcal{F})$-independence relation (except perhaps for the symmetry axiom) then $\mathfrak{i}$ is weakly good.

Definition 8.4. An AEC K is *[fully]* $(<\alpha, \mathcal{F})$*-good* if there exists a [fully] $(<\alpha, \mathcal{F})$-good independence relation $\mathfrak{i}$ with $K_{\mathfrak{i}} = K$. When $\alpha = \infty$ and $\mathcal{F} = [\text{LS}(K), \infty)$, we omit them.

As in the previous section, we give conditions for a generator to induce a good independence relation:

Definition 8.5. $(K, \mathfrak{i})$ is a λ*-generator for a good* $(<\alpha)$*-independence relation* if:

(1) $(K, \mathfrak{i})$ is a λ-generator for a weakly good $(<\alpha)$-independence relation.
(2) There exists $\lambda' \geq \lambda$ such that $K_{\lambda'}^{\text{up}}$ has joint embedding.
(3) Local character: For all $\alpha_0 < \min(\alpha, \lambda)$, there exists an ordering $\trianglelefteq$ such that $(K_{\mathfrak{i}}, \trianglelefteq)$ is a skeleton of K and $\kappa_{\alpha_0}(\mathfrak{i}, \trianglelefteq) = |\alpha_0|^+ + \aleph_0$.

Remark 8.6. If $(K, \mathfrak{i})$ is a λ-generator for a good $(<\alpha)$-independence relation, then it is a λ-generator for a weakly good $(<\alpha)$-independence relation. Moreover if $\alpha < \lambda^+$, then the weak chain local character axiom follows from the local character axiom.

As before, the generator can always be taken to be of a particular form:

Lemma 8.7. Let $(K, \mathfrak{i})$ be a λ-generator for a good $(<\alpha)$-independence relation. Let $\mathfrak{i}' := \mathfrak{i}_{\lambda\text{-ns}}(K)^{<\alpha}$. Then:

(1) $(K, \mathfrak{i}')$ is a λ-generator for a good $(<\alpha)$-independence relation and $<_{\text{univ}}$ is the ordering witnessing local character.
(2) $\text{pre}((K, \mathfrak{i})^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}} = \text{pre}((K, \mathfrak{i}')^{\text{up}}) \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$.

Proof.

(1) By Lemma 6.7 (with K, $\mathfrak{i}'$, $K_{\mathfrak{i}}$ here standing for $K, \mathfrak{i}$, K'' there), $(K, \mathfrak{i}')$ has the local character properties, witnessed by $<_{\text{univ}}$, and the other properties are easy to check.
(2) By Lemma 7.6.

□

As before, we get that a generator for a good independence relation indeed induces a good independence relation.

Theorem 8.8. Let $(K, \mathfrak{i})$ be a λ-generator for a good $(< \alpha)$-independence relation. Then:

(1) K^{up} has joint embedding and no maximal models.
(2) K^{up} is stable in every $\mu \geq \lambda$.
(3) $\mathfrak{i}' := (K, \mathfrak{i})^{\text{up}} \upharpoonright (K^{\text{up}})^{\lambda^+\text{-sat}}$ is a pre-good $(< \alpha, \geq \lambda^+)$-independence relation, except perhaps that $K_{\mathfrak{i}'}$ is not an AEC.
(4) If $\mu \geq \lambda^+$ is such that $(K^{\text{up}})^{\mu\text{-sat}}$ is an AEC with Löwenheim-Skolem number μ, then $\mathfrak{i}' \upharpoonright (K^{\text{up}})^{\mu\text{-sat}}$ is a pre-good $(< \alpha, \geq \mu)$-independence relation.

Proof. Again, this essentially follows from the methods of [Vasa], but we give some details. We show by induction on $\theta \geq \lambda^+$ that $\mathfrak{s}' := \text{pre}(\mathfrak{i}')_{[\lambda^+, \theta]}$ is a good frame, except perhaps that $(K_{\mathfrak{i}'})_{[\lambda^+, \theta]}$ is not an AEC and that symmetry may fail. This gives 3 (use Proposition 4.3.(7)), and (4) together with (1),(2) (use Proposition 5.8) follow.

- $\mathfrak{s}'$ is a weakly good $(< \alpha, [\lambda^+, \theta])$-independence relation: By Theorem 7.5.
- Let $\lambda' \geq \lambda$ be such that $K^{\text{up}}_{\lambda'}$ has joint embedding. By amalgamation, $K^{\text{up}}_{\geq \lambda'}$ has joint embedding. Once it is shown that K^{up} has no maximal models, it will follow that K^{up} has joint embedding (every model of size $\geq \lambda$ extends to one of size λ'). Note that joint embedding is never used in any of the proofs below.
- To prove that $K^{\text{up}}_{[\lambda, \theta]}$ has no maximal models, we can assume without loss of generality that $\alpha = 2$ and (by Lemma 8.7) that $\mathfrak{i} = \mathfrak{i}_{\lambda\text{-ns}}(K)$, with $\kappa_1(\mathfrak{i}, <_{\text{univ}}) = \aleph_0$. By the induction hypothesis (and the assumption that K has no maximal models), $K^{\text{up}}_{[\lambda, \theta)}$ has no maximal models. It remains to see that K^{up}_{θ} has no maximal models. Assume for a contradiction that $M \in K^{\text{up}}_{\theta}$ is maximal. Then it is easy to check that $M \in (K^{\text{up}})^{\theta\text{-sat}}_{\theta}$. Build

$\langle M_i : i < \theta \rangle$ increasing continuous and $a \in |M|$ such that for all $i < \delta$:

(1) $M_i \leq M$.
(2) $M_i <_{\text{univ}} M_{i+1}$.
(3) $M_i \in K^{\text{up}}_{<\theta}$.
(4) $a \notin |M_i|$.

This is enough. Let $M_\theta := \bigcup_{i<\theta} M_i$. Note that $\|M_\theta\| = \theta$ and $a \in |M| \backslash |M_\theta|$, so $M_\theta < M$. By Lemma 2.27, $M_0 <_{\text{univ}} M_\theta$. Thus there exists $f : M \xrightarrow[M_0]{} M_\theta$ and since M is maximal f is an isomorphism. However M is maximal whereas M witnesses that M_θ is not maximal, so M cannot be isomorphic to M_θ, a contradiction.

This is possible. Imitate the proof of [Vasa, Lemma 5.12] (this is where it is useful that the generator is nonsplitting and the local character is witnessed by $<_{\text{univ}}$).

- K^{up} is stable in all $\lambda' \in [\lambda^+, \theta]$: Exactly as in the proof of [Vasa, Theorem 5.6].
- $\mathfrak{s}'$ has base monotonicity, disjointness, and uniqueness because it is weakly good. For all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{s}') = |\alpha_0|^+ + \aleph_0$, $\bar{\kappa}_{\alpha_0}(\mathfrak{s}') = |\alpha_0|^+ + \lambda^{++} = \lambda^{++}$ by Lemma 6.8.
- $\mathfrak{s}'$ has extension: As in [Vasa, Lemma 5.9].

□

Remark 8.9. Our proof of no maximal models above improves on [She09, Conclusion 4.13.3], as it does not use the symmetry property.

9. Canonicity

In [BGKV], we gave conditions under which two independence relations are the same. There we strongly relied on the extension property, but coheir and weakly good frames only have a weak version of it. In this section, we show that if we just want to show two independence relations are the same over sufficiently saturated models, then the proofs become easier and the extension property is not needed. In addition, we obtain an explicit description of the nonforking relation. We conclude that coheir, weakly good frames, and good frames are (in a sense made precise below) canonical. This gives further evidence that these objects are not ad-hoc and answers several questions in [BGKV]. The results of this section are also used in Section 10 to show the equivalence between superstability and strong superstability.

Lemma 9.1 (The canonicity lemma)**.** Let K be an AEC with amalgamation and let $\lambda \geq \mathrm{LS}(K)$ be such that K is stable in λ. Let K' be a dense sub-AC of K such that $K^{\lambda^+\text{-sat}} \subseteq K'$ and K'_λ is dense in K_λ. Let $\mathfrak{i}$, $\mathfrak{i}'$ be $(<\alpha, \geq \lambda)$-independence relation and $K_{\mathfrak{i}} = K_{\mathfrak{i}'} = K'$. Let $\alpha_0 := \min(\alpha, \lambda^+)$.

If:

(1) $\mathrm{pre}(\mathfrak{i})$ and $\mathrm{pre}(\mathfrak{i}')$ have uniqueness.
(2) $\mathfrak{i}$ and $\mathfrak{i}'$ have base monotonicity, the left λ-witness property, and the right λ-model-witness property.
(3) $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) = \bar{\kappa}_{<\alpha_0}(\mathfrak{i}') = \lambda^+$.

Then $\mathrm{pre}(\mathfrak{i}) \upharpoonright K^{\lambda^+\text{-sat}} = \mathrm{pre}(\mathfrak{i}') \upharpoonright K^{\lambda^+\text{-sat}}$, and if in addition both $\mathfrak{i}$ and $\mathfrak{i}'$ have the right λ-witness property, then $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-sat}}$.

Moreover for $M \leq N$ in $K^{\lambda^+\text{-sat}}$, $p \in \mathrm{gS}^{<\alpha}(N)$ does not $\mathfrak{i}$-fork over M if and only if for all $I \subseteq \ell(p)$ with $|I| \leq \lambda$, there exists $M_0 \leq M$ in K'_λ such that p^I does not $\mathfrak{s}_{\lambda\text{-ns}}(K')$-fork over M_0.

Proof. By Fact 2.14, we can assume without loss of generality that K has joint embedding. If $K_{\lambda^+} = \emptyset$, there is nothing to prove so assume $K_{\lambda^+} \neq \emptyset$. Using joint embedding, it is easy to see that K_λ is nonempty and has no maximal models. By the left λ-witness property, we can assume without loss of generality that $\alpha \leq \lambda^+$, i.e. $\alpha = \alpha_0$. By Proposition 7.4, $(K, \mathfrak{i}), (K, \mathfrak{i}')$ are λ-generators for a weakly good $(<\alpha)$-independence relation. By Lemma 7.6, $\mathrm{pre}((K, \mathfrak{i})_{\geq \lambda}) \upharpoonright K^{\lambda^+\text{-sat}} = \mathrm{pre}((K, \mathfrak{i}')_{\geq \lambda}) \upharpoonright K^{\lambda^+\text{-sat}}$.

By Lemma 7.7, for $x \in \{\mathfrak{i}, \mathfrak{i}'\}$, $\mathrm{pre}((K, x)_{\geq \lambda}) \upharpoonright K^{\lambda^+\text{-sat}} = \mathrm{pre}(x) \upharpoonright K^{\lambda^+\text{-sat}}$, so the result follows (the definition of $(K, x)_{\geq \lambda}$ and Lemma 7.6 also give the moreover part). The moreover part of lemma 7.7 says that if $x \in \{\mathfrak{i}, \mathfrak{i}'\}$ has the right λ-witness property, then $(K, x)_{\geq \lambda} \upharpoonright K^{\lambda^+\text{-sat}} = x \upharpoonright K^{\lambda^+\text{-sat}}$, so in case both $\mathfrak{i}$ and $\mathfrak{i}'$ have the right λ-witness property, we must have $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-sat}}$. □

Remark 9.2. If K is an AEC with amalgamation, K' is a dense sub-AC of K such that $K^{\lambda^+\text{-sat}} \subseteq K'$ and K'_λ is dense in K_λ, and $\mathfrak{i}$ is a $(\leq 1, \geq \lambda)$-independence relation with $K_{\mathfrak{i}} = K'$ and base monotonicity, uniqueness, $\bar{\kappa}_1(\mathfrak{i}) = \lambda^+$, then by the proof of Proposition 4.6 and Lemma 5.8 K is stable in any $\mu \geq \mathrm{LS}(K)$ with $\mu = \mu^\lambda$.

Theorem 9.3 (Canonicity of coheir)**.** Let K be an AEC with amalgamation. Let $\kappa = \beth_\kappa > \mathrm{LS}(K)$. Assume K is $(<\kappa)$-tame and short for types of length less than α, $\alpha \geq 2$.

Let $\lambda \geq \kappa_r$ be such that K is stable in λ and $(\alpha_0 + \text{LS}(K))^{<\kappa} \leq \lambda$ for all $\alpha_0 < \min(\lambda^+, \alpha)$. Let $\mathfrak{i}$ be a $(< \alpha, \geq \lambda)$-independence relation so that:

(1) $K' := K_{\mathfrak{i}}$ is a dense sub-AC of K so that $K^{\lambda^+\text{-sat}} \subseteq K'$ and K'_λ is dense in K_λ.
(2) $\text{pre}(\mathfrak{i})$ has uniqueness.
(3) $\mathfrak{i}$ has base monotonicity, the left λ-witness property, and the right λ-model-witness property.
(4) $\bar{\kappa}_{<\min(\lambda^+,\alpha)}(\mathfrak{i}) = \lambda^+$.

Then $\text{pre}(\mathfrak{i}) \upharpoonright K^{\lambda^+\text{-sat}} = \text{pre}(\mathfrak{i}_{\kappa\text{-ch}}(K)^{<\alpha}) \upharpoonright K^{\lambda^+\text{-sat}}$. If in addition $\mathfrak{i}$ has the right λ-witness property, then $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}_{\kappa\text{-ch}}(K)^{<\alpha} \upharpoonright K^{\lambda^+\text{-sat}}$.

Proof. By Fact 2.14, we can assume without loss of generality that K has joint embedding. If $K_{\lambda^+} = \emptyset$, there is nothing to prove so assume $K_{\lambda^+} \neq \emptyset$. By Fact 2.13, K has arbitrarily large models so no maximal models. Let $\mathfrak{i}' := \mathfrak{i}_{\kappa\text{-ch}}(K)^{<\alpha}$. By the proof of Proposition 7.4, $\mathfrak{i}' \upharpoonright K'$ satisfies the hypotheses of Lemma 9.1. Moreover, it has the right $(< \kappa)$-witness property so the result follows. □

Theorem 9.4 (Canonicity of weakly good independence relations)**.** Let K be an AEC with amalgamation and let $\lambda \geq \text{LS}(K)$. Let K' be a dense sub-AC of K such that $K^{\lambda^+\text{-sat}} \subseteq K'$ and K'_λ is dense in K_λ. Let $\mathfrak{i}$, $\mathfrak{i}'$ be weakly good $(< \alpha, \geq \lambda)$-independence relations with $K_{\mathfrak{i}} = K_{\mathfrak{i}'} = K'$.

Then $\text{pre}(\mathfrak{i}) \upharpoonright K^{\lambda^+\text{-sat}} = \text{pre}(\mathfrak{i}') \upharpoonright K^{\lambda^+\text{-sat}}$. If in addition both $\mathfrak{i}$ and $\mathfrak{i}'$ have the right λ-witness property, then $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-sat}}$.

Proof. By definition of a weakly good independence relation, K'_λ is stable in λ. Therefore by Lemma 5.8 K_λ, and hence K, is stable in λ. Now apply Lemma 9.1. □

Theorem 9.5 (Canonicity of good independence relations)**.** If $\mathfrak{i}$ and $\mathfrak{i}'$ are good $(< \alpha, \geq \lambda)$-independence relations with the same underlying AEC K, then $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-sat}}$.

Proof. By Theorem 9.4 (with $K' := K$), $\text{pre}(\mathfrak{i}) \upharpoonright K^{\lambda^+\text{-sat}} = \text{pre}(\mathfrak{i}') \upharpoonright K^{\lambda^+\text{-sat}}$. Since good independence relations have extension, Lemma 4.2 implies $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-sat}}$. □

Recall that [BGKV, Question 6.14] asked if two good λ-frames with the same underlying AEC should be the same. We can make progress toward this question by slightly refining our methods. Note that the results below can be further refined to work for not necessarily type-full frames (that is for two good frames, in Shelah's sense, with the same basic types and the same underlying AEC).

Lemma 9.6. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be good $(< \alpha, \lambda)$-frames with the same underlying AEC K and $\alpha \leq \lambda$. Let K' be the class of λ-limit models of K (recall Definition 2.21). Then $\mathfrak{s} \upharpoonright K' = \mathfrak{s}' \upharpoonright K'$.

Proof sketch. Note that by [She09, Lemma II.4.8] (see [Bon14a, Theorem 9.2] for a detailed proof), $I(K') = 1$. Now refine the proof of Theorem 9.5 by replacing λ^+-saturated models by $(\lambda, |\beta|^+ + \aleph_0)$-limit models for each $\beta < \alpha$. Everything still works since one can use the weak uniqueness and extension properties of nonsplitting (Fact 3.20.(5)). $\square$

Theorem 9.7 (Canonicity of categorical good λ-frames)**.** Let $\mathfrak{s}$ and $\mathfrak{s}'$ be good $(< \alpha, \lambda)$-frames with the same underlying AEC K and $\alpha \leq \lambda$. If K is categorical in λ, then $\mathfrak{s} = \mathfrak{s}'$.

Proof. By Fact 2.24, K has a limit model, and so by categoricity any model of K is limit. Now apply Lemma 9.6. $\square$

Remark 9.8. The proof also gives an explicit description of nonforking: For $M_0 \leq M$ with M_0 a limit model, $p \in \text{gS}(M)$ does not $\mathfrak{s}$-fork over M_0 if and only if there exists $M_0' <_{\text{univ}} M_0$ such that p does not $\mathfrak{s}_{\lambda\text{-ns}}$-fork over M_0'. Note that this is the definition of forking in [Vasa].

Note that Shelah's construction of a good λ-frame in [She09, Theorem II.3.7] relies on categoricity in λ, so Theorem 9.7 establishes that the frame there is canonical. We are still unable to show that the frame built in Theorem 10.14 is canonical in general, although it will be if λ is the categoricity cardinal or if it is weakly successful (by [BGKV, Theorem 6.13]).

10. Superstability

Shelah has pointed out [She09, p. 19] that superstability in abstract elementary classes suffers from schizophrenia, i.e. there are several different possible definitions that turn out to be equivalent in elementary classes but not necessarily so in AECs. The existence of a good $(\geq \lambda)$-frame is a possible definition but it is very hard to check so one would like a definition that implies existence of a good frame but is more tractable.

Shelah claims in chapter IV of his book that solvability[24] ([She09, Definition IV.1.4]) is such a notion, but his justification is yet to appear (in [She]). Essentially, solvability says that certain classes of EM models are superlimits. On the other hand previous work (for example [She99, SV99, GVV]) all rely on a local character property for nonsplitting. This is even made into a definition of superstability in [Gro02, Definition 7.12]. In [Vasa] we gave a similar condition and used it with tameness to build a good frame. We pointed out that categoricity in a cardinal of cofinality greater than the tameness cardinal implied the superstability condition.

We now aim to show the same conclusion under categoricity in a high-enough cardinal of arbitrary cofinality. We also generalize the definition of superstability implicit in [Vasa]:

Definition 10.1 (Superstability)**.** An AEC K is $(\mu, \mathfrak{i})$-*superstable* if $\mu \geq \mathrm{LS}(K)$, K is μ-tame, and $(K_\mu, \mathfrak{i})$ is a μ-generator for a good (≤ 1)-independence relation. When we omit parameter, we mean that there exists a value for the parameter that makes the definition hold. For example, K is *superstable* if it is $(\mu, \mathfrak{i})$-superstable for some μ and some $\mathfrak{i}$.

For technical reasons, we will also use the following version that uses coheir rather than nonsplitting.

Definition 10.2. We say an AEC K is κ-*strongly* $(\mu, \mathfrak{i})$-*superstable* if:

1. $\mathrm{LS}(K) < \kappa \leq \mu$.
2. For some $\kappa_0 < \kappa$, $K_{\geq \kappa_0}$ has amalgamation.
3. K is $(< \kappa)$-tame.
4. K does not have the $(< \kappa)$-order property of length κ.
5. $\bar{\kappa}_1(\mathfrak{i}_{\kappa\text{-ch}}(K)) \leq \mu^+$.
6. $K_{\mathfrak{i}} \subseteq K^{\kappa\text{-sat}}$ and $\mathfrak{i} = \mathfrak{i}_{\kappa\text{-ch}}(K)^{\leq 1} \upharpoonright K_{\mathfrak{i}}$.
7. K is $(\mu, \mathfrak{i})$-superstable.

As before, we may omit some parameters.

Admittedly, the definitions above are very technical and rely on several layers of definitions (the definition of superstability relies on that of a generator for a good independence relation, which in turn relies on the definition of a skeleton). However their generality turns out to be useful when one wants to check that a class is superstable. In any case,

[24] One can ask whether there are any implications between our definition and Shelah's. We leave this to future work.

the reader might be relieved to know that there are simpler equivalent definitions:

Proposition 10.3. Let K be an AEC.

(1) K is μ-superstable if and only if:
  (a) $\mu \geq \text{LS}(K)$.
  (b) $K_{\geq \mu}$ is μ-tame and has amalgamation.
  (c) K_μ is nonempty, is stable in μ, and has no maximal models.
  (d) For some $\mu' \geq \mu$, $K_{\mu'}$ has joint embedding.
  (e) $\kappa_1(\mathfrak{s}_{\mu\text{-ns}}(K_\mu), \leq_{\text{univ}}) = \aleph_0$.
(2) K is κ-strongly μ-superstable if and only if:
  (a) Conditions (1) to (5) in Definition 10.2 hold.
  (b) K_μ is nonempty, is stable in μ, and has no maximal models.
  (c) For some $\mu' \geq \mu$, $K_{\mu'}$ has joint embedding.
  (d) $K_\mu^{\kappa\text{-sat}}$ is dense in K_μ.
  (e) $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \leq_{\text{univ}}) = \aleph_0$.

Proof.

(1) Assume first that the conditions on the right hand side hold. Then one can readily check (using Proposition 7.4) that K is $(\mu, \mathfrak{i}_{\mu\text{-ns}}(K)^{\leq 1})$-superstable, where the local character axiom of the definition of a generator for a good independence relation is witnessed by $\leq_{\text{univ}}$. Conversely, assume K is $(\mu, \mathfrak{i})$-superstable for some $\mathfrak{i}$. By definition, $\mu \geq \text{LS}(K)$ and $(K_\mu, \mathfrak{i})$ is a μ-generator for a good (≤ 1)-independence relation. By Lemma 8.7, $(K_\mu, \mathfrak{i}_{\mu\text{-ns}}(K)^{\leq 1})$ is a μ-generator for a good (≤ 1)-independence relation, and $\leq_{\text{univ}}$ is the ordering witnessing local character. Thus the conditions on the right hand side hold.
(2) Assume first that the conditions on the right hand side hold. Let $\kappa_0 < \kappa$ be such that $K_{\geq \kappa_0}$ has amalgamation. Assume without loss of generality that $\kappa_0 = \text{LS}(K)$ and that $K_{\geq \kappa_0} = K$. By Proposition 7.4, $(K_\mu, \mathfrak{i}_{\kappa\text{-ch}}(K)_\mu^{\leq 1})$ is a μ-generator for a weakly good (≤ 1)-independence relation. By the other conditions, it is actually a μ-generator for a good (≤ 1)-independence relation. Conversely, assume that K is κ-strongly $(\mu, \mathfrak{i})$-superstable for some $\mathfrak{i}$. We check the last two conditions on the right hand side, the others are straightforward. We know that K is $(\mu, \mathfrak{i})$-superstable and $\mathfrak{i} = \mathfrak{i}_{\kappa\text{-ch}}(K)^{\leq 1} \upharpoonright K_{\mathfrak{i}}$. Thus $K_{\mathfrak{i}} \subseteq K_\mu^{\kappa\text{-sat}}$ is dense in K_μ, so (since also $K_{\mathfrak{i}} \subseteq K_\mu$), $K^{\kappa\text{-sat}}$ is dense in K_μ. By Lemma 6.7, $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \leq_{\text{univ}}) \leq \kappa_1(\mathfrak{i}, \trianglelefteq)$ for any $\trianglelefteq$ such that $(K_{\mathfrak{i}}, \trianglelefteq)$ is a skeleton of K (and hence of $K^{\kappa\text{-sat}}$). By assumption

one can find such a $\trianglelefteq$ with $\kappa_1(\mathfrak{i}, \trianglelefteq) = \aleph_0$. Thus

$$\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \leq_{\text{univ}}) = \aleph_0$$

□

Remark 10.4. In Proposition 10.3, one can replace $\leq_{\text{univ}}$ by $\leq_{\delta,\mu}$ for $1 \leq \delta < \mu^+$.

Remark 10.5. Proposition 10.3 shows that our definition of superstability is equivalent (modulo tameness, joint embedding, and amalgamation) to [Gro02, Definition 7.12] and the definitions implicit in [GVV, Vasa].

The next result gives evidence that Definition 10.1 is a reasonable definition of superstability. Note that most of the above already appears implicitly in [Vasa] and essentially restates Theorem 8.8.

Theorem 10.6. Assume K is a $(\mu, \mathfrak{i})$-superstable AEC. Then:

(1) $K_{\geq \mu}$ has joint embedding, no maximal models, and is stable in all $\lambda \geq \mu$.
(2) Let $\lambda \geq \mu^+$ and let $\mathfrak{i}' := (K_\mu, \mathfrak{i})^{\text{up}} \upharpoonright K_{\geq \lambda}^{\mu^+\text{-sat}}$.
   (a) $\mathfrak{i}'$ is a pre-good $(\leq 1, \geq \lambda)$-independence relation, except that $K_{\mathfrak{i}'}$ may not be an AEC.
   (b) If in addition K is *κ-strongly* $(\mu, \mathfrak{i})$-superstable, then $\mathfrak{i}' = \mathfrak{i}_{\kappa\text{-ch}}(K)^{\leq 1} \upharpoonright K_{\geq \lambda}^{\mu^+\text{-sat}}$. That is, the independence relation is $(< \kappa)$-coheir.
   (c) If $\theta \geq \mu^+$ is such that $K' := K_{\geq \lambda}^{\theta\text{-sat}}$ is an AEC with $\text{LS}(K') = \lambda$, then $\mathfrak{i}' \upharpoonright K'$ is a pre-good $(\leq 1, \geq \lambda)$-frame that will be $(< \kappa)$-coheir if K is κ-strongly $(\mu, \mathfrak{i})$-superstable.

Proof. By (the proof of) Theorem 8.8 and Lemma 7.8. □

Remark 10.7. Let T be a complete first-order theory and let $K := (\text{Mod}(T), \preceq)$. Then our definitions of superstability and strong superstability coincide with the classical definition. More precisely for all $\mu \geq |T|$, K is (strongly) μ-superstable if and only if T is stable in all $\lambda \geq \mu$.

Note also that [strong] μ-superstability is monotonic in μ:

Proposition 10.8. If K is [κ-strongly] μ-superstable and $\mu' \geq \mu$, then K is [κ-strongly] μ'-superstable.

Proof. Say K is $(\mu, \mathfrak{i})$-superstable. Let $\mathfrak{i}' := (K, \mathfrak{i})^{\text{up}} \upharpoonright K_{\geq \mu'}^{\mu^+\text{-sat}}$. By Theorem 10.6 and Proposition 7.4, $(K_{\mu'}, \mathfrak{i}')$ is a generator for a good μ'-independence relation, so K is $(\mu', \mathfrak{i}')$-superstable. Similarly, if K is κ-strongly $(\mu, \mathfrak{i})$-superstable then K will be κ-strongly $(\mu', \mathfrak{i}')$-superstable. $\square$

Theorem 10.6.(2b) is the reason we introduced strong superstability. While it may seem like a detail, we are interested in extending our good frame to a frame for types longer than one element and using coheir to do so seems reasonable. Using the canonicity of coheir, we can show that superstability and strong superstability are equivalent if we do not care about the parameter μ:

Theorem 10.9. If K is μ-superstable and $\kappa = \beth_\kappa > \mu$, then K is κ-strongly $(\kappa_r)^+$-superstable.

In particular an AEC is strongly superstable if and only if it is superstable.

Proof. Let $\mu' := (\kappa_r)^+$. We show that K is κ-strongly μ'-superstable by checking the equivalent condition of Proposition 10.3. By Theorem 10.6, $K_{\geq \mu}$ has joint embedding, no maximal models and is stable in all cardinals. By definition, $K_{\geq \mu}$ also has amalgamation. Also, K is μ-tame, hence $(< \kappa)$-tame. By Fact 2.20, K does not have the $(< \kappa)$-order property of length κ. By Fact 3.17:

$$\bar{\kappa}_1(\mathfrak{i}_{\kappa\text{-ch}}(K)) \leq (\text{LS}(K)^{<\kappa})^+ = \kappa^+ \leq (\mu')^+$$

Therefore conditions (1) to (5) in Definition 10.2 hold. Moreover we have already observed that $K_{\mu'}$ is stable in μ' and has joint embedding and no maximal models. Also, $K_{\mu'}^{\kappa\text{-sat}}$ is dense in $K_{\mu'}$ by stability and the fact $\mu' > \kappa$. It remains to check that $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_{\mu'}, \leq_{\text{univ}}) = \aleph_0$.

By Theorem 10.6, there is a $(\leq 1, \geq \mu^+)$-independence relation $\mathfrak{i}'$ such that $K_{\mathfrak{i}'} = K^{\mu^+\text{-sat}}$ and $\mathfrak{i}'$ is good, except that $K_{\mathfrak{i}'}$ may not be an AEC. By Theorem 9.3 (with λ there standing for κ_r here), $\text{pre}(\mathfrak{i}') \upharpoonright K^{\mu'\text{-sat}} = \text{pre}(\mathfrak{i}_{\kappa\text{-ch}}(K)^{\leq 1}) \upharpoonright K^{\mu'\text{-sat}}$. By the proof of Lemma 4.5, $\mathfrak{i}'$ has the right $(< \kappa)$-witness property for members of $K_{\geq \mu^+}$: If $M \in K_{\geq \mu^+}$, $M_0 \leq M$ is in $K^{\mu^+\text{-sat}}$, and $p \in \text{gS}(M)$, then p does not $\mathfrak{i}$-fork over M_0 if and only if $p \upharpoonright B$ does not $\mathfrak{i}$-fork over M_0 for all $B \subseteq |M|$ with $|B| < \kappa$. Therefore by the proof of Theorem 9.3, we actually have that for any $M \in K_{\geq \mu'}$ and $M_0 \leq M$ in $K^{\mu'\text{-sat}}$, $p \in \text{gS}(M)$ does not $\mathfrak{i}'$-fork over M_0 if and only if p does not $\mathfrak{i}_{\kappa\text{-ch}}(K)$-fork over M_0. In particular:

$$\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_{\mu'}) = \kappa_1(\mathfrak{i}'_\mu) = \aleph_0$$

Therefore $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_{\mu'}, \leq_{\text{univ}}) = \aleph_0$, as needed.

□

We now arrive to the main result of this section: categoricity implies strong superstability. We first recall several known consequences of categoricity.

Fact 10.10. Let K be an AEC with no maximal models, joint embedding, and amalgamation. Assume K is categorical in a $\lambda > \text{LS}(K)$. Then:

(1) K is stable in all $\mu \in [\text{LS}(K), \lambda)$.
(2) For $\text{LS}(K) \leq \mu < \text{cf}(\lambda)$, $\kappa_1(\mathfrak{s}_{\mu\text{-ns}}(K_\mu), \leq_{\mu,\omega}) = \aleph_0$.
(3) Assume K does not have the weak κ-order property (see Definition 2.19) and $\text{LS}(K) < \kappa \leq \mu < \lambda$. Then:

$$\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \leq_{\text{univ}}) = \aleph_0$$

(4) If the model of size λ is μ-saturated for $\mu > \text{LS}(K)$, then every member of $K_{\geq \chi}$ is μ-saturated, where $\chi := \min(\lambda, \sup_{\mu_0 < \mu} h(\mu_0))$.

Proof.

(1) Use Ehrenfeucht-Mostowski models (see for example the proof of [Bal09, Theorem 8.21]).
(2) By [She99, Lemma 6.3].
(3) By [BG, Theorem 6.5].
(4) See (the proof of) [BG, Theorem 5.4].

□

The next lemma is useful to obtain joint embedding and no maximal models if we already have amalgamation.

Lemma 10.11. Let K be an AEC with amalgamation. If there exists $\lambda \geq \text{LS}(K)$ such that K_λ has joint embedding, then there exists $\chi < h(\text{LS}(K))$ such that $K_{\geq \chi}$ has joint embedding and no maximal models.

Proof. Write $\mu := h(\text{LS}(K))$. If $K_\mu = \emptyset$, then by Fact 2.13 there exists $\chi < \mu$ such that $K_{\geq \chi} = \emptyset$, so it has has joint embedding and no maximal models. Now assume $K_\mu \neq \emptyset$. In particular, K has arbitrarily large models. By amalgamation, $K_{\geq \lambda}$ has joint embedding, and so no maximal models. If $\lambda < \mu$ we are done so assume $\lambda \geq \mu$. It is enough to show that there exists $\chi < \mu$ such that $K_{\geq \chi}$ has no maximal model since then any model of $K_{\geq \chi}$ embeds inside a model in $K_{\geq \lambda}$ and hence $K_{\geq \chi}$ has joint embedding.

By Fact 2.14, we can write $K = \bigcup_{i \in I} K^i$ where the K^i's are disjoint AECs with $\text{LS}(K^i) = \text{LS}(K)$ and each K^i has joint embedding and amalgamation. Note that $|I| \leq I(K, \text{LS}(K)) \leq 2^{\text{LS}(K)}$. For $i \in I$, let χ_i be the least $\chi < \mu$ such that $K^i_\chi = \emptyset$, or $\text{LS}(K)$ if $K^i_\mu \neq \emptyset$. Let $\chi := \sup_{i \in I} \chi_i$. Note that $\text{cf}(\mu) = \left(2^{\text{LS}(K)}\right)^+ > 2^{\text{LS}(K)} \geq |I|$, so $\chi < \mu$.

Now let $M \in K_{\geq \chi}$. Let $i \in I$ be such that $M \in K^i$. M witnesses that $K^i_\chi \neq \emptyset$ so by definition of χ, K^i has arbitrarily large models. Since K^i has joint embedding, this implies that K^i has no maximal models. Therefore there exists $N \in K^i \subseteq K$ with $M < N$, as desired. □

The next theorem is implicit in [Vasa]. It is really a simple consequence of Fact 10.10.(2).

Theorem 10.12. Let K be an $\text{LS}(K)$-tame AEC with amalgamation and no maximal models. If K is categorical in a λ with $\text{cf}(\lambda) > \text{LS}(K)$, then K is $\text{LS}(K)$-superstable.

Proof. By amalgamation, categoricity, and no maximal models, K has joint embedding. By Fact 10.10.(1), K is stable in $\text{LS}(K)$. Now apply Fact 10.10.(2) and Proposition 10.3 (with Remark 10.4). □

Corollary 10.13. Let K be an $\text{LS}(K)$-tame AEC with amalgamation. If K is categorical in a λ with $\text{cf}(\lambda) \geq h(\text{LS}(K))$, then there exists $\mu < h(\text{LS}(K))$ such that K is μ-superstable.

Proof. By Lemma 10.11, there exists $\mu < h(\text{LS}(K))$ such that $K_{\geq \mu}$ has joint embedding and no maximal models. Now apply Theorem 10.12 to $K_{\geq \mu}$. □

We now remove the restriction on the cofinality and get strong superstability. The downside is that $h(\text{LS}(K))$ is replaced by a fixed point of the beth function above $\text{LS}(K)$.

Theorem 10.14. Let K be an AEC with amalgamation. Let $\kappa = \beth_\kappa > \text{LS}(K)$ and assume K is $(< \kappa)$-tame. If K is categorical in a $\lambda > \kappa$, then:

(1) K is κ-strongly κ-superstable.
(2) K is stable in all cardinals $\geq h(\text{LS}(K))$.
(3) The model of size λ is saturated.

(4) K is categorical in κ.
(5) For $\chi := \min(\lambda, h(\kappa))$, pre $\left(\mathfrak{i}_{\kappa\text{-ch}}(K)^{\le 1}_{\ge \chi}\right)$ is a good $(\le 1, \ge \chi)$-frame with underlying AEC $K_{\ge \chi}$.

Proof. Note that K_λ has joint embedding so by Lemma 10.11, there exists $\chi_0 < h(\text{LS}(K))$ such that $K_{\ge \chi_0}$ (and thus $K_{\ge \kappa}$) has joint embedding and no maximal models. By Fact 10.10.(1), $K_{\ge \chi_0}$ is stable everywhere below λ. Since $\kappa = \beth_\kappa$, Fact 2.20 implies that K does not have the $(< \kappa)$-order property of length κ. Also by Fact 3.17, $\bar{\kappa}_1(\mathfrak{i}_{\kappa\text{-ch}}(K)) \le (\text{LS}(K)^{<\kappa})^+ = \kappa^+$.

Let $\kappa \le \mu < \lambda$. By Fact 10.10.(3), $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \le_{\text{univ}}) = \aleph_0$. Now using Proposition 10.3, K is κ-strongly μ-superstable if and only if $K_\mu^{\kappa\text{-sat}}$ is dense in K_μ. If $\kappa < \mu$, then this holds by stability, so K is κ-strongly μ-superstable. However we want κ-strong κ-superstability. We proceed in several steps.

First, we show K is μ-superstable for *some* $\mu < \lambda$. If $\lambda = \kappa^+$, then this follows directly from Theorem 10.12 with $\mu = \kappa$, so assume $\lambda > \kappa^+$. Then by the previous paragraph K is κ-strongly μ-superstable for $\mu := \kappa^+$.

Second, we prove (2). We have already observed $K_{\ge \chi_0}$ is stable everywhere below λ. By Theorem 10.6, K is stable in every $\mu' \ge \mu$. In particular, it is stable in and above λ, so (2) follows.

Third, we show (3). Since K is stable in λ, we can build a λ_0^+-saturated model of size λ for all $\lambda_0 < \lambda$. Thus the model of size λ is λ_0^+-saturated for all $\lambda_0 < \lambda$, and hence λ-saturated.

Fourth, we prove (4). Since the model of size λ is saturated, it is κ-saturated. By Fact 10.10.(4), every model of size $\sup_{\kappa_0 < \kappa} h(\kappa_0) = \kappa$ is κ-saturated. By uniqueness of saturated models, K is categorical in κ.

Fifth, observe that since every model of size κ is saturated, $K_\kappa^{\kappa\text{-sat}}$ is dense in K_κ. By the second paragraph above, K is κ-strongly κ-superstable so (1) holds.

Finally, we prove (5). We have seen that the model of size λ is saturated, thus κ^+-saturated. By Fact 10.10.(4), every model of size $\ge \chi$ is κ^+-saturated. Now use (1) with Theorem 10.6. $\square$

Remark 10.15. If one just wants to get strong superstability from categoricity, we suspect it should be possible to replace the $\beth_\kappa = \kappa$ hypothesis by something more reasonable (maybe just asking for the categoricity cardinal to be above 2^κ). Since we are only interested in eventual behavior here, we leave this to future work.

As a final remark, we point out that it is always possible to get a good independence relation from superstability if one is willing to restrict the class to sufficiently-saturated models:

Fact 10.16 ([BVa]). Let K be an AEC.

(1) If K is μ-superstable, then there exists $\lambda_0 < h(\mu^+)$ such that for all $\lambda \geq \lambda_0$, $K^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$.
(2) If K is κ-strongly μ-superstable, then whenever $\lambda > (\mu^{<\kappa})^+$, $K^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$.

Corollary 10.17. Let K be an AEC.

(1) If K is μ-superstable, then there exists $\lambda_0 < h(\mu^+)$ such that $K^{\lambda_0\text{-sat}}$ is (≤ 1)-good.
(2) If K is κ-strongly μ-superstable, then $K^{(\mu^{<\kappa})^{+2}\text{-sat}}$ is (≤ 1)-good. Moreover the good frame is induced by $(< \kappa)$-coheir.

Proof. Combine Theorem 10.6.(2c) and Fact 10.16. □

Remark 10.18. If $K^{\lambda\text{-sat}}$ is an AEC with Löwenheim-Skolem number λ, then K_λ has a superlimit (see [She09, Definition 1.13]). Thus we even obtain a good frame in the sense of [She09, Chapter II].

11. Domination

In this section, our aim is to take a sufficiently nice good λ-frame (for types of length 1) and show that it can be extended to types of length $\leq \lambda$. To do this, we will give conditions under which a good λ-frame is *weakly successful*, a key technical property of [She09, Chapter II], see Definition 11.4.

The hypotheses we will work with are:

Hypothesis 11.1.

(1) $\mathfrak{i} = (K, \perp)$ is a $(< \infty, [\mu, \infty))$-independence relation.
(2) $\mathfrak{s} := \text{pre}(\mathfrak{i}^{\leq 1})$ is a type-full good $[\mu, \infty)$-frame.
(3) $\lambda > \mu$ is a cardinal.
(4) For all $n < \omega$:
    (a) $K^{\lambda^{+n}\text{-sat}}$ is an AEC[25] with Löwenheim-Skolem number λ^{+n}.
    (b) $\kappa_{\lambda^{+n}}(\mathfrak{i}) = \lambda^{+n+1}$.
(5) $\mathfrak{i}$ has base monotonicity, $\text{pre}(\mathfrak{i})$ has uniqueness.
(6) $\mathfrak{i}$ has the left and right $(\leq \mu)$-witness properties.

[25] Thus we have a superlimit of size λ^{+n}, see Remark 10.18.

Remark 11.2. We could have given more local hypotheses (e.g. by replacing ∞ by θ or only assuming (4) for n below some fixed $m < \omega$) and made some of the required properties more precise (this is part of what should be done to improve "short" to "diagonally tame" in the main theorem, see the discussion in Section 15).

The key is that we assume there is *already* an independence notion for longer types. However, it is potentially quite weak compared to what we want. The next fact shows that our hypotheses are reasonable.

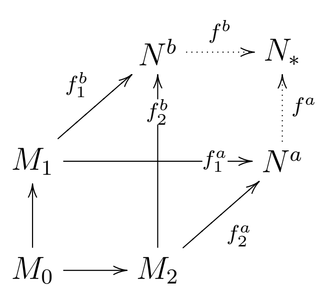
Fact 11.3. Assume K^0 is a fully $(< \kappa)$-tame and short κ-strongly μ_0-superstable AEC. Then for any $\mu \geq (\mu_0^{<\kappa})^{+2}$ and any $\lambda > \mu$ with $\lambda = \lambda^{<\kappa}$, Hypothesis 11.1 holds for $K := (K^0)^{\mu\text{-sat}}$ and $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K^0) \upharpoonright K$.

Proof. By Fact 10.16, for any $\mu' \geq \mu$, $K^{\mu'\text{-sat}}$ is an AEC with $\text{LS}(K^{\mu'\text{-sat}}) = \mu'$. By Theorem 10.6.(2c), $(< \kappa)$-coheir induces a good $(\geq \mu)$-frame for μ-saturated models. The other conditions follow directly from the definition of strong superstability and the properties of coheir (Fact 3.17). For example, the local character condition holds because $\lambda^{<\kappa} = \lambda$ implies $(\lambda^{+n})^{<\kappa} = \lambda^{+n}$ for any $n < \omega$. □

The next technical property is of great importance in Chapter II and III of [She09]. Our definition follows [JS13, Definition 4.1.5] (but as usual, we work only with type-full frames).

Definition 11.4. Let $\mathfrak{t} = (K_{\mathfrak{t}}, \mathop{\smile\!\!\!\!\!\!\mid}\;)$ be a type-full good $\lambda_{\mathfrak{t}}$-frame.

(1) For $M_0 \leq M_\ell$ in K, $\ell = 1, 2$, an *amalgam of M_1 and M_2 over M_0* is a triple (f_1, f_2, N) such that $N \in K_{\mathfrak{t}}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.
(2) Let (f_1^x, f_2^x, N^x), $x = a, b$ be amalgams of M_1 and M_2 over M_0. We say (f_1^a, f_2^a, N^a) and (f_1^b, f_2^b, N^b) are *equivalent over M_0* if there exists $N_* \in K_{\mathfrak{t}}$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:



Note that being "equivalent over M_0" is an equivalence relation ([JS13, Proposition 4.3]).

(3) Let $K^{3,\text{uq}} = K_{\mathfrak{t}}^{3,\text{uq}}$ be the set of triples (a, M, N) such that $M \leq N$ are in K, $a \in |N| \backslash |M|$ and for any $M_1 \geq M$ in K, there exists a unique (up to equivalence over M_0) amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\text{gtp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M. We call the elements of $K^{3,\text{uq}}$ *uniqueness triples.*
(4) $K^{3,\text{uq}}$ has the *existence property* if for any $M \in K_{\mathfrak{t}}$ and any nonalgebraic $p \in \text{gS}(M)$, one can write $p = \text{gtp}(a/M; N)$ with $(a, M, N) \in K^{3,\text{uq}}$. We also talk about the *existence property for uniqueness triples.*
(5) $\mathfrak{s}$ is *weakly successful* if $K^{3,\text{uq}}$ has the existence property.

The uniqueness triples can be seen as describing a version of domination. They were introduced by Shelah for the purpose of starting with a good λ-frame and extending it to a good λ^+-frame. The idea is to first extend the good λ-frame to a nonforking notion for types of models of size λ (and really this is what interests us here, since tameness already gives us a good λ^+-frame). Now, since we already have an independence notion for longer types, we can follow [MS90, Definition 4.21] and give a more explicit version of domination that is exactly as in the first-order case.

Definition 11.5 (Domination). Fix $N \in K$. For $M \leq N$, $B, C \subseteq |N|$, B $\mathfrak{i}$-*dominates* C over M (in N) if for any $N' \geq N$ and any $D \subseteq |N'|$, $B \overset{N'}{\underset{M}{\perp}} D$ implies $B \cup C \overset{N'}{\underset{M}{\perp}} D$.

We say that B $\mathfrak{i}$-*model-dominates* C over M in N if for any $N' \geq N$ and any $M \leq N_0' \leq N'$, $B \overset{N'}{\underset{M}{\perp}} N_0'$ implies $B \cup C \overset{N'}{\underset{M}{\perp}} N_0'$.

Of course, when $\mathfrak{i}$ is clear from context, we omit it.

Model-domination turns out to be the technical variation we need, but of course if $\mathfrak{i}$ has existence, then it is equivalent to domination. We start with a few easy ambient monotonicity properties:

Lemma 11.6. Let $M \leq N$. Let $B, C \subseteq |N|$ and assume B [model-]dominates C over M in N. Then:

(1) If $N' \geq N$, then B [model-]dominates C over M in N'.
(2) If $M \leq N_0 \leq N$ contains $B \cup C$, then B [model-]dominates C over M in N_0.

Proof. We only do the proofs for the non-model variation but of course the model variation is completely similar.

(1) By definition of domination.
(2) Let $N' \geq N_0$ and $D \subseteq |N'|$ be given such that $B \overset{N'}{\underset{M}{\downarrow}} D$. By amalgamation, there exists $N'' \geq N$ and $f : N' \xrightarrow[N_0]{} N''$. By invariance, $B \overset{N''}{\underset{M}{\downarrow}} f[D]$. By definition of domination, $B \cup C \overset{f[N']}{\underset{M}{\downarrow}} f[D]$. By invariance again, $B \cup C \overset{N'}{\underset{M}{\downarrow}} D$, as desired.

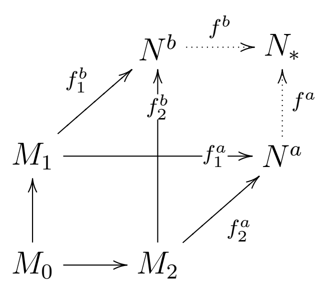
□

The next result is key for us: it ties domination with the notion of uniqueness triples:

Lemma 11.7. Assume $M_0 \leq M_1$ are in K_λ, and $a \in M_1$ model-dominates M_1 over M_0 (in M_1). Then $(a, M_0, M_1) \in K^{3,\text{uq}}_{\mathfrak{s}_\lambda}$.

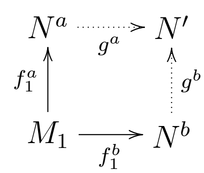
Proof. Let $M_2 \geq M_0$ be in K_λ. First, we need to show that there exists (b, M_2, N) such that $\text{gtp}(b/M_2; N)$ extends $\text{gtp}(a/M_0; M_1)$ and $\text{gtp}(b/M_2; N)$ does not fork over M_0. This holds by the extension property of good frames.

Second, we need to show that any such amalgam is unique: Let (f_1^x, f_2^x, N^x), $x \in \{a, b\}$ be amalgams of M_1 and M_2 over M_0 such that $f_1^x(a) \overset{N^x}{\underset{M_0}{\downarrow}} f_2^x[M_2]$. We want to show that the two amalgams are equivalent: we want $N_* \in K_\lambda$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:



For $x = a, b$, rename f_2^x to the identity to get amalgams $((f_1^x)', \text{id}_{M_2}, (N^x)')$ of M_1 and M_2 over M_0. For $x = a, b$, the amalgams $((f_1^x)', \text{id}_{M_2}, (N^x)')$ and (f_1^x, f_2^x, N^x) are equivalent over M_0, hence we can assume without loss of generality that the renaming has already been done and $f_2^x = \text{id}_{M_2}$

Thus we know that $f_1^x(a) \underset{M_0}{\overset{N^x}{\downarrow}} M_2$ for $x = a, b$. By domination, $f_1^x[M_1] \underset{M_0}{\overset{N^x}{\downarrow}} M_2$. Let $\bar{M}_1$ be an enumeration of M_1. Using amalgamation, we can obtain the following diagram:



This shows $\mathrm{gtp}(f_1^a(\bar{M}_1)/M_0; N^a) = \mathrm{gtp}(f_1^b(\bar{M}_1)/M_0; N^b)$. By uniqueness, $\mathrm{gtp}(f_1^a(\bar{M}_1)/M_2; N^a) = \mathrm{gtp}(f_1^b(\bar{M}_1)/M_2; N^b)$. Let N_* and $f^x : N^x \xrightarrow[M_2]{} N_*$ witness the equality. Since $f_2^x = \mathrm{id}_{M_2}$, $f^b \circ f_2^b = f^b \restriction M_2 = \mathrm{id}_{M_2} = f^a \circ f_2^a$. Moreover, $(f^b \circ f_1^b)(\bar{M}_1) = f^b(f_1^b(\bar{M}_1)) = f^a(f_2^a(\bar{M}_1))$ by definition, so $f^b \circ f_1^b = f^a \circ f_1^a$. This completes the proof. $\square$

Remark 11.8. The converse will hold if $\mathfrak{i}$ has left extension.

Remark 11.9. The relationship of uniqueness triples with domination is already mentioned in [JS13, Proposition 4.1.7], although the definition of domination there is different.

Thus to prove the existence property for uniqueness triples, it will be enough to imitate the proof of [MS90, Proposition 4.22], which gives conditions under which the hypothesis of Lemma 11.7 holds. We first show that we can work inside a monster model.

Lemma 11.10. Let $M \leq N$ and $B \subseteq |N|$. Let $\mathfrak{C} \geq N$ be $\|N\|^+$-model-homogeneous. Then B [model-]dominates N over M in $\mathfrak{C}$ if and only if for any $D \subseteq |\mathfrak{C}|$, [$D$ is a model and] $B \underset{M}{\overset{\mathfrak{C}}{\downarrow}} D$ implies $N \underset{M}{\overset{\mathfrak{C}}{\downarrow}} D$.

Proof. We prove the non-trivial direction for domination. The proof for model-domination is similar. Assume $\mathfrak{C}' \geq \mathfrak{C}$ and $D \subseteq |\mathfrak{C}'|$ is such that $B \underset{M}{\overset{\mathfrak{C}'}{\downarrow}} D$. We want to show that $N \underset{M}{\overset{\mathfrak{C}'}{\downarrow}} D$. Suppose not. Then we can use the $(\leq \mu)$-witness property to assume without loss of generality that $|D| \leq \mu$, and so we can find $N \leq N' \leq \mathfrak{C}'$ containing D with $\|N'\| = \|N\|$ and $B \underset{M}{\overset{N'}{\downarrow}} D$, $N \underset{M}{\overset{N'}{\not\downarrow}} D$. By homogeneity, find $f : N' \xrightarrow[N]{} \mathfrak{C}$.

By invariance, $B \underset{M}{\overset{f[N']}{\mathop{\smile}\limits^{\mid}}} f[D]$ but $N \underset{M}{\overset{f[N']}{\not\mathop{\smile}\limits^{\mid}}} f[D]$. By monotonicity, $B \underset{M}{\overset{\mathfrak{C}}{\mathop{\smile}\limits^{\mid}}} f[D]$ but $N \underset{M}{\overset{\mathfrak{C}}{\not\mathop{\smile}\limits^{\mid}}} f[D]$, a contradiction. □

Lemma 11.11 (Lemma 4.20 in [MS90])**.** Let $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ be increasing in K_λ such that $M_i \le N_i$ for all $i < \lambda^+$. Let $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$.

Then there exists $i < \lambda^+$ such that $N_i \underset{M_i}{\overset{N_{\lambda^+}}{\mathop{\smile}\limits^{\mid}}} M_{\lambda^+}$.

Proof. For each $i < \lambda^+$, let $j_i < \lambda^+$ be least such that $N_i \underset{M_{j_i}}{\overset{N_{\lambda^+}}{\mathop{\smile}\limits^{\mid}}} M_{\lambda^+}$ (exists since $\kappa_\lambda(\mathfrak{i}) = \lambda^+$). Let i^* be such that $j_i < i^*$ for all $i < i^*$ and $\text{cf}(i^*) \ge \mu^+$. By definition of j_i and base monotonicity we have that for all $i < i^*$, $N_i \underset{M_{i^*}}{\overset{N_{\lambda^+}}{\mathop{\smile}\limits^{\mid}}} M_{\lambda^+}$. By the left $(\le \mu)$-witness property, $N_{i^*} \underset{M_{i^*}}{\overset{N_{\lambda^+}}{\mathop{\smile}\limits^{\mid}}} M_{\lambda^+}$. □

Lemma 11.12 (Proposition 4.22 in [MS90])**.** Let $M \in K_\lambda$ be saturated. Let $\mathfrak{C} \ge M$ be saturated of size λ^+. Work inside $\mathfrak{C}$. Write $A \underset{M}{\mathop{\smile}\limits^{\mid}} B$ for $A \underset{M}{\overset{\mathfrak{C}}{\mathop{\smile}\limits^{\mid}}} B$.

- There exists a saturated $N \le \mathfrak{C}$ in K_λ such that $M \le N$, N contains a, and a model-dominates N over M (in $\mathfrak{C}$).
- In fact, if $M^* \le M$ is in $K_{<\lambda}$, $a \underset{M^*}{\mathop{\smile}\limits^{\mid}} M$, and $r \in \text{gS}^{\le\lambda}(M^* a; \mathfrak{C})$, then N can be chosen so that it realizes r.

Proof. Since $\bar{\kappa}_1(\mathfrak{s}) = \mu^+ \le \lambda$, it suffices to prove the second part. Assume it fails.

Claim: For any saturated $M' \ge M$ in K_λ, if $a \underset{M}{\mathop{\smile}\limits^{\mid}} M'$, then the second part fails with M' replacing M.

Proof of claim: By transitivity, $a \underset{M^*}{\mathop{\smile}\limits^{\mid}} M'$. By uniqueness of saturated models, there exists $f : M' \cong_{M^*} M$, which we can extend to an automorphism of $\mathfrak{C}$. Thus we also have $f(a) \underset{M^*}{\mathop{\smile}\limits^{\mid}} M$. By uniqueness, we can assume without loss of generality that f fixes a as well. Since the second part above is invariant under applying f^{-1}, the result follows.

We now construct increasing continuous chains $\langle M_i : i \le \lambda^+\rangle$, $\langle N_i : i \le \lambda^+\rangle$ such that for all $i < \lambda^+$:

(1) $M_0 = M$.
(2) $M_i \le N_i$.
(3) $M_i \in K_\lambda$ is saturated.
(4) $a \underset{M_0}{\perp} M_i$.
(5) $N_i \underset{M_i}{\not\perp} M_{i+1}$.

This is enough: the sequences contradict Lemma 11.11. This is possible: take $M_0 = M$, and N_0 any saturated model of size λ containing M_0 and a and realizing r. At limits, take unions (we are using that $K^{\lambda\text{-sat}}$ is an AEC). Now assume everything up to i has been constructed. By the claim, the second part above fails for M_i, so in particular N_i cannot be model-dominated by a over M_i. Thus (implicitly using Lemma 11.10) there exists $M_i' \ge M_i$ with $a \underset{M_i}{\perp} M_i'$ and $N_i \underset{M_i}{\not\perp} M_i'$. By the witness property, we can assume without loss of generality that $\|M_i'\| \le \lambda$, so using extension and transitivity, we can find $M_{i+1} \in K_\lambda$ saturated containing M_i' so that $a \underset{M_i}{\perp} M_{i+1}$. By monotonicity we still have $N_i \underset{M_i}{\not\perp} M_{i+1}$. Let $N_{i+1} \in K_\lambda$ be any saturated model containing N_i and M_{i+1}. □

Theorem 11.13. $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ is a *weakly successful* type-full good λ-frame.

Proof. Since $\mathfrak{s}_\lambda$ is a type-full good frame, $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ also is. To show it is weakly successful, we want to prove the existence property for uniqueness triples. So let $M \in K_\lambda^{\lambda\text{-sat}}$ and $p \in \mathrm{gS}(M)$ be nonalgebraic. Say $p = \mathrm{gtp}(a/M; N')$. Let $\mathfrak{C}$ be a monster model with $N' \le \mathfrak{C}$. By Lemma 11.12, there exists $N \le \mathfrak{C}$ in $K_\lambda^{\lambda\text{-sat}}$ such that $M \le N$, $a \in |N|$, and a dominates N over M in $\mathfrak{C}$. By Lemma 11.6, a dominates N over M in N. By Lemma 11.7, $(a, M, N) \in K^{3,\text{uq}}_{\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}}$. Now, $p = \mathrm{gtp}(a/M; N') = \mathrm{gtp}(a/M; \mathfrak{C}) = \mathrm{gtp}(a/M; N)$, as desired. □

The term "weakly successful" suggests that there must exist a definition of "successful". Indeed, this is the case:

Definition 11.14 (Definition 10.1.1 in [JS13])**.** A type-full good λ-frame $\mathfrak{t} = (K_{\mathfrak{t}}, \overset{\mathfrak{t}}{\perp})$ is *successful* if it is weakly successful and $\le^{\text{NF}}_{\lambda^+}$ has smoothness: whenever $\langle N_i : i \le \delta\rangle$ is a $\le^{\text{NF}}_{\lambda^+}$-increasing continuous chain of saturated models in $(K_{\mathfrak{t}})_{\lambda^+}$, $N \in (K_{\mathfrak{t}})_{\lambda^+}$ is saturated and $i < \delta$ implies $N_i \leq^{\text{NF}}_{\lambda^+} N$, then $N_\delta \leq^{\text{NF}}_{\lambda^+} N$.

The only thing we need to know about the relation $\leq^{\text{NF}}_{\lambda^+}$ is:

Fact 11.15 (Theorem 4.1 in [Jarb])**.** If $\mathfrak{t} = (K_{\mathfrak{t}}, \overset{\mathfrak{t}}{\downarrow})$ is a weakly successful type-full good λ-frame, $(K_{\mathfrak{t}})_{[\lambda,\lambda^+]}$ has amalgamation and is λ-tame, then $\leq \restriction (K_{\mathfrak{t}})^{\lambda^+\text{-sat}}_{\lambda^+} = \leq^{\text{NF}}_{\lambda^+}$.

Corollary 11.16. $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ is a *successful* type-full good λ-frame.

Proof. By Theorem 11.13, $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ is weakly successful. To show it is successful, it is enough (by Fact 11.15), to see that $\leq$ has smoothness. But this holds since K is an AEC. □

For a good λ-frame $\mathfrak{t}$, Shelah also defines a λ^+-frame $\mathfrak{t}^+$ ([She09, Definition III.1.7]). He then goes on to show:

Fact 11.17 (Claim III.1.9 in [She09])**.** If $\mathfrak{t}$ is a successful good λ-frame, then $\mathfrak{t}^+$ is a good[26] λ^+-frame.

Remark 11.18. This does *not* use the weak continuum hypothesis.

Note that in our case, it is easy to check that:

Fact 11.19. $(\mathfrak{s}_\lambda)^+ = \mathfrak{s}_{\lambda^+} \restriction K^{\lambda^+\text{-sat}}$.

Definition 11.20 (Definition III.1.12 in [She09])**.** Let $\mathfrak{t}$ be a pre-λ-frame.

(1) By induction on $n < \omega$, define $\mathfrak{t}^{+n}$ as follows:
   (a) $\mathfrak{t}^{+0} = \mathfrak{t}$.
   (b) $\mathfrak{t}^{+(n+1)} = (\mathfrak{t}^{+n})^+$.
(2) By induction on $n < \omega$, define "$\mathfrak{t}$ is n-successful" as follows:
   (a) $\mathfrak{t}$ is 0-successful if and only if it is a good λ-frame.
   (b) $\mathfrak{t}$ is $(n+1)$-successful if and only if it is a successful good λ-frame and $\mathfrak{t}^+$ is n-successful.
(3) $\mathfrak{t}$ is ω*-successful* if it is n-successful for all $n < \omega$.

Thus by Fact 11.17, $\mathfrak{t}$ is 1-successful if and only if it is a successful good λ-frame. More generally a good λ-frame $\mathfrak{t}$ is n-successful if and only if $\mathfrak{t}^{+m}$ is a successful good λ^{+m}-frame for all $m < n$.

Theorem 11.21. $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ is an ω*-successful* type-full good λ-frame.

[26]Shelah proves that $\mathfrak{t}^+$ is actually good$^+$. There is no reason to what this means here.

Proof. By induction on $n < \omega$, simply observing that we can replace λ by λ^{+n} in Corollary 11.16. □

We emphasize again that we did *not* use the weak continuum hypothesis (as Shelah does in [She09, Chapter II]). We pay for this by using tameness (in Fact 11.3). Note that all the results of [She09, Chapter III] will apply to such a frame. Even though part of Shelah's point is that ω-successful good λ-frames extend to $(\geq \lambda)$-frames, this is secondary for us (since tameness already implies it). Really, we want to extend the good frame to longer types. We show it is possible in the next section.

12. A fully good long frame

Hypothesis 12.1. $\mathfrak{s} = (K, \mathop{\smile}\limits^{\downarrow})$ is a weakly successful type-full good λ-frame.

This is reasonable since the previous section showed us how to build such a frame. Our goal is to extend $\mathfrak{s}$ to obtain a fully good $(\leq \lambda, \lambda)$-independence relation.

Fact 12.2 (Conclusion II.6.34 in [She09])**.** There exists a relation $\text{NF} \subseteq {}^4K$ satisfying:

(1) $\text{NF}(M_0, M_1, M_2, M_3)$ implies $M_0 \leq M_\ell \leq M_3$ are in K for $\ell = 1, 2$.
(2) $\text{NF}(M_0, M_1, M_2, M_3)$ and $a \in |M_1| \backslash |M_2|$ implies $\text{gtp}(a/M_2; M_3)$ does not $\mathfrak{s}$-fork over M_0.
(3) Invariance: NF is preserved under isomorphisms.
(4) Monotonicity: If $\text{NF}(M_0, M_1, M_2, M_3)$:
  (a) If $M_0 \leq M_\ell' \leq M_\ell$ for $\ell = 1, 2$, then $\text{NF}(M_0, M_1', M_2', M_3')$.
  (b) If $M_3' \leq M_3$ contains $|M_1| \cup |M_2|$, then $\text{NF}(M_0, M_1, M_2, M_3')$.
  (c) If $M_3' \geq M_3$, then $\text{NF}(M_0, M_1, M_2, M_3')$.
(5) Symmetry: $\text{NF}(M_0, M_1, M_2, M_3)$ if and only if $\text{NF}(M_0, M_2, M_1, M_3)$.
(6) Long right transitivity: If $\langle M_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing continuous and $\text{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for all $i < \alpha$, then $\text{NF}(M_0, N_0, M_\alpha, N_\alpha)$.
(7) Full existence: If $M_0 \leq M_\ell$, $\ell = 1, 2$, then for some $M_3 \in K$, $f_\ell : M_\ell \underset{M_0}{\longrightarrow} M_3$, we have $\text{NF}(M_0, f_1[M_1], f_2[M_2], M_3)$.
(8) Uniqueness: If $\text{NF}(M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell)$, $\ell = 1, 2$ and $f_i : M_i^1 \cong M_i^2$ for $i = 0, 1, 2$ and $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then $f_1 \cup f_2$ can be extended to $f_3 : M_3^1 \to M_4^2$, for some M_4^2 with $M_3^2 \leq M_4^2$.

Notation 12.3. We write $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{|}}} M_2$ instead of $\text{NF}(M_0, M_1, M_2, M_3)$. If $\bar{a}$ is a sequence, we write $\bar{a} \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{|}}} M_2$ for $\text{ran}(\bar{a}) \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{|}}} M_2$, and similarly if sequences appear at other places.

Remark 12.4. Shelah's definition of NF ([She09, Definition II.6.12]) is very complicated. It is somewhat simplified in [JS13].

Remark 12.5. Shelah calls such an NF a *nonforking relation which respects* $\mathfrak{s}$ ([She09, Definition II.6.1]). While there are similarities with our definition of a good $(\leq \lambda)$-frame, note that NF is only defined for models while we would like to make it into a relation taking arbitrary sets of size less than or equal to λ on the left hand side.

We start by showing that uniqueness is really the same as the uniqueness property stated for frames. We drop Hypothesis 12.1 for the next lemma.

Lemma 12.6. Let K be an AEC in λ and assume K has amalgamation. The following are equivalent for a relation $\text{NF} \subseteq {}^4K$ satisfying (1), (3), (4) of Fact 12.2:

(1) Uniqueness.
(2) Uniqueness in the sense of frames: If $A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{|}}} M_1$ and $A' \underset{M_0}{\overset{N'}{\mathop{\smile}\limits^{|}}} M_1$ for models A and A', $\bar{a}$ and $\bar{a}'$ are enumerations of A and A' respectively, $p := \text{gtp}(\bar{a}/M_1; N)$, $q := \text{gtp}(\bar{a}'/M_1; N')$, and $p \upharpoonright M_0 = q \upharpoonright M_0$, then $p = q$.

Proof.

- <u>(1) implies (2)</u>: Since $p \upharpoonright M_0 = q \upharpoonright M_0$, there exists $N'' \geq N'$ and $f : N \underset{M_0}{\longrightarrow} N''$ such that $f(\bar{a}) = \bar{a}'$. Therefore by invariance, $\bar{a}' \underset{M_0}{\overset{N''}{\mathop{\smile}\limits^{|}}} f[M_1]$. Let $f_0 := \text{id}_{M_0}$, $f_1 := f^{-1} \upharpoonright f[M_1]$, $f_2 := \text{id}_{A'}$. By uniqueness, there exists $N''' \geq N''$, $g \supseteq f_1 \cup f_2$, $g : N'' \to N'''$. Consider the map $h := g \circ f : N \to N'''$. Then $g \upharpoonright M_1 = \text{id}_{M_1}$ and $h(\bar{a}) = g(\bar{a}') = \bar{a}'$, so h witnesses $p = q$.

- <u>(2) implies (1)</u>: By some renaming, it is enough to prove that whenever $M_2 \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{|}}} M_1$ and $M_2 \underset{M_0}{\overset{N'}{\mathop{\smile}\limits^{|}}} M_1$, there exists $N'' \geq N'$ and

$f : N' \xrightarrow[|M_1| \cup |M_2|]{} N''$. Let $\bar{a}$ be an enumeration of M_2. Let $p := \text{gtp}(\bar{a}/M_1; N)$, $q := \text{gtp}(\bar{a}/M_1; N')$. We have that $p \restriction M_0 = \text{gtp}(\bar{a}/M_1; M_2) = q \restriction M_0$. Thus $p = q$, so there exists $N'' \geq N'$ and $f : N \xrightarrow[M_1]{} N''$ such that $f(\bar{a}) = \bar{a}$. In other words, f fixes M_2, so is the desired map.

□

We now extend NF to take sets on the left hand side. This step is already made by Shelah in [She09, Claim III.9.6], for singletons rather than arbitrary sets. We check that Shelah's proofs still work.

Definition 12.7. Define $\text{NF}'(M_0, A, M, N)$ to hold if and only if $M_0 \leq M \leq N$ are in K, $A \subseteq |N|$, and there exists $N' \geq N$, $N_A \geq M$ with $N_A \leq N'$ and $N_A \underset{M_0}{\overset{N'}{\perp}} M$. We abuse notation and also write $A \underset{M_0}{\overset{N}{\perp}} M$ instead of $\text{NF}'(M_0, A, M, N)$. We let $\mathfrak{t} := (K, \perp)$.

Proposition 12.8.

(1) If $M_0 \leq M_\ell \leq M_3$, $\ell = 1, 2$, then $\text{NF}(M_0, M_1, M_2, M_3)$ if and only if $\text{NF}'(M_0, M_1, M_2, M_3)$.
(2) $\mathfrak{t}$ is a (type-full) pre-$(\leq \lambda, \lambda, \lambda)$-frame.
(3) $\mathfrak{t}$ has base monotonicity, full symmetry, uniqueness, existence, and extension.

Proof. Exactly as in [She09, Claim III.9.6]. □

We now turn to local character. The key is:

Fact 12.9 (Claim III.1.17.2 in [She09])**.** Given $\langle M_i : i \leq \delta \rangle$ increasing continuous, we can build $\langle N_i : i \leq \delta \rangle$ increasing continuous such that for all $i \leq j \leq \delta$, $N_i \underset{M_i}{\overset{N_j}{\perp}} M_j$ and $M_\delta <_{\text{univ}} N_\delta$.

Lemma 12.10. For all $\alpha \leq \lambda$, $\kappa_\alpha(\mathfrak{t}) = |\alpha|^+ + \aleph_0$.

Proof. Let $\langle M_i : i \leq \delta + 1 \rangle$ be increasing continuous with $\delta = \text{cf}(\delta) > |\alpha|$. Let $A \subseteq |M_{\delta+1}|$ have size $\leq \alpha$. Let $\langle N_i : i \leq \delta \rangle$ be as given by Fact 12.9. By universality, we can assume without loss of generality that $M_{\delta+1} \leq N_\delta$. Thus $A \subseteq |N_\delta|$ and by the cofinality hypothesis, there exists $i < \delta$ such that $A \subseteq |N_i|$. In particular, $A \underset{M_i}{\overset{N_\delta}{\perp}} M_\delta$, so $A \underset{M_i}{\overset{M_{\delta+1}}{\perp}} M_\delta$, as needed. □

Remark 12.11. In [JS13] (and later in [JS12, Jara, Jarb]), the authors have considered *semi-good* λ-frames, where the stability condition is replaced by almost stability ($|\mathrm{gS}(M)| \leq \lambda^+$ for all $M \in K_\lambda$), and an hypothesis called the conjugation property is often added. Many of the above results carry through in that setup but we do not know if Lemma 12.10 would also hold.

Finally, we get to the last property: disjointness. The situation is a bit muddy: At first glance, (2) in Fact 12.2 seems to give it to us for free (since we are assuming $\mathfrak{s}$ has disjointness), but unfortunately we are assuming $a \notin |M_2|$ there. We will use a trick similar to the proof of Theorem 7.5. We make the additional hypothesis of categoricity in λ (this is reasonable since one can always restrict oneself to the class of λ-saturated models). Note that disjointness is never used in a crucial way here (but it is always nice to have, as it implies for example disjoint amalgamation when combined with extension).

Lemma 12.12. If K is categorical in λ, then $\mathfrak{t}$ has disjointness and $\mathfrak{t}^{\leq 1} = \mathfrak{s}$.

Proof. The last conclusion follows from the first and Fact 12.2.(2). Now it is of course enough to show that $\mathfrak{t}^{\leq 1}$ has disjointness, so assume $a \underset{M_0}{\overset{N}{\perp}} M$ and $a \in |M|$. We show $a \in |M_0|$. By stability, we can find $\langle N_i : i \leq \omega \rangle$ $<_{\text{univ}}$-increasing continuous in K. By categoricity, we can do this so that $N_\omega = M_0$. By local character and transitivity, there exists $i < \omega$ such that $a \underset{N_i}{\overset{N}{\perp}} M$. Note that $N_i <_{\text{univ}} N_{i+1} \leq N_\omega$, so $N_i <_{\text{univ}} N_\omega = M_0$ by definition of universality. Now go on as in the proof of disjointness in Theorem 7.5. $\square$

What about continuity for chains? The long right transitivity property seems to suggest we can say something, and indeed we can:

Fact 12.13. Assume $\lambda = \lambda_0^{+3}$ and there exists an ω-successful good λ_0-frame $\mathfrak{s}'$ such that $\mathfrak{s} = (\mathfrak{s}')^{+3}$.

Assume δ is a limit ordinal and $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous in K_λ, $\ell \leq 3$. If $M_i^1 \underset{M_i^0}{\overset{M_i^3}{\perp}} M_i^2$ for each $i < \delta$, then $M_\delta^1 \underset{M_\delta^0}{\overset{M_\delta^3}{\perp}} M_\delta^2$.

Proof. By [She09, Claim III.12.2], all the hypotheses at the beginning of each section of Chapter III in the book hold for $\mathfrak{s}$. Now apply Claim III.8.19 in the book. $\square$

Remark 12.14. Why $\lambda_0^{+3} = \lambda$? Shelah's analysis in chapter III of his book proceeds on the following lines: starting with an ω-successful frames $\mathfrak{s}$, we want to show $\mathfrak{s}$ has nice properties like existence of prime triples, weak orthogonality being orthogonality, etc. They are hard to show in general, however it turns out $\mathfrak{s}^+$ has some nicer properties than $\mathfrak{s}$... In general, $\mathfrak{s}^{+(n+1)}$ has even nicer properties than $\mathfrak{s}^{+n}$; and Shelah shows that the frame has all the nice properties he wants after going up three successors.

We obtain:

Theorem 12.15.

(1) If K is categorical in λ, then $\mathfrak{t}$ is a good $(\leq \lambda, \lambda)$-frame.
(2) If $\lambda = \lambda_0^{+3}$ and there exists an ω-successful good λ_0-frame $\mathfrak{s}'$ such that $\mathfrak{s} = (\mathfrak{s}')^{+3}$, then $\mathfrak{t}$ is a fully good $(\leq \lambda, \lambda)$-frame.

Proof. $\mathfrak{t}$ is good by Proposition 12.8, Lemma 12.10, and Lemma 12.12. The second part follows from Fact 12.13 (note that by definition of the successor frame, K will be categorical in λ in that case). □

Remark 12.16. If $\mathfrak{t}$ is [fully] good, cl($\mathfrak{t}$) (see Definition 3.8) will be a [fully] good $(\leq \lambda, \lambda)$-independence relation by Proposition 4.1.

Remark 12.17. In [BVb, Corollary 6.10], it is shown that λ-tameness and amalgamation imply that a good λ-frame extends to a good $(< \infty, \lambda)$-frame. However, the definition of a good frame there is not the same as it does *not* assume that the frame is type-full. Thus the conclusion of Theorem 12.15 is much stronger.

13. Extending the base and right hand side

Hypothesis 13.1.

(1) $\mathfrak{i} = (K, \mathop{\smile}\limits^{\mid})$ is a fully good $(\leq \lambda, \lambda)$-independence relation.
(2) $K' := K^{\text{up}}$ has amalgamation and is λ-tame for types of length less than λ^+.

In this section, we give conditions under which $\mathfrak{i}$ becomes a fully good $(\leq \lambda, \geq \lambda)$-independence relation. In the next section, we will make the left hand side bigger and get a fully good $(< \infty, \geq \lambda)$-independence relation.

Notation 13.2. Let $\mathfrak{i}' := \mathfrak{i}^{\text{up}}$ (recall Definition 6.3). Write $\mathfrak{s} := \text{pre}(\mathfrak{i})$, $\mathfrak{s}' := \text{pre}(\mathfrak{i}')$. We abuse notation and also denote $\mathop{\smile}\limits^{\mid}_{\mathfrak{i}'}$ by $\mathop{\smile}\limits^{\mid}$.

We want to investigate when the properties of $\mathfrak{i}$ carry over to $\mathfrak{i}'$.

Lemma 13.3.

(1) $\mathfrak{i}'$ is a $(<\infty, \geq \lambda)$-independence relation.
(2) K' has joint embedding, no maximal models, and is stable in all cardinals.
(3) $\mathfrak{i}'$ has base monotonicity, transitivity, uniqueness, and disjointness.
(4) $\mathfrak{i}'$ has full model continuity.

Proof.

(1) By Proposition 6.5.
(2) By [BVb, Theorem 1.1], $(\mathfrak{s}')^{\leq 1}$ is a good frame, so in particular K' has joint embedding, no maximal models, and is stable in all cardinals.
(3) See [She09, Claim II.2.11] for base monotonicity and transitivity. Disjointness is straightforward from the definition of $\mathfrak{i}'$, and uniqueness follows from the tameness hypothesis and the definition of $\mathfrak{i}'$.
(4) Assume $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous in K', $\ell \leq 3$, δ is regular, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$, $i < \delta$, and $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\perp}} M_i^2$ for all $i < \delta$. Let $N := M_\delta^3$. By ambient monotonicity, $M_i^1 \overset{N}{\underset{M_i^0}{\perp}} M_i^2$ for all $i < \delta$. We want to see that $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\perp}} M_\delta^2$. Since $\|M_\delta^1\| < \theta$, M_δ^1 and M_δ^0 are in K. Thus it is enough to show that for all $M' \leq M_\delta^2$ in K with $M_\delta^0 \leq M'$, $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\perp}} M'$. Fix such an M'. We consider two cases:
   - <u>Case 1: $\delta < \theta$</u>: Then we can find $\langle M_i' : i \leq \delta \rangle$ increasing continuous in K such that $M_\delta' = M'$ and for all $i < \delta$, $M_0^\delta \leq M_i' \leq M_i^2$. By monotonicity, for all $i < \delta$, $M_i^1 \overset{N}{\underset{M_i^0}{\perp}} M_i'$. By full model continuity in K, $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\perp}} M'$, as desired.
   - <u>Case 2: $\delta \geq \theta$</u>: Since $M_\delta^0, M_\delta^1 \in K$, we can assume without loss of generality that $M_\delta^0 = M_0^0$, $M_\delta^1 = M_\delta^1$. Since δ is regular, there exists $i < \delta$ such that $M' \leq M_i^2$. By

assumption, $M_0^1 \overset{N}{\underset{M_0^0}{\perp}} M_i^2$, so by monotonicity, $M_0^1 \overset{N}{\underset{M_0^0}{\perp}} M'$, as needed.

□

We now turn to local character.

Lemma 13.4. Assume $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^\alpha(M_\delta)$, $\alpha < \lambda^+$ a cardinal and $\delta = \mathrm{cf}(\delta) > \alpha$.

(1) If $\alpha < \lambda$, then there exists $i < \delta$ such that p does not fork over M_i.
(2) If $\alpha = \lambda$ and $\mathfrak{i}$ has the left $(< \mathrm{cf}(\lambda))$-witness property, then there exists $i < \delta$ such that p does not fork over M_i.

Proof.

(1) As in the proof of Lemma 6.8.(2b) Note that weak chain local character holds for free because $\alpha < \lambda$ and $\kappa_\alpha(\mathfrak{i}) = \alpha^+ + \aleph_0$ by assumption.
(2) By the proof of Lemma 6.8.(2b) again, it is enough to see that $\mathfrak{i}$ has weak chain local character: Let $\langle M_i : i < \lambda^+ \rangle$ be increasing in K and let $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$. Let $p \in \mathrm{gS}^\lambda(M_{\lambda^+})$. We will show that there exists $i < \lambda^+$ such that p does not fork over M_i. Say $p = \mathrm{gtp}(\bar{a}/M_{\lambda^+}; N)$ and let $A := \mathrm{ran}(\bar{a})$. Write $A = \bigcup_{j<\mathrm{cf}(\lambda)} A_j$ with $\langle A_j : i < \mathrm{cf}(\lambda) \rangle$ increasing continuous and $|A_j| < \lambda$. By the first part for each $j < \mathrm{cf}(\lambda)$ there exists $i_j < \lambda^+$ such that $A_j \overset{N}{\underset{M_{i_j}}{\perp}} M_{\lambda^+}$. Let $i := \sup_{j<\mathrm{cf}(\lambda)} i_j$. We claim that $A \overset{N}{\underset{M_i}{\perp}} M_{\lambda^+}$. By the $(< \mathrm{cf}(\lambda))$-witness property and the definition of $\mathfrak{i}'$ (here we use that $M_i \in K$), it is enough to show this for all $B \subseteq A$ of size less than $\mathrm{cf}(\lambda)$. But any such B is contained in an A_j, and so the result follows from base monotonicity.

□

Lemma 13.5. Assume $\mathfrak{i}'$ has existence. Then $\mathfrak{i}'$ has right independent amalgamation.

Proof. As in, for example, [Bon14a, Theorem 5.3], using full model continuity. □

Putting everything together, we obtain:

Theorem 13.6. If K is $(< \text{cf}(\lambda))$-tame and short for types of length less than λ^+, then $\mathfrak{i}'$ is a fully pre-good $(\leq \lambda, \geq \lambda)$-independence relation.

Proof. We want to show that $\mathfrak{s}'$ is fully good. The basic properties are proven in Lemma 13.3. By Lemma 4.5, $\mathfrak{i}$ has the left $< (\text{cf}(\lambda))$-witness property. Thus by Lemma 13.4, for any $\alpha < \lambda^+$, $\kappa_\alpha(\mathfrak{i}') = |\alpha|^+ + \aleph_0$. In particular, $\mathfrak{i}'$ has existence, and thus by the definition of $\mathfrak{i}'$ and transitivity in $\mathfrak{i}$, $\bar{\kappa}_\alpha(\mathfrak{i}') = \lambda^+ = |\alpha|^+ + \lambda^+$. Finally by Lemma 13.5, $\mathfrak{i}'$ has independent amalgamation and so by Proposition 4.3.(3), $\mathfrak{i}'$ has extension. $\square$

14. Extending the left hand side

We now enlarge the left hand side of the independence relation built in the previous section.

Hypothesis 14.1.

(1) $\mathfrak{i} = (K, \mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}})$ is a fully good $(\leq \lambda, \geq \lambda)$-independence relation.
(2) K is λ-short for types of length less than λ^+.

Definition 14.2. Define $\mathfrak{i}^{<\infty} = (K, \overset{<\infty}{\mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}}})$ by setting $\overset{<\infty}{\underset{N}{\mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}}}}(M_0, A, B, N)$ if and only if for all $A_0 \subseteq A$ of size less than λ^+, $A_0 \overset{N}{\underset{M_0}{\mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}}}} B$.

Remark 14.3. This notation conflicts with the one introduced in [BVb, Definition 4.3], but the idea is the same: we extend the frame to have longer types. The difference is that $\mathfrak{i}^{<\infty}$ is type-full.

Remark 14.4. We could also have defined $\mathfrak{i}^{<\theta}$ for θ a cardinal and $\mathfrak{i}$ a good $(< \theta, [\lambda, \theta))$-independence relation, but this complicates the notation and we have no use for it here.

Notation 14.5. Write $\mathfrak{i}' := \mathfrak{i}^{<\infty}$. We abuse notation and also write $\mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}}$ for $\overset{<\infty}{\mathop{\smile\kern-0.55em\raise0.3ex\hbox{$\mid$}}}$.

Lemma 14.6.

(1) $\mathfrak{i}'$ is a $(< \infty, \geq \lambda)$-independence relation.
(2) K has joint embedding, no maximal models, and is stable in all cardinals.
(3) $\mathfrak{i}'$ has base monotonicity, transitivity, disjointness, existence, symmetry, the left λ-witness property, and uniqueness.

Proof.

(1) Straightforward.
(2) Because $\mathfrak{i}$ is good.
(3) Base monotonicity, transitivity, disjointness, existence, symmetry, and the left λ-witness property are straightforward. Uniqueness is by the shortness hypothesis.

$\square$

Lemma 14.7. Assume $\kappa \leq \lambda$ is a regular cardinal such that $\mathfrak{i}$ has the left $(< \kappa)$-model-witness property. Then $\mathfrak{i}'$ has full model continuity.

Proof. Let $\langle M_i^\ell : i \leq \delta \rangle$, $\ell \leq 3$ be increasing continuous in K such that $M_i^0 \leq M_i^\ell \leq M_i^3$, $\ell = 1, 2$, and $M_i^1 \underset{M_i^0}{\overset{M_i^3}{\perp}} M_i^2$. Without loss of generality, δ is regular. Let $N := M_\delta^3$. We want to show that $M_\delta^1 \underset{M_\delta^0}{\overset{N}{\perp}} M_\delta^2$. Let $A \subseteq |M_\delta^1|$ have size less than λ^+. Write $\mu := |A|$. By monotonicity, assume without loss of generality that $\lambda + \kappa \leq \mu$. We show that $A \underset{M_\delta^0}{\overset{N}{\perp}} M_\delta^2$, which is enough by definition of $\mathfrak{i}'$. We consider two cases.

- <u>Case 1: $\delta > \mu$</u>: By local character in $\mathfrak{i}$ there exists $i < \delta$ such that $A \underset{M_i^2}{\overset{N}{\perp}} M_\delta^2$. By right transitivity, $A \underset{M_i^0}{\overset{N}{\perp}} M_\delta^2$, so by base monotonicity, $A \underset{M_\delta^0}{\overset{N}{\perp}} M_\delta^2$.
- <u>Case 2: $\delta \leq \mu$</u>: For $i \leq \delta$, let $A_i := A \cap |M_i^1|$. Build $\langle N_i : i \leq \delta \rangle$, $\langle N_i^0 : i \leq \delta \rangle$ increasing continuous in $K_{\leq \mu}$ such that for all $i < \delta$:
  (1) $A_i \subseteq |N_i|$.
  (2) $N_i \leq M_i^1$, $A \subseteq |N_i|$.
  (3) $N_i^0 \leq M_i^0$, $N_i^0 \leq N_i$.
  (4) $N_i \underset{N_i^0}{\overset{N}{\perp}} M_i^2$.

  <u>This is possible</u>. Fix $i \leq \delta$ and assume N_j, N_j^0 have already been constructed for $j < i$. If i is limit, take unions. Otherwise, recall that we are assuming $M_i^1 \underset{M_i^0}{\overset{N}{\perp}} M_i^2$. By Lemma 4.8 (with $A_i \cup \bigcup_{j<i} |N_j|$ standing for A there, this is where we use the $(< \kappa)$-model-witness property), we can find $N_i^0 \leq M_i^0$ and $N_i \leq$

M_i^1 in $K_{\leq \mu}$ such that $N_i^0 \leq N_i$, $N_i \overset{N}{\underset{N_i^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_i^2$, $A_i \subseteq |N_i|$, $N_j \leq N_i$ for all $j < i$, and $N_j^0 \leq N_i^0$ for all $j < i$. Thus they are as desired.

This is enough. Note that $A_\delta = A$, so $A \subseteq |N_\delta|$. By full model continuity in $\mathfrak{i}$, $N_\delta \overset{N}{\underset{N_\delta^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_\delta^2$. By monotonicity, $A \overset{N}{\underset{M_\delta^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_\delta^2$, as desired.

□

We now turn to proving extension. The proof is significantly more complicated than in the previous section. We attempt to explain why and how our proof goes. Of course, it suffices to show independent amalgamation (Proposition 4.3.(3)). We work by induction on the size of the models but land in trouble when all models have the same size. Suppose for example that we want to amalgamate $M^0 \leq M^\ell$, $\ell = 1, 2$ that are all in K_{λ^+}. If M^1 (or, by symmetry, M^2) had smaller size, we could use local character to assume without loss of generality that M^0 is in K_λ and then imitate the usual directed system argument (as in for example [Bon14a, Theorem 5.3]).

Here however it seems we have to take at least two resolutions at once so we fix $\langle M_i^\ell : i < \lambda^+ \rangle$, $\ell = 0, 1$, satisfying the usual conditions. Letting $p := \text{gtp}(M^1/M^0; M^1)$ and its resolution $p_i := \text{gtp}(M_i^1/M_i^0; M^1)$, it is natural to build $\langle q_i : i < \lambda^+ \rangle$ such that q_i is the nonforking extension of p_i to M^2. If everything works, we can take the direct limit of the q_is and get the desired nonforking extension of p. However with what we have said so far it is not clear that q_{i+1} is even an extension of q_i! In the usual argument, this is the case since both p_i and p_{i+1} do not fork over the same domain but we cannot expect it here. Thus we require in addition that $M_i^1 \overset{M^1}{\underset{M_i^0}{\mathop{\smile\!\!\!\!\!\!|}}} M^0$ and this turns out to be enough for successor steps. To achieve this extra requirement, we use Lemma 4.8, which relies on local character.

We can prove this local character from extension, so we end up proving both extension and local character together by induction on the size of the base model. Unfortunately, we also do not know how to go through limit steps without making one extra hypothesis:

Definition 14.8 (Type-locality).

(1) Let δ be a limit ordinal, and let $\bar{p} := \langle p_i : i < \delta \rangle$ be an increasing chain of Galois types, where for $i < \delta$, $p_i \in \text{gS}^{\alpha_i}(M)$ and $\langle \alpha_i : i \leq \delta \rangle$ are increasing continuous. We say $\bar{p}$ is *type-local* if whenever $p, q \in \text{gS}^{\alpha_\delta}(M)$ are such that $p^{\alpha_i} = q^{\alpha_i} = p_i$ for all $i < \delta$, then $p = q$.
(2) We say K is *type-local* if every $\bar{p}$ as above is type-local.
(3) We say K is *densely type-local above* λ if for every $\lambda_0 > \lambda$, $M \in K_{\lambda_0}$, $p \in \text{gS}^{\lambda_0}(M)$, there exists $\langle N_i : i \leq \delta \rangle$ such that:
   (a) $\delta = \text{cf}(\lambda_0)$.
   (b) For all $i < \delta$, $N_i \in K_{<\lambda_0}$.
   (c) $\langle N_i : i \leq \delta \rangle$ is increasing continuous.
   (d) $N_\delta \geq M$ is in K_{λ_0}.
   (e) Letting $q_i := \text{gtp}(N_i / M; N_\delta)$ (seen as a member of $\text{gS}^{\alpha_i}(M)$, where of course $\langle \alpha_i : i \leq \delta \rangle$ are increasing continuous), we have that q_δ extends p and $\langle q_j : j < i \rangle$ is type-local for all limit $i \leq \delta$.

   We say K is *densely type-local* if it is densely type-local above λ for some λ.

We suspect that dense type-locality should hold in our context, see the discussion in Section 15 for more. The following lemma says that increasing the elements in the resolution of the type preserves type-locality.

Lemma 14.9. Let δ be a limit ordinal. Assume $\bar{p} := \langle p_i : i < \delta \rangle$ is an increasing chain of Galois types, $p_i \in \text{gS}^{\alpha_i}(M)$ and $\langle \alpha_i : i \leq \delta \rangle$ are increasing continuous. Assume $\bar{p}$ is type-local and assume $p_\delta \in \text{gS}^{\alpha_\delta}(M)$ is such that $p^{\alpha_i} = p_i$ for all $i < \delta$. Say $p = \text{gtp}(\bar{a}_\delta / M; N)$ and let $\bar{a}_i := \bar{a}_\delta \restriction \alpha_i$ (so $p_i = \text{gtp}(\bar{a}_i / M; N)$).

Assume $\langle \bar{b}_i : i \leq \delta \rangle$ are increasing continuous sequences such that $\bar{a}_\delta = \bar{b}_\delta$ and $\bar{a}_i$ is an initial segment of $\bar{b}_i$ for all $i < \delta$. Let $q_i := \text{gtp}(\bar{b}_i / M; N)$. Then $\bar{q} := \langle q_i : i < \delta \rangle$ is type-local.

Proof. Say $\bar{b}_i$ is of type β_i. So $\langle \beta_i : i \leq \delta \rangle$ is increasing continuous and $\alpha_\delta = \beta_\delta$.

If $q \in \text{gS}^{\beta_\delta}(M)$ is such that $q^{\beta_i} = q_i$ for all $i < \delta$, then $q^{\alpha_i} = (q_i)^{\alpha_i} = p_i$ for all $i < \delta$ so by type-locality of $\bar{p}$, $p = q$, as desired. □

Before proving Lemma 14.12, let us make precise what was meant above by "direct limit" of a chain of types. It is known that (under some set-theoretic hypotheses) there exists an AEC with a chain of Galois types that has no upper bound, see [BS08, Theorem 3.3]. However a *coherent* chain of types (see below) always has an upper bound. We generalize Definition 5.1 in [Bon14a] (which is implicit already in [She01, Claim 0.32.2] or [GV06a, Lemma 2.12]) to our purpose.

Definition 14.10. Let δ be an ordinal. An increasing chain of types $\langle p_i : i < \delta \rangle$ is said to be *coherent* if there exists a sequence $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$ and maps $f_{i,j} : N_i \to N_j$, $i \leq j < \delta$, so that for all $i \leq j \leq k < \delta$:

(1) $f_{j,k} \circ f_{i,j} = f_{i,k}$.
(2) $\text{gtp}(\bar{a}_i / M_i; N_i) = p_i$.
(3) $\langle M_i : i < \delta \rangle$ and $\langle N_i : i < \delta \rangle$ are increasing.
(4) $M_i \leq N_i$, $\bar{a}_i \in {}^{<\infty} N_i$.
(5) $f_{i,j}$ fixes M_i.
(6) $f_{i,j}(\bar{a}_i)$ is an initial segment of $\bar{a}_j$.

We call the sequence and maps above a *witnessing sequence* to the coherence of the p_i's.

Given a witnessing sequence $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$ with maps $f_{i,j} : N_i \to N_j$, we can let N_δ be the direct limit of the system $\langle N_i, f_{i,j} : i \leq j < \delta \rangle$, $M_\delta := \bigcup_{i<\delta} M_i$, and $\bar{a}_\delta := \bigcup_{i<\delta} f_{i,\delta}(\bar{a}_i)$ (where $f_{i,\delta} : N_i \to N_\delta$ is the canonical embedding). Then $p := \text{gtp}(\bar{a}/M_\delta; N_\delta)$ extends each p_i. Note that p depends on the witness but we sometimes abuse language and talk about "the" direct limit (where really some witnessing sequence is fixed in the background).

Finally, note that full model continuity also applies to coherent sequences. More precisely:

Proposition 14.11. Assume $\mathfrak{i}$ has full model continuity. Let $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$, $\langle f_{i,j} : N_i \to N_j, i \leq j < \delta \rangle$ be witnesses to the coherence of $p_i := \text{gtp}(\bar{a}_i / M_i; N_i)$. Assume that for each $i < \delta$, $\bar{a}_i$ enumerates a model M_i' and that $\langle M_i^0 : i < \delta \rangle$ are increasing such that $M_i^0 \leq M_i$, $M_i^0 \leq M_i'$, and p_i does not fork over M_i^0. Let p be the direct limit of the p_is (according to the witnessing sequence). Then p does not fork over $M_\delta^0 := \bigcup_{i<\delta} M_i^0$.

Proof. Use full model continuity inside the direct limit. $\square$

Lemma 14.12. Assume K is densely type-local above λ, and assume $\kappa \leq \lambda$ is a regular cardinal such that K is $(< \kappa)$-tame for types of all lengths. Then:

(1) For all cardinals θ:
  (a) $\kappa_\theta(\mathfrak{i}') = \theta^+ + \aleph_0$.
  (b) $\bar{\kappa}_\theta(\mathfrak{i}') = \theta^+ + \lambda^+$.

(2) $\mathfrak{i}'$ has extension.

Proof. First note that by Lemma 4.5 and symmetry, $\mathfrak{i}$ has the left $(<\kappa)$-model-witness property. By Lemma 14.7, $\mathfrak{i}'$ has full model continuity. Let $\mathfrak{i}'' := \text{cl}(\text{pre}(\mathfrak{i}'))$ and let $\lambda_0 \geq \lambda$ be a cardinal. We prove the following statements simultaneously by induction on λ_0:

(1) $\text{pre}(\mathfrak{i}')$ has extension for base models in K_{λ_0}.
(2) $\underset{\mathfrak{i}'}{\perp}$ and $\underset{\mathfrak{i}''}{\perp}$ agree for base models in K_{λ_0}.
(3) For all cardinals $\theta \leq \lambda_0$, (1a) and (1b) hold.

This is enough: then $\mathfrak{i}' = \mathfrak{i}''$ (because (2) holds for all λ_0 so the two relations agree on all base models) so $\mathfrak{i}'$ has extension by definition of $\mathfrak{i}''$.

So let $\lambda_0 \geq \lambda$ and assume that (1), (2), and (3) above hold for all $\lambda_0' < \lambda_0$.

Assume that we have shown (1) for λ_0. We show how (2) and (3) for λ_0 follow. First, by Proposition 4.3.(7) and Remark 4.4, $\text{pre}(\mathfrak{i}')$ has full symmetry over base models in $K_{\leq \lambda_0}$. Thus (over base models in $K_{\leq \lambda_0}$ again) $\mathfrak{i}''$ has full symmetry and it also has extension by definition. By Lemma 4.5, $\mathfrak{i}''$ has (over base models in $K_{\leq \lambda_0}$) the left λ-witness property. Thus $\mathfrak{i}'$ and $\mathfrak{i}''$ both have the left λ-witness property over base models of size $\leq \lambda_0$, but then (since $\mathfrak{i} = \text{cl}(\text{pre}(\mathfrak{i}))$: it has extension), it follows that (2) holds for λ_0. Now let's prove (3): if $\lambda_0 = \lambda$, this is true because $\mathfrak{i}$ is good. If $\lambda_0 > \lambda$, Lemma 4.7 (with α there standing for λ_0^+ here) gives us that $\bar{\kappa}_\theta(\mathfrak{i}') = \theta^+ + \lambda^+$ for all $\theta \leq \lambda_0$. Now if $\theta \leq \lambda$, we have already argued that $\kappa_\theta(\mathfrak{i}') = \theta^+ + \aleph_0$. If $\theta > \lambda$, then by Proposition 4.3.(5), $\kappa_\theta(\mathfrak{i}') \leq \bar{\kappa}_\theta(\mathfrak{i}') = \theta^+ + \lambda^+ = \theta^+$. By definition of κ_θ, we also have that $\kappa_\theta(\mathfrak{i}) \geq \theta^+$, so $\kappa_\theta(\mathfrak{i}') = \theta^+ = \theta^+ + \aleph_0$, as needed.

It remains to show (1) for λ_0, assuming (1), (2), and (3) below λ_0. By Proposition 4.3.(3), it is enough to prove independent amalgamation.

Let $M^0 \leq M^\ell$, $\ell = 1, 2$ be in K with $\|M^0\| = \lambda_0$. We want to find $q \in \text{gS}^{\lambda_0}(M^2)$ a nonforking extension of $p := \text{gtp}(M^1/M^0; M^1)$. Let $\lambda_\ell := \|M^\ell\|$ for $\ell = 1, 2$.

Assume we know the result when $\lambda_0 = \lambda_1 = \lambda_2$. Then we can work by induction on (λ_1, λ_2): if they are both λ_0, the result holds by assumption. If not, we can assume by symmetry that $\lambda_1 \leq \lambda_2$, find an increasing continuous resolution of M^2, $\langle M_i^2 \in K_{<\lambda_2} : i < \lambda_2 \rangle$ and do a directed system argument as in [Bon14a, Theorem 5.3] (using full model continuity and the induction hypothesis).

Now assume that $\lambda_0 = \lambda_1 = \lambda_2$. If $\lambda_0 = \lambda$, we get the result by extension in $\mathfrak{i}$, so assume $\lambda_0 > \lambda$. Let $\delta := \text{cf}(\lambda_0)$. By dense type-locality, we can assume (extending M^1 if necessary) that there exists $\langle N_i : i \leq \delta \rangle$ an increasing continuous resolution of M^1 with $N_i \in K_{<\lambda_0}$ for $i < \delta$ so that $\langle \text{gtp}(N_j / M^0; M^1) : j < i \rangle$ is type-local for all limit $i \leq \delta$.

Step 1. Fix increasing continuous $\langle M_i^\ell : i \leq \delta \rangle$ for $\ell < 2$ such that for all $i < \delta$, $\ell < 2$:

(1) $M^\ell = M_\delta^\ell$.
(2) $M_i^\ell \in K_{<\lambda_0}$.
(3) $N_i \leq M_i^1$.
(4) $M_i^0 \leq M_i^1$.
(5) $M_i^1 \underset{M_i^0}{\overset{M^1}{\downarrow}} M^0$.

This is possible by repeated applications of Lemma 4.8 (as in the proof of Lemma 14.7), starting with $M^1 \underset{M^0}{\overset{M^1}{\downarrow}} M^0$ which holds by existence. Note that we have used (3) (below λ_0) here.

Step 2. Fix enumerations of M_i^1 of order type α_i such that $\langle \alpha_i : i \leq \delta \rangle$ is increasing continuous, $\alpha_\delta = \lambda_0$ and $i < j$ implies that M_i' appears as the initial segment up to α_i of the enumeration of M_j'. For $i \leq \delta$, let $p_i := \text{gtp}(M_i^1 / M_i^0; M^1)$ (seen as an element of $\text{gS}^{\alpha_i}(M_i^0)$). We want to find $q \in \text{gS}^{\lambda_0}(M^2)$ extending $p = p_\delta$ and not forking over M^0. Note that since for all $j < \delta$, $N_j \leq M_j^1$, we have by Lemma 14.9 that $\langle \text{gtp}(M_j^1 / M^0; M^1) : j < i \rangle$ is type-local for all limit $i \leq \delta$.

Build an increasing, coherent $\langle q_i : i \leq \delta \rangle$ such that for all $i \leq \delta$,

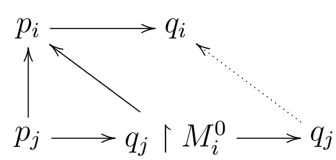
(1) $q_i \in \text{gS}^{\alpha_i}(M^2)$.
(2) $q_i \restriction M_i^0 = p_i$.
(3) q_i does not fork over M_i^0.

This is enough: then q_δ is an extension of $p = p_\delta$ that does not fork over $M_\delta^0 = M^0$.

This is possible: We work by induction on $i \leq \delta$. While we do not make it explicit, the sequence witnessing the coherence is also built inductively in the natural way (see also [Bon14a, Proposition 5.2]): at base and successor steps, we use the definition of Galois types. At limit steps, we take direct limits.

Now fix $i \leq \delta$ and assume everything has been defined for $j < i$.

- **Base step:** When $i = 0$, let $q_0 \in \mathrm{gS}^{\alpha_0}(M^2)$ be the nonforking extension of p_0 to M^2_0 (exists by extension below λ_0).
- **Successor step:** When $i = j+1$, $j < \delta$, let q_i be the nonforking extension (of length α_i) of p_i to M^2. We have to check that q_i indeed extends q_j (i.e. $q_i^{\alpha_j} = q_j$). Note that $q_j \upharpoonright M^0$ does not fork over M^0_j so by step 1 and uniqueness, $q_j \upharpoonright M^0 = \mathrm{gtp}(M^1_j/M^0; M^1)$. In particular, $q_j \upharpoonright M^0_i = \mathrm{gtp}(M^1_j/M^0_i; M^1)$. Since q_i extends p_i, $q_i \upharpoonright M^0_i = \mathrm{gtp}(M^1_i/M^0_i; M^1)$ so $q_i^{\alpha_j} \upharpoonright M^0_i = \mathrm{gtp}(M^1_j/M^0_i; M^1) = q_j \upharpoonright M^0_i$. By base monotonicity, q_j does not fork over M^0_i so by uniqueness $q_i^{\alpha_j} = q_j$. A picture is below.



- **Limit step:** Assume i is limit. Let q_i be the direct limit of the coherent sequence $\langle q_j : j < i \rangle$. Note that $q_i \in \mathrm{gS}^{\alpha_i}(M^2)$ and by Proposition 14.11, q_i does not fork over M^0_i. It remains to see $q_i \upharpoonright M^0_i = p_i$.

  For $j < i$, let $p'_j \in \mathrm{gS}^{\alpha_j}(M^0_i)$ be the nonforking extension of p_j to M^0_i. By step 1, $p'_j = \mathrm{gtp}(M^1_j/M^0_i; M^1)$. Thus $\langle p'_j : j < i \rangle$ is type-local. By an argument similar to the successor step above, we have that for all $j < i$, $p_i^{\alpha_j} = p'_j$. Moreover, for all $j < i$, $q_i^{\alpha_j} \upharpoonright M^0_j = p_j$ and q_j does not fork over M^0_j so by uniqueness, $q_i^{\alpha_j} \upharpoonright M^0_i = (q_i \upharpoonright M^0_i)^{\alpha_j} = p'_j$. By type-locality, it follows that $q_i \upharpoonright M^0_i = p_i$, as desired.

□

Putting everything together, we get:

Theorem 14.13. If:

(1) For some regular $\kappa \leq \lambda$, K is $(< \kappa)$-tame for types of all lengths.
(2) K is densely type-local above λ.

Then $\mathfrak{i}'$ is a fully good $(< \infty, \geq \lambda)$-independence relation.

Proof. Lemma 14.6 gives most of the properties of a good independence relation. By Lemma 4.5, $\mathfrak{i}$ has the left $(< \kappa)$-witness property. By Lemma 14.7, $\mathfrak{i}'$ has full model continuity. By Lemma 14.12, $\mathfrak{i}'$ has extension and the local character properties. □

15. The main theorems

Recall (Definition 8.4) that an AEC K is fully good if there is a fully good independence relation with underlying class K. Intuitively, a fully good independence relation is one that satisfies all the properties of forking in a superstable first-order theory. We are finally ready to show that densely type-local fully tame and short superstable classes are fully good, at least on a dense subclass of saturated models[27]:

Theorem 15.1. Let K be a fully $(< \kappa)$-tame and short abstract elementary class. Assume that K is densely type-local above κ.

(1) If K is μ-superstable, $\kappa = \beth_\kappa > \mu$, and $\lambda := (\kappa^{<\kappa})^{+7}$, then $K^{\lambda\text{-sat}}$ is fully good.
(2) If K is κ-strongly μ-superstable and $\lambda := (\mu^{<\kappa})^{+6}$, then $K^{\lambda\text{-sat}}$ is fully good.
(3) If K has amalgamation, $\kappa = \beth_\kappa > \text{LS}(K)$, and K is categorical in a $\mu > \lambda_0 := (\kappa^{<\kappa})^{+5}$, then $K_{\geq \lambda}$ is fully good, where $\lambda := \min(\mu, h(\lambda_0))$.

Proof.

(1) By Theorem 10.9 and Proposition 10.8, K is κ-strongly $(\kappa^{<\kappa})^+$-superstable. Now apply (2).
(2) By Fact 11.3, Hypothesis 11.1 holds for $\mu' := (\mu^{<\kappa})^{+2}$, λ there standing for $(\mu')^+$ here, and $K' := K^{\mu'\text{-sat}}$. By Theorem 11.21, there is an ω-successful type-full good $(\mu')^+$-frame $\mathfrak{s}$ on $K^{(\mu')^+\text{-sat}}$. By Theorem 12.15 and Remark 12.16, $\mathfrak{s}^{+3}$ induces a fully good $(\leq \lambda, \lambda)$-independence relation $\mathfrak{i}$ on $K^{(\mu')^{+4}\text{-sat}} = K^{\lambda\text{-sat}}$. By Theorem 13.6, $\mathfrak{i}' := \text{cl}(\text{pre}(\mathfrak{i}_{\geq \lambda}))$ is a fully good $(\leq \lambda, \geq \lambda)$-independence relation on $K^{\lambda\text{-sat}}$. By Theorem 14.13, $(\mathfrak{i}')^{<\infty}$ is a fully good $(< \infty, \geq \lambda)$-independence relation on $K^{\lambda\text{-sat}}$. Thus $K^{\lambda\text{-sat}}$ is fully good.
(3) By Theorem 10.14, K is κ-strongly κ-superstable. By (2), $K^{\lambda_0^+\text{-sat}}$ is fully good. By Fact 10.10.(4), all the models in $K_{\geq \lambda}$ are λ_0^+-saturated, hence $K_{\geq \lambda}^{\lambda_0^+\text{-sat}} = K_{\geq \lambda}$ is fully good.

□

We now discuss the necessity of the hypotheses of the above theorem. It is easy to see that a fully good AEC is superstable. Moreover, the

[27] The number 7 in (1) is possibly the largest natural number ever used in a statement about abstract elementary classes!

existence of a relation $\perp$ with disjointness and independent amalgamation directly implies disjoint amalgamation. An interesting question is whether there is a general framework in which to study independence without assuming amalgamation, but this is out of the scope of this paper. To justify full tameness and shortness, one can ask:

Question 15.2. Let K be a fully good AEC. Is K fully tame and short?

If the answer is positive, we believe the proof to be nontrivial. We suspect however that the shortness hypothesis of our main theorem can be weakened to a condition that easily holds in all fully good classes. In fact, we propose the following:

Definition 15.3. An AEC K is *diagonally* $(< \kappa)$*-tame* if for any $\kappa' \geq \kappa$, K is $(< \kappa')$-tame for types of length less than κ'. K is *diagonally* κ*-tame* if it is diagonally $(< \kappa^+)$-tame. K is *diagonally tame* if it is diagonally $(< \kappa)$-tame for some κ.

It is easy to check that if $\mathfrak{i}$ is a good $(< \infty, \geq \lambda)$-independence relation, then $K_{\mathfrak{i}}$ is diagonally λ-tame. Thus we suspect the answer to the following should be positive:

Question 15.4. In Theorem 15.1, can "fully $(< \kappa)$-tame and short" be replaced by "diagonally $(< \kappa)$-tame?

Finally, we believe the dense type-locality hypothesis can be removed[28]. Indeed, chapter III of [She09] has many results on getting models "generated" by independent sequences. Since independent sequences exhibit a lot of finite character (see also [BVb]), we suspect the following conjecture should be true. Note that if it holds, then one can remove the dense type-locality hypothesis from Theorem 15.1.

Conjecture 15.5. Let K be a fully $(< \kappa)$-tame and short abstract elementary class. If K is κ-strongly μ-superstable and $\lambda := (\mu^{<\kappa})^{+6}$, then $K^{\lambda\text{-sat}}$ is densely type-local above λ.

We end with some results that do not need dense type-locality. Note that we can replace categoricity by superstability or strong superstability as in the proof of Theorem 15.1.

[28]In fact, the result was initially announced without this hypothesis but Will Boney found a mistake in our proof of Lemma 14.12. This is the only place where type-locality is used

Theorem 15.6. Let K be a fully $(<\kappa)$-tame and short AEC with amalgamation. Assume:

$$\text{LS}(K) < \kappa = \beth_\kappa < \lambda = \beth_\lambda \le \lambda'$$

If K is categorical in λ', then:

(1) There exists an ω-successful type-full good λ-frame $\mathfrak{s}$ with $K_{\mathfrak{s}} = K_\lambda$. Furthermore, the frame is induced by $(<\kappa)$-coheir: $\mathfrak{s} = \text{pre}(\mathfrak{i}_{\kappa\text{-ch}}(K)_\lambda^{\le 1})$.
(2) $K_{\ge\lambda}$ is $(\le\lambda)$-good.
(3) $K^{\lambda^{+3}\text{-sat}}$ is fully $(\le\lambda^{+3})$-good.

Proof. Note that $\lambda = \lambda^{<\kappa}$ since it is strong limit. By Fact 11.3 and Theorem 11.21, there is an ω-successful type-full good λ-frame $\mathfrak{s}$ with $K_{\mathfrak{s}} = K_\lambda^{\lambda\text{-sat}}$. Now (by Theorem 10.14 if $\lambda' > \lambda$), K is categorical in λ. Thus $K^{\lambda\text{-sat}} = K_{\ge\lambda}$. Theorem 12.15 gives the last two parts. $\square$

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