

# An Analysis of the Transition Zone between the various Scaling Regimes in the Small-World Model

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## Abstract

We analyse the so-called small-world network model (originally devised by Strogatz and Watts), treating it, among other things, as a case study of non-linear coupled difference or differential equations. We derive a system of evolution equations containing more of the previously neglected (possibly relevant) non-linear terms. As an exact solution of this entangled system of equations is out of question we develop a (as we think, promising) method of enclosing the “exact” solutions for the expected quantities by upper and lower bounds, which represent solutions of a slightly simpler system of differential equation. Furthermore we discuss the relation between difference and differential equations and scrutinize the limits of the spreading idea for random graphs. We then show that there exists in fact a “broad” (with respect to scaling exponents) crossover zone, smoothly interpolating between linear and logarithmic scaling of the diameter or average distance. We are able to corroborate earlier findings in certain regions of phase or parameter space (as e.g. the finite size scaling ansatz) but find also deviations for other choices of the parameters. Our analysis is supplemented by a variety of numerical calculations, which, among other things, quantify the effect of various approximations being made. With the help of our analytical results we manage to calculate another important network characteristic, the (fractal) dimension, and provide numerical values for the case of the small-world network.

Catchwords: Small-World Networks, Non-linear Difference Equations

# 1 Introduction

As part of a broader interest in complex systems, the analysis of large networks of interacting agents or simply certain degrees of freedom is currently under intense study. Recently a presumably far-reaching core-concept came to the fore, called the *small-world effect*, (for an incomplete list of references see, for example [1] to [8]). To put it briefly, the presence of a surprisingly small number of random edges, inserted in an initially quite regular graph, may have drastic effects as to the average distance between nodes or the expected diameter of the network. These additional random edges, called short cuts, may typically connect regions which have been quite a distance apart in the original regular graph, thus effecting a drastic shrinkage of average distance or diameter in certain regions of parameter space.

It is perhaps noteworthy that we detected a similar phenomenon in quite a different area of modern physics (quantum space-time physics) at almost the same time, being completely unaware of similar findings in other fields of natural science. We called this phenomenon a *microscopic wormhole structure* ([9],[10]).

To understand this smallworld effect in more quantitative terms, a simple model, the so-called *Strogatz-Watts-model*, was investigated in more detail in [11],[12],[13] and a little bit later also in [14].

In its most tractable form it consists of  $N$  linearly ordered vertices (nodes) with periodic boundary condition (i.e. node  $x_N$  is linked to node  $x_1$ ). In general each node may also be linked to its regular neighbors up to order  $k$ . The generic case is already given for  $k = 1$  (i.e. nearest neighbors only or  $\mathbb{Z}_N$ ).

To mimic the random-rewiring of edges of the original Strogatz-Watts-model, it is convenient to superimpose the given regular graph by a random graph, living over the same set of vertices. While we prefer to introduce the so-called *edge-probability*  $p$ , that is, the independent probability for the existence of an edge between two nodes, as is usually done in the random graph framework (see e.g. [15],[16]), some authors (for certain reasons, which come from the original idea of rewiring existing links) made a different choice, referring the probability of a random edge (or *shortcut*) to the number of nodes,  $N$ , in the graph (for the case  $k = 1$ !). The relation between these two probabilities,  $p$  and  $\phi$  is described at the beginning of the following section.

Important (random) graph characteristics to be employed in detecting the small-world effect are the (*expected*) *diameter* of the graph and the *mean-distance* between pairs of nodes. A little bit surprisingly, it turns out to be possible to estimate or calculate these quantities in the Strogatz-Watts-model as functions of the two parameters,  $N$  and  $p$  or  $\phi$ . This is remarkable as one has to deal with two coupled non-linear difference or differential equations for the variables  $f(n)$  and  $g(n)$  described in the following section or the figure caption to figure 1. The degree of nonlinearity varies of course depending on the extent of approximations being made.

The general observation is that, depending on the number of shortcuts in the network, there exist several regimes in the parameter space. To put it more succinctly, we solve, on the one hand, the equations of the model for fixed  $p$  and  $N$ . On the other hand, it is interesting to study the limit  $N \rightarrow \infty$  with  $p(N)$  now a function vanishing for large  $N$ . That is, in this latter case, we analyse the different asymptotic regimes around the “point” ( $N = \infty, p = 0$ ). In the first case one can study the change of behavior of the network for either  $N$  fixed and changing  $p$  or vice versa.

For very small  $p$  (more precisely, very few shortcuts) the average distance, for example, scales linearly with  $N$ . For still quite small  $p$  one expects a (transition like) threshold or, rather, a threshold region, above which the average distance (or the diameter) scales roughly like in a *sparse* random graph, i.e. more or less logarithmically. There was a certain debate about the nature of this transition zone. We show in the following that instead of a threshold one actually has a

relatively “broad” *cross-over* region in which the scaling changes in a smooth way from linear through  $\sim N^\epsilon[(1 - \epsilon) \ln N + O(1)]$  (in first order) to  $\ln N$  depending on  $p$ . More precisely, if we scale  $p$  with  $N$  and choose  $N$  large the corresponding values of the edge probability are  $p \sim N^{-2}$  (linear),  $p \sim N^{-(1+\epsilon)}$  (intermediate),  $p \sim N^{-1}$  (logarithmically), respectively. The original threshold was (in our units) conjectured to occur for  $p \sim N^{-2}$ .

Another interesting conjecture, which was then corroborated both numerically and by plausibility arguments, was a *finite-size-scaling ansatz* for the shape of the functional dependence of the average distance,  $L$ , on  $N$  and  $p$ . We were able to confirm this ansatz modulo some deviations which occurred in a certain region of the parameter space.

At the end of the paper we introduce a (fractal-like) notion of dimension for networks and calculate the *dimension* of the small-world-network.

To briefly characterize our own approach, we include, on the one hand, more possibly relevant terms making hence the evolution equations more complicated. In contrast to using then approximate solutions we manage to derive upper and lower bound comparison difference equations (differential equations) for the “exact” solutions, which allow us to enclose them from above and below. By this method we are able to compare the reliability of the various (approximate) results produced in the literature, relate them to our exact bounds and represent them in a single diagram. Last but not least we were quite scrupulous to compensate for the possible quantitative errors (coming from *overcounting*) which are easily introduced by being too cavalier as to the (thumb rule like) *spreading argument* frequently evoked for random graphs.

We expect that our method of providing comparison difference or differential equations for complicated non-linear equations, which, on their side, are presumably not solvable, may represent a strategy which might prove useful in a more general context.

## 2 The Description of the Small World Model

We start from the graph  $\mathbb{Z}_N$ , i.e.  $N$  nodes on a line with periodic boundary condition; that is, node  $x_N$  is linked to node  $x_1$ .

Remark: To make the red thread of our analysis better visible, we treat for the time being only the nearest neighbor model. A node  $x_i$  is only connected to  $x_{i\pm 1}$  (i.e.  $k = 1$ , or coordination number  $z = 2$ ). The more general case is a straightforward generalisation and can be reduced to the case  $k = 1$  by a renormalization step, cf. [12].

In a next step we superimpose this graph with a true random graph, living on the same  $N$  nodes and having independent *edge probability*  $p$  (cf. for example [15] or [16]). This entails that the expected number of random edges in our model is  $p \cdot N(N - 1)/2$  (the average vertex degree in the respective random graph) and the expected number of random edges being incident with a fixed but arbitrary node,  $x_i$ , is  $p \cdot (N - 1)$ . Note that with this definition it may happen that some of the nearest regular neighbors of a node  $x_0$ , can now also be linked to  $x_0$  by a random edge. This plays however no role in the global analysis and could of course be avoided but makes the numerical analysis more compact.

The above  $p$  should be compared with the probability,  $\phi$ , occurring in [12] or [13]. The latter one is referred to the existing number of regular (non-random) edges, that is  $k \cdot N$ , or  $N$  for  $k = 1$ . The reason for this derives from the original model in which existing edges were randomly rewired. Thus, for  $k = 1$ ,  $\phi$  leads to an expected number of random edges in the graph equal to  $\phi \cdot N$  instead of

$p \cdot N(N - 1)/2$  in our model ( $N$  large). The two probabilities are hence related by

$$p = 2\phi/(N - 1) \quad (1)$$

if we refer them to the same global expected number of shortcuts in the superimposed random graph.

We are in particular interested in the *small world* effect. What is usually studied is the *mean distance*,  $L(G)$ , between two arbitrarily selected nodes,  $x_i$  and  $x_j$ . Note that graphs are discrete *metric spaces* in a natural way, the distance  $d(x_i, x_j)$  being given by the length of a shortest path, connecting them (number of consecutive edges). If the individual realisations of graphs or networks belong to a sample (probability) space, an averaging has to be performed both over the selected pairs of nodes and the sample space (cf. [12] or [10]).

This quantity is closely related to another important graph characteristic, the (*expected*) *diameter*, which we will study in the following. Choosing an arbitrary start node,  $x_0$ , the graph metric allows to define *l-neighborhoods*,  $U_l(x_0)$ , with

$$U_l(x_0) = \{x_i, d(x_0, x_i) \leq l\} \quad (2)$$

and their respective *boundaries*, defined by

$$\Gamma_l(x_0) = \{x_i, d(x_0, x_i) = l\} \quad (3)$$

With  $|\Gamma_l(x_0)|$  denoting the number of nodes lying in  $\Gamma_l(x_0)$ , the sequence of this values is called the *distance degree sequence* relative to node  $x_0$  and is denoted by  $dds(x_0)$ . When tabulating this for the full node set we arrive at the *distance distribution*  $dd(G) = \{D_1, D_2, \dots\}$  with  $D_l$  the number of pairs of nodes having distance equal to  $l$  ([17] or [10]). We have the following formula for the mean distance:

$$L(G) = M^{-1} \cdot \sum_{l=1}^D l \cdot D_l \quad (4)$$

with  $M = N(N - 1)/2$  being the number of different pairs of nodes. The number  $D = D(G)$ , that is, the maximal distance occurring in this counting is called the *diameter* of the graph.

Evidently,  $L(G)$  and  $D(G)$  cannot be expected to be the same numerically but in the generic situation one may surmise that they are closely related and scale in the same way for, say,  $N \rightarrow \infty$  (being motivated by the qualitative picture of *spreading* in a random graph). While the precise analytic calculation of the degree sequence  $dd(G)$ , the mean distance or the diameter is a quite ambitious task in the random graph framework (see for example [15]), the qualitative behavior can be inferred as follows.

If the edge probability,  $p$ , is sufficiently low, a randomly selected node,  $x_0$ , has on average  $p \cdot N$  neighbors and roughly  $p^2 N^2$  second neighbors and so on as long as the number of vertices being reached is not too large compared to the total number  $N$ . If this latter condition does no longer hold, the probability increases that one meets a given vertex twice. Hence, due to this overcounting the true numbers are systematically smaller, the deviations becoming appreciable when  $N$  is approached.

In the sequel we therefore employ the following strategy. Instead of calculating the exact distance degree sequence or the exact diameter of our small world model, we calculate, among other things, the number of steps necessary to reach the fraction  $\alpha \cdot N$  of nodes with  $\alpha$  preferably chosen as  $1/2$ . In this way we hope to avoid the problems of overcounting at least to a large degree, while, on the other hand, we expect the scaling behavior of the respective quantities to be more or less the same as for the true numbers.

### 3 The Derivation of the Evolution Equations

As we remarked at the end of the preceding section, we want to estimate the expected number of steps necessary to reach, for example, half of the number of vertices, starting from a fixed but arbitrary vertex,  $x_0$ . We expect that this quantity displays the same  $N$ - and  $p$ -dependence as the mean distance or the diameter of the network or graph under discussion, avoiding at the same time the problem of overcounting or, on the other hand, of very complex equations when approaching the total number of vertices,  $N$ .

As has been done in [13], we choose the following two variables.

**Definition 3.1**  $f(n)$  denotes the expected number of nodes, not reached after  $n$  steps, starting from an arbitrary but fixed node,  $x_0$  (“free nodes”).  $g(n)$  denotes the number of gaps, that is, the number of (connected) segments of nodes, lying on the original  $\mathbb{Z}_N$ , not yet reached and which are separated by the segments of nodes already reached after  $n$  steps (cf. figure 1).

We note that our evolution equations describe the evolution of mean- or expected values. In some respects this approach hence shares some characteristics with what one calls mean-field theory in statistical mechanics (cf. also [13]). However, we think, the approximations being made by us are not so drastic as in typical mean-field models, where, among other things, Hamiltonians are typically strongly modified (frequently almost linearized). This is not the case in the small-world model which, in particular in our approach, contains strongly non-linear terms which encode at least part of the fluctuation content in integrated form (see, for example, the discussion about the inclusion of terms incorporating the effects of very small gaps around eqn (10)). In a sense, what we call “full equations” in the following rather describe the behavior of a “typical” or generic small-world graph. So it does not come as a terrible surprise that the evolution equations for the expected values are in relatively good agreement with what follows from real numerical simulations of the model.

On the other hand, statistical fluctuations and correlation are not really treated by us while this could be done in principle as the underlying probability space is explicitly given, that is, the regular graph  $\mathbb{Z}_N$  superimposed with a random graph for which probability theory is well established. One therefore may make the slightly vague statement that the small-world model is, depending on the degree of approximations, of an intermediate character.

To arrive at equations which are not only asymptotically correct or are only good in a restricted region of parameter or phase space, we try to include as many relevant terms as possible (under the proviso that the resulting coupled and non-linear difference or differential equations are still solvable). We start with the difference equation, describing the expected change of the number of gaps between consecutive steps.

For  $n = 0$  we have exactly one gap, comprising all the nodes except  $x_0$ , that is we have  $g(0) = 1$ . The number of gaps increases only due to the consecutive inclusion of shortcuts with increasing  $n$ , connecting pairs of nodes in a random manner and being parametrized by the edge probability  $p$ . The main contribution in consecutive steps,  $n \rightarrow (n + 1)$ , comes from the term  $+2pg(n)f(n)$ . We will further explain it after the introduction of equation (6). There exists however another contribution which acts in the opposite direction and which becomes relevant when already many gaps do exist. This term reads  $-2g(n)^2/f(n)$  and is of a purely combinatorial (more involved) character to be explained below when discussing equation (6).

The initial condition for  $f$  reads  $f(0) = N - 1$ . For  $k = 1$  each gap of free nodes shrinks by two in the next step provided the gap comprises more than one node.

Neglecting in a first step this latter possibility, the first contribution is hence of the form  $-2g(n)$ . Then there is a contribution coming from new shortcuts of the form  $-2pg(n)f(n)$ . The overcounting in the first term (neglection of one-node gaps) has now to be compensated by a term  $+g(n)^2/f(n)$ . The emergence of this and the corresponding term in equation (5) will be explained in greater detail below.

**Observation 3.2**

$$g(n + 1) - g(n) = 2pg(n)f(n) - 2g(n)^2/f(n) \tag{5}$$

$$f(n + 1) - f(n) = -2g(n) - 2pg(n)f(n) + g(n)^2/f(n) \tag{6}$$

We furthermore have the following apriori bounds which immediately follow from the meaning of the respective variables in our model system:

**Lemma 3.3** We have  $g(n) \leq N/2$  and  $g(n) \leq f(n)$ .

Proof: Each gap is followed by a non-empty string of nodes being already covered, hence the first inequality. The second one follows as each gap contains at least one node. □

The occurrence of the term  $2pg(n)f(n)$  can be understood as follows. In each step,  $n \rightarrow (n + 1)$ , the two endpoints of each of the  $g(n)$  gaps may become the source of new shortcuts to the remaining  $f(n)$  free nodes, the expected number being  $pf(n)$ . This leads hence to a term of the above form in both equations. One can even be a little bit more precise if one wants to. New gaps are not created if the shortcuts end at free nodes which are adjacent to nodes already reached. There are on average  $2g(n)$  of them. That is, in the equation describing the evolution of gaps the correct term is  $2pg(n)(f(n) - 2g(n))$ . The equation describing the evolution of  $f(n)$  is not altered. This additional correction term is always negative and we could in principle incorporate it in the following. It will make the whole numerics slightly nastier without making a big effect. So we will largely neglect this term but will incorporate it into what we call the "full equations", see (18).

The other quadratic nonlinear terms are slightly more intricate and of a more stochastic nature. While in equation (6) gaps containing only one node will contribute only one instead of two nodes in the difference equation, in equation (5) gaps vanish in the next step if at level  $n$  they contain only one or two nodes. The probability for the existence of such gaps will now be calculated. We begin with the case of one-gaps. We will solve this problem with the help of the well-known partitioning problem of a given set into disjoint subsets. We associate the set of  $g$  gaps and  $f$  free nodes with  $f$  balls to be distributed over  $g$  boxes. In general there exist

$$\binom{f + (g - 1)}{g - 1} = \binom{f + (g - 1)}{f} \tag{7}$$

combinations (see [21] or [22]). In effect we calculate the number of different words of length  $f + (g + 1)$  consisting of  $(g + 1)$  bars and  $f$  dots, under the proviso that each word begins and ends with a bar and that each consecutive pair of bars is divided by a non-empty string of dots as in our case no box can be empty. This implies that we can place exactly one ball in each box and perform the above calculation, (7), which represents the number of partions without the constraint of non-empty boxes, for the remaining number of  $(f - g)$  balls, yielding

$$A := \binom{f - 1}{g - 1} \tag{8}$$

configurations. This is the cardinality of the set of elementary events in our probability space.

To calculate the expected number of gaps containing only one or two nodes, we introduce the following random variables,  $Y_1, Y_2$  over the probability space of words, we associated with the random graphs  $G_j$ :

$$Y_i(G_j) := \#\{\text{gaps of length } i\} \quad (9)$$

in each of the above  $A$  configurations (graphs),  $G_j$ . Before we proceed a short remark as to the probabilities of the individual configurations should be in order.

In our model probability space a regular graph is superimposed by a random graph with edge probability  $p$ . In our above calculation we deal with fixed numbers  $f$  and  $g$ . As the gaps arise due to the existence of random edges, the gaps are expected to be randomly scattered over the regular graph  $\mathbb{Z}_N$  in basically the same way as pairs of nodes are linked by random edges, that is, almost independently. This then should also essentially hold for the number of gaps, met after  $n$  steps. Furthermore, this reasoning should not be affected in a serious way by the possible annihilation of gaps for large step-number  $n$ , as long as we stay away from the regime where the *spreading argument* for random graphs is no longer correct. From this we see that it is a reasonable strategy to remain below the value  $N/2$  with the step number  $n$ . We hence conclude that in the indicated regime each configuration should have the independent probability  $A^{-1}$ .

We therefore have

$$E(Y_i) = A^{-1} \cdot \sum_j Y_i(G_j) \quad (10)$$

For the one-gaps we can represent  $Y_1$  by more elementary random variables,  $y_k$  with  $k$  running from 1 to  $g$ , enumerating the existing  $g$  gaps and  $y_k = 1$  if gap ( $k$ ) contains only one element and zero else. This yields

$$Y_1(G_j) = \sum_k y_k(G_j) \quad (11)$$

and

$$E(Y_1) = \sum_k E(y_k) = g \cdot \binom{f-2}{g-2} / \binom{f-1}{g-1} \quad (12)$$

where  $\binom{f-2}{g-2}$  is the number of configurations with only one element in gap ( $k$ ). Correspondingly we get for the expected number of 2-gaps:

$$E(Y_2) = g \cdot \binom{f-3}{g-2} / \binom{f-1}{g-1} \quad (13)$$

For the one-gaps this yields

$$E(Y_1) = g \cdot g/f \quad (14)$$

For the two-gaps we have

$$E(Y_2) = g \cdot \frac{(g-1) \cdot (f-g)}{(f-1) \cdot (f-2)} \quad (15)$$

and for  $f$  large, i.e.  $f \approx (f-1) \approx (f-2)$ :

$$E(Y_2) \approx g \cdot (g/f - 1/f - g^2/f^2 + g/f^2) \quad (16)$$

With our apriori estimate  $g \leq f$  and as we start from the initial conditions  $f(0) = (N-1) \approx N$ ,  $g(0) = 1$  in most of our phase space the dominant contribution comes from the term  $g \cdot g/f$  also in the case of two-gaps.

**Conclusion 3.4** The expected number of one- or two-gaps is approximately

$$E(\#(1\text{- or }2\text{-gaps})) \approx 2g^2/f \quad (17)$$

(Note that in our probability space the possibility of being a 1-gap or a 2-gap is mutually exclusive). This result explains the occurrence of the correction terms in our evolution equations. Without these approximations, the equations (5) and (6) would read

$$g(n+1) - g(n) = 2pg(f-2g) - \frac{g \cdot (g-1)}{f-1} - \frac{g \cdot (g-1) \cdot (f-g)}{(f-1) \cdot (f-2)} \quad (18)$$

$$f(n+1) - f(n) = -2g - 2pgf + \frac{g \cdot (g-1)}{f-1} \quad (19)$$

with  $f$  and  $g$  instead of  $f(n)$  and  $g(n)$  on the rhs. We'll refer to these as "full equations" and compare them with the simplified ones ((5) and (6)) in figure 2. Note in particular that we approximated the last term occurring in 18 by  $-g^2/f$  in 5, neglecting the positive higher order contribution, being essentially of the form  $+g^3/f^2$ .

A brief comment is in order as to the corresponding formulas derived in [13] (cf. their formulas (3),(6) or (A10),(A11)). We decided to neglect all terms in (16) except the leading one,  $g^2/f$ , which is reasonable in our view. In [13], in the corresponding equation an additional term of the type  $g/f$  occurs (derived by a different argument). On the other hand, more important in our view is equation (6), which comprises three terms in our approach (including a nasty non-linear one,  $g^2/f$ ), while in [13] only the first one,  $-2g$ , occurs on the rhs. This makes the corresponding equations of course much easier to solve but may only be a good approximation in a restricted regime of parameter space (see the brief discussion at the end of section 5). We discuss and compare the numerical results in section 6. One can see that the solution of [13] is similar to our *lower bound*-equation for  $f$  (eqn (45)), which is reasonable as in our lower bound for  $f$  the quadratic term is largely suppressed (cf. the following section and figure 3).

## 4 Solution Strategies

The above system of evolution equations contains non-linearly coupled quantities and can be solved only in very exceptional lucky circumstances. Instead of making more or less uncontrollable approximations, we develop the following strategy. We try to enclose the above exact equations by *comparison equations* bounding the exact solutions,  $f(n)$ ,  $g(n)$ , from above and below, the corresponding variables being denoted by

$$\overline{f}(n), \underline{f}(n), \overline{g}(n), \underline{g}(n) \quad (20)$$

The problem is that, on the one hand, the comparison equations have to be so chosen that they can be rigorously solved and, on the other hand, these bounds have to be quite good so that we are able to infer something relevant also for the enclosed exact equations in particular for the scaling limit  $N$  very large and  $p$  a vanishing function of  $N$ . A central role will be played by the value  $n_*$  for which we have reached on average  $N/2$  of the nodes when starting from an arbitrary but fixed node  $x_0$  (or more generally  $\alpha N$  nodes with  $0 < \alpha < 1$ ). In other words, the range of  $n$ -values we are using is restricted by

$$n \in [0, n_*] \quad \text{so that} \quad (N-1) = f(0) \geq f(n) \geq \alpha N \quad (21)$$

We now study the rhs of the equations (5), (6). For one, we assume that these equations have been solved for our initial conditions  $g(0) = 1$ ,  $f(0) = N-1$ , so

that  $g(n)$ ,  $f(n)$  now represent particular functions of  $n$ . On the other hand we can regard the rhs (dropping the dependence on the variable  $n$ ) as functions on the phase space, spanned by the possible values of the variables  $g$ ,  $f$ . In our assumed range of possible parameters and variables we have the estimate

$$2pgN - 2g^2/N \geq 2pgf - 2g^2/f \geq 2pg \cdot (\alpha N) - 2g^2/(\alpha N) \quad (22)$$

The idea is now to introduce the comparison difference equations

$$\bar{g}(n+1) - \bar{g}(n) := 2pN\bar{g}(n) - 2\bar{g}(n)^2/N \quad (23)$$

and

$$\underline{g}(n+1) - \underline{g}(n) := 2p(\alpha N)\underline{g}(n) - 2\underline{g}(n)^2/(\alpha N) \quad (24)$$

with the initial conditions

$$g(0) = \bar{g}(0) = \underline{g}(0) = 1 \quad (25)$$

For the initial differences we have

$$\bar{g}(1) - \bar{g}(0) > g(1) - g(0) > \underline{g}(1) - \underline{g}(0) \quad (26)$$

Our strategy is now to use these comparison equations to learn something about the true equations. Unfortunately matters are not so transparent for difference equations as compared to differential equations. The reason is that they are only given at discrete points and may (therefore) display a more complex behavior (see for example Hoelder's theorem and extensions thereof in [18],p.283 or [19],p.220). Due to these problems we will in the following go over to the corresponding differential equations, being however aware of the fact that it does not seem to be an easy task to provide good error estimates, in particular as the differences in our context are not infinitesimal (as to this interesting question of principle cf. the discussion in [23]). What is furthermore remarkable is the observation (see below) that we can prove a useful theorem in the case of differential equations the analogue of which, as far as we can see, we cannot prove for difference equations (at least with the same methods).

The corresponding differential equations read:

$$g'(x) = 2pg(x)f(x) - 2g(x)^2/f(x) \quad (27)$$

$$f'(x) = -2g(x) - 2pg(x)f(x) + g(x)^2/f(x) \quad (28)$$

with  $g(0) = 1$ ,  $f(0) = (N-1)$ ,  $x \in [0, x_*]$  so that  $(N-1) \geq f(x) \geq \alpha N$ . The comparison differential equations with respect to  $g(x)$  are

$$\bar{g}'(x) = 2pN\bar{g}(x) - 2\bar{g}(x)^2/N \quad (29)$$

$$\underline{g}'(x) = 2p(\alpha N)\underline{g}(x) - 2\underline{g}(x)^2/(\alpha N) \quad (30)$$

with  $\bar{g}(0) = \underline{g}(0) = 1$ .

As to equation (28) we proceed as follows (we note that we in fact experimented with different possibilities; the one we are presenting below seems to be the most appropriate one). We mentioned above the apriori bound  $g \leq f$ . For the rhs of equation (28) we then have:

$$-2g - 2pgf + g^2/f \leq -2g - 2pgf + g = -g - 2pgf \quad (31)$$

On the other hand we have also

$$-2g - 2pgf + g^2/f > -2g - 2pgf \quad (32)$$

Therefore our comparison differential equations for  $f$  are

$$\overline{f}'(x) = -\underline{g}(x) - 2p\underline{g}(x)\overline{f}(x) \quad (33)$$

$$\underline{f}'(x) = -2\overline{g}(x) - 2p\overline{g}(x)\underline{f}(x) \quad (34)$$

with  $\overline{f}(0) = N$ ,  $\underline{f}(0) = N - 2$ .

Remark: As  $N$  is supposed to be very large, it does not make a big difference for the numerical calculations to let all the initial values be equal to  $N$ . The above choice makes the analytic argument a little bit simpler (see below).

To further exploit these comparison equations we proceed as follows. Note that we have succeeded in decoupling the equations for  $f$  and  $g$ . We can solve the differential equations for  $\overline{g}$  and  $\underline{g}$  and plug the solutions into the  $\overline{f}$ - and  $\underline{f}$ -equation. We assume that together with the comparison differential equations the original differential equations for  $g$  and  $f$  have been solved for the mentioned initial conditions. In the differential equation for  $g$  with initial condition  $g(0) = 1$  we can then regard the corresponding  $f$ -solution as an external function with  $g(x)$  solving this “new” differential equation (together with the given initial condition). We compare the solution  $g(x)$  of this latter equation with the solutions of the differential equations for  $\overline{g}$ ,  $\underline{g}$  respectively. We have

$$\overline{g}(0) = \underline{g}(0) = g(0) = 1 \quad \text{and} \quad \underline{g}'(0) < g'(0) < \overline{g}'(0) \quad (35)$$

We now prove the following result.

**Proposition 4.1** Let  $y_1(x)$  and  $y_2(x)$  be solutions of the two differential equations

$$y_1'(x) = F_1(y_1, x), \quad y_2'(x) = F_2(y_2, x) \quad (36)$$

on the interval  $[0, x_*]$  with  $y_1(0) \leq y_2(0)$ . Let  $F_1, F_2$  fulfil

$$F_1(y, x) < F_2(y, x) \quad (37)$$

on the domain  $[0, x_*] \times I_y$ ,  $I_y$  a suitable  $y$ -interval and with both  $y_1(x), y_2(x)$  staying in this domain. Then

$$y_1(x) \leq y_2(x) \quad \text{on} \quad [0, x_*] \quad (38)$$

Proof: From the assumptions it follows that  $y_1(x) < y_2(x)$  in some open interval  $(0, \varepsilon)$ . If  $y_1(x) > y_2(x)$  for some  $x$ , there exists an  $r > 0$  (by continuity) with  $y_1(r) = y_2(r)$  and  $y_1(x) > y_2(x)$  in an open interval  $(r, r + \varepsilon')$ . But this is a contradiction since

$$F_1(y_1(r), r) < F_2(y_1(r), r) \quad (39)$$

hence again implying that  $y_1(x) < y_2(x)$  in an open interval  $(r, r + \varepsilon'')$ . We conclude that  $y_1(x) \leq y_2(x)$  on  $[0, x_*]$ .  $\square$

It sometimes happens that we have  $F_1(y, x) < F_2(y, x)$  on the open interval  $(0, x_*)$  but  $F_1(y(0), 0) = F_2(y(0), 0)$  for some value  $y(0)$ . We then can prove the following corollary:

**Corollary 4.2** Making the same assumptions as before except for  $F_1(y(0), 0) = F_2(y(0), 0)$  instead of  $F_1(y(0), 0) < F_2(y(0), 0)$ . We assume that there exists a parameter  $\lambda$  so that on the closed interval we have

$$F_1(y, x; \lambda) < F_2(y, x) \quad (40)$$

for  $\lambda > 0$  and  $F_1(y, x; 0) = F_1(y, x)$ , the dependence on  $\lambda$  being continuous or differentiable. Then the parameter dependent solutions converge pointwise towards the solutions for  $\lambda = 0$  (cf. [20]). For  $\lambda > 0$  our above result applies. So, by continuity it applies also in the limit  $\lambda \rightarrow 0$ .

Note that in our case the parameter  $\lambda$  is the parameter taking the values  $N, N - 1$  etc.

Remark: We surmise that such comparison results are known in the large literature about differential equations but we were unable to find a reference.

**Conclusion 4.3** What we have now shown is that under the assumptions being made,  $\bar{g}$ ,  $\underline{g}$  and  $\bar{f}$ ,  $\underline{f}$  are upper and lower bounds of the corresponding solutions  $g$ ,  $f$  of the original differential equations.

## 5 The Quantitative Results

With our  $\alpha$  being now either 1 or  $\frac{1}{2}$  we can express both the upper and lower bound by a single equation:

$$\tilde{g}'(x) = 2\alpha p N \tilde{g}(x) - \frac{2\tilde{g}(x)^2}{\alpha N} \quad (41)$$

with  $\tilde{g} = \bar{g}$  for  $\alpha = 1$  and  $\tilde{g} = \underline{g}$  for  $\alpha = \frac{1}{2}$ . This nonlinear differential equation (of Bernoulli type) can be transformed into a linear one with the help of the transformation  $z := \tilde{g}^{-1}$  and yields with the initial condition  $\tilde{g}(0) = 1$  the solution

$$\tilde{g}(x) = \left( \left( 1 - \frac{1}{p\alpha^2 N^2} \right) e^{-2p\alpha N x} + \frac{1}{p\alpha^2 N^2} \right)^{-1} \quad (42)$$

Inserting these solutions into the corresponding upper and lower bound equations for  $f(x)$

$$\bar{f}'(x) = -2p \underline{g}(x) \bar{f}(x) - \underline{g}(x) \quad (43)$$

$$\underline{f}'(x) = -2p \bar{g}(x) \underline{f}(x) - 2\bar{g}(x) \quad (44)$$

and introducing the parameter  $\beta \in \{1, 2\}$  so that  $\tilde{f}' = -2p\tilde{g}\tilde{f} - \beta\tilde{g}$  we obtain (after a simple variable transformation,  $\tilde{f} \rightarrow \tilde{f} + 1/2p$ ,  $\tilde{f} \rightarrow \tilde{f} + 1/p$ , and separation of variables) the following result, using  $f(0) = N$  as initial condition instead of  $N - 1$ :

$$\tilde{f}(x) = \frac{2pN + \beta}{2p} \left( 1 - \frac{1}{p\alpha^2 N^2} + \frac{1}{p\alpha^2 N^2} e^{2p\alpha N x} \right)^{-p\alpha N} - \frac{\beta}{2p} \quad (45)$$

We are interested in the value  $\tilde{x}_*$  of  $x$  for which  $\tilde{f} = \frac{1}{2}N$ . Solving for  $\tilde{x}_*$  yields

$$\tilde{x}_* = \frac{1}{2\alpha p N} \ln \left( \left[ \left( \frac{2pN + \beta}{pN + \beta} \right)^{\frac{1}{p\alpha N}} - 1 \right] p\alpha^2 N^2 + 1 \right) \quad (46)$$

the lower bound being assumed for  $\alpha = 1, \beta = 2$  the upper bound for  $\alpha = \frac{1}{2}$  and  $\beta = 1$ .

Remark: The argument of the logarithm is always larger than one, as  $((2pN + \beta)/(pN + \beta))^{1/p\alpha N} > 1$  and  $p\alpha^2 N^2 > 0$ . We furthermore want to stress the important point that a too poor approximation of, say, the upper bound,  $\bar{f}(n)$ , may easily lead to a function which does not decay sufficiently. In that case our estimates would have been useless. We see however that our above choice is strong enough.

We now want to investigate the scaling regime  $N \rightarrow \infty$ ,  $p = cN^{-1-\epsilon}$  with  $\epsilon \in [0, 1]$ ,  $c > 0$ , with the boundary cases  $p \propto N^{-1}$  and  $p \propto N^{-2}$  being particularly

interesting. Inserting these choices into the preceding equation we get

$$\tilde{x}_* = \frac{N^\epsilon}{2\alpha c} \ln \left( \left[ \left( \frac{2cN^{-\epsilon} + \beta}{cN^{-\epsilon} + \beta} \right)^{\frac{N^\epsilon}{c\alpha}} - 1 \right] c\alpha^2 N^{1-\epsilon} + 1 \right) \quad (47)$$

For  $\epsilon = 0$  we have

$$\tilde{x}_* = \frac{1}{2\alpha c} \ln N + C_1(\alpha, \beta, c) \quad (48)$$

with  $C_1$  being of the precise form:

$$C_1 = (2\alpha c)^{-1} \cdot \ln \left( c\alpha^2 \left[ \left( \frac{2c + \beta}{c + \beta} \right)^{1/c\alpha} - 1 \right] + 1/N \right) \quad (49)$$

which becomes asymptotically independent of  $N$  for large  $N$ .

For  $\epsilon \neq 0$  we have (developing the logarithm up to the first order)

$$\left( \frac{2cN^{-\epsilon} + \beta}{cN^{-\epsilon} + \beta} \right)^{\frac{N^\epsilon}{c\alpha}} = \exp \frac{N^\epsilon}{c\alpha} \left[ \ln \left( \frac{2c}{\beta N^\epsilon} + 1 \right) - \ln \left( \frac{c}{\beta N^\epsilon} + 1 \right) \right] \quad (50)$$

$$= \exp \frac{N^\epsilon}{c\alpha} \left[ \frac{c}{\beta N^\epsilon} + O(N^{-2\epsilon}) \right] \quad (51)$$

$$= \exp \left( \frac{1}{\beta\alpha} + O(N^{-\epsilon}) \right) \quad (52)$$

and get

$$\tilde{x}_* = \frac{N^\epsilon \left( (1-\epsilon) \ln N + \ln \left( [\exp(1/\alpha\beta) - 1] c\alpha^2 + 1/N^{1-\epsilon} \right) \right)}{2\alpha c} \quad (53)$$

which obviously also describes the boundary cases  $\epsilon = 0$  and  $\epsilon = 1$  (provided we would include the neglected term  $O(N^{-\epsilon})$  which is now  $O(1)$ ). This behavior is valid both for the upper and lower bound of  $f$ . As the  $x_*$  for the true  $f$  has to lie between the respective values for the upper and lower bound we infer that it has the same scaling behavior for  $N \rightarrow \infty$ .

**Conclusion 5.1** We infer that between  $p \propto N^{-1}$  and  $p \propto N^{-2}$  there exists a broad transition zone with the scaling of the diameter or mean distance exactly interpolating between these two boundary cases,  $x_* \approx \ln N$  and  $x_* \approx N$  (up to now we have only studied the scaling of  $x_*$ , a parameter which is of course closely related to the above mentioned graph characteristics; see below).

In our above calculations we dealt with the expected value,  $x_*$ , at which the expected number of free nodes drops to the value  $N/2$ . We argued above that the corresponding (exact) formulas for the value, at which this number assumes the value zero, would be much more complicated. To make nevertheless a statement about this value we apply the following (plausibility) argument (which, however, should not be viewed as a rigorous proof). Put differently, we will provide an argument which is expected to hold only for *expectation values* or *typical* nodes. Let  $X$  be an arbitrary initial vertex and  $X'$  a vertex with largest possible distance to  $X$  in a given realisation of the network. The expected radius of the  $N/2$ -neighborhoods for both vertices is equal to  $x_*$ . In case the corresponding  $x_*$ -neighborhoods  $U_*(X)$  and  $U_*(X')$  of  $X$  and  $X'$  are not disjoint the distance between  $X$  and  $X'$  must be less than  $2x_*$ . If these neighborhoods are disjoint the graph  $G$  is a disjoint union of  $U_*(X)$  and  $U_*(X')$  and hence the distance between  $X$  and  $X'$  must be exactly  $2x_*$ .

As all these arguments apply only to the generic case, we can associate this value with the *expected diameter* of our network, denoted by  $D$  and get the estimate

$$x_* \leq D \leq 2x_* \quad (54)$$

Remark: We again emphasize that this argument is only correct in an averaged sense in which all nodes are assumed to stand on the same footing. It is of course easy to design particular graphs where this estimate does not hold. Take for example a graph having a densely entangled neighborhood around some node  $x$  from which a long one-dimensional string emanates. In this case the diameter is of course much larger than  $2x_*$ . We think one could prove something rigorous at this place, which, on the other hand, may be a little bit tedious and unnecessarily blow up the paper.

On the other hand we infer from equation (4) that the average distance,  $L$  fulfils

$$L \leq D \quad (55)$$

Taking again a typical node,  $X$ , (so that its neighborhood  $U_*$  has approximately  $N/2$  members), we can approximate the mean distance  $L$  by  $(N-1)^{-1} \cdot \sum l \cdot |\Gamma_l|$  and get the estimate

$$L \geq (N-1)^{-1} \cdot \left( \sum_1^{x_*-1} l \cdot |\Gamma_l| + N/2 \cdot x_* \right) \geq x_*/2 \quad (56)$$

as

$$\sum_{l \geq x_*} l \cdot |\Gamma_l| \geq x_* \cdot \sum_{l \geq x_*} |\Gamma_l| \geq x_* \cdot N/2 \quad (57)$$

We thus get

$$x_*/2 \leq L \leq D \leq 2x_* \quad (58)$$

We can now insert the respective parameters,  $\alpha$  and  $\beta$  in our expressions for  $\tilde{x}_*$ , thus yielding

$$x_*^{\alpha=1, \beta=2}/2 \leq x_*/2 \leq L \leq D \leq 2x_* \leq 2x_*^{\alpha=\frac{1}{2}, \beta=1} \quad (59)$$

With our numerical expressions for the upper and lower bounds we get

$$\frac{1}{4pN} \ln \left( \left[ \left( \frac{2pN+2}{pN+2} \right)^{\frac{1}{pN}} - 1 \right] pN^2 + 1 \right) \leq L \leq \frac{2}{pN} \ln \left( \left[ \left( \frac{2pN+1}{pN+1} \right)^{\frac{2}{pN}} - 1 \right] \frac{pN^2}{4} + 1 \right) \quad (60)$$

For very small ( $\epsilon > 1$  such that  $pN^2 \rightarrow 0$ ) or vanishing  $p$  the inequality reduces to  $0.16N \leq L \leq 3.19N$ , with  $L = 0.25N$  for the true  $L$  in the case  $p = 0$ . For  $p = cN^{-1-\epsilon}$ ,  $c$  of order one and  $\epsilon \in [0, 1]$ , and a reasonable number of shortcuts ( $pN^2 > 1$ ),  $L$  displays a behavior already exemplified for  $x_*$ :

$$L \approx \frac{C_2 N^\epsilon}{c} ((1-\epsilon) \ln N + \ln c + C_1(c)) \quad (61)$$

with  $C_1$  constant for  $\epsilon \neq 0$ . For very strongly connected graphs ( $\epsilon \in [-1, 0)$ ), these bounds become invalid, as  $L$  tends to one and the differential equations will no longer approximate the difference equations well enough.

We want to come back to the question of the importance of the non-linear quadratic terms in our evolution equations (5) and (6). One may be led to the wrong conclusion that they are always marginal because initially  $g$  is very small compared to the huge  $f$ . But one should note that  $g$  grows very fast for certain choices of the parameter  $p$ . To get some feeling we take, for example,  $\bar{g}$  and ask for what values of  $x \bar{g}^2/N$  is of order  $\bar{g}$ .

This is the case if  $\bar{g} \approx N$ . The result strongly depends on the value of  $p$ . Inserting  $p = c/N$  in the equation for  $\bar{g}$  and solving for  $x$  we get  $x \approx (2c)^{-1} \cdot \ln N$ . Hence the quadratic contribution becomes appreciable when  $x$  approaches the regime where  $f$  drops to  $N/2$ , that is, the regime we are interested in. Put differently, it is dangerous to neglect this term on apriori grounds.

On the other hand, taking for example  $p = 1/N^2$ , and making the same calculation we infer that the non-linear term remains negligible in the domain we are interested in. For  $\epsilon > 0$  we get of course intermediate results.

The effects of neglecting the even smaller combinatorial terms which appear in the full eqns (18) and (19) can be seen in figure 2: Here the neglect has a greater effect for the case  $p = 1/N^2$  than for  $p = 1/N$ , as the total number  $g$  of gaps in the first case is smaller and therefore nearer to 1 than in the second case. Nevertheless, these discrepancies are still negligible.

## 6 Comparison with former results

Barthélemy and Amaral, and later also Newman and Watts ([11],[12]) conjectured a scaling behavior for  $L$  of the form

$$L = N \cdot F(pN^2) \quad (62)$$

with some universal function  $F$  with  $F(y) \rightarrow \frac{1}{4}$  for  $y \rightarrow 0$  and  $F(y) \rightarrow C \ln(y)/y$  for  $y \rightarrow \infty$ . Our bounds do not scale exactly in this way, but at least do so approximately for large  $N$  and  $\epsilon > 0$ . In this regime the  $pN$ -dependent term in the logarithm tends to the constant  $\exp(1/\alpha\beta)$  and our bounds assume the form

$$L = N \cdot F(pN^2) \quad \text{with} \quad \frac{1}{4y} \ln \left( \left[ e^{1/2} - 1 \right] y + 1 \right) \leq F(y) \leq \frac{2}{y} \ln \left( \frac{e^2 - 1}{4} y + 1 \right) \quad (63)$$

On the other hand, for large  $pN$  and  $\epsilon < 0$  or  $\epsilon = 0$  with large  $c$ , this scaling behavior breaks down and our estimate for the average distance scales like  $(\ln N)/pN$ . This estimate makes however only sense for  $\ln N > pN$ , as  $L \geq 1$ . Thus the case  $\epsilon < 0$  isn't described correctly by this formula. For  $\epsilon = 0$ ,  $L$ , according to this scaling-ansatz, simply scales like  $(\ln N)/c$ , without any correction term of the form  $(\ln c)/c$ , which occurs in our above presumably more exact result. So, although  $L$  correctly scales with  $\ln N$  in both cases (depicting a random graph), there exist certain deviations for large  $c$ .

In [13], Newman, Moore and Watts found the following expression for their universal scaling function

$$F_{NMW}(y) = \frac{1}{2\sqrt{y^2 + 2y}} \tanh^{-1} \sqrt{\frac{y}{y+2}}. \quad (64)$$

For  $p \propto N^{-2}$  ( $\epsilon = 1$ ) this function is a constant. On the other hand, for  $p = cN^{-1-\epsilon}$  with  $0 \leq \epsilon < 1$ , the argument  $y = pN^2$  is large for large  $N$  and

$$L = N F_{NMW}(pN^2) \approx \frac{\ln p N^2 + \ln 2}{4 p N} \approx \frac{N^\epsilon ((1 - \epsilon) \ln N + \ln c + \ln 2)}{4 c} \quad (65)$$

In figure 3  $F_{NMW}$  and the scaling functions deriving from our bounds are shown, indicating that the NMW-ansatz complies with them. Note however that this is only valid for  $\epsilon > 0$  or  $\epsilon = 0$  with a not too large  $c$  in  $p = c/p$ .

In [14], Barbour and Reinert made a rigorous analysis of the probability distribution for the distance function, getting the following result: Let  $X$  and  $X'$  be

some randomly chosen vertices on  $G$ ,  $\rho = pN$  and  $S := pN^2$  (to be identified with  $L\rho$  in [14]) then

$$P \left[ d(X, X') > \frac{\ln S}{2\rho} + \frac{x}{\rho} \right] = \int_0^\infty \frac{e^{-y} dy}{1 + y e^{2x}} + O \left( \frac{e^x (1 + e^{2x}) \ln^2 S}{\sqrt{S}} \right) \quad (66)$$

for all  $-\frac{1}{2} \ln S \leq x < \frac{1}{4} \ln S$ .  $L$  is the mean distance resulting from this distribution for  $d(X, X')$ . To make things simpler, we instead treat the median of it, which can be easily approximated by the special choice  $x = 0$ . Neglecting the error term we obtain

$$P \left[ d(X, X') > \frac{\ln S}{2\rho} \right] \approx 0,596 \quad (67)$$

stating  $L \approx L_{\text{median}} \approx \ln(pN^2)/2pN$  or, with  $p = cN^{-1-\epsilon}$ ,

$$L \approx \frac{N^\epsilon ((1 - \epsilon) \ln N + \ln c)}{2c}. \quad (68)$$

The corresponding universal scaling function reads  $F_{BR} = (\ln y)/2y$ , and is also depicted in figure 3. For  $pN^2 > 2$  (at least one expected shortcut) this lies within our bounds. Below this, the error term of the probability  $P$  rises above one and  $F_{BR}$  loses its meaning.

## 7 Dimension of the Small World

In [24], two related dimensional concepts (of a fractal type) were introduced for infinite graphs (note the close connection to the *distance degree sequence*, discussed in [10]) and a number of its properties proved. We learned later that this concept occurred already earlier in the literature but, as far as we can see, its interesting properties were never systematically studied (see for example [25]). A technically different but physically related concept was exploited by Dhar ([26]), see also [27]. Furthermore Ising models on such irregular spaces were studied recently, an early source being [28]. The reason to deal with infinite graphs is that only in the limit  $N \rightarrow \infty$  the global notion of dimension becomes independent of local (model dependent) aspects like e.g. coordination numbers of, say, lattices, all having the same embedding dimension. One of the two definitions reads:

**Definition 7.1** Let  $G$  be an arbitrary graph with  $N$  vertices and  $U_l(x)$  the  $l$ -neighborhood of the vertex  $x \in G$ . Then we define the *dimension* of  $G$  (relative to  $x$ ) as

$$\dim_x(G) := \lim_{l \rightarrow \infty} \frac{\ln \#U_l(x)}{\ln l} \quad (69)$$

(provided the limit exists; in general we have to deal with  $\liminf$  and  $\limsup$ ).

In [24] it was shown, that this notion of dimension (also called the “internal scaling dimension”) is independent of the initial vertex  $x$  (under a mild technical assumption). For finite (but large) and connected graphs we can instead employ the following graph characteristic:

$$\dim_{\text{approx}}(G) = \ln N / \ln \text{diam}(G) \quad (70)$$

For real networks this value has been analysed in [29]. There one can see, that the approximate dimension might tend to underestimate other notions of dimension, because  $\ln \#U_l(x)$  saturates for large  $x$  (an effect which is however pretty obvious as the spreading argument does of course no longer hold in that regime). On

the other hand, in our case we expect  $\ln \#U_i(x)$  not to saturate before reaching  $N/2 =: \#U_i(x_*)$  (cf. figure 2). As  $x_* \leq \text{diam}G \leq 2x_*$  (see however the discussion after eqn(54)), we get

$$\frac{\ln N - \ln 2}{\ln x_* + \ln 2} \leq \dim_{\text{approx}}(G) \leq \frac{\ln N}{\ln x_*}. \quad (71)$$

Thus, for large enough  $N$  and  $x_*$  the approximate dimension doesn't deviate too much when staying below  $N/2$  and won't suffer from saturating-effects.

Applying this concept to the Small World Model for large  $N$  we get

$$\dim_{\text{approx}}(G) = \frac{\ln N}{\ln \text{diam}(G)} = \frac{\ln N}{\ln C_1 N^\epsilon (C_2 + \ln N^{1-\epsilon})} \quad (72)$$

which is  $\approx \frac{1}{\epsilon}$  for  $\epsilon > 0$ , the constants  $C_1$  and  $C_2$  being independent of  $N$  (depending only on  $c, \alpha$  and  $\beta$ ). For  $\epsilon = 1$  ( $p \sim N^{-2}$ ) we get the value one, which is reasonable for this rarefied linear case. For the opposite case,  $\epsilon = 0$  ( $p \sim N^{-1}$ ), we have

$$\dim_{\text{approx}}(G) \approx \ln N / \ln \ln N \quad (73)$$

which diverges for  $N \rightarrow \infty$ .

In [12], Newman et.al. introduced a renormalization process for the Small World Model. This process divides the graph into segments of length 2 and interprets these segments again as vertices in a new Small World Model with size  $N' = N/2$  and edge-probability<sup>1</sup>  $p' = 4p$ . With  $p = cN^{-1-\epsilon}$  this substitution yields  $p' = c'N'^{-1-\epsilon}$  with  $c' = 2^{1-\epsilon}c$ . Hence our  $\dim_{\text{approx}} = 1/\epsilon$  is constant under this renormalization. A similar phenomenon was observed in the renormalization process for infinite graphs (with globally bounded node degree) introduced in [9].

## 8 Conclusion

We found the mean distance  $L$  of a Small World Model with  $N \gg 1$  nodes and edge-probability  $p = cN^{-1-\epsilon}$  to be bounded from above and below by two expressions of the form  $C_2 N^\epsilon (\ln N^{1-\epsilon} + C_1)$  (the constants depending on whether the upper or lower bound is taken). This implies a broad transition zone, in which the mean distance drops from a linear growth to a logarithmic one, permitting each power law  $L \sim N^\epsilon$  for  $\epsilon \in (0, 1)$ . Furthermore,  $1/\epsilon$  can be regarded as an approximative dimension of the corresponding graph. Our results partly corroborate earlier work but lead also to certain numerical deviations.

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<sup>1</sup>Note that our notation differs from that in [12].

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# Figures

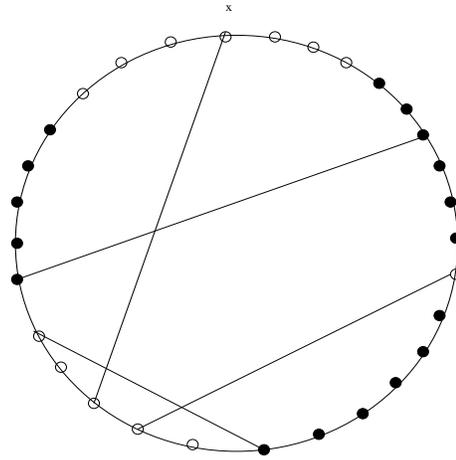


Figure 1:

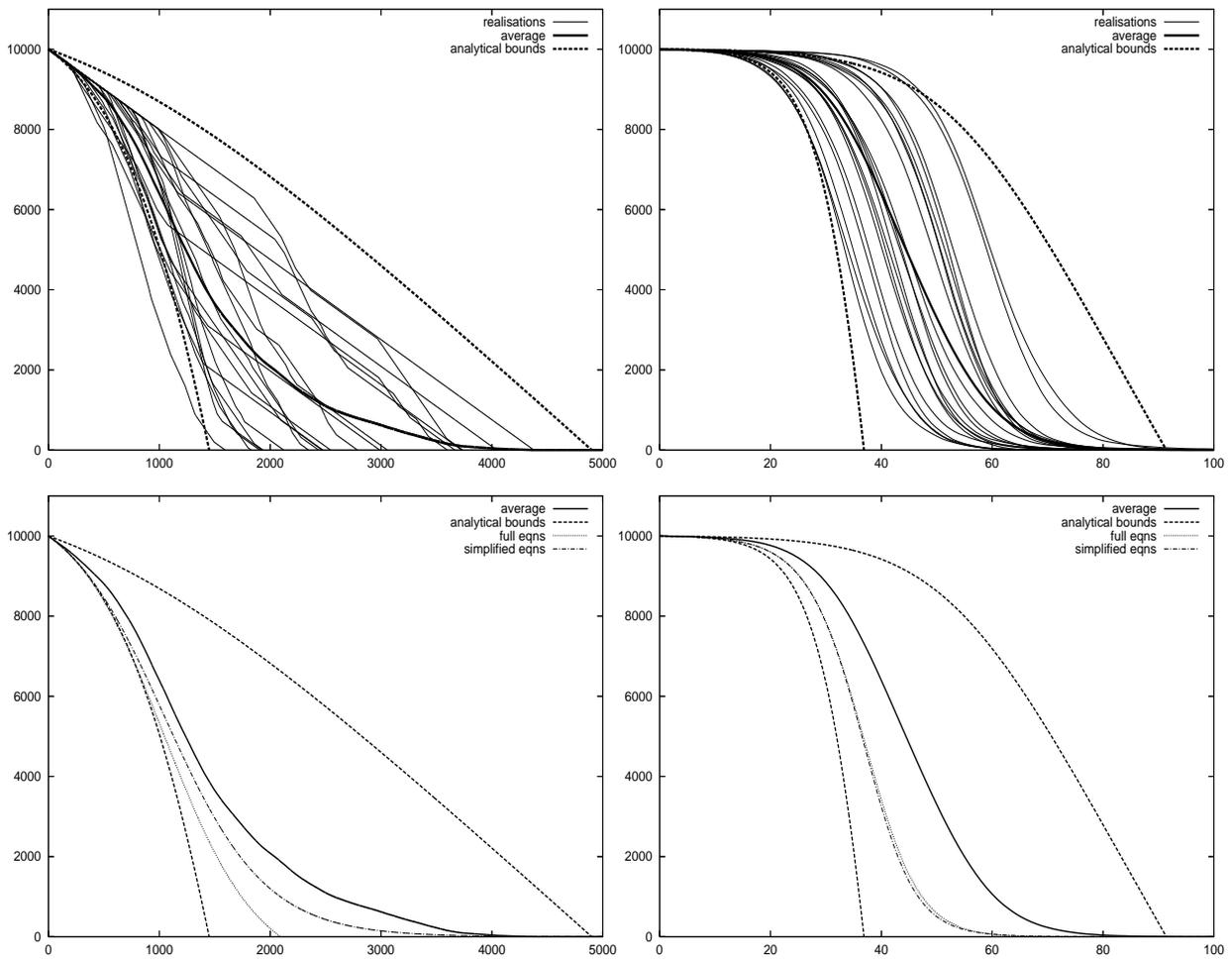


Figure 2:

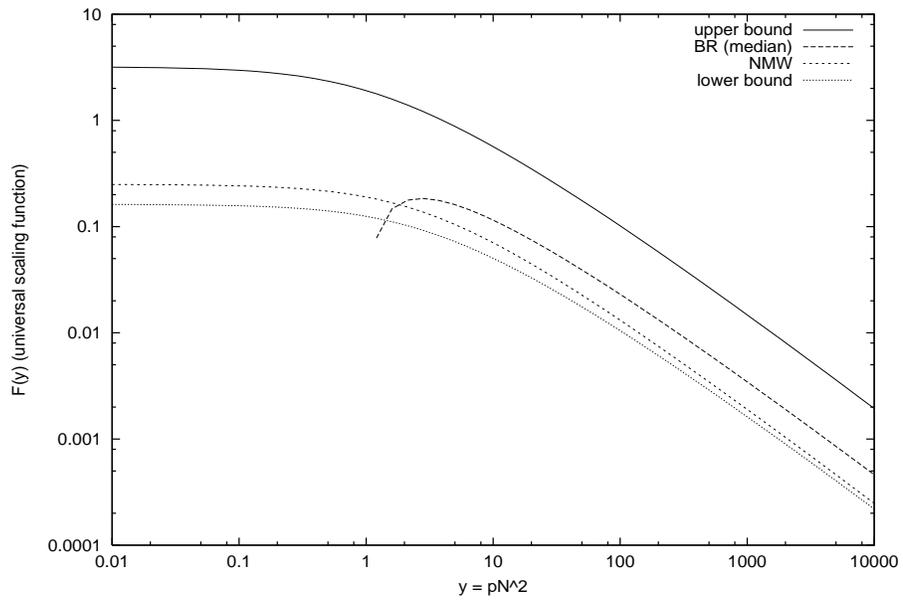


Figure 3:

## Figure Captions

Caption Figure 1: The smallworld model for  $k = 1$ . The number of nodes is  $N = 30$ . In this particular realisation we have inserted four additional shortcuts. The unfilled nodes are the vertices which can be reached by  $\leq 3$  steps starting from node  $x$ , the step-number will be denoted by  $n$  in the following. The black nodes are the vertices not reached after three steps, their cardinality being denoted by  $f(n)$  (free nodes). This set consists of three connected subsets which are separated by the subsets of nodes already reached. The number of segments of free nodes is denoted by  $g(n)$  (gaps).

Caption Figure 2: The function  $f(n)$  for two examples. Left side:  $N = 10^4, p = 10^{-7}, \epsilon = \frac{3}{4}$ ; right side:  $N = 10^4, p = 10^{-5}, \epsilon = \frac{1}{4}$  (with  $p$  normalized to  $N-(1+\epsilon)$ ). The upper diagrams show 20 realisations, their mean value (averaged over fixed  $n$ ) and the analytical bounds of (45). The lower diagrams show these bounds and the averaged curve again, together with the numerical solutions of the difference eqns (18,19) and (5, 6). In both cases the averaged curve exceeds the solution of the difference eqns (notice however the perhaps surprisingly large standard deviation), but doesn't top the upper analytical bounds. The solutions of the full eqns (18, 19) only differ notable from the simplified ones (5, 6) in the left, nearly linear case, the neglect of the additional combinatorial terms was hence justified. Yet, what can't be seen in these diagrams is that, in contrast to the full ones, the solution of the simplified eqns does not sink to zero, and even crosses the upper boundary for large  $n$ . This is however neither surprising nor important, as our boundary eqns are only guaranteed to hold in the interval  $f(n) \in [N/2, N]$ , thus we omit an additional logarithmical diagram.

Caption Figure 3: Universal scaling functions