*American Journal of Applied Mathematics and Statistics, 2016, Vol. 4, No. 3, 94-98*
Available online at http://pubs.sciepub.com/ajams/4/3/5

DOI:10.12691/ajams-4-3-5

# Transmuted Laplace Distribution: Properties and Applications

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**Abstract** New parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models and this could be done in various ways. In this article, we generalize the Laplace distribution using the quadratic rank transmutation map studied by Shaw et al. (2007) to develop a transmuted Laplace distribution (TLD). We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. To show that the TLD distribution can be a better model than one based on the LD distribution we use a real data set of number of million revolutions before failure for each of the 23 ball bearings in the life tests and The usefulness of the transmuted Laplace distribution for modeling reliability data is illustrated.

***Keywords:*** *Laplace distribution, maximum likelihood estimation, moments, order statistics, likelihood ratio test*

**Cite This Article:** Dina H. Abdel Hady, and Rania, M. Shalaby, "Transmuted Laplace Distribution: Properties and Applications." *American Journal of Applied Mathematics and Statistics*, vol. 4, no. 3 (2016): 94-98. doi: 10.12691/ajams-4-3-5.

## 1. Introduction

This template, created in MS Word 2003 and saved as "Word 2003 – doc" for the PC, provides authors with most of the formatting specifications needed for preparing electronic versions of their papers. All standard paper components have been specified for three reasons: 1) ease of use when formatting individual papers, 2) automatic compliance to electronic requirements that facilitate the concurrent or later production of electronic products, and 3) Margins, column widths, line spacing, and type styles are built-in; examples of the type styles are provided throughout this document. Some components, such as multi-leveled equations, graphics, and tables are not prescribed, although the various table text styles are provided. The formatter will need to create these components, incorporating the applicable criteria that follow. Use the styles, fonts and point sizes as defined in this template, but do not change or redefine them in any way as this will lead to unpredictable results. You will not need to remember shortcut keys. Just a mouse-click at one of the menu options will give you the style that you want. In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

The Laplace distribution is named after Pierre-Simon Laplace (1749-1827), who obtained the likelihood of the Laplace distribution is maximized when the location parameter is set to be the median. The Laplace distribution is also known as the law of the difference between two exponential random variables.

In this article we present a new generalization of Laplace distribution called the transmuted Laplace distribution.

**Definition** 1: A random variable X is said to have transmuted distribution if its cumulative distribution function (cdf) is given by

$$F(x)=(1+\lambda)G(x)-\lambda\left[G(x)\right]^2, |\lambda|\leq 1 \qquad (1)$$

where G(x) is the cdf of the base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable.

Many transmuted distributions are proposed. A new generalization of Weibull distribution called the transmuted Weibull distribution [4]. [13] proposed and studied the various structural properties of the transmuted Rayleigh distribution. [11] introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by [2]. [16] introduce transmuted Pareto distribution. Transmuted Generalized Linear Exponential Distribution introduced by [9] among other. Aryal et al. (2009) studied the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. In the present study we will provide mathematical formulation of the transmuted Laplace distribution and some of its properties.

## 2. Transmuted Laplace Distribution

The Laplace distribution, also called the double exponential distribution, is the differences between two independent variates with identical exponential distributions [1].

**Definition 2:** A random variable X is said to have the Laplace distribution with parameter β if its probability density is defined as:

$$g(x)=\frac{1}{2\beta}\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\beta>0. \qquad (2)$$

The corresponding cumulative distribution function (c.d.f.) isc:

$$G(x)=\frac{1}{2}\left\{\mathrm{sgn}(x)\left[1-Exp\left(-\frac{1}{\beta}|x|\right)\right]+1\right\} \qquad (3)$$

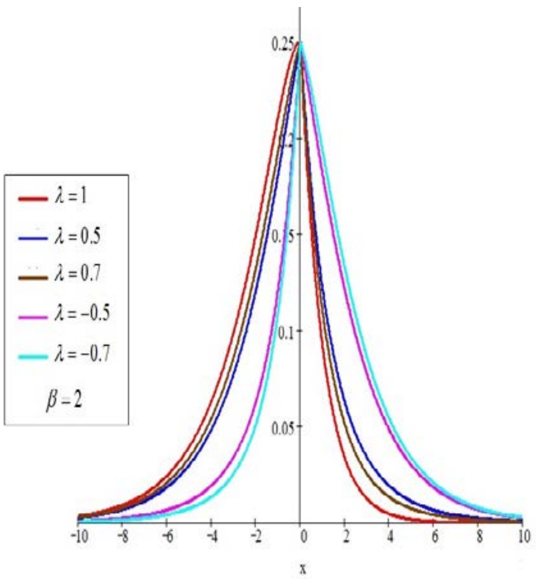
Now using (1) and (3), we have the cdf of a transmuted Laplace distribution

$$\begin{aligned}F(x)&=\frac{1}{2}\left\{\mathrm{sgn}(x)\left[1-\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\right]+1\right\}\\&\times\left\{(1+\lambda)-\frac{\lambda}{2}\left\{\mathrm{sgn}(x)\left[1-\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\right]+1\right\}\right\}.\end{aligned} \qquad (4)$$

Hence, the pdf of transmuted Laplace distribution with parameter $\lambda$ is:

$$\begin{aligned}f(x)&=\frac{1}{2\beta}\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\\&\left\{1+\lambda\, sgn(x)\left[\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)-1\right]\right\}.\end{aligned} \qquad (5)$$

Note that the transmuted Laplace distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions. The Laplace distribution is clearly a special case for $\lambda=0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted Laplace distribution for selected values of the parameters $\lambda$ and $\beta = 2$.




Using (4) and (5), the hazard rate function of transmuted Laplace distribution is:

$$h(X)=\frac{f(x)}{1-F(X)}$$

$$=\frac{\left\{\begin{array}{l}\frac{1}{2\beta}\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\\ \left\{1+\lambda\, sgn(x)\left[\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)-1\right]\right\}\end{array}\right\}}{\left\{\begin{array}{l}\left[1-\frac{1}{2}\left\{\mathrm{sgn}(x)\left[1-\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\right]+1\right\}\right]\\ \left\{(1+\lambda)-\frac{\lambda}{2}\left\{\mathrm{sgn}(x)\left[1-\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\right]+1\right\}\right\}\end{array}\right\}}.$$

## 3. Moments

Now let us consider the different moments of the transmuted Laplace distribution.

Suppose X denote the transmuted Laplace distribution random variable with parameter $\lambda$ and $\beta$, then :

$$E\left(X^r\right)=\int_{-\infty}^{\infty}X^r f(x)\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\left\{\begin{array}{l}X^r\frac{1}{2\beta}\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)\\ \left\{1+\lambda\, sgn(x)\left[\mathrm{ExpExp}\left(-\frac{1}{\beta}|x|\right)-1\right]\right\}\end{array}\right\}\mathrm{d}x$$

$$=\int_{-\infty}^{0}x^r\frac{1}{2\beta}e^{\frac{x}{\beta}}\left\{1+\lambda-\lambda e^{\frac{x}{\beta}}\right\}\mathrm{d}x$$

$$+\int_{0}^{\infty}x^r\frac{1}{2\beta}e^{-\frac{x}{\beta}}\left\{1-\lambda+\lambda e^{-\frac{x}{\beta}}\right\}\mathrm{d}x.$$

After simplification, the $r^{th}$ moment of (TLD) is:

$$E\left(X^r\right)=\Gamma(r+1)\left\{\begin{array}{l}\frac{(-\beta)^r(1+\lambda)}{2}-\frac{\beta^r(1-\lambda)}{2}\\ -\frac{(-\beta)^r\lambda}{2^{r+2}}+\frac{\beta^r\lambda}{2^{r+2}}\end{array}\right\}. \qquad (6)$$

Therefore putting $r=1$, we obtain the mean as

$$E(x)=\frac{-3}{4}\lambda\,\beta. \qquad (7)$$

And putting $r=2$ we obtain the second moment as

$$E\left(x^2\right)=2\beta^2. \qquad (8)$$

Then the variance of (TLD) is

$$V(X)=E\left(X^2\right)-\{E(X)\}^2=\beta^2\left\{2-\frac{9\lambda^2}{16}\right\}.$$

The moment generating function of (TLD) can readily obtained as:

$$\mathrm{M_X}(\mathrm{t}) = \mathrm{E}\left(\mathrm{e}^{\mathrm{tx}}\right) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}} f(\mathrm{x})\mathrm{dx}$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} \mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)$$

$$\left\{1+\lambda\ sgn(x)\left[\mathrm{Exp}\left(-\frac{1}{\beta}|x|\right)-1\right]\right\}\mathrm{dx}$$

$$= \int_{-\infty}^{0} e^{tx} \frac{1}{2\beta} e^{\frac{x}{\beta}} \left\{1+\lambda-\lambda e^{\frac{x}{\beta}}\right\}\mathrm{d}x$$

$$+\int_{0}^{\infty} e^{tx} \frac{1}{2\beta} e^{-\frac{x}{\beta}} \left\{1-\lambda+\lambda e^{-\frac{x}{\beta}}\right\}\mathrm{d}x.$$

After simplification, the moment generating function of (TLD) is:

$$M_X(t) = \frac{1-\lambda\beta t}{1-\beta^2 t^2} + \frac{\lambda\beta t}{4-\beta^2 t^2}. \tag{9}$$

Note That

$$M_X(0) = 1.$$

Mean and variance of (TLD) can be found by using (6).

$$E(X) = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \frac{-3}{4}\lambda\ \beta,$$

$$E\left(X^2\right) = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = 2\beta^2.$$

These results are the same results previously obtained in (7) and (8), and can also reach to the same value of the variance as the previous mentioned, can also find skewness and kurtosis by the calculation of moments with degrees higher than the second degree easily.

## 4. Order Statistics

In statistics, the k$^{th}$ order statistic of a statistical sample is equal to its k$^{th}$ smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max((X_1, X_2, \ldots, X_n).$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$Range((X_1, X_2, \ldots, X_n) = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F(x)$ and pdf $f(x)$ then the pdf of $X_{(j)}$ is given by

$$F_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}$$

$$= \frac{n!}{(j-1)!(n-j)!} f(x)\sum_{i=1}^{n-j}\binom{n-j}{i}(-1)^{n-j-i}\left[F(x)\right]^{n-i-1}$$

For $j = 1, 2, \ldots, n$. The pdf of the j$^{th}$ order statistic for (TLD) is given by

**a) At $x < 0$**

$$F_{X_{(j)}}(x) = \frac{n!}{\beta(j-1)!(n-j)!}\left(1+\lambda-\lambda e^{\frac{x}{\beta}}\right)$$
$$\times\sum_{i=1}^{n-j}\binom{n-j}{i}(-1)^{n-j-i}\left(\frac{1}{2}\right)^{n-i} e^{\frac{(n-i)x}{\beta}}\left[1+\lambda-\frac{\lambda}{2}e^{\frac{x}{\beta}}\right]^{n-i-1}. \tag{10}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$F_{X_{(n)}}(x) = \frac{n}{2\beta}\left(1+\lambda-\lambda e^{\frac{x}{\beta}}\right)e^{\frac{x}{\beta}}\left[1+\lambda-\frac{\lambda}{2}e^{\frac{x}{\beta}}\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(n)}$ is given by

$$F_{X_{(j)}}(x) = \frac{n}{\beta}\left(1+\lambda-\lambda e^{\frac{x}{\beta}}\right)$$
$$\times\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-i-1}\left(\frac{1}{2}\right)^{n-i} e^{\frac{(n-i)x}{\beta}}\left[1+\lambda-\frac{\lambda}{2}e^{\frac{x}{\beta}}\right]^{n-i-1}.$$

**b) At x>0**

$$F_{X_{(j)}}(x) = \frac{n!}{2\beta(j-1)!(n-j)!}\left(1-\lambda+\lambda e^{\frac{-x}{\beta}}\right)e^{\frac{-x}{\beta}}$$
$$\times\sum_{i=1}^{n-j}\binom{n-j}{i}(-1)^{n-j-i}\left[1-\frac{(1-\lambda)}{2}e^{\frac{-x}{\beta}}-\frac{\lambda}{4}e^{\frac{-x}{\beta}}\right]^{n-i-1}.$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by:

$$F_{X_{(n)}}(x) = \frac{n}{2\beta}\left(1-\lambda+\lambda e^{\frac{-x}{\beta}}\right)e^{\frac{-x}{\beta}}$$
$$\left[1-\frac{(1-\lambda)}{2}e^{\frac{-x}{\beta}}-\frac{\lambda}{4}e^{\frac{-x}{\beta}}\right]^{n-1}.$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$F_{X_{(1)}}(x) = \frac{n}{2\beta}\left(1-\lambda+\lambda e^{\frac{-x}{\beta}}\right)e^{\frac{-x}{\beta}}$$
$$\times\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-i-1}\left[1-\frac{(1-\lambda)}{2}e^{\frac{-x}{\beta}}-\frac{\lambda}{4}e^{\frac{-x}{\beta}}\right]^{n-i-1}.$$

## 5. Maximum Likelihood Estimators

In this section we discuss the maximum likelihood estimators (MLE's) and inference for the TLD ($\beta, \lambda$) distribution. Let $x_1, \ldots, x_n$ be a random sample of size n from TLD ($\beta, \lambda$) then the likelihood function can be written as

$$L(\theta) = \prod_{i=1}^{n_1}\left[\frac{1}{2\beta}e^{\frac{x}{\beta}}\left\{1+\lambda-\lambda e^{\frac{x}{\beta}}\right\}\right] \prod_{i=1}^{n_2}\left[\frac{1}{2\beta}e^{-\frac{x}{\beta}}\left\{1-\lambda+\lambda e^{-\frac{x}{\beta}}\right\}\right] \quad (11)$$

Where $n_1$ is number of the negative observations and $n_2$ is number of the positive observations. By accumulation taking logarithm of equation (11), and the log- likelihood function can be written as

$$l(\theta) = -n\ln(2\beta) - \frac{1}{\beta}\sum_{i=1}^{n}|x| + \sum_{i=1}^{n_1}\ln(1+\lambda-\lambda e^{\frac{x}{\beta}}) + \sum_{i=1}^{n_2}\ln(1-\lambda+\lambda e^{-\frac{x}{\beta}}) \quad (12)$$

Differentiating equation (12) with respect to β and λ then equating it to zero. The normal equations become

$$\frac{\partial l(\theta)}{\partial\beta} = \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}|x| + \sum_{i=1}^{n_1}\left(\frac{\frac{\lambda x}{\beta^2}e^{\frac{x}{\beta}}}{1+\lambda-\lambda e^{\frac{x}{\beta}}}\right) + \sum_{i=1}^{n_2}\left(\frac{\frac{\lambda x}{\beta^2}e^{\frac{-x}{\beta}}}{1-\lambda+\lambda e^{\frac{-x}{\beta}}}\right) = 0 \quad (13)$$

$$\frac{\partial l(\theta)}{\partial\lambda} = \sum_{i=1}^{n_1}\left(\frac{1-e^{\frac{x}{\beta}}}{1+\lambda-\lambda e^{\frac{x}{\beta}}}\right) + \sum_{i=1}^{n_2}\left(\frac{e^{\frac{-x}{\beta}}-1}{1-\lambda+\lambda e^{\frac{-x}{\beta}}}\right) = 0 \quad (14)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above nonlinear system of equations (13, 14) to zero and solve them simultaneously. These solutions will yield the ML estimators for $\hat{\beta}$ and $\hat{\lambda}$. For the two parameters transmuted Laplace distribution TLD ($\beta, \lambda$) pdf, all the second order derivatives exist.

Under certain regularity conditions, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, I^{-1}(\theta))$ (here $\xrightarrow{d}$ stands for convergence in distribution), where $I(\theta)$ denotes the information matrix given by

$$(\theta) = E\left(\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta}\right).$$

This information matrix $I(\theta)$ may be approximated by the observed information matrix

$$I(\hat{\theta}) = \left.\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta}\right|_{\theta=\hat{\theta}}.$$

Then, using the approximation $\sqrt{n}(\hat{\theta}-\theta) \sim N\left(0, I^{-1}(\hat{\theta})\right)$ one can carry out tests and find confidence regions for functions of some or all parameters in $\theta$.

Approximate two sided 100(1 − α) % confidence intervals for $\beta$ and λ are, respectively, given by

$$\hat{\beta} \pm z_{\alpha/2}\sqrt{I_{11}^{-1}(\theta)}$$

and

$$\hat{\lambda} \pm z_{\alpha/2}\sqrt{I_{22}^{-1}(\theta)}$$

where $z_\alpha$ is the upper $\alpha^{th}$ quantile of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing on some transmuted LD sub-models. For example, we can use the LR test statistic to check whether the TLD distribution for a given data set is statistically superior to the LD distribution. In any case, hypothesis tests of the type $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ can be performed using a LR test. In this case, the LR test statistic for testing $H_0$ versus $H_1$ is $\omega = 2(\ell(\hat{\theta}; x) - \ell(\hat{\theta}_0; x))$, where $\hat{\theta}$ and $\hat{\theta}_0$ are the MLEs under $H_1$ and $H_0$, respectively. The statistic $\omega$ is asymptotically (as n → ∞) distributed as $\chi_k^2$ where k is the length of the parameter vector θ of interest. The LR test rejects $H_0$ if $\omega > \chi_{k,\alpha}^2$ where $\chi_{k,\alpha}^2$ denotes the upper $100\alpha\%$ quantile of the $\chi_k^2$ distribution.

## 6. Applications

In this section, we use a real data set to show that the TLD distribution can be a better model than one based on the LD distribution. The data set given in Table 1 taken from Lawless(1986) page 228. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and they are:

**Table 1. The number of million revolutions before failure for each of the 23 ball**

| |
|---|
| 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, |
| 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, |
| 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40 |

We will use these data minus the overall average for the experiment; this average was 68 to fit the data with both Laplace distribution (LD) and transmuted Laplace (TLD).

**Table 2. Estimated parameters of the Laplace and transmuted Laplace distributions for the data**

| Model | Parameter Estimate | Standard Error | $-l(.;x)$ |
|---|---|---|---|
| Laplace | $\hat{\beta} = 28.306$ | 0.28 | 115.79 |
| Transmuted Laplace | $\hat{\beta} = 28.099$<br>$\hat{\lambda} = 0.025$ | 0.36<br>0.112 | 183.046 |

The variance covariance matrix of the MLEs under the transmuted Laplace distribution is computed as

$$I^{-1}\left(\hat{\theta}\right)=\begin{bmatrix} 0.4 & -0.009 \\ -0.009 & 0.013 \end{bmatrix}.$$

Thus, the variances of the MLE of $\beta$ and λ are $var(\hat{\beta})$ = 0.41 and $var(\hat{\lambda})$ = 0.0132,. Therefore, 95% confidence intervals for $\beta$ and λ are [26.895, 29.339], and [−0.198, 0.248] respectively.

The LR test statistic to test the hypotheses $\mathrm{H}_0: \lambda = 0$ versus $\mathrm{H}_1 : \lambda \neq 0$ is $\omega = 134.426 > 3.841 = \chi^2_{1,0.05}$, so we reject the null hypothesis.

# 7. Conclusion

Here we propose a new model, the so-called the transmuted Laplace distribution which extends the Laplace distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood; also the information matrix is derived. An application of TLD distribution to real data shows that the new distribution can be used quite effectively to provide better fits than LD distribution.

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