

Twistor theory and differential equations

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, UK

E-mail: m.dunajski@damtp.cam.ac.uk

Received 31 January 2009, in final form 17 March 2009

Published 16 September 2009

Online at stacks.iop.org/JPhysA/42/404004

Abstract

This is an elementary and self-contained review of twistor theory as a geometric tool for solving nonlinear differential equations. Solutions to soliton equations such as KdV, Tzitzeica, integrable chiral model, BPS monopole or Sine–Gordon arise from holomorphic vector bundles over $T\mathbb{C}P^1$. A different framework is provided for the dispersionless analogues of soliton equations, such as dispersionless KP or $SU(\infty)$ Toda system in 2+1 dimensions. Their solutions correspond to deformations of (parts of) $T\mathbb{C}P^1$, and ultimately to Einstein–Weyl curved geometries generalizing the flat Minkowski space. A number of exercises are included and the necessary facts about vector bundles over the Riemann sphere are summarized in the appendix.

PACS number: 02.30.Ik

(Some figures in this article are in colour only in the electronic version)

1. Introduction

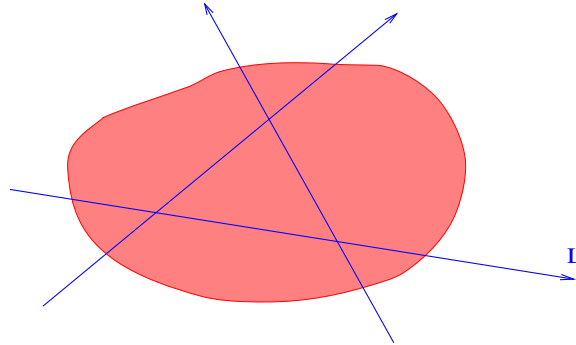
Twistor theory was created by Penrose [19] in 1967. The original motivation was to unify general relativity and quantum mechanics in a non-local theory based on complex numbers. The application of twistor theory to differential equations and integrability has been an unexpected spin off from the twistor programme. It has been developed over the last 30 years by the Oxford school of Penrose and Atiyah with the crucial early input from Ward [24, 25] and Hitchin [10, 11] and further contributions from Lionel Mason, George Sparling, Paul Tod, Nick Woodhouse and others.

The twistor approach to integrability is the subject of the monograph [18] as well as the forthcoming book [6]. This short paper is supposed to give a self-contained introduction to the subject. The approach will be elementary—explicit calculations will be used in place of (often very elegant) abstract geometric constructions. Filling in the gaps in these calculations should be within the reach of a first year research student.

1.1. Motivation-integral geometry

Twistor theory is based on projective geometry and as such has its roots in the 19th century Klein correspondence. It can also be traced back to other areas of mathematics. One such area is the subject now known as integral geometry (the relationship between twistor theory and integral geometry has been explored by Gindikin [8]).

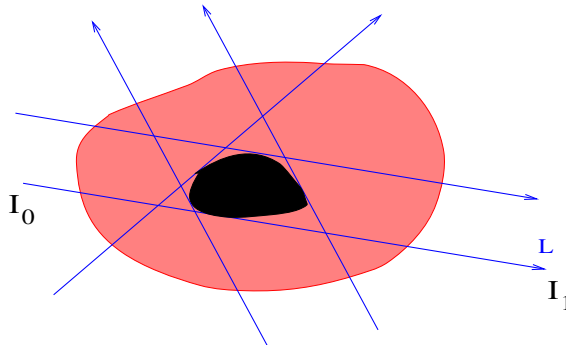
Radon transform. Integral geometry goes back to Radon [23] who considered the following problem: let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function with suitable decay conditions at ∞ (for example a function of compact support as shown below)



and let $L \subset \mathbb{R}^2$ be an oriented line. Define a function on the space of oriented lines in \mathbb{R}^2 by

$$\phi(L) := \int_L f. \tag{1.1}$$

Radon has demonstrated that there exists an inversion formula $\phi \rightarrow f$. Radon's construction can be generalized in many ways and it will become clear that Penrose's twistor theory is its far reaching generalization. Before moving on, it is however worth remarking that an extension of Radon's work has led to Nobel Prize awarded (in medicine) for pure mathematical research! It was given in 1979 to Cormack [2], who unaware of Radon's results had rediscovered the inversion formula for (1.1), and had explored the set-up allowing the function f to be defined on a non-simply connected region in \mathbb{R}^2 with a convex boundary. If one only allows the lines which do not pass through the black region



or are tangent to the boundary of this region, the original function f may still be reconstructed from its integrals along such lines (this is called the *support theorem* (see [9] for details)). In the application to computer tomography, one takes a number of 2D planar sections of 3D objects and relates the function f to the (unknown) density of these objects. The input data given to a radiologist consist of the intensity of the incoming and outgoing x-rays passing

through the object with intensities I_0 and I_1 respectively

$$\phi(L) = \int_L \frac{dI}{I} = \log I_1 - \log I_0 = - \int_L f,$$

where $dI/I = -f(s) ds$ is the relative infinitesimal intensity loss inside the body on an interval of length ds .

The Radon transform then allows to recover f from this data, and the generalization provided by the support theorem becomes important if not all regions in the object (for example patient's heart) can be x-rayed.

John transform. The inversion formula for the Radon transform (1.1) can exist because both \mathbb{R}^2 and the space of oriented lines in \mathbb{R}^2 are two dimensional. Thus, at least naively, one function of two variables can be constructed from another such function (albeit defined on a different space). This symmetry does not hold in higher dimensions, and this underlines the following important result of John [12]. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function (again, subject to some decay conditions which makes the integrals well defined) and let $L \subset \mathbb{R}^3$ be an oriented line. Define $\phi(L) = \int_L f$, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds, \tag{1.2}$$

where (α, β) parametrize the four-dimensional space \mathbb{T} of oriented lines in \mathbb{R}^3 . (Note that this parametrization misses out the lines parallel to the plane $x_3 = \text{const}$. The whole construction can be done invariantly without choosing any parametrization, but here we choose the explicit approach for clarity.) The space of oriented lines is four dimensional, and $4 > 3$ so expect one condition on ϕ . Differentiating under the integral sign yields the ultrahyperbolic wave equation

$$\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0,$$

and John has shown that all smooth solutions to this equation arise from some function on \mathbb{R}^3 . This is a feature of twistor theory: an unconstrained function on twistor space (which in this case is identified with \mathbb{R}^3) yields a solution to a differential equation on spacetime (in this case locally \mathbb{R}^4 with a metric of $(2, 2)$ signature). After the change of coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$ the equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

which may be relevant to physics two times! The integral formula given in the following section corrects the 'wrong' signature to that of the Minkowski space and is a starting point of twistor theory.

Penrose transform. In 1969, Penrose gave a formula for solutions to the wave equation in the Minkowski space [20]:

$$\phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z+t) + (x+iy)\lambda, (x-iy) - (z-t)\lambda, \lambda) d\lambda. \tag{1.3}$$

Here $\Gamma \subset \mathbb{CP}^1$ is a closed contour and the function f is holomorphic on \mathbb{CP}^1 except some number of poles. Differentiating the RHS verifies that

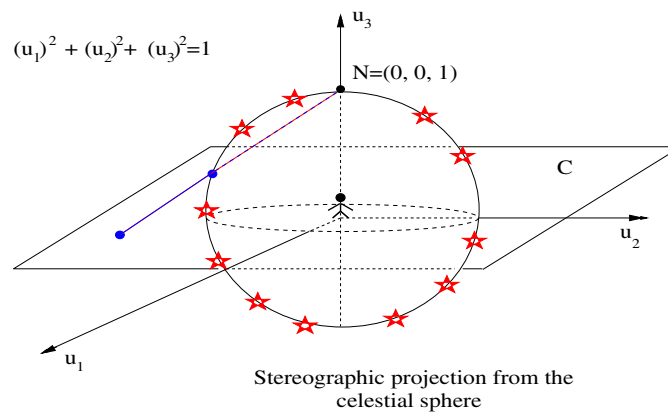
$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Despite the superficial similarities, the Penrose formula is mathematically much more sophisticated than John's formula (1.2). One could modify a contour and add a holomorphic

function inside the contour to f without changing the solution ϕ . The proper description uses sheaf cohomology which considers equivalence classes of functions and contours (see, e.g. [28]).

1.2. Twistor programme

Penrose’s formula (1.3) gives real solutions to the wave equation in the Minkowski space from holomorphic functions of three arguments. According to the twistor philosophy, this appearance of complex numbers should be understood at a fundamental, rather than technical, level. In quantum physics, the complex numbers are regarded as fundamental: the complex wavefunction is an element of a complex Hilbert space. In twistor theory, Penrose aimed to bring classical physics to an equal footing, where the complex numbers play a role from the start. This already takes place in special relativity, where the complex numbers appear on the celestial sphere visible to an observer on a night sky.



The two-dimensional sphere is the simplest example of a non-trivial complex manifold (see the appendix for more details). Stereographic projection from the north pole $(0, 0, 1)$ gives a complex coordinate

$$\lambda = \frac{u_1 + iu_2}{1 - u_3}.$$

Projecting from the south pole $(0, 0, -1)$ gives another coordinate

$$\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}.$$

On the overlap $\tilde{\lambda} = 1/\lambda$. Thus the transition function is holomorphic and this makes S^2 into a complex manifold $\mathbb{C}\mathbb{P}^1$ (the Riemann sphere). The double covering $SL(2, \mathbb{C}) \xrightarrow{2:1} SO(3, 1)$ can be understood in this context. If the worldlines of two observers travelling with relative constant velocity intersect at a point in spacetime, the celestial spheres these observers see are related by a Möbius transformation

$$\lambda \rightarrow \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta},$$

where the unit-determinant matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C})$$

corresponds to the Lorentz transformation relating the two observers.

The celestial sphere is a past light cone of an observer \mathbf{O} which consists of light rays through an event \mathbf{O} at a given moment. In the twistor approach, the light rays are regarded as more fundamental than events in spacetime. The five-dimensional space of light rays \mathcal{PN} in the Minkowski space is a hypersurface in a three-dimensional complex manifold $\mathcal{PT} = \mathbb{CP}^3 - \mathbb{CP}^1$ called the projective twistor space. (**Exercise:** Why is \mathcal{PN} five dimensional? Show that as a real manifold $\mathcal{PN} \cong S^2 \times \mathbb{R}^3$.)

Let $(Z^0, Z^1, Z^2, Z^3) \sim (cZ^0, cZ^1, cZ^2, cZ^3), c \in \mathbb{C}^*$ with $(Z^2, Z^3) \neq (0, 0)$ be homogeneous coordinates of a twistor (a point in \mathcal{PT}). The twistor space and the Minkowski space are linked by the incidence relation

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}, \tag{1.4}$$

where $x^\mu = (t, x, y, z)$ are coordinates of a point in the Minkowski space. (**Exercise:** Show that if two points in the Minkowski space are incident with the same twistor, then they are null separated.) Define the Hermitian inner product

$$\Sigma(Z, \bar{Z}) = Z^0 \bar{Z}^2 + Z^1 \bar{Z}^3 + Z^2 \bar{Z}^0 + Z^3 \bar{Z}^1$$

on the non-projective twistor space $\mathcal{T} = \mathbb{C}^4 - \mathbb{C}^2$. The signature of Σ is $(+ + - -)$ so that the orientation-preserving endomorphisms of \mathcal{T} preserving Σ form a group $SU(2, 2)$. This group has 15 parameters and is locally isomorphic to the conformal group $SO(4, 2)$ of the Minkowski space. We divide the twistor space into three parts depending on whether Σ is positive, negative or zero. This partition descends to the projective twistor space. In particular, the hypersurface

$$\mathcal{PN} = \{[Z] \in \mathcal{PT}, \Sigma(Z, \bar{Z}) = 0\} \subset \mathcal{PT}$$

is preserved by the conformal transformations of the Minkowski space which can be verified directly using (1.4).

Fixing the coordinates x^μ of a spacetime point in (1.4) gives a plane in the non-projective twistor space $\mathbb{C}^4 - \mathbb{C}^2$ or a projective line \mathbb{CP}^1 in \mathcal{PT} . If the coordinates x^μ are real, this line lies in the hypersurface \mathcal{PN} . Conversely, fixing a twistor in \mathcal{PN} gives a light ray in the Minkowski space.

So far only the null twistors (points in \mathcal{PN}) have been relevant in this discussion. General points in \mathcal{PT} can be interpreted in terms of the complexified Minkowski space \mathbb{C}^4 where they correspond to null two-dimensional planes with a self-dual tangent bi-vector. This, again, is a direct consequence of (1.4) where now the coordinates x^μ are complex. There is also an interpretation of non-null twistors in the real Minkowski space, but this is far less obvious [19]: the Hermitian inner product Σ defines a vector space \mathcal{T}^* dual to the non-projective twistor space. The elements of the corresponding projective space \mathcal{PT}^* are called dual twistors. Now take a non-null twistor $Z \in \mathcal{PT}$. Its dual $\bar{Z} \in \mathcal{PT}^*$ corresponds to a projective two plane \mathbb{CP}^2 in \mathcal{PT} . (**Exercise:** Use (1.4) to find an explicit equation for this plane.) A holomorphic two-plane intersects with the hypersurface \mathcal{PN} in a real three-dimensional locus. This locus corresponds to a three-parameter family of light rays in the real Minkowski space. This family representing a single twistor is called the Robinson congruence. A picture of this configuration which appears on the front cover of [22] shows a system of twisted oriented circles in the Euclidean space \mathbb{R}^3 , the point being that any light-ray is represented by a point in \mathbb{R}^3 together with an arrow indicating the direction of the ray's motion. This configuration originally gave rise to a name 'twistor'.

Finally, we can give a twistor interpretation of the contour integral formula (1.3). Consider a function $f = f(Z^0/Z^2, Z^1/Z^2, Z^3/Z^2)$ which is holomorphic on an intersection of two open sets covering \mathcal{PT} (one of these sets is defined by $Z^2 \neq 0$ and the other by $Z^3 \neq 0$)

and restrict this function to a rational curve (1.4) in \mathcal{PN} . Now integrate f along a contour in this curve. This gives (1.3) with $\lambda = Z^3/Z^2$. (**Exercise:** Explain why f , when viewed as a function on the non-projective twistor space, must be homogeneous of degree -2 in Z^α . Find a solution ϕ to the wave equation corresponding to $f = (A_\alpha Z^\alpha)^{-1}(B_\beta Z^\beta)^{-1}$, where $\alpha, \beta = 0, \dots, 3$ and (A_α, B_β) are constant complex numbers.)

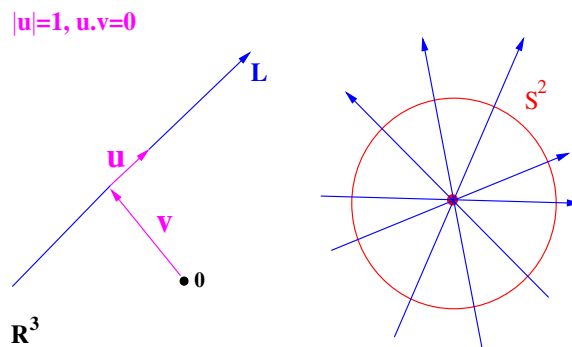
To sum up, the spacetime points are derived objects in twistor theory. They become ‘fuzzy’ after quantization. This may provide an attractive framework for quantum gravity, but it must be said that despite 40 years of research the twistor theory is still waiting to have its major impact on physics. It has however had a surprisingly major impact on pure mathematics: ranging from representation theory and differential geometry to solitons, instantons and integrable systems.

This ends the ‘historical’ part of the paper. The rest of the paper is intended to give a ‘down-to-earth’ introduction to the calculations done in twistor theory. Rather than using the twistors of a (3+1)-dimensional Minkowski space, we shall focus on mini-twistors which arise in the (2+1)-dimensional Minkowski space or in \mathbb{R}^3 . This ‘mini-twistor theory’ is in many ways simpler but still sufficient in applications to (2+1)- and three-dimensional integrable systems and their reductions. The mini-twistor space \mathbb{T} (from now on called the twistor space) is the holomorphic tangent bundle to the Riemann sphere. The difference between the Lorentzian and Euclidean signatures of the corresponding spacetime is encoded in the anti-holomorphic involution on \mathbb{T} which, when restricted to rational curves, becomes the antipodal map in the Euclidean case and the equator-fixing conjugation in the Lorentzian case. We shall study the Euclidean theory in the following section and the Lorentzian theory in section 3.

2. Non-Abelian monopoles and Euclidean mini-twistors

It is well known that the problem of finding harmonic functions in \mathbb{R}^2 can be solved ‘in one line’ by introducing complex numbers: any solution of a two-dimensional Laplace equation $\phi_{xx} + \phi_{yy} = 0$ is a real part of a function holomorphic in $x + iy$. This technique fails when applied to the Laplace equation in three dimensions as \mathbb{R}^3 cannot be identified with \mathbb{C}^n for any n .

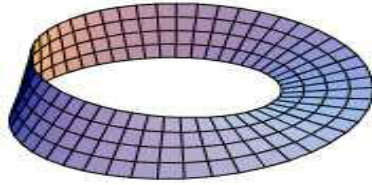
Following Hitchin [10] we shall associate a two-dimensional complex manifold with the three-dimensional Euclidean space. Define the twistor space \mathbb{T} to be the space of oriented lines in \mathbb{R}^3 . Any oriented line is of form $\mathbf{v} + s\mathbf{u}$, $s \in \mathbb{R}$ where \mathbf{u} is a unit vector giving the direction of the line and \mathbf{v} is orthogonal to \mathbf{u} and joins the line with some chosen point (say the origin) in \mathbb{R}^3 .



Thus

$$\mathbb{T} = \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3, \mathbf{u} \cdot \mathbf{v} = 0\}$$

and the dimension of \mathbb{T} is 4. For each fixed $\mathbf{u} \in S^2$ this space restricts to a tangent plane to S^2 . The twistor space is the union of all tangent planes—the tangent bundle TS^2 . This is a topologically non-trivial manifold: locally it is diffeomorphic to $S^2 \times \mathbb{R}^2$ but globally it is twisted in a way analogous to the Möbius strip.



Reversing the orientation of lines induces a map $\tau : \mathbb{T} \rightarrow \mathbb{T}$ given by

$$\tau(\mathbf{u}, \mathbf{v}) = (-\mathbf{u}, \mathbf{v}).$$

The points $\mathbf{p} = (x, y, z)$ in \mathbb{R}^3 correspond to two spheres in \mathbb{T} given by τ -invariant maps

$$\mathbf{u} \rightarrow (\mathbf{u}, \mathbf{v}(\mathbf{u}) = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u})\mathbf{u}) \in \mathbb{T} \tag{2.1}$$

which are sections of the projection $\mathbb{T} \rightarrow S^2$.

Twistor space as a complex manifold

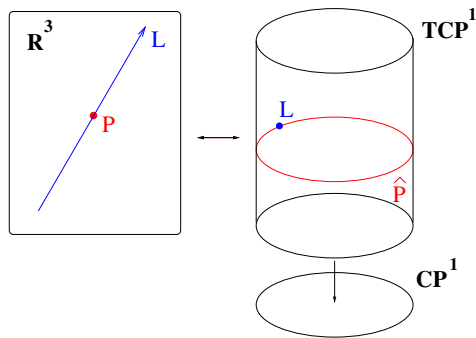
Introduce the local holomorphic coordinates on an open set $U \subset \mathbb{T}$ where $\mathbf{u} \neq (0, 0, 1)$ by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3,$$

and analogous complex coordinates $(\tilde{\lambda}, \tilde{\eta})$ in an open set \tilde{U} containing $\mathbf{u} = (0, 0, 1)$. On the overlap

$$\tilde{\lambda} = 1/\lambda, \quad \tilde{\eta} = -\eta/\lambda^2.$$

(Exercise: Work out its details.) This endows \mathbb{T} with a structure of complex manifold $T\mathbb{CP}^1$. It is a holomorphic tangent bundle to the Riemann sphere (see the appendix).



In the holomorphic coordinates, the line orientation reversing involution τ is given by

$$\tau(\lambda, \eta) = \left(-\frac{1}{\lambda}, -\frac{\bar{\eta}}{\lambda^2}\right). \tag{2.2}$$

This is an antipodal map lifted from a two-sphere to the total space of the tangent bundle. The formula (2.1) implies that the points in \mathbb{R}^3 are τ -invariant holomorphic maps $\mathbb{CP}^1 \rightarrow T\mathbb{CP}^1$ given by

$$\lambda \rightarrow (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \tag{2.3}$$

(Exercise: Verify that (2.3) follows from (2.1).)

Harmonic functions and Abelian monopoles

Finally we can return to our original problem. To find a harmonic function at $P = (x, y, z)$

- (1) Restrict a twistor function $f(\lambda, \eta)$ defined on $U \cap \tilde{U}$ to a line (2.3) $\hat{P} = \mathbb{CP}^1 = S^2$.
- (2) Integrate along a closed contour

$$\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy)) d\lambda. \tag{2.4}$$

- (3) Differentiate under the integral to verify

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

This formula was already known to Whittaker [29] in 1903, albeit Whittaker’s formulation does not make any use of complex numbers and his formula is given in terms of a real integral.

A small modification of this formula can be used to solve a first-order linear equation for a function ϕ and a magnetic potential $\mathbf{A} = (A_1, A_2, A_3)$ of the form

$$\nabla \phi = \nabla \wedge \mathbf{A}.$$

This is the Abelian monopole equation. Geometrically, the one-form $A = A_j dx^j$ is a connection on a $U(1)$ principal bundle over \mathbb{R}^3 , and ϕ is a section of the adjoint bundle. Taking the **curl** of both sides of this equation implies that ϕ is harmonic, and conversely given a harmonic function ϕ locally one can always find a one-form A (defined up to addition of a gradient of some function) such that the Abelian monopole equation holds. (**Exercise:** Find an integral formula for the one-form A analogous to (2.4). This question is best handled using the spinor formalism introduced in section 3.1.)

2.1. Non-Abelian monopoles and Hitchin correspondence

Replacing $U(1)$ by a non-Abelian Lie group generalizes this picture to some equations on \mathbb{R}^3 in the following way: let (A_j, ϕ) be anti-Hermitian traceless n by n matrices on \mathbb{R}^3 . Define the non-Abelian magnetic field

$$F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.$$

The non-Abelian monopole equation is a system of nonlinear PDEs

$$\frac{\partial \phi}{\partial x^j} + [A_j, \phi] = \frac{1}{2} \varepsilon_{jkl} F_{kl}. \tag{2.5}$$

These are three equations for three unknowns as (A, ϕ) are defined up to gauge transformations

$$A \longrightarrow gAg^{-1} - dg g^{-1}, \quad \phi \longrightarrow g\phi g^{-1}, \quad g = g(x, y, t) \in SU(n), \tag{2.6}$$

and one component of A (say A_1) can always be set to zero.

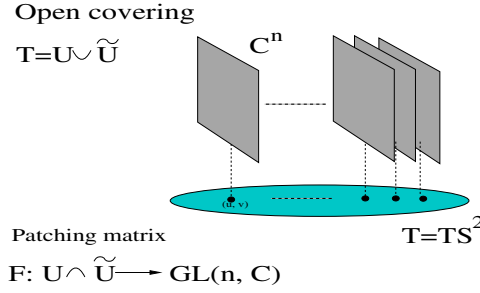
The twistor solution to the monopole equation consists of the following steps [10]:

- Given $(A_j(\mathbf{x}), \phi(\mathbf{x}))$ solve a matrix ODE along each oriented line $\mathbf{x}(s) = \mathbf{v} + s\mathbf{u}$

$$\frac{dV}{ds} + (u^j A_j + i\phi)V = 0.$$

Space of solutions at $p \in \mathbb{R}^3$ is a complex vector space \mathbb{C}^n .

- This assigns a complex vector space \mathbb{C}^n to each point of \mathbb{T} , thus giving rise to a complex vector bundle over \mathbb{T} with patching matrix $F(\lambda, \bar{\lambda}, \eta, \bar{\eta}) \in GL(n, \mathbb{C})$.



- The monopole equation (2.5) on \mathbb{R}^3 holds if and only if this vector bundle is holomorphic, i.e. the Cauchy–Riemann equations

$$\frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial F}{\partial \bar{\eta}} = 0$$

hold.

- Holomorphic vector bundles over $T\mathbb{C}P^1$ are well understood. Take one and work backwards to construct a monopole. We shall work through the details of this reconstruction (albeit in complexified settings) in the proof of theorem 3.1.

3. The Ward model and Lorentzian mini-twistors

In this section, we shall demonstrate how mini-twistor theory can be used to solve nonlinear equations in 2+1 dimensions. Let $A = A_\mu dx^\mu$ and ϕ be a one-form and a function respectively on the Minkowski space $\mathbb{R}^{2,1}$ with values in a Lie algebra of the general linear group. They are defined up to gauge transformations (2.6) where g takes values in $GL(n, \mathbb{R})$.

Let $D_\mu = \partial_\mu + A_\mu$ be a covariant derivative, and define $D\phi = d\phi + [A, \phi]$. The Ward model is a system of PDEs (2.5) where now the indices are raised using the metric on $\mathbb{R}^{2,1}$. If the metric and the volume form are chosen to be

$$h = dx^2 - 4 du dv, \quad \text{vol} = du \wedge dx \wedge dv$$

where the coordinates (x, u, v) are real the equations become

$$D_x \phi = \frac{1}{2} F_{uv}, \quad D_u \phi = F_{ux}, \quad D_v \phi = F_{xv}, \tag{3.1}$$

where $F_{\mu\nu} = [D_\mu, D_\nu]$. These equations arise as the integrability conditions for an overdetermined system of linear Lax equations

$$L_0 \Psi = 0, \quad L_1 \Psi = 0, \quad \text{where} \quad L_0 = D_u - \lambda(D_x + \phi), \quad L_1 = D_x - \phi - \lambda D_v, \tag{3.2}$$

and $\Psi = \Psi(x, u, v, \lambda)$ takes values in $GL(n, \mathbb{C})$. We shall follow [27] and ‘solve’ the system by establishing a one-to-one correspondence between its solutions and certain holomorphic vector bundles over the twistor space \mathbb{T} . This construction is of interest in soliton theory as many known integrable models arise as symmetry reduction and/or choosing a gauge in (3.1). To this end, we note a few examples of such reductions. See [18] for a much more complete list.

- Choose the unitary gauge group $G = U(n)$. The integrability conditions for (3.2) imply the existence of a gauge $A_v = 0$, and $A_x = -\phi$, and a matrix $J : \mathbb{R}^{2,1} \rightarrow U(n)$ such that

$$A_u = J^{-1} \partial_u J, \quad A_x = -\phi = \frac{1}{2} J^{-1} \partial_x J.$$

With this gauge choice equations (3.1) become the integrable chiral model

$$\partial_v (J^{-1} \partial_u J) - \partial_x (J^{-1} \partial_x J) = 0. \tag{3.3}$$

This formulation breaks the Lorentz invariance of (3.1) but it allows the introduction of a positive definite energy functional. See [26] where more details can be found.

- Solutions to equation (3.1) with the gauge group $SL(2, \mathbb{R})$ which are invariant under a null translation given by a Killing vector K such that the matrix $K \lrcorner A$ is nilpotent are characterized by the KdV equation [17].
- The direct calculation shows that the Ward equations with the gauge group $SL(3, \mathbb{R})$ are solved by the ansatz

$$\phi = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -e^\psi & 0 & 0 \end{pmatrix},$$

$$A = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ e^\psi & 0 & 0 \end{pmatrix} dx + \begin{pmatrix} \psi_u & 0 & 0 \\ 1 & -\psi_u & 0 \\ 0 & 1 & 0 \end{pmatrix} du + \begin{pmatrix} 0 & e^{-2\psi} & 0 \\ 0 & 0 & e^\psi \\ 0 & 0 & 0 \end{pmatrix} dv \tag{3.4}$$

iff $\psi(u, v)$ satisfies the Tzitzéica equation

$$\frac{\partial^2 \psi}{\partial u \partial v} = e^\psi - e^{-2\psi}. \tag{3.5}$$

This reduction can also be characterized in a gauge invariant manner using the Jordan normal forms for the Higgs fields; see [4] for details. (**Exercise:** Show that (3.5) follows from (3.1). What can you say about the gauge field corresponding to the trivial solution $\psi = 0$?)

3.1. Null planes and Ward correspondence

The geometric interpretation of the Lax representation (3.2) is the following. For any fixed pair of real numbers (η, λ) the plane

$$\eta = v + x\lambda + u\lambda^2 \tag{3.6}$$

is null with respect to the Minkowski metric on $\mathbb{R}^{2,1}$, and conversely all null planes can be put in this form if one allows $\lambda = \infty$. The two vector fields

$$\delta_0 = \partial_u - \lambda \partial_x, \quad \delta_1 = \partial_x - \lambda \partial_v \tag{3.7}$$

span this null plane. Thus the Lax equations (3.2) imply that the generalized connection (A, ϕ) is flat on null planes. This underlies the twistor approach [27], where one works in a complexified Minkowski space $M = \mathbb{C}^3$, and interprets (η, λ) as coordinates in a patch of the twistor space $\mathbb{T} = T\mathbb{C}P^1$, with $\eta \in \mathbb{C}$ being a coordinate on the fibres and $\lambda \in \mathbb{C}P^1$ being an affine coordinate on the base. We shall adopt this complexified point of view from now on.

It is convenient to make use of the spinor formalism based on the isomorphism

$$TM = \mathbb{S} \odot \mathbb{S},$$

where \mathbb{S} is the rank 2 complex vector bundle (spin bundle) over M and \odot is the symmetrized tensor product. The fibre coordinates of this bundle are denoted by (π^0, π^1) and the sections $M \rightarrow \mathbb{S}$ are called spinors. We shall regard \mathbb{S} as a symplectic bundle with an anti-symmetric product

$$\kappa \cdot \rho = \kappa^0 \rho^1 - \kappa^1 \rho^0 = \varepsilon(\kappa, \rho)$$

on its sections. The constant symplectic form ε is represented by a matrix

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This gives an isomorphism between \mathbb{S} and its dual bundle, and thus can be used to ‘rise and lower the indices’ according to $\kappa_A = \kappa^B \varepsilon_{BA}$, $\kappa^A = \varepsilon^{AB} \kappa_B$, where $\varepsilon_{AB} \varepsilon^{CB}$ is an identity endomorphism.

Rearrange the spacetime coordinates (u, x, v) of a displacement vector as a symmetric two-spinor

$$x^{AB} := \begin{pmatrix} u & x/2 \\ x/2 & v \end{pmatrix},$$

such that the spacetime metric is

$$h = -2 dx_{AB} dx^{AB}.$$

The twistor space of M is the two-dimensional complex manifold $\mathbb{T} = T\mathbb{C}\mathbb{P}^1$. Points of \mathbb{T} correspond to null 2-planes in M via the incidence relation

$$x^{AB} \pi_A \pi_B = \omega. \tag{3.8}$$

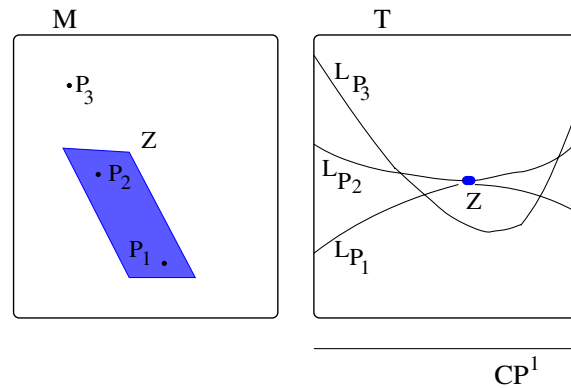
Here (ω, π_0, π_1) are homogeneous coordinates on \mathbb{T} as $(\omega, \pi_A) \sim (c^2 \omega, c \pi_A)$, where $c \in \mathbb{C}^*$. In the affine coordinates $\lambda := \pi_0/\pi_1$, $\eta := \omega/(\pi_1)^2$ equation (3.8) gives (3.6).

The projective spin space $P(\mathbb{S})$ is the complex projective line $\mathbb{C}\mathbb{P}^1$. The homogeneous coordinates are denoted by $\pi_A = (\pi_0, \pi_1)$, and the two-set covering of $\mathbb{C}\mathbb{P}^1$ lifts to a covering of the twistor space \mathbb{T} :

$$U = \{(\omega, \pi_A), \pi_1 \neq 0\}, \quad \tilde{U} = \{(\omega, \pi_A), \pi_0 \neq 0\}. \tag{3.9}$$

The functions $\lambda = \pi_0/\pi_1$, $\tilde{\lambda} = 1/\lambda$ are the inhomogeneous coordinates in U and \tilde{U} , respectively. It then follows that $\lambda = -\pi^1/\pi^0$.

Fixing (ω, π_A) gives a null plane in M . An alternative interpretation of (3.8) is to fix x^{AB} . This determines ω as a function of π_A , i.e. a section of $\mathbb{T} \rightarrow \mathbb{C}\mathbb{P}^1$ when factored out by the relation $(\omega, \pi_A) \sim (c^2 \omega, c \pi_A)$. These are embedded rational curves with self-intersection number 2, as infinitesimally perturbed curve $\eta + \delta\eta$ with $\delta\eta = \delta v + \lambda \delta x + \lambda^2 \delta u$ generically intersects (3.6) at two points. Two curves intersect at one point if the corresponding points in M are null separated. This defines a conformal structure on M .



The space of holomorphic sections of $\mathbb{T} \rightarrow \mathbb{CP}^1$ is $M = \mathbb{C}^3$ (see the appendix). The real spacetime \mathbb{R}^{2+1} arises as the moduli space of those sections that are invariant under the conjugation

$$\tau(\omega, \pi_A) = (\bar{\omega}, \bar{\pi}_A), \tag{3.10}$$

which corresponds to real x^{AB} . The points in \mathbb{T} fixed by τ correspond to real null planes in $\mathbb{R}^{2,1}$. (**Exercise:** Show that as a complex manifold \mathbb{T} is biholomorphic with a cone in \mathbb{CP}^3 with its vertex removed, where the points in M correspond to the conic sections omitting the vertex. Demonstrate that allowing the conic sections passing through the vertex of the cone results in a compactification of the complexified Minkowski space $\bar{M} = M + \mathbb{CP}^2 = \mathbb{CP}^3$.)

The following result makes the mini-twistors worthwhile.

Theorem 3.1. (Ward [27]) *There is a one-to-one correspondence between:*

- (1) *The gauge equivalence classes of complex solutions to (3.1) in the complexified Minkowski space M with the gauge group $GL(n, \mathbb{C})$.*
- (2) *Holomorphic rank n vector bundles E over the twistor space \mathbb{T} which are trivial on the holomorphic sections of $T\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$.*

Proof. Let (A, ϕ) be a solution to (3.1). Therefore we can integrate a pair of linear PDEs $L_0V = L_1V = 0$, where L_0, L_1 are given by (3.2). This assigns an n -dimensional vector space to each null plane Z in a complexified Minkowski space, and so to each point $Z \in \mathbb{T}$. It is a fibre of a holomorphic vector bundle $\mu : E \rightarrow \mathbb{T}$. The bundle E is trivial on each section, since we can identify fibres of $E|_{L_p}$ at Z_1, Z_2 because covariantly constant vector fields at null planes Z_1, Z_2 coincide at a common point $p \in M$.

Conversely, assume that we are given a holomorphic vector bundle E over \mathbb{T} which is trivial on each section. Since $E|_{L_p}$ is trivial and $L_p \cong \mathbb{CP}^1$, the Birkhoff–Grothendieck theorem (appendix) gives

$$E|_{L_p} = \mathcal{O} \oplus \mathcal{O} \oplus \dots \oplus \mathcal{O},$$

and the space of sections of E restricted to L_p is \mathbb{C}^n . This gives us a holomorphic rank n vector bundle \hat{E} over the complexified three-dimensional Minkowski space. We shall give a concrete method of constructing a pair (A, ϕ) on this bundle which satisfies (3.1).

Let us cover the twistor space with two open sets U and \tilde{U} as in (3.9). Let

$$\chi : \mu^{-1}(U) \rightarrow U \times \mathbb{C}^n, \quad \tilde{\chi} : \mu^{-1}(\tilde{U}) \rightarrow \tilde{U} \times \mathbb{C}^n$$

be local trivializations of E , and let $F = \tilde{\chi} \circ \chi^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic patching matrix for a vector bundle E over $T\mathbb{CP}^1$ defined on $U \cap \tilde{U}$. Restrict F to a section (3.8) where the bundle is trivial, and therefore F can be split (compare (A.1) in the appendix):

$$F = \tilde{H}H^{-1}, \tag{3.11}$$

where the matrices H and \tilde{H} are defined on $M \times \mathbb{CP}^1$ and are holomorphic in π^A around $\pi^A = o^A = (1, 0)$ and $\pi^A = \iota^A = (0, 1)$ respectively. As a consequence of $\delta_A F = 0$ the splitting matrices satisfy

$$H^{-1}\delta_A H = \tilde{H}^{-1}\delta_A \tilde{H} = \pi^B \Phi_{AB}, \tag{3.12}$$

for some $\Phi_{AB}(x^\mu)$ which does not depend on λ . This is because the rhs and lhs are homogeneous of degree 1 in π^A and holomorphic around $\lambda = 0$ and $\lambda = \infty$, respectively. (**Exercise:** Prove it starting from the Liouville theorem which says that any function holomorphic on \mathbb{CP}^1 must be constant.) Decomposing

$$\Phi_{AB} = \Phi_{(AB)} + \varepsilon_{AB}\phi$$

gives a one-form $A = \Phi_{AB} dx^{AB}$ and a scalar field $\phi = (1/2)\epsilon^{AB}\Phi_{AB}$ on the complexified Minkowski space, i.e.

$$\Phi_{AB} = \begin{pmatrix} A_u & A_x + \phi \\ A_x - \phi & A_v \end{pmatrix}.$$

The Lax pair (3.2) becomes

$$L_A = \delta_A + H^{-1}\delta_A H,$$

where $\delta_A = \pi^B \partial_{AB}$, so that

$$L_A(H^{-1}) = -H^{-1}(\delta_A H)H^{-1} + H^{-1}(\delta_A H)H^{-1} = 0,$$

and $\Psi = H^{-1}$ is a solution to the Lax equations regular around $\lambda = 0$. Let us show explicitly that (3.1) holds. Differentiating (3.12) with respect to δ^A yields

$$\delta^A(H^{-1}\delta_A H) = -(H^{-1}\delta^A H)(H^{-1}\delta_A H)$$

which holds for all π^A if

$$D_{A(C}\Phi^A_{B)} = 0, \tag{3.13}$$

where $D_{AC} = \partial_{AC} + \Phi_{AC}$. This is the spinor form of the Yang–Mills–Higgs system (3.1). \square

- To single out the Euclidean reality conditions leading to non-Abelian monopoles (2.5) on \mathbb{R}^3 with the gauge group $SU(n)$, the vector bundle E must be compatible with the involution (2.2). This comes down to $\det F = 1$ and

$$F^*(Z) = F(\tau(Z)),$$

where $Z \in \mathbb{T}$ and $*$ denotes the Hermitian conjugation.

- To single out the Lorentzian reality conditions, the bundle must be invariant under the involution (3.10). Below we shall demonstrate how the gauge choices leading to the integrable chiral model (3.3) can be made at the twistor level.

Let

$$h := H(x^\mu, \pi^A = o^A), \quad \tilde{h} := \tilde{H}(x^\mu, \pi^A = \iota^A)$$

so that

$$\Phi_{A0} = h^{-1}\partial_{A0}h, \quad \Phi_{A1} = \tilde{h}^{-1}\partial_{A1}\tilde{h}.$$

The splitting matrices are defined up to a multiple by an inverse of a non-singular matrix $g = g(x^\mu)$ independent of π^A

$$H \rightarrow Hg^{-1}, \quad \tilde{H} \rightarrow \tilde{H}g^{-1}.$$

(Exercise: Show that this corresponds to the gauge transformation (2.6) of Φ_{AB} .)

We choose g such that $\tilde{h} = \mathbf{1}$ so

$$\Phi_{A1} = \iota^A \Phi_{AB} = 0$$

and

$$\Phi_{AB} = -\iota_B o^C h^{-1}\partial_{AC}h,$$

i.e.

$$A_x + \phi = A_v = 0.$$

This is the Ward gauge with $J(x^\mu) = h$. In this gauge, the system (3.13) reduces to

$$\partial^A_{A1}\Phi_{A0} = 0$$

which is (3.3). The solution is given by

$$J(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0),$$

where $\Psi = H^{-1}$ is a solution to the Lax pair.

- In the Abelian case $n = 1$ the patching matrix becomes a function defined on the intersection of two open sets, and we can set $F = \exp(f)$ for some f . The nonlinear splitting (A.1) reduces to the additive splitting of f which can be carried out explicitly using the Cauchy integral formula. The Higgs field is now a function that satisfies the wave equation and is given by the formula

$$\phi = \oint_{\Gamma} \frac{\partial f}{\partial \omega} \rho \cdot d\rho,$$

where Γ is a real contour in a rational curve $\omega = x^{AB} \pi_A \pi_B$. If the Euclidean reality conditions are chosen, we recover the Whittaker formula (2.4).

- **Exercise:** Find the patching matrix for the holomorphic rank 3 bundle $E \rightarrow \mathbb{T}$ corresponding to the one-soliton solution to the Tzitzeica equation (3.5). (*Note:* the solution to this exercise remains unknown to the author.)

4. Dispersionless systems and deformed mini-twistors

There is a class of integrable systems in 2+1 and three dimensions which do not fit into the framework described in the previous section. They do not arise from (3.1) and there is no finite-dimensional Riemann–Hilbert problem analogous to (3.11) which leads to their solutions. These dispersionless integrable systems admit Lax representations which do not involve matrices, like (3.2), but instead consist of vector fields. This leads to curved geometries in the following way. Consider a Lax pair

$$L_0 = W - \lambda V + f_0 \frac{\partial}{\partial \lambda}, \quad L_1 = V - \lambda \tilde{W} + f_1 \frac{\partial}{\partial \lambda}, \quad (4.1)$$

where (W, \tilde{W}, V) are vector fields on a complex three-manifold M (which generalizes the complexified Minkowski space) and (f_0, f_1) are cubic polynomials in $\lambda \in \mathbb{CP}^1$. Assume that the distribution spanned by the Lax pair is integrable in the sense of Frobenius, i.e.,

$$[L_0, L_1] = \alpha L_0 + \beta L_1$$

for some α, β . The twistor space \mathbb{T} is defined to be the quotient of the total space of the projective spin bundle $P(\mathbb{S}) \rightarrow M$ by this distribution, i.e.

$$\mathbb{T} = M \times \mathbb{CP}^1 / (L_0, L_1).$$

This is a deformation of $T\mathbb{CP}^1$ (or its region as in general the construction is local in M so \mathbb{T} is taken to be a tubular neighbourhood of a rational curve corresponding to $p \in M$) which arises if L_0, L_1 are given by (3.7).

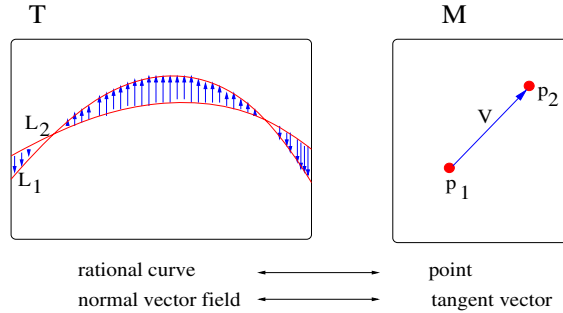
The twistor space is a complex surface containing a three-parameter family of rational curves \mathbb{CP}^1 with self-intersection number 2. In general \mathbb{T} does not fibre holomorphically over \mathbb{CP}^1 which is a consequence of the presence of $\partial/\partial\lambda$ terms in the Lax pair (4.1).

Conversely, given such a complex manifold \mathbb{T} one defines M to be the moduli space of rational curves in \mathbb{T} (Kodaira theorems [13] guarantee that M exists and is three complex dimensional). One can show [11] that M comes equipped with the geometric structure consisting of a conformal structure $[h]$ and a compatible torsion-free connection ∇ . The details are as follows: the points of M correspond to rational curves with self-intersection 2 in the complex surface \mathbb{T} and points in \mathbb{T} correspond to null surfaces in M . Recall that the normal bundle $N(L) \rightarrow L$ to a submanifold $L \subset \mathbb{T}$ is defined by

$$N(L) = \cup_{Z \in L} N_Z(L),$$

where $N_Z = (T_Z \mathbb{T}) / (T_Z L)$ is a quotient vector space. If $L_p \subset \mathbb{T}$ is the curve corresponding to $p \in M$ then the elements of $T_p M$ correspond to sections of the normal bundle $N(L_p)$ and

as a holomorphic line bundle $N(L_p) \cong \mathcal{O}(2)$ (see the appendix). The conformal structure on M arises as we define the null vectors at p in M to be the sections of the normal bundle $N(L_p)$ which vanish at some point to the second order. A section of $\mathcal{O}(2)$ has a form $V^{AB}\pi_A\pi_B$ (see the appendix), thus the vanishing condition $(V^{01})^2 - V^{00}V^{11}$ is quadratic and defines $[h]$. If p_1, p_2 are two points in M which are not null separated, then the corresponding curves in \mathbb{T} intersect at two points. If p_1 and p_2 are infinitesimally close, and thus are joined by a vector starting from p_1 , then the corresponding section of $N(L_1)$ will vanish at two points.



To define the connection ∇ , we define a direction at $p \in M$ to be a one-dimensional space of sections of $\mathcal{O}(2)$ which vanish at two points Z_1 and Z_2 in L_p . The one-dimensional family of $\mathcal{O}(2)$ curves in \mathbb{T} passing through Z_1 and Z_2 gives a geodesic curve in M in a given direction and defines ∇ . In the limiting case $Z_1 = Z_2$, these geodesics are null with respect to $[h]$. This compatibility means that for any choice of $h \in [h]$

$$\nabla h = \omega \otimes h,$$

for some one-form ω on M . This condition is invariant under the conformal rescalings of h if

$$h \longrightarrow c^2 h, \quad \omega \longrightarrow \omega + 2d(\ln(c)),$$

where c is a nonzero function on M . Therefore, the null geodesics for $[h]$ are also geodesic of ∇ and thus the pair $([h], \nabla)$ gives a Weyl structure on M . The Weyl structures coming from a twistor space satisfy a set of equations generalizing Einstein equations. This is because the special surfaces in M corresponding to points in \mathbb{T} are totally geodesic with respect to ∇ (if a geodesic is tangent to a surface at some point then it lies on that surface). The integrability conditions for the existence of totally geodesic surfaces are equivalent to the conformally invariant Einstein–Weyl equations

$$R_{(jk)} = \Lambda h_{(jk)},$$

where $R_{(jk)}$ is the symmetrized Ricci tensor of the connection ∇ and Λ is some function on M .

The Einstein–Weyl equations admit a Lax formulation with the Lax pair given by (4.1): if the distribution spanned by (4.1) is integrable then there exists a one-form ω such that the metric h given by

$$h = V \otimes V - 2(W \otimes \tilde{W} + \tilde{W} \otimes W) \tag{4.2}$$

and ω satisfy the Einstein–Weyl equations. Any Einstein–Weyl structure arises from such a Lax pair [3].

An example of a dispersionless system which fits into this construction is the interpolating integrable system [5]

$$u_y + w_x = 0, \quad u_t + w_y - c(uw_x - wu_x) + buu_x = 0, \tag{4.3}$$

where $u = u(x, y, t)$, $w = w(x, y, t)$ and (b, c) are constants. It admits a Lax pair

$$L_0 = \frac{\partial}{\partial t} + (cw + bu - \lambda cu - \lambda^2) \frac{\partial}{\partial x} + b(w_x - \lambda u_x) \frac{\partial}{\partial \lambda},$$

$$L_1 = \frac{\partial}{\partial y} - (cu + \lambda) \frac{\partial}{\partial x} - bu_x \frac{\partial}{\partial \lambda}.$$

A linear combination of L_0, L_1 is of form (4.1). The Einstein–Weyl structure associated with (4.3) is

$$h = (dy - cu dt)^2 - 4(dx - (cw + bu) dt) dt,$$

$$\omega = -cu_x dy + (4bu_x + c^2uu_x - 2cu_y) dt.$$

(Exercise: Verify that (4.3) arises as $[L_0, L_1] = 0$ from the given Lax pair. Use (4.2) to construct the given metric h from (u, w) .) Setting $c = 0, b = 1$ gives the dispersionless Kadomtsev–Petviashvili equation. On the twistor level, this limit is characterized [3] by the existence of a preferred section of $\kappa^{-1/4}$ where κ is the canonical bundle of holomorphic two-forms on \mathbb{T} . Another interesting limit is $(b = 0, c = -1)$, where the corresponding twistor space fibres holomorphically over \mathbb{CP}^1 .

There are several approaches to dispersionless integrable systems in 2+1 dimensions: the Krichever algebro–geometric approach, the hydrodynamic reductions developed by Ferapontov and his collaborators, the Cauchy problem of Manakov–Santini and the $\bar{\partial}$ –formulation of Konopelchenko and Martinez Alonso to name a few (see [15, 14, 1, 7, 16]). The Einstein–Weyl geometry and the associated deformed mini-twistor theory provide another framework which is coordinate independent, and geometric as the solutions are parametrized by complex manifolds with embedded rational curves.

5. Summary and outlook

Twistor theory arose as a non-local attempt to unify general relativity and quantum mechanics. In this theory, a spacetime point is a derived object corresponding to a rational curve in some complex manifold. The mathematics behind twistor theory has its roots in the 19th century projective geometry of Plücker and Klein, but it can also be traced back to the integral geometry of Radon and John developed in the first half of the 20th century. While the twistor programme is yet to have its big impact on physics (however see [30]), it has led to methods of solving linear and nonlinear differential equations. In the linear case, one gets nice geometrical interpretations of the integral formulae of Whittaker and John. The twistor methods of solving nonlinear integrable PDEs are genuinely new and lead to parametrizing ‘all’ solutions by unconstrained holomorphic data. In the case of the Ward model and its reductions (as well as the anti-self-dual Yang–Mills equations [24] not discussed in this paper), the solutions correspond to holomorphic vector bundles trivial on twistor lines. The solutions of dispersionless integrable models (as well as anti-self-dual conformal equations [21] and heavenly equations) correspond to holomorphic deformations of the complex structure underlying the twistor space.

It is unlikely that all integrable equations fit into one of the (rather rigid) frameworks (3.2) or (4.1) presented in this paper. It should however be possible to extend these frameworks, while keeping their essential features, to incorporate those integrable systems which so far have resisted the twistor approach.

Acknowledgment

I thank Prim Plansangkate for carefully reading the manuscript and correcting several errors.

Appendix

Riemann sphere

Two-dimensional sphere $S^2 \subset \mathbb{R}^3$ is a one-dimensional complex manifold with local coordinates defined by stereographic projection. Let $(u_1, u_2, u_3) \in S^2$. Define two open subsets covering S^2

$$U = S^2 - \{(0, 0, 1)\}, \quad \tilde{U} = S^2 - \{(0, 0, -1)\}$$

and introduce complex coordinates λ and $\tilde{\lambda}$ on U and \tilde{U} respectively by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3}, \quad \tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}.$$

The domain of λ is the whole sphere less the North pole; the domain of $\tilde{\lambda}$ is the whole sphere less the South pole. On the overlap $U_0 \cap U_1$ we have $\tilde{\lambda} = 1/\lambda$ which is a holomorphic function. The resulting complex manifold is called \mathbb{CP}^1 . It also arises as the quotient of \mathbb{C}^2 by the equivalence relation

$$(\pi_0, \pi_1) \sim (c\pi_0, c\pi_1) \quad \text{for some } c \in \mathbb{C}^*.$$

The homogeneous coordinates π_A label the points uniquely, up to an overall nonzero complex scaling factor. In this approach, the complex manifold structure on \mathbb{CP}^1 is introduced by using the {inhomogeneous coordinates}. On the open set U in which $\pi_1 \neq 0$, we define $\lambda = \pi_0/\pi_1$ and on the open set \tilde{U} with $\pi_0 \neq 0$ we set $\tilde{\lambda} = \pi_1/\pi_0$ so that $\tilde{\lambda} = 1/\lambda$ on the overlap.

Holomorphic vector bundles

A holomorphic vector bundle of rank n over a complex manifold \mathbb{T} is a complex manifold E , and a holomorphic projection $\pi : E \rightarrow \mathbb{T}$ such that

- For each $z \in \mathbb{T}$, $\pi^{-1}(z)$ is an n -dimensional complex vector space.
- Each point $z \in \mathbb{T}$ has a neighbourhood U_α and a homeomorphism χ_α such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\chi_\alpha} & U_\alpha \times \mathbb{C}^n \\ \pi \searrow & & \swarrow \\ & U_\alpha & \end{array}$$

is commutative.

- The patching matrix $F_{\alpha\beta} := \chi_\beta \circ \chi_\alpha^{-1} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{C})$ is a holomorphic map to the space of invertible $n \times n$ matrices.

The product $E = \mathbb{T} \times \mathbb{C}^n$ is called a trivial vector bundle. The bundle is trivial, iff there exist holomorphic splitting matrices $H_\alpha : U_\alpha \rightarrow GL(n, \mathbb{C})$ such that

$$F_{\alpha\beta} = H_\beta H_\alpha^{-1}. \tag{A.1}$$

We shall give examples of holomorphic line bundles (i.e. vector bundles with $n = 1$) over \mathbb{CP}^1 . First define a tautological line bundle

$$\mathcal{O}(-1) = \{(\lambda, (\pi_0, \pi_1)) \in \mathbb{CP}^1 \times \mathbb{C}^2 | \lambda = \pi_0/\pi_1\}.$$

Representing the Riemann sphere as the projective line gives the projection $\mathbb{C}^2 \rightarrow \mathbb{CP}^1$. The fibre above the point with coordinate $[\pi]$ is the one-dimensional line $c\pi$ through the origin in \mathbb{C}^2 containing the point (π_0, π_1) . The transition function for this bundle is $F = \lambda$. (**Exercise:** Show it). Other line bundles can be obtained by algebraic operations:

$$\mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m}, \quad \mathcal{O}(m) = \mathcal{O}(-m)^*, \quad \mathcal{O} = \mathcal{O}(-1) \otimes \mathcal{O}(1), \quad m \in \mathbb{N}.$$

The transition function for $\mathcal{O}(m)$ is $F = \lambda^{-m}$ on $U \cap \tilde{U} \cong \mathbb{C}^*$.

The line bundles $\mathcal{O}(m)$ for any $m \in \mathbb{Z}$ are building blocks for all other vector bundles over the Riemann sphere. This is a consequence of the Birkhoff–Grothendieck theorem which states that a rank n holomorphic vector bundle $E \rightarrow \mathbb{CP}^1$ is isomorphic to a direct sum of line bundles $\mathcal{O}(m_1) \oplus \dots \oplus \mathcal{O}(m_n)$ for some integers m_i .

Holomorphic sections

A holomorphic section of a vector bundle E over \mathbb{T} is a holomorphic map $s : \mathbb{T} \rightarrow E$ such that $\pi \circ s = \text{id}_{\mathbb{T}}$. The local description is given by a collection of holomorphic maps $s_\alpha : U_\alpha \rightarrow \mathbb{C}^n$

$$z \longrightarrow (z, s_\alpha(z)), \quad \text{for } z \in U_\alpha.$$

with the transition rule $s_\beta(z) = F_{\alpha\beta}(z)s_\alpha(z)$.

A global holomorphic section of the line bundle $\mathcal{O}(m)$ is given by functions s and \tilde{s} on \mathbb{C} holomorphic in λ and $\tilde{\lambda}$ respectively and is related by

$$s(\lambda) = \lambda^m \tilde{s}(\tilde{\lambda})$$

on the overlap \mathbb{C}^* . Expanding these functions as power series in their respective local coordinates, and using the fact that $\tilde{\lambda} = \lambda^{-1}$ and hence the space of holomorphic sections of $\mathcal{O}(m)$ is \mathbb{C}^{m+1} if $m > 0$. There are no global holomorphic sections if $m < 0$. A global holomorphic section of $\mathcal{O}(m)$, $m \geq 0$ is the same as a global function on \mathbb{C}^2 homogeneous of degree m (a polynomial). If $m > 0$ such a function is of the form

$$f([\pi]) = V^{AB\dots C} \pi_A \pi_B \dots \pi_C$$

for some symmetric object $V^{AB\dots C}$.

Holomorphic vector fields on \mathbb{CP}^1 are sections of the holomorphic tangent bundle $T\mathbb{CP}^1$. Using

$$\frac{\partial}{\partial \lambda} = -\lambda^{-2} \frac{\partial}{\partial \tilde{\lambda}}$$

and absorbing the minus signs into the local trivializations, we deduce that $T\mathbb{CP}^1 = \mathcal{O}(2)$. (**Exercise:** Consider a general section of $\mathcal{O}(2) \rightarrow \mathbb{CP}^1$ given by the local form (3.6) where (v, x, u) and (η, λ) are complex. Show that this section is invariant under (2.2) if $x \in \mathbb{R}$ and $u = -\bar{v}$. Thus deduce (2.3).)

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