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On the Construction of Families of type Π_1 Subfactors Each Containing a Middle Subfactors

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Abstract In this article we are going to construct a family of type Π_1 subfactors each containing a middle subfactor. As a result of the above construction we show that the set of the indices of hyperfinite irreducible subfactors contains the interval [37.0037,∞).

Keywords: *subfactors, von Neumann algebras, Jones Index, lattice, relative commutants*

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1. Introduction and Priliminaries

In the next chapter we are going to use locally trivial subfactors to construct a set of middle subfactors. The important of locally trivial subfactos was indicated by S.Popa at [9]. The simplest locally trivial subfactors are those having only two orthogonal projections in their relative commutant. It is well known that these subfactors are isomorphic to Jones subfactors.. Also it is easy easy to show that these kind of subfactors do not posses middle subfactors. For the locally trivial subfactors that their relative commutant have dimension larger than two we believe that the above result is still valid ie'there are no middle subfactors. Suppose we are Given a pair $L \subset M$ of subfactors that are limiting algebras of a tower of commuting squares.

Then using the results in [1] we can show that if the inclusion graphs of the corresponding finite C^* algebras are A graphs, then the subfactors $L \subset M$ do not have middle subfactors. One of the problems in the index theory, is to find the set of all the values for the indices of hyperfinite irreducible subfactors. Using the above constructions, we are going to show that the above set contains the interval (37.0037,∞).

2. Main Results

For a given pair of subfactors $N \subset M$, With $[M{:}N] = \lambda^{-1} < \infty$. Let e be a projection in M that induces the expectation N onto $Q = (e)' \cap N$. Let $P_1, P_2 = 1\text{-}P_1$ be a partition of unity in Q. Using standard argu-ments as in [5], there exists an isomorphism Φ, taking NP_1 onto NP_2. Let $L(P_1)$ be the set of all the elements of the form $L(P_1) = (x + \Phi(x);\ x \in NP_1)$. Then it is well known that $L(P_1)$, is a locally trivial subfactor of N. Also by (Lemma2.2.1) [5], $[N : L_P] = 1{=}tr(P_1) + 1{=}tr(P_2)$. Where tr is a unique normalized trace on M. Suppose P1, does not communicate with $e_0 = e$. Set $y = P_1eP_2 \neq 0$. Note that the relative commutant of $L(P_1)$, inside M is spanned by the projection P_1. Since y, does not communicate with P_1, we have that the algebra, $H(P_1) = < L(P_1);\ Y >$ is a middle subfac-tor which is strictly larger than $L(P_1)$ If under certain conditions $H(P_1)$ becomes strictly smaller than M, then $H(P_1)$ becomes a proper middle sub-factor. In this case by the above arguments it is easy to see that the inclusions $H(P_1) \supset L(P_1)$ and $H(P_1) \subset M$ are irreducible inclusion of subfactors. Let us denote $r_1 = [\ H(P_1) : L(P_1)]$ and $r_2 = [M : H(P_1)]$. Let IR, be the set of all irreducible subfactors of finite index. Let us denote by IIR, the set of indices of all subfactors in IR. Then by (Proposition2.1.15) [5] r_1r_2 is in IIR. It is easy to check that the set of of the elements f of the form $f = \Sigma_{j \in J} u_j g z_j e g w_j$, with $g = E_Q(P_2)$, $z_j \in Q$, $u_j = P_1x_jP_1$, $w_j = P_1y_jP_1$, $x_j \in N$ $y_j \in N$ where J is a set of indices is dense in $(H(P_1))P_1$. Let e_1 be a projection in N, such that $E_Q(e_1) = \lambda$. Then e_1 induces the expectation of Q onto the subfactor $Q_1 = (e)' \cap Q$. Next for a number r, $r \in IIR$, construct an irreducible subfac-tor Q_2, $Q_2 \subseteq Q_1$, with $[Q_1 : Q_2] = r$. We can define the projection e_3, Using (corollary 1.8) [13], there exits a projection e_3 in Q_1, such that e_3 induces the expectation of Q_2 onto The subfactor Q_3, with $Q_3 = (e_3)' \cap Q_2$. This process will induce the following tonnel, $M \supset N \supset Q \supset Q_1 \supset Q_2 \supset Q_3$. Let us set $P_1 = qe_1e_3$ with q a projection in Q_3. Now we can check that the following set of elements, f of the form,

$$f = P_1(1-\lambda)^2 e_0 \left(\sum_{j \in J} x_j z_j y_j \right) P_1 = (1-\lambda)^2 P_1 e_0 z P_1,$$

with z = $\sum_{j \in J} x_j z_j y_j$, with x_j, z_j, y_j as in the above and J a set of indices will be a dense subset of $(H(P_1))P_1$. In particular assuming now that $H(P_1) = M$ implies that the above set of elements are dense in MP_1. Furthermore as we mentioned in the above for any number $r \in IIR$; $e_3 \in Q_1$ can be chosen such that $tr(e_3) = r$. For example suppose

$tr(e_1) = tr(e_0) = .5$. Then it is easy to see that there exists a unitary $V \in (e_3)' \cap N = L$, with L a type π_1 Von Neumann algebra, such that $V e_1 V^* = 1 - e_1$. Then we can express the f from the above as $f = \lambda(1-\lambda)^2 e_1e_3ze_1e_3 + (e_1e_3e_0V e_1e_3)(e_1e_3V^*ze_1e_3)$. For a given real number S, let $[S]$, be the largest integer which is smaller or equal to $[S]$. Let us set $S_r = S - [S]$. Let us assume now that $H(P_1) = M$, We will get the following results.

Lemma 1 Keeping the same notations as in the above let $\lambda^{-1} = \left[\lambda^{-1}\right] + \lambda_r^{-1}$. Then there exist unitary operators U, U_2, U_3 in L and projection $p \le e_1$, such that f can be expresses as in the following, $f = \left(1-\lambda^2\right)\left(e_1e_3e_0qe_1q^* ze_1e_3 + e_1e_3e_0Ue_1q^*U^* ze_1e_3 + \sigma_1 e_1e_3e_0U_2e_1U_2^* ze_1e_3 + \sigma_2 e_1e_3e_0U_3e_1U_3^* ze_1e_3\right)$. With σ1, will be equal to 1 if $\left[\lambda^{-1}\right]$ odd integer and equal to zero otherwise. Simi-larly σ2 is equal to 1 if λ^{-1} is not an integer and equal to zero otherwise.

Proof First suppose $tr(e_0)^{-1} = \lambda^{-1} = 2n$, for some positive integer n, then let $f_1 = e_1, f_2, \ldots\ldots, f_{2n}$ be a partition of unity by orthogonal projections in L, such that for any odd integer $k < 2n$, there exists a unitary $V_k \in L_{f_k + f_{k+1}}$, with $f_k = V_k f_{k+1} V_k^*$. Using our definition of $f_1 = e_1$, this implies, $f = (1-\lambda)^2 \left(\sum_{i=1}^{n} \begin{pmatrix} e_1e_3e_0 f_{2i-1} ze_1e_3 \\ +e_1e_3e_0V_i f_{2i-1} V_i^* ze_1e_3 \end{pmatrix}\right)$.

Let $U = \sum_{i=1}^{i=n} V_i$. Then U is a unitary in L. Set $g = \sum_{i=1}^{i=n} f_{2i-1}$. The we have, $f = (1-\lambda)^2 \left(e_1e_3e_0gze_1e_3 + e_1e_3e_0UgU^* ze_1e_3\right)$. Next for each $1 \le i \le n$, there exists a unitary $m_i \in L_{e_1 + f_{2i-1}}$ such that $m_ie_im_i^* = f_{2i-1}$. Hence $g = \sum_{i=1}^{i=n} m_ie_im_i^*$. Next since for $i \ne j$, $\left(m_ie_im_i^*\right)\left(m_je_jm_j^*\right) = 0$. Thus we get $tr\left(m_ie_1m_i^*m_je_1m_j^*\right) = 0$. For $i \ne j$, let us define $y = e_1m_j^*m_ie_1$. Then it is easy to check $tr\left(yy^*\right) = 0$. This implies that $y = 0$. Another useful relation that we will need later is the following equality, $e_1qq^*e_1 = ne_1$ that can be checked easily. Let us define the operator h, with $h = \left(\sum_i m_i\right)e_1\left(\sum_k m_k^*\right) = qe_1q^*$ with, $1 \le i, k \le n$ and $q = \sum_i m_i$. Then using the above relations we can see that h is a projection, $tr(h) = tr(g)$, and UhU^* is orthogonal to h. Hence we get, $tr(h) = tr(UhU^*) = .5$. This will implies that f can be expressed as, $f = (1-\lambda)^2\left(e_1e_3e_0qe_1q^* ze_1e_3 + e_1e_3e_0Uqe_1q^*U^* ze_1e_3\right)$

Suppose $\lambda^{-1} = 2n+1$. Then we have the following partition of unity, $e = f_1, f_2, \ldots\ldots, f_{2n}, f_{2n+1}$. Where the above projections have equal traces. Furthermore ther exists a unitary $U_2 \in L_{e_1 + f_{2n+1}}$ such that f can be expressed as, $f = (1-\lambda)^2 \begin{pmatrix} e_1e_3e_0qe_1q^* ze_1e_3 + e_1e_3e_0Uqe_1q^*U^* ze_1e_3 \\ +e_1e_3e_0U_2qe_1q^*U_2^* ze_1e_3 \end{pmatrix}$. then we have the following partition of unity by the following projections, $e = f_1, f_2, f_3, \ldots\ldots, f_{2n}, \sigma_1 f_{2n+1}, \sigma_2 f_0$. Where σ$_1$ and σ$_2$, can only take values 0 or 1, depending if the corresponding projections f_{2n+1} and f_0 are or are not equal to zero. Furthermore for $k \ne 0$, all non zero projections fk's, have equal traces and $tr(f_0) < \lambda$. Hence generally f can be expressed as, $f = (1-\lambda)^2 \begin{pmatrix} e_1e_3e_0qe_1q^* ze_1e_3 \\ +e_1e_3e_0Uqe_1q^*U^* ze_1e_3 \\ +\sigma_1 e_1e_3e_0U_2e_1U_2^* ze_1e_3 \\ +\sigma_2 e_1e_3e_0U_3e_1pU_3^* ze_1e_3 \end{pmatrix}$.

Where U and U_2 are as in the above, U_3 a unitary in $L_{e_1+f_0}$ and p, is a sub projection of e_1 and is in L_{e_1+f}. At this point note that we can extend U_2 and U_3 to be unitaries in L. Finally we can express f, as

$$f = (1-\lambda)^2 \begin{pmatrix} \left(e_1e_3e_0qe_1e_3\right)\left(e_1e_3q^* ze_1e_3\right) \\ +\left(e_1e_3e_0Uqe_1e_3\right)\left(e_1e_3q^*U^* ze_1e_3\right) \\ +\sigma_1\left(e_1e_3e_0U_2e_1e_3\right)\left(e_1e_3U_2^* ze_1e_3\right) \\ +\sigma_2\left(e_1e_3e_0U_3e_1e_3p\right)\left(e_1e_3U_3^* ze_1e_3\right) \end{pmatrix}.$$

Now let us set the following notations. $n_1 = e_1e_3e_0qe_1e_3$, $n_2 = e_1e_3e_0Uqe_1e_3$, $n_3 = e_1e_3e_0U_2e_1e_3$, $n_4 = e_1e_3e_0U_3e_1e_3$ and $G = N_{e_1,e_3}$.

Lemma 2 Keeping the same notations as in the above, and assuming that σ$_1$ and σ$_2$ both different from zero and without loss of generality, we have the following equalities. $E_{N_{e_1,e_3}}\left(n_k^*n_k\right) = e_1e_3,\ j \ne k,\ E_{N_{e_1,e_3}}\left(n_j^*n_k\right) = 0,\ k = 1,1,3,4.$

Proof Note that since e_3 commues with all the above operators, drop-ping e_3 from all the operations does not makes any different from the final outcome. So in the following operations we ignore the existence of e_3. In particular we can identify N_{e_1} with $G = N_{e_1}$. Since the proof of the above equalities are very similar, we only show some of the equalities. We have $n_1n_1^* = e_1e_0qe_1q^*e_0e_1$. Hence we get, $E_G\left(n_1n_1^*\right) = \lambda^2e_1qq^*e_1 = n\lambda^2e_1$. Next $n_2^*n_1 = e_1q^*U^*e_0e_1e_0qe_1 = \lambda e_1q^*U^*e_0qe_1$. Thus $E_G\left(n_2^*n_1\right) = \lambda^2e_1q^*U^*qe_1 = \lambda^2e_1q^*U^*qe_1$. But we have, $e_1e_1qq^*e_1/n = e_1qq^*e_1/n = e_1$. This implies, $E_G\left(n_2^*n_1\right) = n^{-2}\lambda^2e_1q^*qe_1q^*U^*qe_1q^*qe_1 = n^{-2}\lambda^2e_1q^*U^*U\left(qe_1q^*\right)U^*\left(qe_1q^*\right)qe_1$. But $U\left(qe_1q^*\right)U^*$ is orthogonal qe_1q^* which implies that $E_G\left(n_2^*n_1\right) = 0.$. Fur-thermore $n_3^*n_3 = e_1U_2^*e_0e_1e_1e_0U_2e_1 = \lambda e_1U_2^*e_0qe_1$. Hence we get, $E_G\left(n_3n_3^*\right) = \lambda^2e_1$. Next $n_3^*n_1 = e_1U_2^*e_0e_1e_1e_0qe_1$

$= \lambda e_1 U_2^* e_0 q e_1$. This implies, $E_G(n_3 n_1) = \lambda^2 e_1 U_2^* q e_1$ $= e_1 U_2^* q e_1 q^* q e_1 / n = U_2^* U_2 e_1 U_2^* q^* e_1 q^* q e_1 / n$. But using the above relations, $(U_2 e_1 U_2^*)(q e_1 q^*) = 0$, which implies $E_G(n_3^* n_1) = 0$. Now let us calculate $E_G(n_1^* n_4)$. $E_G(n_1^* n_4) = E_G(e_1 q^* e_0 e_1 e_1 e_0 U_3 e_1 p) = \lambda^2 e_1 q^* U_3 e_1 p$ $= \lambda^2 e_1 q^* U_3 e_1 p U_3^* U_3$. Now using the above relations we can show that $e_1 q^* U_3 p U_3^* = 0$, hence $E_G(n_1^* n_4) = 0$.

Assuming that Me_1e_3 is acting standardly on $H = [Me_1e_3]$ and $L(P_1) = M$, the above lemma implies that the operator identity can be spanned by at most four orthogonal projections in G each of trace less or equal to λ. Hence by Remarks(1.4) [13], we get the following Corollary,

Corollary 3 Keeping the same notations as before, for [M:N] > 4, $H(P_1)$ is a proper middle subfactor.

As before let *IIR* represents the set of all indices of irreducible hyperfinite subfactors.

Suppose $H(P_1)$ is a proper middle subfactor, ie' $L(P_1) \subset H(P_1) \subset M$, where the inclusions are restrict. Let us denote $r_1 = [H(P_1):L(P_1)]$ and $r_2 = [M:L(P_1)]$. Also using the fundamental property of the index of subfactors, $r_1r_2 = r = [M:L(P_1)] = [M : N][N : L(P_1)]$. But $[M : N] = \lambda^{-1}$ and $[N:L(P_1)] = (tr(P_1)(tr(P_2))^{-1}$ Hence $r = r_1r_2 = \lambda^{-1}(tr(P_1)tr(P_2))^{-1}$ And by the results of [5], $r \in IIR$. Now notice that by the result of S.Popa in "Subfactors and classification in von Neumann algebras" Corollary(4.4) of the above article indicate the gap in *IIR* between the values 4 and $2+\sqrt{5}$. In fact $2+\sqrt{5} \simeq 4.026$, corresponds to the square of the norm of Coxeter graph E_{10} and there exists a subfactor of such an index. By its definition $P_1 = qe_1e_3$ and we had $\lambda = tr(e_1)$ $\lambda_1 = tr(e_3)$. Hence $tr(P_1) = tr(q)\lambda\lambda_1$. Let us set $c = tr(q)$, and $\omega = 2+\sqrt{5}$ then c can take any value in the interval [0,1]. Note also that $(\lambda_1)^{-1}$ can take any value in *IIR* larger than .5. Let us denote, $\alpha = \lambda\lambda_1$, $\beta = c\alpha$, then $\beta = tr(P_1)$. Note that $(\alpha)^{-1}$ can take any value in *IIR* larger than or equal to 2ω. Hence if we set $\beta = c\lambda\lambda_1$, then we have $0 \le \beta \le 1/(2\omega)$. Since q can be taken to be any projection in Q_3, $tr(P_1)$, can get any value in the interval $[0, 1/(2\omega)]$. This implies the following theorem.

Theorem 4 Keeping the same notations as in the above, suppose $\lambda^{-1} = \omega$ and $(\lambda_1)^{-1} = 2$. Let $N \supset M$, $[M:N]=\lambda^{-1}$ be a pair of irreducible subfactors and lets define a projection $p = qe_1e_3$, for some projection q in Q_3. Then $H(P_1) = \langle L(P_1), p_1ep_2 \rangle$ is a proper irreducible subfactor of M. In particular letting q to vary in Q_3, we will get that *IIR* includes the interval $[\omega/(1/2\omega)(1-1/2\omega), \infty]$ $\simeq [37.0037, \infty)$.

At this end note that by the above arguments there exists a function Φ, acting on the above interval

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