A Posteriori Error Estimates of Residual Type for Second Order Quasi-Linear Elliptic PDEs

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Abstract

We derived a posteriori error estimates for the Dirichlet problem with vanishing boundary for quasi-linear elliptic operator:

$$\begin{aligned} -\nabla\cdot(\alpha(x,\nabla u)\nabla u) &= f(x) && \text{in } \Omega\subset\mathbb{R}^2,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is assumed to be a polygonal bounded domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$, and α is a bounded function which satisfies the strictly monotone assumption. We estimated the actual error in the H^1-norm by an indicator η which is composed of L^2-norms of the element residual and the jump residual. The main result is divided into two parts; the upper bound and the lower bound for the error. Both of them are accompanied with the data oscillation and the α-approximation term emerged from nonlinearity. The design of the adaptive finite element algorithm were included accordingly.

Keywords: Adaptive Finite Element, Quasi-Linear Elliptic PDEs

1. Introduction

A posteriori error estimation began playing role in analyzing the accuracy of the numerical solution with a pioneering work of Babuška and Rheinboldt (Babuška, I., 1978). A local estimator not only shows us how good the approximation performs, but sometimes also acts as an indicator used to determine whether that local mesh should be refined. From this usage, a new mesh will be created with the expectation that it will improve the accuracy of the approximation in efficiency way, without increasing the degree of polynomials used in the approximation. All of this ensemble forms the following procedure:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

In principle, the local estimator, or indicator, should be derived elementwise from the problem residual which is computed from the discrete solution and the given data of the problem. Thus, after solving, the indicators will be output from the submodule ESTIMATE. All elements with higher value of the indicators than the user's tolerance must be marked. Those marked elements then must be divided by some appropriate strategies. The new discrete problem with the resulted finer mesh is now ready to be solved again. The adaptive algorithm iterates the above procedure until the overall error is determined small enough. Applying finite element method in the step SOLVE allows us to call this process as the adaptive finite element method (AFEM).

The introductory principles of adaptive finite elements and additional references can be found in the books by Ainsworth and Oden (Ainsworth, M., 2000), and Verfürth (Verfürth, R., 1996). For the linear case we refer to the works of Morin, Nochetto, and Siebert (Morin, P., 2000), where the convergence for second order elliptic equations with piecewise constant coefficients and without lower order terms were investigated by using a technique originated by Dörfler (Dörfler, W., 1996). They also introduced the notion of data oscillation meant to quantify information missed in projecting the residual with discrete functions which is a process associated with the finite element method. Thereafter, Mekchay and Nochetto extended these results for general second order linear elliptic PDEs (Mekchay, K., 2005), and (in cooperation with Morin) for the Laplace-Beltrami operator on surfaces (Mekchay, K., 2011). Recently, Garau, Morin, and Zuppa (Garau, E. M., 2011) designed an adaptive finite element algorithm for solving quasi-linear elliptic problems based on a Kačanov iteration. They estimated the problem residual instead of the actual error, which need a practical way to deal with the negative norm in the dual space H^{-1}. The quasi-optimal convergence rate of the algorithm were proved in (Garau, E. M., in press).

The objective of this article is to obtain a posteriori error estimates for the Dirichlet problem with vanishing boundary for quasi-linear elliptic operator:

$$\begin{aligned} -\nabla\cdot(\alpha(x,\nabla u)\nabla u) &= f(x) \qquad \text{in } \Omega\subset\mathbb{R}^2, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is assumed to be a polygonal bounded domain in $\mathbb{R}^2$, $f\in L^2(\Omega)$, and α is a bounded function which satisfies the monotonic properties (see assumptions (3)-(4) below) for admission of a unique weak solution. We estimated the actual error in the H^1-norm by an indicator η which is composed of L^2- norms of the element residual and the jump residual.

This paper is organized as follows. In §2, we give the weak formulation of (1) and its corresponding discrete problem, together with some assumptions imposed to α for admission of a unique weak solution. The analysis of a posteriori error estimation is described in §3, which is divided into two parts; the upper bound and the lower bound for the error. Then we discuss about the adaptive algorithm in the last section.

2. Problem Formulations

By $L^2(\Omega)$, we denote the usual Lebesgue space with norm

$$\|f\|_0 = \left(\int_\Omega |f(x)|^2 dx\right)^{1/2}.$$

The Sobolev space of functions $u\in L^2(\Omega)$ with weak derivatives $\nabla u\in L^2(\Omega)$ is denoted by $H^1(\Omega)$ with semi norm

$$|u|_1 = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2},$$

and norm

$$\|u\|_1 = \left(\|u\|_0^2 + |u|_1^2\right)^{1/2}.$$

Normally, $\|u\|_{0,\omega}$ and $\|u\|_{1,\omega}$ represent L^2-norm and H^1-norm restricted on the subdomain ω, respectively. According to the boundary condition of Dirichlet type, $H_0^1(\Omega)$ is a subset of $H^1(\Omega)$ composed of functions vanishing on $\partial\Omega$.

We multiply the PDE (1) by a smooth test function $\phi\in C^\infty(\Omega)$ and integrate by parts over Ω to admit the weak formulation: find $u\in H_0^1(\Omega)$ such that

$$A(u;u,\phi) = L(\phi) \quad \forall\phi\in H_0^1(\Omega), \tag{2}$$

where $A(u;v,\phi) = \int_\Omega \alpha(\cdot,\nabla u)\nabla v\cdot\nabla\phi$ and $L(\phi) = \int_\Omega f\phi$. Let us define a nonlinear vector field $\vec{a}$ on $\Omega\times\mathbb{R}^n$ by

$$\vec{a}(x,p) = \alpha(x,p)p. \tag{3}$$

According to the monotonicity methods described in (Evans, L. C., 1998), to guarantee the existence of a unique weak solution of (2), the vector field $\vec{a}$ is assumed to be strictly monotone in the second variable; that is

$$(\vec{a}(p)-\vec{a}(q))\cdot(p-q) \geq \theta|p-q|^2, \tag{4}$$

for all $x\in\Omega$, for all $p,q\in\mathbb{R}^2$ and for some constant $\theta>0$. Some examples of problems falling into the case are given by:
(I) The equations of prescribed mean curvature:

$$\alpha(x,\nabla u) := [1+\|\nabla u\|^2]^{-1/2}.$$

(II) The p-Laplacian:

$$\alpha(x,\nabla u) := \|\nabla u\|^{p-2}, \quad p>1.$$

(III) The subsonic flow of a irrotational, ideal, compressible gas:

$$\alpha(x,\nabla u) := \left[1-\frac{\gamma-1}{2}\|\nabla u\|^2\right]^{-\frac{1}{\gamma-1}}, \quad \gamma>1.$$

Consider $V_h \subset H_0^1(\Omega)$, a class of continuous piecewise linear functions over the shape regular conforming triangulation $\mathcal{T}_h$ of Ω, i.e.

$$V_h := \{v \in H_0^1(\Omega) \mid v_{|T} \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h\}, \tag{5}$$

where $\mathcal{P}_1(T)$ is the set of linear polynomial on T. Note that all T in $\mathcal{T}_h$ are triangular elements and $\Omega = \bigcup_{T\in\mathcal{T}_h} T$. The Lagrange basis functions $\{\Phi_i\}$ satisfy

$$\Phi_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for } x_j \in \Omega.$$

The discrete problem corresponding to (2) is then constructed as: find $u_h \in V_h$ such that

$$A(u_h; u_h, \phi_h) = L(\phi_h), \qquad \forall \phi_h \in V_h. \tag{6}$$

3. A Posteriori Error Analysis

Before we get to the analysis, we would like to introduce some symbols associated with geometric information of the triangulation. We define d_T the diameter of triangle T and d_S the length of side S. Let $\mathcal{S}_h$ denote the set of all interior sides of the triangulation $\mathcal{T}_h$.

Consider $A(u; e_h, v)$ where $e_h = u - u_h$ is the error. Note that we use the abbreviations $\alpha = \alpha(\cdot, u)$ and $\alpha_h = \alpha(\cdot, u_h)$ whenever convenient. By means of Green's identity, we obtain this formula

$$\begin{aligned} A(u; e_h, v) &= \int_\Omega fv - \int_\Omega \alpha_h \nabla u_h \cdot \nabla v - \int_\Omega (\alpha - \alpha_h)\nabla u_h) \cdot \nabla v \\ &= \sum_{T\in\mathcal{T}_h} \left[\int_T (f + \nabla \cdot (\alpha_h \nabla u_h))v - \int_{\partial T} (\alpha_h \nabla u_h) \cdot \nu_T v \right] - \int_\Omega (\alpha - \alpha_h)\nabla u_h) \cdot \nabla v \end{aligned} \tag{7}$$

for all $v \in H_0^1(\Omega)$. Here ν_T is the unit outward normal vector of T.

Let the functionals R_T and J_S represent the element residual

$$R_T(u_h) := f + \nabla \cdot (\alpha_h \nabla u_h), \quad \text{in } T \in \mathcal{T}_h \tag{8}$$

and the jump residual

$$J_S(u_h) := -[(\alpha_h^+ \nabla u_h^+) \cdot \nu^+ + (\alpha_h^- \nabla u_h^-) \cdot \nu^-] =: [\![(\alpha_h \nabla u_h)]\!]_S \cdot \nu_S, \quad \text{on } S \in \mathcal{S}_h, \tag{9}$$

where S is the side shared by two triangles, T^+ and T^- with the unit outward normal vectors ν^+ and ν^-, respectively (see Figure 2), and $\nu_S := \nu^-$. Equation (7) then turns into:

$$A(u; e_h, v) = \sum_{T\in\mathcal{T}_h} \int_T R_T(u_h)v + \sum_{S\in\mathcal{S}_h} \int_S J_S(u_h)v + \int_\Omega (\alpha_h - \alpha)\nabla u_h \cdot \nabla v, \tag{10}$$

for all $v \in H_0^1(\Omega)$. Let us define the local error indicator as

$$\eta_h^2(T) = d_T^2 \|R_T(u_h)\|_{0,T}^2 + \frac{d_T}{2} \sum_{S\in\partial T} \|J_S(u_h)\|_{0,S}^2, \tag{11}$$

and the global estimator as

$$\eta_h^2(\Omega) = \sum_{T\in\mathcal{T}_h} \eta_h^2(T). \tag{12}$$

3.1 Upper Bound

From the error representation (10), we obtain the upper bound for the error as follow.

Theorem 1 (Upper bound) *Let u_h be the approximate solution of the model problem with the error e_h. Then*

$$\|e_h\|_1 \leq C\eta_h(\Omega) + \|(\alpha_h - \alpha)\nabla u_h\|_0,$$

where the constant C depends only on the shape regularity of the triangulation $\mathcal{T}_h$ of Ω.

In order to prove Theorem 1, we need the following lemma constructed by Clément (Clément, P., 1975).

Lemma 2 (Clément's interpolations) *Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω. Then there exists a linear mapping $I_h : H^1(\Omega) \to V_h$ such that*

$$\begin{aligned} \|v - I_h v\|_{m,T} &\leq Cd_T^{1-m}\|v\|_{1,\omega_T} \text{ for } v \in H^1(\Omega),\ m = 0, 1,\ T \in \mathcal{T}_h \\ \|v - I_h v\|_{0,S} &\leq Cd_T^{1/2}\|v\|_{1,\omega_T} \text{ for } v \in H^1(\Omega),\ S \in \partial T,\ T \in \mathcal{T}_h, \end{aligned} \tag{13}$$

where the constant C depends only on the shape regularity of the triangulation $\mathcal{T}_h$, and ω_T is the patch of all elements that share at least one vertex with T, see Figure 1.

Proof of Theorem 1 Let $\mathcal{I}_h : H_0^1(\Omega) \to V_h$ be the L^2-projection. If we use $\mathcal{I}_h v$ as a test function in (7), by (6) we can easily show that

$$A(u; e_h, I_h v) = \int_\Omega (\alpha_h - \alpha)\nabla u_h \cdot \nabla I_h v. \tag{14}$$

Substitute $\mathcal{I}_h v$ again in place of v in (10), we obtain

$$\sum_{T\in\mathcal{T}_h} \int_T R_T(u_h)\mathcal{I}_h v + \sum_{S\in\mathcal{S}_h} \int_S J_S(u_h)\mathcal{I}_h v = 0. \tag{15}$$

Thus it is reasonable to rewrite (10) as

$$\begin{aligned} A(u; e_h, v) &= \sum_{T\in\mathcal{T}_h} \int_T R_T(u_h)(v - \mathcal{I}_h v) + \sum_{S\in\mathcal{S}_h} \int_S J_S(u_h)(v - \mathcal{I}_h v) \\ &\quad + \int_\Omega (\alpha_h - \alpha)\nabla u_h \cdot \nabla v. \end{aligned} \tag{16}$$

Thanks to the Clément's interpolations (Lemma 2) and Cauchy-Schwarz inequality, there holds

$$\begin{aligned} A(u; e_h, v) &\leq C\left[\sum_{T\in\mathcal{T}_h} d_T\|R_T(u_h)\|_{0,T}\|v\|_{1,\omega_T} + \sum_{S\in\mathcal{S}_h} d_T^{1/2}\|J_S(u_h)\|_{0,S}\|v\|_{1,\omega_S}\right] \\ &\quad + \|(\alpha_h - \alpha)\nabla u_h\|_0\|\nabla v\|_0 \\ &\leq C\|v\|_1\left[\sum_{T\in\mathcal{T}_h} d_T^2\|R_T(u_h)\|_{0,T}^2 + \sum_{S\in\mathcal{S}_h} d_T\|J_S(u_h)\|_{0,S}^2\right]^{1/2} \\ &\quad + \|(\alpha_h - \alpha)\nabla u_h\|_0\|v\|_1 \end{aligned} \tag{17}$$

for some generic constant $C > 0$ depending only on regularity of the triangulation. Here ω_S denotes the patch of two elements sharing the side S, see Figure 1.

Substituting e_h in place of v in (17) results in

$$A(u; e_h, e_h) \leq C\eta_h(\Omega)\|e_h\|_1 + \|(\alpha_h - \alpha)\nabla u_h\|_0\|e_h\|_1. \tag{18}$$

The monotonic assumption (4) allows us to claim that $A(u; e_h, e_h) \geq \theta\|e_h\|_1^2$, for a positive constant θ. We then finally obtain the upper estimate for the error

$$\|e_h\|_1 \leq C\eta_h(\Omega) + \|(\alpha_h - \alpha)\nabla u_h\|_0. \tag{19}$$

Remark 1 Theorem 1 tells us that the error is controlled by the error estimator $\eta_h(\Omega)$ and the oscillation from nonlinear term $\|(\alpha_h - \alpha)\nabla u_h\|_0$. Then it is helpful in designing a stopping criterion for the AFEM, if $\|(\alpha_h - \alpha)\nabla u_h\|_0$ is small enough. Since α is not computable, we need some further analysis to handle with $\|(\alpha_h - \alpha)\nabla u_h\|_0$. For example, let us assume that $\vec{a}$ is Lipschitz continuous, i.e. there is a constant c such that

$$\|\vec{a}(p) - \vec{a}(q)\| \leq c\|p - q\|, \tag{20}$$

for all $p, q \in V_h$ and for any norm $\|\cdot\|$ defined on V_h. Consider the α-approximation term:

$$\begin{aligned} \|(\alpha_h - \alpha)\nabla u_h\|_0 &= \|(\alpha_h\nabla u_h - \alpha\nabla u) + (\alpha\nabla u - \alpha\nabla u_h)\|_0, \\ &\leq c\|\nabla(u - u_h)\|_0 + \|\alpha\nabla(u - u_h)\|_0. \end{aligned}$$

Since α is a bounded function, there is a real number $M < \infty$ such that

$$\|(\alpha_h - \alpha)\nabla u_h\|_0 \leq (c + M)\|e_h\|_1.$$

If $(c + M)$ is lower than 1, the α-approximation term in (19) can be ignored.

3.2 Lower Bound

In order to obtain a similar lower bound for the error, we need to estimate the indicator $\eta_h(T)$ locally on T. The idea is to estimate the two components of $\eta_h(T)$: $\|R_T(u_h)\|_{0,T}$ and $\|J_S\|_{0,S}$ in terms of $\|e_h\|_1$. From now on we write $R_T(u_h)$ as R_T and $J_S(u_h)$ as J_S in short. With the idea of bubble functions introduced by (Verfürth, R., 1996), we obtain the following local lower bound.

Theorem 3 (Local lower bound) *Let u_h be the approximate solution of the model problem with the error e_h. Then*

$$\eta_h^2(T) \leq C\left\{\|e_h\|^2_{1,\tilde{\omega}_T} + osc_h^2(\tilde{\omega}_T) + \|(\alpha_h - \alpha)\nabla u_h\|^2_{0,\tilde{\omega}_T}\right\},$$

where $\tilde{\omega}_T$ is the patch of elements sharing a common side with T (see Figure 1), and the oscillation on T is defined by

$$osc_h^2(T) := d_T^2\|\bar{R}_T - R_T\|^2_{0,T} + \frac{d_T}{2}\sum_{S\in\partial T}\|\bar{J}_S - J_S\|^2_{0,S}. \tag{21}$$

The oscillation on a subset $\omega \in \Omega$ is defined by

$$osc_h^2(\omega) := \sum_{T\in\mathcal{T}_h, T\subset\omega} osc_h^2(T). \tag{22}$$

Here, the constant C depends only on the shape regularity of the triangulation $\mathcal{T}_h$.

Before giving the proof of Theorem 3, we introduce here the notions of bubble functions used for estimations of the interior and edge residuals.

For each $T \in \mathcal{T}_h$, we define ψ_T to be a polynomial function on T vanishing on ∂T and $0 \leq \psi_T \leq 1 = \max\psi_T$.

For each $S \in \mathcal{S}_h$, we define also χ_S to be a polynomial function on ω_S, as denoted in (17), vanishing on $\partial\omega_S$ and $0 \leq \chi_S \leq 1 = \max\chi_S$.

The proof of Theorem 3 relies on the properties on bubble functions as stated in the following Lemmas which are proved in (Ainsworth, M., 2000).

Lemma 4 *Let $P(T) \subset H^1(T)$ be a finite dimensional subspace and let ψ_T denote the interior bubble function over the element T. Then there exists a constant C such that for all $v \in P(T)$*

$$C^{-1}\|v\|^2_{0,T} \leq \int_T \psi_T v^2 \leq C\|v\|^2_{0,T} \tag{23}$$

and

$$C^{-1}\|v\|_{0,T} \leq \|\psi_T v\|_{0,T} + d_T|\psi_T v|_{1,T} \leq C\|v\|_{0,T} \tag{24}$$

where the constant C is independent of v and d_T.

Lemma 5 *Let $P(\omega_S) \subset H^1(\omega_S)$ be a finite dimensional subspace. Let $S \subset \partial T$ be an edge and let χ_S be the corresponding edge bubble function. Then there exists a constant C such that for all $v \in P(\omega_S)$*

$$C^{-1}\|v\|^2_{0,S} \leq \int_S \chi_S v^2 \leq C\|v\|^2_{0,S} \tag{25}$$

and

$$d_T^{-1/2}\|\chi_S v\|_{0,T} + d_T^{1/2}|\chi_S v|_{1,T} \leq C\|v\|_{0,S} \tag{26}$$

where the constant C is independent of v and d_T.

Proof of Theorem 3

(a) Estimation of interior residual:

Let $\bar{R}_T$ be a polynomial approximation to R_T on T. Ones can see that $\mathrm{supp}(\psi_T\bar{R}_T) \subset T$. It may be extended to the rest of the domain as a continuous function by defining its value outside the element to zero. The resulting extended function then belongs to $H^1_0(\Omega)$ and thus can be used as a test function in the residual equation (10),

$$A(u_h; e_h, \psi_T\bar{R}_T) = \int_T R_T(\psi_T\bar{R}_T) + \int_T(\alpha_h - \alpha)\nabla u_h\cdot\nabla(\psi_T\bar{R}_T). \tag{27}$$

Notice that

$$\int_T \psi_T \bar{R}_T^2 = \int_T \psi_T \bar{R}_T(\bar{R}_T - R_T) + \int_T \psi_T \bar{R}_T R_T. \tag{28}$$

The right hand side may be bounded with the aid of the Cauchy-Schwarz inequality. Property (24) of the interior bubble function combined with (27) help us derive

$$\begin{aligned}\int_T \psi_T \bar{R}_T^2 &\leq \|\psi_T \bar{R}_T\|_{0,T}\|\bar{R}_T - R_T\|_{0,T} + \|\alpha_h \nabla e_h\|_{0,T}|\psi_T \bar{R}_T|_{1,T} \\ &\quad + \|(\alpha_h - \alpha)\nabla u_h\|_{0,T}|\psi_T \bar{R}_T|_{1,T} \qquad (29) \\ &\leq C\|\bar{R}_T\|_{0,T}\left\{\|\bar{R}_T - R_T\|_{0,T} + d_T^{-1}\|e_h\|_{1,T} + d_T^{-1}\|(\alpha_h - \alpha)\nabla u_h\|_{0,T}\right\},\end{aligned}$$

for some generic constant $C > 0$. Applying property (23) of the interior bubble function together with the triangle inequality leads to the bound for the element residual

$$\|R_T\|_{0,T} \leq C\left\{\|\bar{R}_T - R_T\|_{0,T} + d_T^{-1}\|e_h\|_{1,T} + d_T^{-1}\|(\alpha_h - \alpha)\nabla u_h\|_{0,T}\right\}. \tag{30}$$

(b) Estimation of edge residual:

We first extend the jump residual J_S, defined originally on S, constantly along the normal direction of S such that it is defined on ω_S, and denote this extension by $Ex(J_S)$. Similarly, let $\bar{J}_S$ be a polynomial approximation to the jump $Ex(J_S)$ on the patch ω_S. In the same manner, the function $\chi_S \bar{J}_S$ having $\text{supp}(\chi_S \bar{J}_S) \subset \omega_s$ can be extended to the whole domain and can be used as a choice of v in (10):

$$A(u_h; e_h, \chi_S \bar{J}_S) = \int_{\omega_S} \chi_S \bar{J}_S R_T + \int_S \chi_S \bar{J}_S J_S + \int_{\omega_S} (\alpha_h - \alpha)\nabla u_h \cdot \nabla \chi_S \bar{J}_S. \tag{31}$$

Now consider

$$\int_S \chi_S \bar{J}_S^2 = \int_S \chi_S \bar{J}_S(\bar{J}_S - J_S) + \int_S \chi_S \bar{J}_S J_S. \tag{32}$$

Each of the right hand side terms is dealt with the edge bubble function's properties and the scaled trace inequality for side $S \in \partial T$. We arrive at

$$\begin{aligned}\int_S \chi_S \bar{J}_S^2 &\leq \|\chi_S \bar{J}_S\|_{0,S}\|\bar{J}_S - J_S\|_{0,S} + \|\alpha \nabla e_h\|_{0,\omega_S}|\chi_S \bar{J}_S|_{1,\omega_S} + \|\chi_S \bar{J}_S\|_{0,\omega_S}\|R_T\|_{0,\omega_S} \\ &\quad + \|(\alpha_h - \alpha)\nabla u_h\|_{0,\omega_S}|\chi_S \bar{J}_T|_{1,\omega_S}, \\ \|\bar{J}_S\|_{0,S}^2 &\leq C\|\bar{J}_S\|_{0,S}\left\{\|\bar{J}_S - J_S\|_{0,S} + d_T^{-1/2}\|e_h\|_{1,\omega_S} + d_T^{1/2}\|R_T\|_{0,\omega_S}\right. \\ &\quad \left. + d_T^{-1/2}\|(\alpha_h - \alpha)\nabla u_h\|_{0,\omega_S}\right\}, \qquad (33)\end{aligned}$$

for some generic constant $C > 0$. As a consequence of (30) and the triangle inequality, we have the estimate to the jump residual

$$\begin{aligned}\|J_S\|_{0,S} &\leq C\left\{\|\bar{J}_S - J_S\|_{0,S} + d_T^{-1/2}\|e_h\|_{1,\omega_S} + d_T^{1/2}\|\bar{R}_T - R_T\|_{0,\omega_S}\right. \\ &\quad \left. + d_T^{-1/2}\|(\alpha_h - \alpha)\nabla u_h\|_{0,\omega_S}\right\}. \qquad (34)\end{aligned}$$

The asserted estimate for $\eta_h(T)^2$ is thus obtained by the combination of the square of (30) and the sum squares of (34).

$$\eta_h^2(T) \leq C\left\{\|e_h\|_{1,\tilde{\omega}_T}^2 + \text{osc}_h^2(\tilde{\omega}_T) + \|(\alpha_h - \alpha)\nabla u_h\|_{0,\tilde{\omega}_T}^2\right\}. \tag{35}$$

Remark 2 Ones can see from Theorem 3 that, if $\text{osc}(\tilde{\omega}_T)$ and $\|(\alpha_h - \alpha)\nabla u_h\|_{0,\tilde{\omega}_T}$ is sufficiently small relative to the error, the error e_h will be large when the indicator $\eta_h(T)$ is large. This is inline with the idea of marking strategy that mark those elements if their local indicators are higher than user's tolerance. By Remark 1, the α-approximation term can be absorbed to the error. However, the oscillation still needs to be controlled by a reasonable marking.

4. Adaptive Algorithm

For a given conforming shape regular triangulation $\mathcal{T}_0$ along with input data α and f, the adaptive finite element algorithm proceeds as the following:

Pick an initial guess u_0 with $u_0 = 0$ on $\partial\Omega$, choose $0 < \theta_E < 1$, and set $h = 1$.

1. u_h=SOLVE($\mathcal{T}_h, u_{h-1}, \alpha, f$).
2. $\{\eta_h(T), \text{osc}_h(T)\}_{T\in\mathcal{T}_h}$ = ESTIMATE($\mathcal{T}_h, u_h, \alpha, f$).
3. $\hat{\mathcal{T}}_h$ = MARK($\mathcal{T}_h, \{\eta_h(T), \text{osc}_h(T)\}_{T\in\mathcal{T}_h}$).
4. $\mathcal{T}_{h+1}$ = REFINE($\mathcal{T}_h, \hat{\mathcal{T}}_h$)
5. Set $h = h + 1$ and go to Step 1.

Next, we discuss about the detail of each procedure at the h^{th}-iteration.

The procedure SOLVE: A relatively simple way to solve nonlinear problem is the Kacanov method (Han, W.,1997) which is kind of linearlization. Given the initial approximation $U_0 = u_{h-1}$, we merely solve a sequence of linear problems

$$A(U_{k-1}; U_k, \phi_h) = L(\phi_h), \qquad \forall\phi_h \in V_h, \tag{36}$$

for $k = 1, 2, \ldots$, instead of the nonlinear problem (6), until U_K and U_{k-1} are close enough. Then the procedure outputs the approximate solution $u_h = U_K$. Nevertheless, some drawbacks of this fixed-point method are time consuming and requiring of some suitable assumptions for converging.

The procedure ESTIMATE: This procedure returns all quantities required in the procedure mark which are $\eta_h(T)$ and $\text{osc}_h(T)$ for all $T \in \mathcal{T}_h$. However, if necessary the approximation of the oscillation of the nonlinear term may be included.

The procedure MARK: To construct a subset $\hat{\mathcal{T}}_h$ of $\mathcal{T}_h$, we use the Marking Strategy E introduced by Dörfler (Dörfler, W., 1996) which guarantees the error reduction.

Marking Strategy E: Given a parameter $0 < \theta_E < 1$, construct a minimal subset $\hat{\mathcal{T}}_h$ of $\mathcal{T}_h$ such that

$$\sum_{T\in\hat{\mathcal{T}}_h} \eta_h^2(T) \geq \theta_E^2 \eta_h^2(\Omega), \tag{37}$$

and mark all elements in $\hat{\mathcal{T}}_h$ for refinement.

Another strategy used to control the oscillation reduction is the Marking Strategy O introduced by Morin et al. (Morin, P., 2000).

Marking Strategy O: Given a parameter $0 < \theta_O < 1$ and the subset $\hat{\mathcal{T}}_h \subset \mathcal{T}_h$ produced by Marking Strategy E, enlarge $\hat{\mathcal{T}}_h$ to a minimal set such that

$$\sum_{T\in\hat{\mathcal{T}}_h} \text{osc}_h^2(T) \geq \theta_O^2 \text{osc}_h^2(\Omega), \tag{38}$$

and mark all elements in $\hat{\mathcal{T}}_h$ for refinement.

Remark 3 In light of the investigation in (Cascon, J. M., 2008), the rate of convergence for separate marking is suboptimal except for some range of marking parameters θ_E and θ_O. Observing that the indicator dominates oscillation, it is sufficient to use only the marking strategy E.

The procedure REFINE: This procedure takes the triangulation $\mathcal{T}_h$ and the subset $\hat{\mathcal{T}}_h$ of marked elements as inputs. In order to preserve the shape regularity, all elements in $\hat{\mathcal{T}}_h$ will be refine by newest vertex bisection rule for at least n times ($n \geq 1$). Of course, some more elements outside $\hat{\mathcal{T}}_h$ are also bisected to obtain a new conforming triangulation $\mathcal{T}_{h+1}$. Notice that the resulting spaces are nested, i.e., $V_h \subset V_{h+1}$.

References

Ainsworth, M., & Oden, J. T. (2000). *A Posteriori Error Estimation in Finite Element Analysis*. New York: John Wiley.

Babuška, I., & Rheinboldt, W. C. (1978). A posteriori error estimates for the finite element method. *International Journal for Numerical Methods in Engineering, 12*, 1597-1615.

Braess, D. (2001). *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press.

Cascon, J. M., Kreuzer, C., Nochetto, R. H., & Siebert, K. G. (2008). Quasi-optimal convergence rate for an adaptive finite element method. *SIAM Journal on Numerical Analysis, 46*, 2524-2550. http://dx.doi.org/10.1137/07069047x

Clément, P. (1975). Approximation by finite element functions using local regularization. *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, 9* (2), 77-84.

Dörfler, W. (1996). A convergent adaptive algorithm for Poisson's equation. *SIAM Journal on Numerical Analysis, 33*, 1106-1124. http://dx.doi.org/10.1137/0733054

Evans, L. C. (1998). *Partial Differential Equations*. Rhode Island: American Mathematical Society.

Garau, E. M., Morin, P., & Zuppa, C. (2011). Convergence of an adaptive Kačanov FEM for quasi-linear problems. *Applied Numerical Mathematics, 61*, 512-529. http://dx.doi.org/10.1016/j.apnum.2010.12.001

Garau, E. M., Morin, P., & Zuppa, C. (in press). Quasi-Optimal convergence rate of an AFEM for quasi-linear problems.

Han, W., Jensen, S., & Shimansky, I. (1997). The Kačanov method for some nonlinear problems. *Applied Numerical Mathematics, 24*, 57-79. http://dx.doi.org/10.1016/s0168-9274(97)00009-3

Mekchay K., & Nochetto, R. H. (2005). Convergence of adaptive finite element methods for general second order linear elliptic PDEs. *SIAM Journal on Numerical Analysis, 43*, 1803-1827. http://dx.doi.org/10.1137/04060929x

Mekchay, K., Morin, P., & Nochetto, R. H. (2011). AFEM for the Laplace-Beltrami operator on graphs: Design and conditional contraction property. *Mathematics of Computation, 80*, 625-648. http://dx.doi.org/10.1090/s0025-5718-2010-02435-4

Morin, P., Nochetto, R. H., & Siebert, K. G. (2000). Data oscillation and convergence of adaptive FEM. *SIAM Journal on Numerical Analysis, 38* (2), 466-488. http://dx.doi.org/10.1137/s0036142999360044

Morin, P., Nochetto, R. H., & Siebert, K. G. (2002). Convergence of adaptive finite element methods. *SIAM Review, 44*, 631-658. http://dx.doi.org/10.1137/s0036144502409093

Verfürth, R. (1996). *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Technique*. Chichester: Wiley-Teubner.

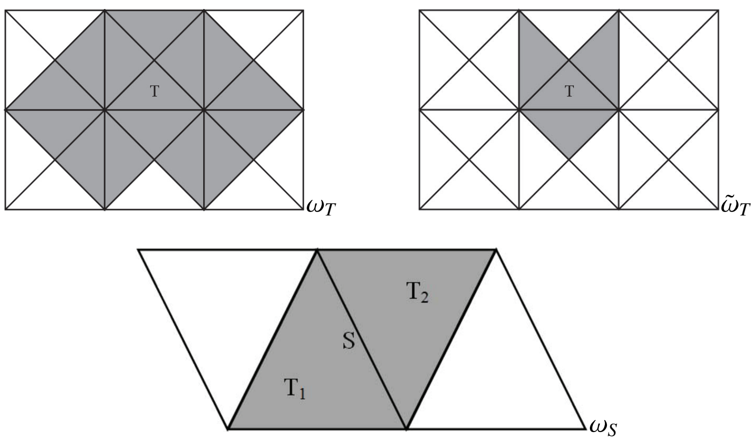

Figure 1. The neighborhood of T (ω_T), the patch of T $(\tilde{\omega}_T)$, and the patch of S (ω_S) are shown by the shaded area

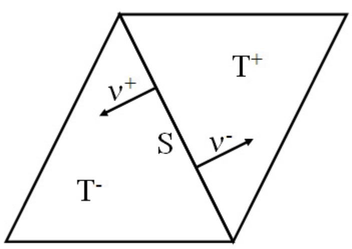

Figure 2. The unit outward normal vector ν^+ and ν^- of T^+ and T^- on the common side S