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Mountain Pass solutions for non-local elliptic operators ☆

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ABSTRACT

The purpose of this paper is to study the existence of solutions for equations driven by a non-local integrodifferential operator with homogeneous Dirichlet boundary conditions. These equations have a variational structure and we find a non-trivial solution for them using the Mountain Pass Theorem. To make the nonlinear methods work, some careful analysis of the fractional spaces involved is necessary. We prove this result for a general integrodifferential operator of fractional type and, as a particular case, we derive an existence theorem for the fractional Laplacian, finding non-trivial solutions of the equation

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

As far as we know, all these results are new.

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Contents

1. Introduction

One of the most celebrated applications of the Mountain Pass Theorem (see [1,5,6,8]) consists in the construction of non-trivial solutions of semilinear equations of the type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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In this framework, the solutions are constructed with a variational method by a minimax procedure on the associated energy functional.

We think that a natural question is whether or not these Mountain Pass techniques may be adapted to the fractional analogue of Eq. (1.1), namely

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Here, $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n. \tag{1.2}$$

For instance, when $s = 1/2$, the above operator is the square root of (minus) the Laplacian (the minus sign is needed to make the operator positive definite, see [4] and references therein for a basic introduction to the fractional Laplace operator).

Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

The literature on non-local operators and on their applications is, therefore, very interesting and, up to now, quite large (see, e.g., [4] for an elementary introduction to this topic and a for a – still not exhaustive – list of related references).

The purpose of this paper is to study an equation driven by the non-local operator $\mathcal{L}_K$ defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \big(u(x+y) + u(x-y) - 2u(x)\big) K(y) \, dy, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function with the properties that

$$\gamma K \in L^1\big(\mathbb{R}^n\big), \quad \text{where } \gamma(x) = \min\big\{|x|^2, 1\big\}; \tag{1.4}$$

$$\text{there exists } \lambda > 0 \quad \text{such that} \quad K(x) \geqslant \lambda |x|^{-(n+2s)} \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}; \tag{1.5}$$

$$K(x) = K(-x) \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.6}$$

A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$, see (1.2).

In this paper we deal with the following equation

$$\mathcal{L}_K u + f(x,u) = 0 \quad \text{in } \Omega \tag{1.7}$$

in the case of homogeneous Dirichlet boundary conditions, that is $u = 0$ in $\mathbb{R}^n \setminus \Omega$. We remark that the Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $\mathcal{L}_K$.

More precisely, we study the following problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y) \, dx \, dy = \int_{\Omega} f\big(x, u(x)\big) \varphi(x) \, dx \quad \forall \varphi \in X_0, \\ u \in X_0. \end{cases} \tag{1.8}$$

In our setting (1.8) represents the weak formulation of (1.7) (for this, it is convenient to assume (1.6)).

Here $s \in (0,1)$ is fixed, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary, and X is the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto \big(g(x) - g(y)\big)\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dx\,dy\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \big\{g \in X\colon\ g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

We stress that

$$C_0^2(\Omega) \subseteq X_0, \tag{1.9}$$

see, e.g., [7, Lemma 11] (for this we need condition (1.4)), and so X and X_0 are non-empty.

Finally, we suppose that the right-hand side of Eq. (1.7) is a Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ verifying the following conditions

$$\text{there exist } a_1, a_2 > 0 \text{ and } q \in \left(2, 2^*\right),\ 2^* = 2n/(n-2s), \quad \text{such that}$$

$$\left|f(x,t)\right| \leqslant a_1 + a_2|t|^{q-1} \quad \text{a.e. } x \in \Omega,\ t \in \mathbb{R}; \tag{1.10}$$

$$\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0 \quad \text{uniformly in } x \in \Omega; \tag{1.11}$$

$$\text{there exist } \mu > 2 \text{ and } r > 0 \quad \text{such that} \quad \text{a.e. } x \in \Omega,\ t \in \mathbb{R},\ |t| \geqslant r, \quad 0 < \mu F(x,t) \leqslant t f(x,t), \tag{1.12}$$

where the function F is the primitive of f with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau. \tag{1.13}$$

As a model for f we can take the function $f(x,t) = a(x)|t|^{q-2}t$, with $a \in L^\infty(\Omega)$ and $q \in (2, 2^*)$.

When dealing with partial differential equations driven by the Laplace operator (or, more generally, by uniformly elliptic operators) with homogeneous Dirichlet boundary conditions, assumption (1.12) is the standard superquadraticity condition on F (see, for instance, [1,6,8]).

Finally, we note that in the model case $f(x,t) = |t|^{q-2}t$ with $q \in (2, 2^*)$, assumption (1.12) is trivially satisfied for $\mu = q$. The exponent 2^* here plays the role of a critical Sobolev exponent (see, e.g. [4, Theorem 6.5]).

The main result of the present paper is an existence theorem for equations driven by general integrodifferential operators of non-local fractional type, as stated here below.

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying* (1.4)–(1.6) *and let f be a Carathéodory function verifying* (1.10), (1.11) *and* (1.12).

Then, problem (1.8) *admits a Mountain Pass type solution $u \in X_0$ which is not identically zero.*

In fact we can find a non-trivial non-negative (non-positive) solution of problem (1.8) (see Corollary 13 for more details).

In the non-local framework, the simplest example we can deal with is given by the fractional Laplacian, according to the following result:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Consider the following equation*

$$\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega f\big(x, u(x)\big)\varphi(x)\,dx \tag{1.14}$$

for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying (1.10), (1.11) *and* (1.12), *then problem* (1.14) *admits a Mountain Pass type solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.*

We observe that (1.14) is the equation

$$(-\Delta)^s u = f(x,u) \quad \text{in } \Omega \tag{1.15}$$

written in the distributional sense (see [4] and references therein for further details on the fractional Laplacian).

When $s = 1$, Eq. (1.15) reduces to a standard semilinear Laplace partial differential equation: in this sense Theorem 2 may be seen as the fractional version of the classical result in [8, Theorem 6.2] (see also [1,6]). This classical result is indeed the paradigmatic application of the Mountain Pass Theorem for elliptic partial differential equations, and we think that Theorem 2 may be seen as its natural extension to the non-local fractional setting.

In fact, the non-local analysis that we perform in this paper in order to use Mountain Pass Theorem is quite general and may be suitable for other goals too. Our proof will check that the classical geometry of the Mountain Pass Theorem is respected by the non-local framework. For this, we will develop a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation. As a technical remark, we notice that we do not make use of the extension theory of [3] (this allows us to avoid singular/degenerate elliptic operators in a higher dimension, and to deal with more general types of operators).

As far as we know the results presented here are new. The paper is organized as follows. In Section 2 we collect some preliminary observations. In Section 3 we prove Theorem 1 performing the classical Mountain Pass Theorem, while in Section 4, as an application, we discuss the case of an equation driven by the fractional Laplacian operator and we prove Theorem 2.

2. Some preliminary results

In this section we prove some preliminary results which will be useful in the sequel.

2.1. Preliminary estimates on the nonlinearity

Here, we use the structural assumptions on f to deduce some bounds from above and below for the nonlinear term and its primitive. This part is quite standard and does not take into account the non-local features of the problem: the reader familiar with these nonlinear analysis estimates may go directly to Section 2.2.

Lemma 3. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions* (1.10) *and* (1.11)*. Then, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$*

$$\left|f(x,t)\right| \leqslant 2\varepsilon|t| + q\delta(\varepsilon)|t|^{q-1} \tag{2.1}$$

and so, as a consequence,

$$\left|F(x,t)\right| \leqslant \varepsilon|t|^2 + \delta(\varepsilon)|t|^q, \tag{2.2}$$

where F is defined as in (1.13)*.*

Proof. By assumption (1.11) for any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that for any $t \in \mathbb{R}$ with $|t| < \sigma$ and a.e. $x \in \Omega$ we get

$$\left|f(x,t)\right| \leqslant 2\varepsilon|t|. \tag{2.3}$$

Moreover, by (1.10) there exists $\delta = \delta(\sigma) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ with $|t| \geqslant \sigma$ we have

$$\left|f(x,t)\right| \leqslant q\delta(\sigma)|t|^{q-1}. \tag{2.4}$$

Combining (2.3) and (2.4) it easily follows (2.1). Using the definition of F (see (1.13)), we also get (2.2). □

Lemma 4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let F be as in* (1.13)*. Assume that condition* (1.12) *holds true. Then, there exist two positive measurable functions $m = m(x)$ and $M = M(x)$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$*

$$F(x,t) \geqslant m(x)|t|^\mu - M(x). \tag{2.5}$$

If, in addition, f satisfies conditions (1.10) *and* (1.11)*, then $m, M \in L^\infty(\Omega)$.*

Proof. Let $r > 0$ be as in (1.12): then, a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ with $|t| \geqslant r > 0$

$$\frac{tf(x,t)}{F(x,t)} \geqslant \mu.$$

Suppose $t > r$. Dividing by t and integrating both terms in $[r,t]$ we obtain

$$F(x,t) \geqslant \frac{F(x,r)}{r^\mu} t^\mu.$$

With the same arguments it is easy to prove that if $t < -r$ then it holds

$$F(x,t) \geqslant \frac{F(x,-r)}{r^\mu} |t|^\mu,$$

so that for any $t \in \mathbb{R}$ with $|t| \geqslant r$ we get

$$F(x,t) \geqslant m(x)|t|^\mu, \tag{2.6}$$

where $m(x) = r^{-\mu} \min\{F(x,r), F(x,-r)\}$.

Since the function $t \mapsto F(\cdot,t)$ is continuous in $\mathbb{R}$, by the Weierstrass Theorem, it is bounded for any $t \in \mathbb{R}$ such that $|t| \leqslant r$, say

$$\left|F(x,t)\right| \leqslant \widetilde{M}(x) \quad \text{in } \left\{|t| \leqslant r\right\}, \tag{2.7}$$

where $\widetilde{M}(x) = \max\{|F(x,t)|: \ |t| \leqslant r\}$. Formula (2.5) follows from (2.6) and (2.7) taking $M(x) = \widetilde{M}(x) + m(x)r^\mu$.

Note that m and M are measurable functions, since $x \mapsto F(x,\cdot)$ does. Moreover, m and M are positive, being $F(x,t) > 0$ a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ such that $|t| \geqslant r$ (see (1.12)).

Now, suppose that f satisfies conditions (1.10) and (1.11). Then, by (2.2) with $\varepsilon = 1$ we get

$$m(x) = r^{-\mu} \min\{F(x,r), F(x,-r)\} \leqslant r^{-\mu}\left(|r|^2 + \delta(1) r^q\right) \in L^\infty(\Omega)$$

and

$$M(x) = \max\{F(x,t)\colon\ |t| \leqslant r\} \leqslant |r|^2 + \delta(1) r^q \in L^\infty(\Omega),$$

so that the assertion follows. $\square$

2.2. *The functional analytic setting*

Now, before going on, we need some preliminary results on X and X_0. In the sequel we denote $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$, where

$$\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2n}, \tag{2.8}$$

and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$.

The space X is endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \left(\int_Q \left|g(x) - g(y)\right|^2 K(x-y)\,dx\,dy\right)^{1/2}. \tag{2.9}$$

It is easy to check that $\|\cdot\|_X$ is a norm on X. We only show that if $\|g\|_X = 0$, then $g = 0$ a.e. in $\mathbb{R}^n$. Indeed, by $\|g\|_X = 0$ we get $\|g\|_{L^2(\Omega)} = 0$, which implies that

$$g = 0 \quad \text{a.e. in } \Omega, \tag{2.10}$$

and

$$\int_Q \left|g(x) - g(y)\right|^2 K(x-y)\,dx\,dy = 0. \tag{2.11}$$

By (2.11) we deduce that $g(x) = g(y)$ a.e. $(x,y) \in Q$, that is g is constant a.e. in $\mathbb{R}^n$, say $g = c \in \mathbb{R}$ a.e. in $\mathbb{R}^n$. By (2.10) it easily follows that $c = 0$, so that $g = 0$ a.e. in $\mathbb{R}^n$.

In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}. \tag{2.12}$$

We remark that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.9) and (2.12) are not the same, because $\Omega \times \Omega$ is strictly contained in Q (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

For further details on the fractional Sobolev spaces we refer to [4] and to the references therein.

Lemma 5. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (1.4)–(1.6)*. Then the following assertions hold true*:

a) *if $v \in X$, then $v \in H^s(\Omega)$. Moreover*

$$\|v\|_{H^s(\Omega)} \leqslant c(\lambda)\|v\|_X;$$

b) *if $v \in X_0$, then $v \in H^s(\mathbb{R}^n)$. Moreover*

$$\|v\|_{H^s(\Omega)} \leqslant \|v\|_{H^s(\mathbb{R}^n)} \leqslant c(\lambda)\|v\|_X.$$

In both cases $c(\lambda) = \max\{1, \lambda^{-1/2}\}$, where λ is given in (1.5)*.*

Proof. Let us prove part a). By (1.5) we get

$$\int_{\Omega\times\Omega} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant \frac{1}{\lambda} \int_{\Omega\times\Omega} \left|v(x) - v(y)\right|^2 K(x-y)\,dx\,dy$$
$$\leqslant \frac{1}{\lambda} \int_Q \left|v(x) - v(y)\right|^2 K(x-y)\,dx\,dy < +\infty,$$

since $v \in X$. The first assertion is proved.

For part b) note that $v \in X$ and $v = 0$ a.e. in $\mathcal{C}\Omega$. As a consequence,

$$\|v\|_{L^2(\mathbb{R}^n)} = \|v\|_{L^2(\Omega)} < +\infty,$$

and

$$\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \int_Q \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$
$$\leqslant \frac{1}{\lambda}\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy < +\infty.$$

Hence $v \in H^s(\mathbb{R}^n)$. The estimate on the norm easily follows (we remark that, even if $v = 0$ on $\mathcal{C}\Omega$, it is not true that $\|v\|_{H^s(\Omega)} = \|v\|_{H^s(\mathbb{R}^n)}$, since in the latter norm the interaction between Ω and $\mathcal{C}\Omega$ gives, in general, a positive contribution). □

Lemma 6. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (1.4)–(1.6)*. Then*

a) *there exists a positive constant c, depending only on n and s, such that for any $v \in X_0$*

$$\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant c\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

where 2^* is given in (1.10);

b) *there exists a constant $C > 1$, depending only on n, s, λ and Ω, such that for any $v \in X_0$*

$$\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy \leqslant \|v\|^2_X \leqslant C\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy,$$

that is

$$\|v\|_{X_0} = \left(\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy\right)^{1/2} \tag{2.13}$$

is a norm on X_0 equivalent to the usual one defined in (2.9).

Proof. Let v be in X_0. By Lemma 5, we know that $v \in H^s(\mathbb{R}^n)$ and so, using [4, Theorem 6.5] (here with $p = 2$), we get

$$\|v\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant c\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

where c is a positive constant depending only on n and s. Since $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, we get assertion a).

For part b), we note that by (2.9) it easily follows that

$$\|v\|^2_X \geqslant \int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy.$$

Moreover, using the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (being Ω bounded and $2 < 2^* = 2n/(n-2s)$), part a) and assumption (1.5) we get

$$\|v\|^2_X = \left(\|v\|_{L^2(\Omega)} + \left(\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy\right)^{1/2}\right)^2$$
$$\leqslant 2\|v\|^2_{L^2(\Omega)} + 2\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy$$
$$\leqslant 2|\Omega|^{(2^*-2)/2^*}\|v\|^2_{L^{2^*}(\Omega)} + 2\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy$$

$$\leqslant 2c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+2\int_Q\big|v(x)-v(y)\big|^2K(x-y)\,dx\,dy$$
$$\leqslant 2\left(\frac{c|\Omega|^{(2^*-2)/2^*}}{\lambda}+1\right)\int_Q\big|v(x)-v(y)\big|^2K(x-y)\,dx\,dy.$$

Hence, assertion b) follows by taking $C=2(\frac{c}{\lambda}|\Omega|^{(2^*-2)/2^*}+1)>1$.

In order to show that (2.13) is a norm on X_0 it is enough to prove that if $\|g\|_{X_0}=0$, then $g=0$ a.e. in $\mathbb{R}^n$. Indeed, by $\|g\|_{X_0}=0$ we get

$$\int_Q\big|g(x)-g(y)\big|^2K(x-y)\,dx\,dy=0,$$

so that $g(x)=g(y)$ a.e. $(x,y)\in Q$, that is g is constant a.e. in $\mathbb{R}^n$, say $g=c\in\mathbb{R}$ a.e. in $\mathbb{R}^n$. Since $g=0$ a.e. in $\mathbb{R}^n\setminus\Omega$, it easily follows that $c=0$, so that $g=0$ a.e. in $\mathbb{R}^n$.

The last argument is indeed similar to the one in (2.9)–(2.11), with a technical difference: there g was 0 in Ω, while here g is 0 in $\mathbb{R}^n\setminus\Omega$. An alternative proof could be the following: if $\|g\|_{X_0}=0$, then $\|g\|_X=0$ by Lemma 6-a). Hence $g=0$ a.e. in $\mathbb{R}^n$, since $\|\cdot\|_X$ is a norm. □

From now on, we take (2.13) as norm on X_0.

Lemma 7. $(X_0,\|\cdot\|_{X_0})$ *is a Hilbert space.*

Proof. First of all, let us consider the map

$$X_0\times X_0\ni(u,v)\mapsto\langle u,v\rangle:=\int_Q\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x-y)\,dx\,dy.$$

Thanks to the properties of the integrals and the positivity of K, it is easy to see that $\langle\cdot,\cdot\rangle$ is a scalar product in $X_0\times X_0$, which induces the norm defined in (2.13).

In order to show that X_0 is a Hilbert space, it remains to prove that X_0 is complete with respect to the norm $\|\cdot\|_{X_0}$. For this, let u_j be a Cauchy sequence in X_0. Thus, for any $\varepsilon>0$ there exists ν_ε such that if $i,j\geqslant\nu_\varepsilon$ then

$$\varepsilon\geqslant\|u_i-u_j\|^2_{X_0}\geqslant\frac{1}{C}\|u_i-u_j\|^2_X\geqslant\frac{1}{C}\|u_i-u_j\|^2_{L^2(\Omega)},\tag{2.14}$$

thanks to Lemma 6-b).

Hence, being $L^2(\Omega)$ complete, there exists $u_\infty\in L^2(\Omega)$ such that $u_j\to u_\infty$ in $L^2(\Omega)$ as $j\to+\infty$.

Since $u_j=0$ a.e. in $\mathbb{R}^n\setminus\Omega$, we may define $u_\infty:=0$ in $\mathbb{R}^n\setminus\Omega$, and then $u_j\to u_\infty$ in $L^2(\mathbb{R}^n)$ as $j\to+\infty$. So, there exists a subsequence u_{j_k} in X_0, such that $u_{j_k}\to u_\infty$ a.e. in $\mathbb{R}^n$ (see [2, Theorem IV.9]).

Therefore, by Fatou Lemma and the first inequality in (2.14) with $\varepsilon=1$, we have that

$$\begin{aligned}\int_Q\big|u_\infty(x)-u_\infty(y)\big|^2K(x-y)\,dx\,dy&\leqslant\liminf_{k\to+\infty}\int_Q\big|u_{j_k}(x)-u_{j_k}(y)\big|^2K(x-y)\,dx\,dy\\&=\liminf_{k\to+\infty}\|u_{j_k}\|^2_{X_0}\\&\leqslant\liminf_{k\to+\infty}\big(\|u_{j_k}-u_{\nu_1}\|_{X_0}+\|u_{\nu_1}\|_{X_0}\big)^2\\&\leqslant\big(1+\|u_{\nu_1}\|_{X_0}\big)^2\\&<+\infty,\end{aligned}$$

so that $u_\infty\in X_0$.

Now, it remains to show that the whole sequence converges to u_∞ in X_0. For this, let us take $i\geqslant\nu_\varepsilon$. By the first inequality in (2.14), Lemma 6-b) and Fatou Lemma we get

$$\begin{aligned}\|u_i-u_\infty\|^2_{X_0}&\leqslant\|u_i-u_\infty\|^2_X\\&\leqslant\liminf_{k\to+\infty}\left[\left(\int_Q\big|u_i(x)-u_{j_k}(x)-u_i(y)+u_{j_k}(y)\big|^2K(x-y)\,dx\,dy\right)^{1/2}+\|u_i-u_{j_k}\|_{L^2(\Omega)}\right]^2\end{aligned}$$

$$\begin{aligned}
&= \liminf_{k\to+\infty} \|u_i - u_{j_k}\|_X^2 \\
&\leqslant C \liminf_{k\to+\infty} \|u_i - u_{j_k}\|_{X_0}^2 \\
&\leqslant C\varepsilon,
\end{aligned}$$

that is $u_i \to u_\infty$ in X_0 as $i \to +\infty$. Hence, Lemma 7 is proved. □

Finally, we prove a convergence property for bounded sequences in X_0.

Lemma 8. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (1.4)–(1.6) and let v_j be a bounded sequence in X_0. Then, there exists $v_\infty \in L^\nu(\mathbb{R}^n)$ such that, up to a subsequence,*

$$v_j \to v_\infty \quad \text{in } L^\nu\big(\mathbb{R}^n\big)$$

as $j \to +\infty$, for any $\nu \in [1, 2^*)$.

Proof. By Lemma 5-b), $v_j \in H^s(\mathbb{R}^n)$ and so $v_j \in H^s(\Omega)$. More precisely, by Lemma 5-b), Lemma 6-b) and the definition of X_0, we see that

$$\|v_j\|_{H^s(\Omega)} \leqslant \|v_j\|_{H^s(\mathbb{R}^n)} \leqslant c(\lambda)\|v_j\|_X \leqslant c(\lambda)\sqrt{C}\|v_j\|_{X_0},$$

with $c(\lambda)$ and C depending only on n, s, λ and Ω. Hence, v_j is bounded in $H^s(\Omega)$, and so in $L^2(\Omega)$.

Then, by [4, Corollary 7.2] and our assumptions on Ω, there exists $v_\infty \in L^m(\Omega)$ such that, up to a subsequence,

$$v_j \to v_\infty \quad \text{in } L^m(\Omega)$$

as $j \to +\infty$, for any $m \in [1, 2^*)$. Since v_j vanishes outside Ω, we can define $v_\infty := 0$ in $\mathbb{R}^n \setminus \Omega$ and obtain that the convergence occurs in $L^m(\mathbb{R}^n)$. □

3. Mountain Pass solutions in a non-local framework: proof of Theorem 1

For the proof of Theorem 1, we observe that problem (1.8) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\int_Q \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy - \int_\Omega F\big(x, u(x)\big)\,dx.$$

Note that the functional $\mathcal{J}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\big\langle \mathcal{J}'(u), \varphi\big\rangle = \int_Q \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy - \int_\Omega f\big(x, u(x)\big)\varphi(x)\,dx.$$

Thus, critical points of $\mathcal{J}$ are solutions to problem (1.8). In order to find these critical points, we will make use of the Mountain Pass Theorem (see [1,6,8]). For this, we have to check that the functional $\mathcal{J}$ has a particular geometric structure (as stated, e.g., in conditions (1°)–(3°) of [8, Theorem 6.1]) and satisfies the Palais–Smale compactness condition (see, for instance, [8, page 70]).

We start by proving the necessary geometric features of the functional $\mathcal{J}$.

Proposition 9. *Let f be a Carathéodory function satisfying conditions (1.10) and (1.11). Then, there exist $\rho > 0$ and $\beta > 0$ such that for any $u \in X_0$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}(u) \geqslant \beta$.*

Proof. Let u be a function in X_0. By (2.2) we get that for any $\varepsilon > 0$

$$\begin{aligned}
\mathcal{J}(u) &\geqslant \frac{1}{2}\int_Q \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy - \varepsilon\int_\Omega \big|u(x)\big|^2\,dx - \delta(\varepsilon)\int_\Omega \big|u(x)\big|^q\,dx \\
&\geqslant \frac{1}{2}\int_Q \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy - \varepsilon\|u\|_{L^2(\Omega)}^2 - \delta(\varepsilon)\|u\|_{L^q(\Omega)}^q \\
&\geqslant \frac{1}{2}\int_Q \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|_{L^{2^*}(\Omega)}^2 - |\Omega|^{(2^*-q)/2^*}\delta(\varepsilon)\|u\|_{L^{2^*}(\Omega)}^q, \qquad (3.1)
\end{aligned}$$

thanks to the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously (being Ω bounded and $\max\{2, q\} = q < 2^*$).

Using (1.5) and Lemma 6-a), we deduce from (3.1) that for any $\varepsilon > 0$

$$\begin{aligned}
\mathcal{J}(u) &\geqslant \frac{1}{2}\int_{Q}\left|u(x)-u(y)\right|^2 K(x-y)\,dx\,dy \\
&\quad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}\left(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{q/2} \\
&\geqslant \left(\frac{1}{2}-\frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\lambda}\right)\int_{Q}\left|u(x)-u(y)\right|^2 K(x-y)\,dx\,dy \\
&\quad - \frac{\delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}}{\lambda}\left(\int_{Q}\left|u(x)-u(y)\right|^2 K(x-y)\,dx\,dy\right)^{q/2}. \qquad (3.2)
\end{aligned}$$

Choosing $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < \lambda$, by (3.2) and Lemma 6-b) it easily follows that

$$\begin{aligned}
\mathcal{J}(u) &\geqslant \alpha\int_{Q}\left|u(x)-u(y)\right|^2 K(x-y)\,dx\,dy\left[1-\kappa\left(\int_{Q}\left|u(x)-u(y)\right|^2 K(x-y)\,dx\,dy\right)^{(q/2)-1}\right] \\
&\geqslant \alpha\|u\|_{X_0}^2\left(1-\kappa\|u\|_{X_0}^{q-2}\right),
\end{aligned}$$

for suitable positive constants α and κ.

Now, let $u \in X_0$ be such that $\|u\|_{X_0} = \rho > 0$. Since $q > 2$ by assumption, we can choose ρ sufficiently small (i.e. ρ such that $1-\kappa\rho^{q-2} > 0$), so that

$$\inf_{\substack{u\in X_0\\ \|u\|_{X_0}=\rho}} \mathcal{J}(u) \geqslant \alpha\rho^2\left(1-\kappa\rho^{q-2}\right) =: \beta > 0.$$

Hence, Proposition 9 is proved. □

Proposition 10. *Let f be a Carathéodory function satisfying conditions* (1.10)–(1.12)*. Then, there exists $e \in X_0$ such that $e \geqslant 0$ a.e. in $\mathbb{R}^n$, $\|e\|_{X_0} > \rho$ and $\mathcal{J}(e) < \beta$, where ρ and β are given in Proposition* 9*.*

Proof. We fix $u \in X_0$ such that $\|u\|_{X_0} = 1$ and $u \geqslant 0$ a.e. in $\mathbb{R}^n$: we remark that this choice is possible thanks to (1.9) (alternatively, one can replace any $u \in X_0$ with its positive part, which belongs to X_0 too, thanks to [7, Lemma 12]).

Now, let $t > 0$. By Lemma 4 we have

$$\begin{aligned}
\mathcal{J}(tu) &= \frac{1}{2}\int_{Q}\left|tu(x)-tu(y)\right|^2 K(x-y)\,dx\,dy - \int_{\Omega} F\big(x, tu(x)\big)\,dx \\
&\leqslant \frac{t^2}{2} - t^{\mu}\int_{\Omega} m(x)\left|u(x)\right|^{\mu}dx + \int_{\Omega} M(x)\,dx.
\end{aligned}$$

Since $\mu > 2$, passing to the limit as $t \to +\infty$, we get that $\mathcal{J}(tu) \to -\infty$, so that the assertion follows taking $e = Tu$, with T sufficiently large. □

Propositions 9 and 10 give that the geometry of the Mountain Pass Theorem is fulfilled by $\mathcal{J}$. Therefore, in order to apply such Mountain Pass Theorem, we have to check the validity of the Palais–Smale condition: this will be accomplished in the forthcoming Propositions 11 and 12.

Proposition 11. *Let f be a Carathéodory function satisfying conditions* (1.10)–(1.12)*. Let $c \in \mathbb{R}$ and let u_j be a sequence in X_0 such that*

$$\mathcal{J}(u_j) \to c \qquad (3.3)$$

and

$$\sup\{|\langle \mathcal{J}'(u_j), \varphi\rangle|: \varphi \in X_0, \|\varphi\|_{X_0} = 1\} \to 0 \tag{3.4}$$

as $j \to +\infty$.

Then u_j is bounded in X_0.

Proof. For any $j \in \mathbb{N}$ by (3.4) and (3.3) it easily follows that there exists $\kappa > 0$ such that

$$\left|\left\langle \mathcal{J}'(u_j), \frac{u_j}{\|u_j\|_{X_0}}\right\rangle\right| \leqslant \kappa, \tag{3.5}$$

and

$$|\mathcal{J}(u_j)| \leqslant \kappa. \tag{3.6}$$

Moreover, by Lemma 3 applied with $\varepsilon = 1$ we have that

$$\left|\int_{\Omega\cap\{|u_j|\leqslant r\}} \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x))u_j(x)\right) dx\right| \leqslant \left(r^2 + \delta(1)r^q + \frac{2}{\mu} r + \frac{q}{\mu}\delta(1)r^{q-1}\right)|\Omega| =: \tilde{\kappa}. \tag{3.7}$$

Also, thanks to (1.12) and (3.7) we get

$$\begin{aligned}
\mathcal{J}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'(u_j), u_j\rangle &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \frac{1}{\mu}\int_\Omega \left(\mu F(x, u_j(x)) - f(x, u_j(x))u_j(x)\right) dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \int_{\Omega\cap\{|u_j|\leqslant r\}} \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x))u_j(x)\right) dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \tilde{\kappa}.
\end{aligned} \tag{3.8}$$

As a consequence of (3.5) and (3.6) we also have

$$\mathcal{J}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'(u_j), u_j\rangle \leqslant \kappa(1 + \|u_j\|_{X_0})$$

so that, by (3.8) for any $j \in \mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_*(1 + \|u_j\|_{X_0})$$

for a suitable positive constant κ_*. Hence, the assertion of Proposition 11 is proved. □

Proposition 12. *Let f be a Carathéodory function satisfying conditions* (1.10)–(1.12)*. Let u_j be a sequence in X_0 such that u_j is bounded in X_0 and* (3.4) *holds true. Then there exists $u_\infty \in X_0$ such that, up to a subsequence, $\|u_j - u_\infty\|_{X_0} \to 0$ as $j \to +\infty$.*

Proof. Since u_j is bounded in X_0 and X_0 is a reflexive space (being a Hilbert space, by Lemma 7), up to a subsequence, still denoted by u_j, there exists $u_\infty \in X_0$ such that

$$\begin{aligned}
&\int_Q (u_j(x) - u_j(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy \\
&\quad \to \int_Q (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy \quad \text{for any } \varphi \in X_0
\end{aligned} \tag{3.9}$$

as $j \to +\infty$. Moreover, by Lemma 8, up to a subsequence,

$$\begin{aligned}
&u_j \to u_\infty \quad \text{in } L^q(\mathbb{R}^n), \\
&u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n
\end{aligned} \tag{3.10}$$

as $j \to +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \text{ for any } j \in \mathbb{N} \tag{3.11}$$

(see, for instance [2, Theorem IV.9]).

By (1.10), (3.9)–(3.11), the fact that the map $t \mapsto f(\cdot, t)$ is continuous in $t \in \mathbb{R}$ and the Dominated Convergence Theorem we get

$$\int_\Omega f\big(x, u_j(x)\big)u_j(x)\,dx \to \int_\Omega f\big(x, u_\infty(x)\big)u_\infty(x)\,dx \tag{3.12}$$

and

$$\int_\Omega f\big(x, u_j(x)\big)u_\infty(x)\,dx \to \int_\Omega f\big(x, u_\infty(x)\big)u_\infty(x)\,dx \tag{3.13}$$

as $j \to +\infty$. Moreover, by (3.4) we have that

$$0 \leftarrow \big\langle \mathcal{J}'(u_j), u_j\big\rangle = \int_Q \big|u_j(x) - u_j(y)\big|^2 K(x-y)\,dx\,dy - \int_\Omega f\big(x, u_j(x)\big)u_j(x)\,dx$$

so that, by (3.12) we deduce that

$$\int_Q \big|u_j(x) - u_j(y)\big|^2 K(x-y)\,dx\,dy \to \int_\Omega f\big(x, u_\infty(x)\big)u_\infty(x)\,dx \tag{3.14}$$

as $j \to +\infty$. Furthermore,

$$0 \leftarrow \big\langle \mathcal{J}'(u_j), u_\infty\big\rangle = \int_Q \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big)K(x-y)\,dx\,dy - \int_\Omega f\big(x, u_j(x)\big)u_\infty(x)\,dx \tag{3.15}$$

as $j \to +\infty$. By (3.9), (3.13) and (3.15) we obtain

$$\int_Q \big|u_\infty(x) - u_\infty(y)\big|^2 K(x-y)\,dx\,dy = \int_\Omega f\big(x, u_\infty(x)\big)u_\infty(x)\,dx. \tag{3.16}$$

Thus, (3.14) and (3.16) give that

$$\int_Q \big|u_j(x) - u_j(y)\big|^2 K(x-y)\,dx\,dy \to \int_Q \big|u_\infty(x) - u_\infty(y)\big|^2 K(x-y)\,dx\,dy,$$

so that

$$\|u_j\|_{X_0} \to \|u_\infty\|_{X_0} \tag{3.17}$$

as $j \to \infty$.

Finally we have that

$$\begin{aligned}\|u_j - u_\infty\|_{X_0}^2 &= \|u_j\|_{X_0}^2 + \|u_\infty\|_{X_0}^2 - 2\int_Q \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big)K(x-y)\,dx\,dy \\ &\to 2\|u_\infty\|_{X_0}^2 - 2\int_Q \big|u_\infty(x) - u_\infty(y)\big|^2 K(x-y)\,dx\,dy = 0\end{aligned}$$

as $j \to +\infty$, thanks to (3.9) and (3.17). Then, the assertion of Proposition 12 is proved. □

Proof of Theorem 1. Since Propositions 9–12 hold true, the Mountain Pass Theorem (for instance, in the form given by [8, Theorem 6.1]; see also [1,6]) gives that there exists a critical point $u \in X_0$ of $\mathcal{J}$. Moreover

$$\mathcal{J}(u) \geqslant \beta > 0 = \mathcal{J}(0),$$

so that $u \not\equiv 0$.

Thus, the assertion of Theorem 1 follows. □

As in the classical case of the Laplacian (see [8, pages 103–104]), one can determine the sign of the Mountain Pass type solutions. Indeed, about problem (1.8) the following result holds true.

Corollary 13. *Let all the assumptions of Theorem* 1 *be satisfied. Then, problem* (1.8) *admits a non-negative solution* $u_+ \in X_0$ *and a non-positive solution* $u_- \in X_0$ *that are of Mountain Pass type and that are not identically zero.*

Proof. In order to prove the existence of a non-negative (non-positive) solution of problem (1.8) it is enough to introduce the functions

$$F_\pm(x,t) = \int_0^t f_\pm(x,\tau)\,d\tau,$$

with

$$f_+(x,t) = \begin{cases} f(x,t) & \text{if } t \geqslant 0,\\ 0 & \text{if } t<0 \end{cases} \quad \text{and} \quad f_-(x,t) = \begin{cases} 0 & \text{if } t>0,\\ f(x,t) & \text{if } t \leqslant 0. \end{cases}$$

Note that $f_\pm$ satisfy conditions (1.10) and (1.11), while assumption (1.12) is verified by f_+ and F_+ a.e. in Ω and for any $t>r$, and by f_- and F_- a.e. in Ω and for any $t<-r$.

Let $\mathcal{J}_\pm : X_0 \to \mathbb{R}$ be the functional defined as follows

$$\mathcal{J}_\pm(u) = \frac{1}{2}\int_Q \big|u(x)-u(y)\big|^2 K(x-y)\,dx\,dy - \int_\Omega F_\pm\big(x,u(x)\big)\,dx.$$

It is easy to see that the functional $\mathcal{J}_\pm$ is well defined, is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\big\langle \mathcal{J}_\pm'(u),\varphi\big\rangle = \int_Q \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy - \int_\Omega f_\pm\big(x,u(x)\big)\varphi(x)\,dx. \tag{3.18}$$

Moreover, $\mathcal{J}_\pm$ satisfies Propositions 10–12 (because we can take $e \geqslant 0$ in Proposition 10) and $\mathcal{J}_\pm(0)=0$. Hence, by the Mountain Pass Theorem, there exists a non-trivial critical point $u_\pm \in X_0$ of $\mathcal{J}_\pm$.

We claim that u_+ is non-negative in $\mathbb{R}^n$. Indeed, we define $\varphi := (u_+)^-$, where v^- is the negative part of v, i.e. $v^- = \max\{-v,0\}$. We remark that, since $u_+ \in X_0$, we have that $(u_+)^- \in X_0$, by [7, Lemma 12], and therefore we can use φ in (3.18). In this way, we get

$$\begin{aligned}
0 &= \big\langle \mathcal{J}_\pm'(u_+),(u_+)^-\big\rangle \\
&= \int_Q \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy - \int_\Omega f_+\big(x,u_+(x)\big)(u_+)^-(x)\,dx \\
&= \int_Q \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy \\
&= \big\|(u_+)^-\big\|_{X_0}^2,
\end{aligned}$$

thanks to the definition of f_+ and of negative part. Thus, $\|(u_+)^-\|_{X_0}=0$, so that $u_+ \geqslant 0$ a.e. in $\mathbb{R}^n$, which is the assertion.

With the same arguments it is easy to show that u_- is non-positive in $\mathbb{R}^n$. □

4. An equation driven by the fractional Laplacian: proof of Theorem 2

Theorem 2 follows from Theorem 1 by choosing

$$K(x) = |x|^{-(n+2s)}$$

and by recalling that $X_0 \subseteq H^s(\mathbb{R}^n)$ by Lemma 5-b).

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