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ELECTRONIC STRUCTURE OF MATERIALS CENTRE

Optimal current loop systems for producing uniform magnetic fields

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Abstract

This article presents magnetic field uniformity design data for several alternative current loop systems. Universal field symmetry properties of the class of current loop systems that is being considered are elucidated. A common property of the five loop systems that are investigated in detail is that they are all in a sense optimal. This 'Nth order' optimality criterion is defined and discussed. Parameters of selected Nth order current loop systems are quoted. Computations of the field uniformity of these loop systems are presented in graphical form, as 'isogauss' contours, and in tabular form, as the 'normalised volumes' enclosed by the isogauss contours. Information is provided about a current loop system that was actually constructed on the basis of the design data presented here.

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I. INTRODUCTION

There is a widely recognised need in contemporary science for large volume, easily accessible, spatially uniform and low to moderate strength magnetic fields. One example from biology and medicine is controlled investigations of the physiological effects of magnetic fields (Kirschvink [1]). From the earth and planetary sciences one has the use of extremely sensitive ($\sim 10^{-9}$T) proton precession magnetometers to measure planetary magnetic fields (Serson [2]). An example from physics and chemistry is the requirement for cancellation of the terrestrial magnetic field in sensitive electron spectrometers (Storer *et al* [3]).

Such uniform magnetic fields are produced by systems of current loops. This article presents a quantitative and comparitive survey of the magnetic field uniformity of several optimal current loop configurations. The designs considered here were candidates for the terrestrial magnetic field cancellation coils for the electron coincidence spectrometer described by Storer *et al* [3]. A similar survey, motivated by biological applications, has been undertaken by Kirschvink [1]. Serson [2] undertakes a formal analysis of the magnetic field uniformity, distinguished by the inclusion of an analysis of current loop perturbations and nonidealities.

II. GENERAL SYMMETRIC CURRENT LOOP SYSTEMS

Bloom *et al* [4] and Garrett [5,6] have articulated techniques for devising systems of coaxial current loops that produce an arbitrarily uniform local magnetic field.

Consider a system of coaxial, circular current loops that are symmetrically disposed about a plane normal to the axis of the loops, the origin being defined as the intersection of the axis and plane of symmetry. Such a system of two identical current loops is illustrated in Figure 1, where the axis of symmetry is chosen to be the z-axis. A system with the same symmetry properties consisting of an arbitrary even number of current loops can be constructed by superimposing the required number of such two loop systems, all with the same orientation and centre, but otherwise disparate parameters. By collapsing one of the two loop systems so that the two loops coincide in the plane of symmetry (this corresponds to setting $h = 0$ in Figure 1), a system with the required symmetry consisting of an arbitrary odd number of current loops can also be constructed from the basic two loop system of Figure 1.

One may deduce the following universal symmetry properties of the magnetic field of current loop systems with the symmetry extant in Figure 1. The ϕ component of the magnetic field vanishes everywhere, that is,

$$B_\phi(\mathbf{r}) = 0 \quad \forall\, \mathbf{r} \ . \tag{1}$$

For arbitrary azimuthal angle ϕ_0,

$$\begin{aligned} B_\rho(\rho,\phi,z) &= B_\rho(\rho,\phi-\phi_0,z) \ , \\ B_z(\rho,\phi,z) &= B_z(\rho,\phi-\phi_0,z) \ . \end{aligned} \tag{2}$$

Since the rotation angle ϕ_0 is arbitrary, (2) implies that the cylindrical components of the magnetic field do not depend on the azimuthal angle coordinate ϕ. Any point on the z-axis ($\rho = 0$) is invariant under a rotation about the z-axis, and since the magnetic field vector at that point is invariant under that rotation, it follows that the magnetic field on the z-axis must be in the z-direction, that is,

$$\mathbf{B}(0,\phi,z) = B_z(\rho=0,z)\,\hat{\mathbf{z}} \ . \tag{3}$$

Corresponding components of the magnetic field at equal but opposite z values are related by,

$$\begin{aligned} B_\rho(\rho,\phi,-z) &= -B_\rho(\rho,\phi,z) \ , \\ B_z(\rho,\phi,-z) &= B_z(\rho,\phi,z) \ . \end{aligned} \tag{4}$$

Equation (4) expresses the property that $B_\rho(\mathbf{r})$ is an odd function of z, whereas $B_z(\mathbf{r})$ is an even function of z. It therefore follows that in the xy-plane ($z = 0$) only the z component of the magnetic field is non-zero, that is,

$$\mathbf{B}(\rho,\phi,0) = B_z(\rho,z=0)\,\hat{\mathbf{z}} \ . \tag{5}$$

Since $B_z(\mathbf{r})$ is an even function of z, all of its odd partial derivatives with respect to z are odd functions of z, vanishing at $z = 0$, that is,

$$\left.\frac{\partial^m B_z}{\partial z^m}\right|_0 = 0 \quad , \quad m = 1,3,5,7,\ldots \ . \tag{6}$$

Furthermore, (3) implies that the ρ component of the magnetic field is identically zero everywhere along the z-axis ((1) states that there is no ϕ component anywhere to consider), so all partial derivatives of $B_\rho(\mathbf{r})$ with respect to z at $\rho = 0$ vanish. This much is deducible from symmetry considerations alone, and applies to a general current loop system that has the requisite symmetry properties.

III. CURRENT LOOP SYSTEMS OF DEFINITE ORDER

Specialising beyond the universal property (6), Bloom *et al* [4] indicate that the parameters of the current loop system can be specifically chosen so that all of the axial derivatives of the magnetic field at the origin, up to the $(N-1)$th derivative, where N is an even number that depends upon the number of free parameters, can be made identically zero, the lowest non-zero derivative being the Nth derivative of the field component along the axis of symmetry. One can therefore introduce the following definition:

Definition 1 *An* **Nth order loop system** *is one for which the lowest non zero axial derivative of the magnetic field at the centre of symmetry is the Nth, that is,*

$$B_z(0) \neq 0 \ ,$$
$$\left.\frac{\partial^m \mathbf{B}}{\partial z^m}\right|_0 = 0 \quad , \quad m = 1,\ldots,(N-1) \ ,$$
$$\left.\frac{\partial^N B_z}{\partial z^N}\right|_0 \neq 0 \ ,$$

where the axis of symmetry is the z-axis and the centre of symmetry is the point $\mathbf{0}$.

Garrett [5] contends that for an Nth order current loop system,

$$\left.\frac{\partial^m \mathbf{B}}{\partial \rho^m}\right|_0 = \left.\frac{\partial^m \mathbf{B}}{\partial z^m}\right|_0 \qquad m = 1, 2, 3, 4, \ldots \ , \tag{7}$$

which, by Definition 1, implies that the lowest non-zero derivative of the magnetic field with respect to ρ at $\mathbf{0}$ is also the Nth derivative of the z field component. This is suggestive of the tendency of the magnetic field at the centre of symmetry to be equally uniform in all directions. Bloom *et al* [4] demonstrate the theoretically interesting and practically useful property that an Nth order loop system can be constructed to satisfy the 'isotropy' condition (7), from regular polygonal current loops with at least N sides . If the polygons have fewer sides, although Definition 1 can be satisfied, condition (7) will not simultaneously be satisfied, thereby producing a prolate region of uniform magnetic field (longer along $\hat{\mathbf{z}}$, shorter along $\hat{\rho}$). Garrett [6] tabulates the parameters for a multitude of optimal circular Nth order loop systems for many (even) values of N.

Smythe [7, Section 7.10] derives the formula for the cylindrical components of the magnetic field of a single circular current loop in vacuum, in terms of *complete elliptic integrals of the first and second kind* (Abramowitz and Stegun [8, Chapter 17]). The magnetic field of a system of current loops is a superposition of the field of individual current loops. Thus, the computational apparatus for the magnetic field distribution of any system of circular current loops is available.

Every current loop system to be considered in this article has specific numerical values for all physical dimensions and loop currents. Denote the magnetic field of the current loop system as directly considered, by $\mathbf{B}^{\text{direct}}(\rho, \phi, z)$. Scale the current loop system by multiplying all characteristic distances by c_d, and multiplying all characteristic currents by c_i. It follows directly from the Biot-Savart law, that the magnetic field scales according to the relation,

$$\mathbf{B}^{\text{scaled}}(c_d\rho, \phi, c_d z) = \frac{c_i}{c_d}\mathbf{B}^{\text{direct}}(\rho, \phi, z) \ . \tag{8}$$

Consequently, any generic current loop system can be made arbitrarily large, with an arbitrarily large central magnetic field.

Various Nth order current loop systems are to be considered, with the intention of assessing the extent of field uniformity associated with each loop system. A quantitative measure of the magnetic field uniformity is provided by the scalar expressing the magnitude of the discrepancy between the vector field at an arbitrary point and the vector field at the centre of symmetry, that is,

$$B^{\text{discrepancy}}(\mathbf{r}) \equiv \|\mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{0})\| \ . \tag{9}$$

It follows from equations (3) and (4) that,

$$B^{\text{discrepancy}}(\rho, \phi, -z) = B^{\text{discrepancy}}(\rho, \phi, z) \ . \tag{10}$$

The scalar field $B^{\text{discrepancy}}(\mathbf{r})$ is calculated as a function of the cylindrical coordinates ρ and z (independent of ϕ) for every current loop system that is considered. Selected 'isogauss' contours in the ρz-plane are extracted from this field, and the results for each of the current loop systems that are investigated are graphed in Figure 2. The isogauss contour values in Figure 2 are 0.100%, 0.215%, 0.464%, 1.000%, 2.154%, 4.642% and 10.000%, of the central field value $B(\mathbf{0})$. The (ρ, z) coordinates of each constituent current loop forming the system are indicated on the graphs by a '$\otimes$' symbol. Contours and current loops are only plotted for $z \geq 0$, since the resultant graph is reflected in the ρ-axis and rotated about the z-axis to give the complete isogauss surface, and the complete current loop system.

For each isogauss surface, the enclosed volume is expressed relative to the volume of the figure of revolution formed by rotating about the z-axis the polygon whose vertices are the loop positions in the ρz-plane. This normalises the volume of a region of given uniformity relative to the volume of the current loop system that produces it, thereby enabling fairer comparison between different current loop systems. These 'normalised volumes', together with the central magnetic field magnitude, are tabulated in Table I.

IV. SPECIFIC Nth ORDER LOOP SYSTEMS

Helmholtz coils (Helmholtz [9]), which are the ubiquitous paradigm for uniform magnetic field production by current loops, are a 4th order current loop system consisting of two symmetrical loops with the following characteristic parameter values (Figure 1 defines the parameters):

$$\begin{aligned} I_1 &= 1.0\,\mathrm{A} \\ a_1 &= 1.0\,\mathrm{m} \\ h_1 &= 0.5\,\mathrm{m} \ . \end{aligned} \tag{11}$$

Maxwell coils (Maxwell [10, Article 715]) are a 6th order current loop system consisting of three loops with the prescribed symmetry of Section II, and the following characteristic parameter values:

$$\begin{aligned} I_1 &= 1.0\,\mathrm{A} & I_2 &= 49/64\,\mathrm{A} \\ a_1 &= 1.0\,\mathrm{m} & a_2 &= \sqrt{4/7}\,\mathrm{m} \\ h_1 &= 0.0\,\mathrm{m} & h_2 &= \sqrt{3/7}\,\mathrm{m} \ , \end{aligned} \tag{12}$$

where I_1 is the total current through the loop lying in the plane of symmetry ($z = 0$), and is not subject to doubling although the mirror image of the loop coincides with the loop itself.

Garrett coils (Garrett [6, Table IV]) are an 8th order current loop system consisting of four loops with the required symmetry properties. Characteristic parameter values are:

$$\begin{aligned} I_1 &= 1.0\,\mathrm{A} & I_2 &= 0.024533\,\mathrm{A} \\ a_1 &= 1.0\,\mathrm{m} & a_2 &= 0.265226\,\mathrm{m} \\ h_1 &= 0.434681\,\mathrm{m} & h_2 &= 0.434681\,\mathrm{m} \ . \end{aligned} \tag{13}$$

The distinctive feature of Garrett coils is that both loop pairs have the same axial coordinate ($h_1 = h_2$).

Barker coils (Barker [11], Garrett [6], also attributed to Sauter and Sauter [12]), are an 8th order current loop system composed of four loops with the requisite symmetry. Characteristic parameter values are:

$$
\begin{aligned}
I_1 &= 1.0\,\text{A} & I_2 &= 2.26044\,\text{A} \\
a_1 &= 1.0\,\text{m} & a_2 &= 1.0\,\text{m} \\
h_1 &= 0.243186\,\text{m} & h_2 &= 0.940733\,\text{m}\ .
\end{aligned}
\tag{14}
$$

Barker coils are distinguished by the equal radii of both constituent loop pairs ($a_1 = a_2$).

Braunbek coils (Braunbek [13], Garrett [6]) are an 8th order current loop system composed of four loops with the required symmetry, and with the following characteristic parameter values:

$$
\begin{aligned}
I_1 &= 1.0\,\text{A} & I_2 &= 1.0\,\text{A} \\
a_1 &= 1.0\,\text{m} & a_2 &= 0.763899\,\text{m} \\
h_1 &= 0.278028\,\text{m} & h_2 &= 0.845664\,\text{m}\ .
\end{aligned}
\tag{15}
$$

The notable feature of Braunbek coils is the equal currents through both loop pairs ($I_1 = I_2$).

V. CONCLUSION

The regions of magnetic field uniformity, to within prescribed tolerances, of these five optimal current loop systems, may be ascertained from the isogauss contours graphed in Figure 2, and the tabulated normalised volumes enclosed by isogauss surfaces of Table I. On the basis of these quantitative results, the Braunbek coil configuration was chosen to cancel the terrestrial magnetic field in the electron coincidence spectrometer of Storer *et al* [3].

Specifications for the Braunbek coil current loop system that is implemented for the spectrometer, are obtained by applying the scaling factors $c_d = 0.943$ and $c_i = 32.60$ of (8), to the characteristic parameter values in (15), to give the following parameter values for the constructed current loop system:

$$
\begin{aligned}
I_1 &= 32.60\,\text{A} & I_2 &= 32.60\,\text{A} \\
a_1 &= 0.943\,\text{m} & a_2 &= 0.720\,\text{m} \\
h_1 &= 0.262\,\text{m} & h_2 &= 0.797\,\text{m}\ .
\end{aligned}
\tag{16}
$$

The loop currents are achieved by winding each loop with 36 turns of enamelled copper wire (diameter=0.76mm), with the wire current being 0.906 A. The frame of the coil system is constructed entirely of extruded aluminium rod. From (8) and Table I, the resulting central magnetic field is 5.60×10^{-5} Tesla, which is the magnitude of the Earth's magnetic field at the site of the spectrometer. The axis of the loop system is skewed from the vertical by 19°, thus bringing it into alignment with the Earth's field.

The residual magnetic field present in the sensitive areas of the spectrometer was reduced to below 2.0×10^{-6} Tesla. Experience has shown that this is sufficiently low for effective operation of the electron spectrometer. Furthermore, the Braunbek coils are particularly compact, being only slightly larger than the spectrometer vacuum chamber. In summary, the Braunbek coils proved to be a compact, efficient and satisfactory solution to the terrestrial magnetic field cancellation problem.

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FIGURES

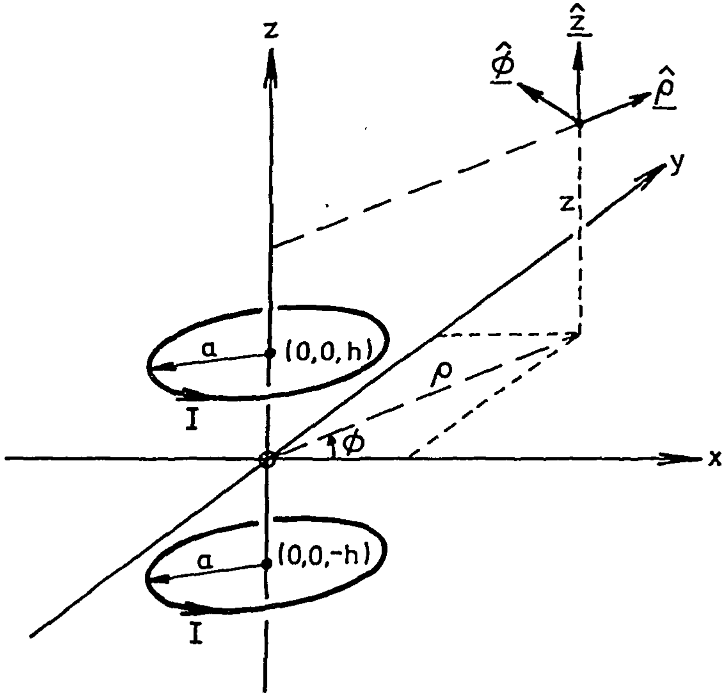
FIG. 1. Two identical coaxial current loops and their coordinate system.

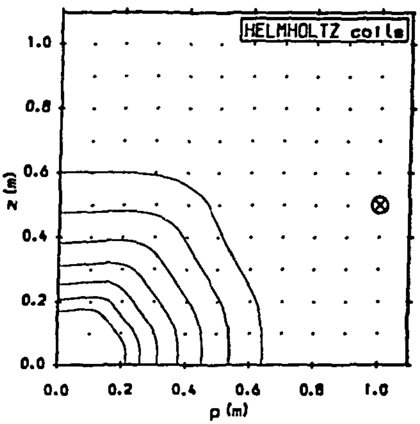
FIG. 2. $B^{\mathrm{discrepancy}}(\rho, z)$ = 0.100%, 0.215%, 0.464%, 1.000%, 2.154%, 4.642% and 10.000% of $B(\mathbf{0})$, isogauss contours, progressing contiguously from the innermost to the outermost contour, for (a)Helmholtz, (b)Maxwell, (c)Garrett, (d)Barker and (e)Braunbek optimal current loop configurations. The positions of the symmetrical current loop pairs are also indicated. A plane of symmetry is normal to the z-axis and includes the ρ-axis; the z-axis is an axis of symmetry.

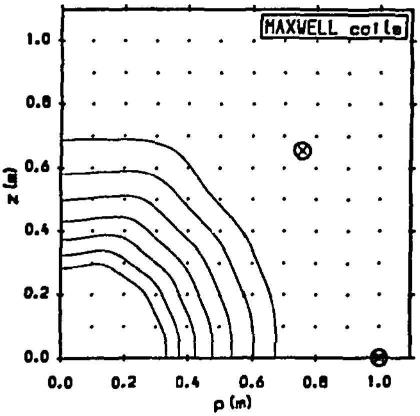
TABLES

system	Helmholtz	Maxwell	Garrett	Barker	Braunbek
no. of loops	2	3	4	4	4
order	4	6	8	8	8
$B(\mathbf{0})$ (10^{-7}T)	8.992	11.78	9.857	22.50	16.19
vol. (m^3)	3.14	3.19	2.73	5.91	4.54
0.100% n.v.	1.3%	4.8%	5.7%	9.7%	9.8%
0.215% n.v.	2.3%	7.0%	7.6%	12.9%	13.0%
0.464% n.v.	4.0%	10.2%	10.4%	17.0%	17.2%
1.000% n.v.	7.1%	14.8%	14.2%	22.5%	22.7%
2.154% n.v.	12.4%	21.4%	19.7%	29.9%	30.0%
4.642% n.v.	21.6%	30.9%	28.1%	39.8%	39.6%
10.000% n.v.	37.4%	44.6%	41.9%	53.5%	52.3%

TABLE I. Characteristics of various optimal current loop systems. "x% n.v." is the volume enclosed by the $B^{\text{discrepancy}}(\rho, z) = x\%$ of $B(\mathbf{0})$ isogauss surface, normalised to the volume enclosed by the loop system (this volume is "vol. (m^3)").


z
$\hat{z}$
$\hat{\phi}$
$\hat{\rho}$
y
z
a
(0,0,h)
ρ
I
ϕ
x
a
(0,0,-h)
I


HELMHOLTZ coils
1.0
0.8
0.6
0.4
0.2
0.0
z (m)
0.0
0.2
0.4
0.6
0.8
1.0
ρ (m)


MAXWELL coils
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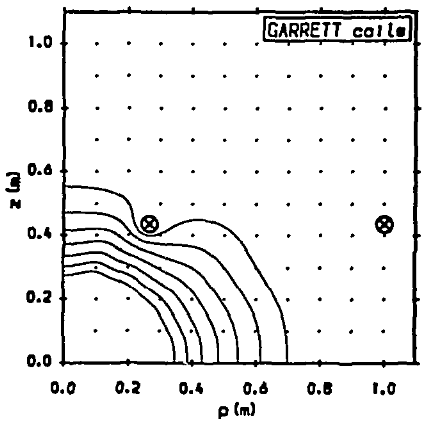
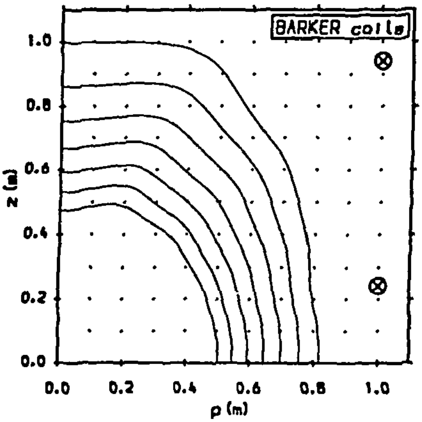
0.8
0.6
0.4
0.2
0.0
z (m)
0.0
0.2
0.4
0.6
0.8
1.0
ρ (m)


GARRETT coils
1.0
0.8
0.6
0.4
0.2
0.0
z (m)
0.0
0.2
0.4
0.6
0.8
1.0
ρ (m)


BARKER coils
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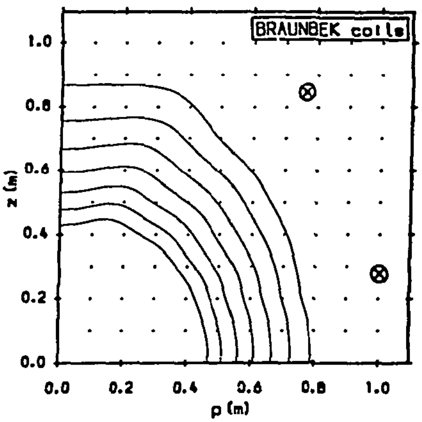
0.8
0.6
0.4
0.2
0.0
z (m)
0.0
0.2
0.4
0.6
0.8
1.0
ρ (m)


BRAUNBEK coils
1.0
0.8
0.6
0.4
0.2
0.0
z (m)
0.0
0.2
0.4
0.6
0.8
1.0
ρ (m)