

Table of contents of Bourbaki's
Elements of mathematics

assembled by Ulrich Thiel¹

Note. This table is incomplete in so far as it only contains the table of contents of a few books of the series. But all these tables are complete. Version from August 25, 2011.

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