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Q_K Classes in Clifford Analysis

M. A. Bakhit[*]

Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia
*Corresponding author: mabakhit@jazanu.edu.sa

Abstract In this paper, we define the classes Q_K of quaternion-valued functions, then we characterize quaternion Bloch functions by quaternion Q_K functions in the unit ball of $\mathbb{R}^3$. Further, some important basic properties of these functions are also considered.

Keywords: *Clifford analysis, quaternion Bloch space, Q_K spaces*

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1. Introduction

1.1. Analytic Function Spaces

In [17], Wulan and Wu introduced the so called Q_K spaces. These spaces consist of analytic functions on the unit open complex disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ such that

$$|f'(z)|^2 K(g(z,a))dxdy < \infty,$$

where $K: [0,\infty) \to [0,\infty)$ is a non-decreasing and righ-continuous function.

Green's function $g(z,a)$ in the unit disk with logarithmic singularity at $a \in \mathbb{D}$ is given by

$$g(z,a) = \ln \frac{1}{|\varphi_a(z)|},$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Moreover, $f \in Q_{K,0}$ if

$$\lim_{|a|\to 1} \int_D |f'(z)|^2 K(g(z,a))dxdy = 0.$$

For more results of Q_K spaces see [5,6,11] and [16]. It is known that the spaces Q_K are Banach spaces under the norm

$$\|f\|_K = \|f\|_{QK} + |f(0)|$$

for every $f \in Q_K$ and $a \in \mathbb{D}$. Moreover, it is known that the Green's function $g(z,a)$ can be replaced by the weight function $1 - |\varphi_a(z)|^2$.

There are a number of ways we can further generalize the Q_K spaces; see [4] and [14] for example.

Remark 1.1

If $K(t) = t^p, 0 \le p < \infty$, then $Q_K = Q_p$ see [5]. In particular, if $K(t) = 1$, then Q_K is the Dirichlet space $\mathcal{D}$. Moreover, if $K(t) = t$, then Q_K coincides with BMOA, the space of analytic functions of bounded mean oscillation.

Two magnitudes $A > 0$ and $B > 0$ are similar, denoted by $A \approx B$ if there exist two non-negative real constants C_1 and C_2 such that, $C_1 A \le B \le C_2 A$.

1.2. Quaternion Function Spaces

In this paper we will work in $\mathbb{H}$, the skew field of quaternions, that is, each element $z \in \mathbb{H}$, can be written in the form

$$a := a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where $a_k \in \mathbb{R}, k = 0,1,2,3$ and $1, e_1, e_2, e_3$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1 e_2 &= -e_2 e_1 = e_3, \\ e_2 e_3 &= -e_3 e_2 = e_1, \\ e_3 e_1 &= -e_1 e_3 = e_2. \end{aligned}$$

The product is extended by linearity. The quaternionic conjugation $\bar{a}$ is given by $\bar{a} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$ and we have the property

$$a\bar{a} = \bar{a}a = |a|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Therefore, if $a \in \mathbb{H} \setminus \{0\}$, the quaternion

$$a^{-1} := \bar{a} / |a|^2.$$

Also, the norm satisfies $|ab| = |a||b|$ for each $a, b \in \mathbb{H}$.

We identify each point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with a quaternion x of the form $x = x_0 + x_1 e_1 + x_2 e_2$.

Let $\mathbb{B} \in \mathbb{R}^3$ be the unit ball in the real three-dimensional space, with boundary $S = \partial\mathbb{B}$. For $r > 0$ and $a \in \mathbb{R}^3$, we denote by $\mathbb{B}(a,r)$ the ball with center a and radius r.

Let Ω be a domain in $\mathbb{R}^3$, then we will consider $\mathbb{H}$-valued functions defined in Ω (depending on $x = (x_1, x_2, x_3)$):

$$f:\Omega \to \mathbb{H}.$$

The notation $C^p(\Omega;\mathbb{H}), p \in \mathbb{N} \cup \{0\}$, has the usual component-wise meaning. On $C^1(\Omega;\mathbb{H})$ we define ageneralized Cauchy-Riemann operator D by

$$Df = \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2},$$

and it's conjugate operator by

$$\bar{D}f = \frac{\partial f}{\partial x_0} - e_1 \frac{\partial f}{\partial x_1} - e_2 \frac{\partial f}{\partial x_2}.$$

The solutions of $Df = 0, x \in \Omega$, are called (left) hyperholomorphic (or monogenic) functions and generalize the class of holomorphic functions from the one-dimensional complex function theory. For more details about quaternionic analysis and general Clifford analysis, we refer to [1], [8] and [15] and others.

We denote by $\mathcal{M}(\mathbb{B})$ the class of hyperholomorphic (or monogenic) functions on $\mathbb{B}$. For $a \in \mathbb{B}$ the Möbius transform $\varphi_a(x)$: $\mathbb{B} \longrightarrow \mathbb{B}$ is defined by

$$\varphi_a(x) = \frac{a-x}{1-\bar{a}x}.$$

Furthermore, let

$$g(z,a) = \frac{1}{|\varphi_a(z)|} - 1$$

be a multiple scalar of the fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$, i.e. $g(z,a)$ is the modified Green's function in quaternion sense.

For $a \in \mathbb{B}$ and $0 < R < 1$ the pseudo-hyperbolic ball $U(a,R)$ is defined by

$$U(a,R)\left\{x : |\varphi_a(z)| < R\right\}.$$

This is an Euclidean ball, with center and radius given respectively by:

$$\frac{\left(1-R^2\right)a}{1-R^2|a|^2}, \frac{\left(1-|a|^2\right)R}{1-R^2|a|^2}.$$

Let $\alpha > 0$, the α-Bloch space $\mathcal{B}^\alpha$ of quaternion valued functions given by (see [2,9]):

$$\mathcal{B}^\alpha(f) := \left\{ f \in \mathcal{M}(\mathbb{B}) : \sup_{a \in \mathbb{B}} \left|\bar{D}f(x)\right| \left(1-|x|^2\right)^\alpha \right\}.$$

The space $\mathcal{B}^{\frac{3}{2}}$ is called the quaternion Bloch space $\mathcal{B}$. The little quaternion α-Bloch space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}^\alpha$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|a| \to 1} \left|\bar{D}f(x)\right| \left(1-|x|^2\right)^\alpha = 0.$$

The quaternion Dirichlet space $\mathcal{D}$ is given by:

$$\mathcal{D}(f) := \left\{ f \in \mathcal{M}(\mathbb{B}) : \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 dx < \infty \right\}.$$

Let $K:(0,\infty) \to [0,\infty)$ be a non-decreasing function. Define $I_{K,g}(f(a)) : \mathbb{B} \to [0,\infty)$ as

$$I_{K,g}(f(a)) = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K(g(x,a))dx.$$

The spaces Q_K of quaternion valued functions given by

$$\mathcal{Q}_K(f) := \left\{ f \in \mathcal{M}(\mathbb{B}) : I_{K,g}(f(a)) < \infty \right\}.$$

Moreover, the little quaternion $Q_{K,0}$ space consists of those $f \in \mathcal{M}(\mathbb{B})$ for which

$$\lim_{|a| \to 1} \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx = 0.$$

Remark 1.2

Obviously, the quaternion Q_K spaces are not Banach spaces, also are not linear spaces. Nevertheless, if we consider a small neighborhood of the origin N_ε, with an arbitrary but fixed $\varepsilon > 0$, then we can add the L_1-norm of the function f over N_ε to the seminorms, so Q_K spaces will become Banach spaces.

Remark 1.3

It should be remarked that if we put $K(t) = t^p, p < 3$, then $Q_K = Q_p$ (see [7]). Also, if $K(t) = 1$, then $Q_K = \mathcal{D}$, the quaternion Dirichlit space.

Let $K:(0,\infty) \to [0,\infty)$ be a non-decreasing function, consider the following problems:

1. What conditions must K have in order that Q_K to be non-trivial?
2. Which properties of K_1 and K_2 imply that $Q_{K_1} = Q_{K_2}$?
3. For which a necessary and sufficient conditions on K so that $Q_K = \mathcal{B}$?

The main aim of this paper is to study these Q_K spaces and their relations to the above mentioned quaternionic Bloch space. We shall develop a general theory for quaternionic Q_K spaces which answers these questions and gives most basic properties of Q_K and $Q_{K,0}$ spaces. Our results are extensions of the results due to Essén and Wulan (see [5]) in quaternion sense.

The concept may be generalized in the context of Clifford analysis to arbitrary real dimensions. We will restrict us for simplicity to $\mathbb{R}^3$ and quaternion-valued functions as (the lowest non-commutative case) a model case. For more studies on quaternion function spaces, we refer to [2,3,7,10] and others.

We will need the following lemma in the sequel (see [12], Lemma 2.2, if $p = 2$):

Lemma 1.1

Let $f \in \mathcal{M}(\mathbb{B})$ and let $0 < R < 1$. Then for every $a \in \mathbb{B}$, we have

$$\left|\bar{D}f(x)\right|^2 \le \frac{C\left(1-|a|^2\right)^{-3}}{R^3\left(1-R^2\right)^4} \int_{U(a,r)} \left|\bar{D}f(x)\right|^2 dx, \qquad (1)$$

where $C = \frac{768}{\pi}$.

Remark 1.4

If we change the variables $x = \varphi_a(w)$ (the Jacobian determinant $\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3$ has no singularities). In quaternion sense, the problem is that, $\bar{D}f(x)$ is hyperholomorphic, but after the change of variables $\overline{D}f(\varphi_a(w))$ is not hyperholomorphic.

But we know from [13] that $\frac{1-\overline{w}a}{|1-\overline{a}w|^2}\overline{D}f(\varphi_a(w))$ is again hyperholomorphic. So, we can solve this problem by the following lemma (see [10], Lemma 2.2):

Lemma 1.2

Let $f \in \mathcal{M}(\mathbb{B})$ and let $f_a = f \circ \varphi_a$ and let $\Psi_{f_a}: \mathbb{B} \to \mathbb{H}$ given by

$$\Psi_{f_a}(x) \leq \frac{1-\overline{x}a}{|1-\overline{a}x|^2}\overline{D}f(\varphi_a(x)). \tag{2}$$

Then $\Psi_{f_a} \in \mathcal{M}(\mathbb{B})$ and $|\Psi_{f_a}|^2$ is a subharmonic function.

We also refer to [15] who studied this problem for the four-dimensional case already in 1979.

2. Q_K –spaces in Clifford Analysis

In this section, relations between Q_K and Bloch spaces, which have been attracted considerable attention are given in quaternion sense. Our results are extensions of the results due to Essen and Wulan (see [5]) in quaternion sense. We consider some essential properties of Q_K spaces of quaternion-valued functions as basic scale properties.

For a non-decreasing function $K: (0,\infty) \to [0,\infty)$ we say that the space Q_K is trivial if Q_K contains only constant functions. Whether the space Q_K is trivial or not depends on the integral

$$\int_0^1 K\left(\frac{1-r}{r}\right) r^2 dr. \tag{3}$$

Proposition 2.1

(i) If the integral (3) is divergent, then the space Q_K is trivial.

(ii) If the integral (3) is convergent, then $Q_K \subset \mathcal{B}$.

Proof:

(i) For $a \in \mathbb{B}, f \in \mathcal{M}(\mathbb{B})$ and $f_a = f \circ \varphi_a$. Let $\Psi_{f_a}: \mathbb{B} \to \mathbb{H}$ given by (2). Then Ψ_{f_a} is a hyperholomorphic function and $|\Psi_{f_a}|^2$ is a subharmonic function. By Lemma 2.1, after a change of variables $x = \varphi_a(w)$, we have $|\Psi_{f_a}(0)| = |\overline{D}f(a)|(1-|a|^2)^3$. Assume that there exists $f \in Q_K$ such that $\Psi_{f_a}(0) \neq 0$ for some $a \in \mathbb{B}$.

By subharmonicity of $|\Psi_{f_a}|^2$, we have

$$\begin{aligned} \infty &\geq \int_{\mathbb{B}} |\overline{D}f(x)|^2 K(g(x,a))dx \\ &= \int_{\mathbb{B}} |\Psi_{f_a}(y)|^2 K\left(\frac{1-|y|}{|y|}\right)\frac{(1-|a|^2)^3}{|1-\overline{a}y|^2}dy \\ &\geq 2\pi |\Psi_{f_a}(0)|^2 \int_0^1 K\left(\frac{1-r}{r}\right) r^2 dr. \end{aligned} \tag{4}$$

Thus the integral (3) must be convergent and we have proved (i).

(ii) Conversely, if the integral (3) is convergent and $f \in Q_K$, it follows from the inequality (4) that $\mathcal{B}(f) < \infty$, i.e., we have $Q_K \subset \mathcal{B}$. This completes the proof.

The convergence of (3) is related to the growth order of K. The log-order of the real-valued function $K(r)$ is defined as

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log K(r)}{\log r}.$$

If $0 < \rho < \infty$, the log-type of the quaternion-valued function $K(r)$ is defined as

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log^+ K(r)}{r^\rho}.$$

We always assume that the non-decreasing function K is differentiable and satisfies $K(t) = K(1) > 0$ if $t \geq 1$ and $K(2t) \approx K(t)$ if $t \geq 0$. We assume also that the integral (3) is convergent, otherwise, Q_K contains constant functions only.

The following result was proved in [3]:

Proposition 2.2

If the log-order ρ and the log-type σ of a non-decreasing function $K(r)$ satisfy one of the following conditions:

(1) $\rho > 1$,

(2) $\rho = 1$ and $\sigma > 3$.

Then the space Q_K is trivial.

Remark 2.1

In the critical case $\rho = 1$ and $\sigma = 3$, Q_K may be trivial or nontrivial.

From now on and through the remainder of Sections 2 and 3 we assume that the function $K: (0,\infty) \to [0,\infty)$ is non-decreasing and that the integral (3) is convergent.

Theorem 2.1

Assume that $K_1(1) > 0$ and set

$$K_2(r) = \begin{cases} K_1(r), & 0 < r \leq 1; \\ K_1(1), & 1 \leq r < \infty. \end{cases}$$

Then $Q_{K_1} = Q_2$.

Proof:

Since K_1 is non-decreasing and $K_2 \leq K_1$, it is clear that $Q_{K_1} \subset Q_{K_2}$.It remains to prove that $Q_{K_2} \subset Q_{K_1}$. We note that

$$\begin{aligned} &g(x,a) > 1; x \in U(a,1/2), \\ &g(x,a) \leq 1; x \in \mathbb{B} \setminus U(a,1/2). \end{aligned}$$

Thus $K_1(g(x,a)) \leq K_2(g(x,a))$ in $\mathbb{B} \setminus U(a,1/2)$. It suffices to deal with integrals over $U(a,1/2)$.

Now we let $f \in Q_{K_2}$ then for $a \in \mathbb{B}$, we have

$$\begin{aligned} &\int_{U(a,1/2)} |\overline{D}f(x)|^2 K_1(g(x,a))dx \\ &\leq [\mathcal{B}(f)]^2 \int_{U(a,1/2)} (1-|x|^2)^{-3} K_1(g(x,a))dx \\ &\leq [\mathcal{B}(f)]^2 \int_0^{1/2} (1-r^2)^{-3} K_1\left(\frac{1-r}{r}\right) r^2 dr. \end{aligned}$$

By condition (3), the last integral above is convergent. This shows that $f \in Q_{K_1}$ and Theorem 3.1 is proved.

The significance of Theorem 3.1 is that the space Q_K only depends on the behavior of $K(r)$ for r close to 0. In particular, when studying Q_K spaces, we can always assume that $K(r) = K(1)$ for $r \geq 1$. However, we do not make this assumption in our main theorems.

Proposition 2.3

Let $K: (0,\infty) \to [0,\infty)$. Then, a monogenic function $f \in \mathcal{M}(\mathbb{B})$ belongs to the Bloch space $\mathcal{B}$ if and only if there exists an $R \in (0,1)$ such that $K\left(\frac{1-R}{R}\right) > 0$ and

$$\sup_{a\in\mathbb{B}} \int_{U(a,R)} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx < \infty. \quad (5)$$

Proof:

If $f \in \mathcal{B}$, by the argument in the proof of Theorem 3.1, the supremum in (5) is finite for any $R \in (0,1)$.

Conversely, if the supremum in (5) is finite, then

$$\sup_{a\in\mathbb{B}} \int_{U(a,R)} \left|\bar{D}f(x)\right|^2 dx \le \frac{1}{K\left(\frac{1-R}{R}\right)} \sup_{a\in\mathbb{B}} \int_{U(a,R)} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx < \infty.$$

The following result gives a characterization of the quaternion Bloch space $\mathcal{B}$ by quaternion $\mathcal{Q}_K$ spaces.

Theorem 2.2

Let $K: (0,\infty) \to [0,\infty)$, then $\mathcal{Q}_K = \mathcal{B}$ if and only if

$$\int_0^1 (1-r^2)^{-3} K\left(\frac{1-r}{r}\right) r^2 dr < \infty. \quad (6)$$

Proof:

Let us first assume that (6) holds. For $\alpha > 0$, we have

$$(1-|x|^2)^{\alpha} \left|\bar{D}f(x)\right| \le \mathcal{B}^{\alpha}(f).$$

Then, for $\alpha = \frac{3}{2}$, we deduce that

$$\int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx \le \left[\mathcal{B}(f)\right]^2 \int_{\mathbb{B}} (1-|x|^2)^{-3} K\left(g(x,a)\right) dx \le \left[\mathcal{B}(f)\right]^2 \int_{\mathbb{B}} \left(1-\left|\varphi_a(x)\right|^2\right)^{-3} K\left(\frac{1-|x|}{|x|}\right) J^3(a,x) dx.$$

Here, we used that the Jacobian determinant is

$$J(a,x) = \frac{1-|a|^2}{|1-\bar{a}x|^2}.$$

Now, using the equality

$$1-\left|\varphi_a(x)\right|^2 = \frac{\left(1-|a|^2\right)\left(1-|x|^2\right)}{\left|1-\bar{a}x\right|^2},$$

we obtain that,

$$\int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx \le \left[\mathcal{B}(f)\right]^2 \int_0^1 \left(1-r^2\right)^{-3} K\left(\frac{1-r}{r}\right) r^2 dr.$$

Then, we have $\mathcal{B} \subset \mathcal{Q}_K$.

To prove that $\mathcal{Q}_K \subset \mathcal{B}$, we assume that $f \in \mathcal{B}$. For a fixed $R \in (0,1)$ let

$$E(a,R) = \left\{x \in \mathbb{B} : |x-a| < R|1-a|\right\}.$$

Then, we have

$$\int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx \ge \int_{U(a,R)} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx \ge K\left(\frac{1-R}{R}\right) \int_{U(a,R)} \left|\bar{D}f(x)\right|^2 dx \ge K\left(\frac{1-R}{R}\right) \int_{E(a,R)} \left|\bar{D}f(x)\right|^2 dx.$$

By Lemma 1.1, we obtain

$$\int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx \ge \frac{R^3\left(1-R^2\right)^4}{C\left(1-|a|^2\right)^{-3}} K\left(\frac{1-R}{R}\right) \left|\bar{D}f(a)\right|^2,$$

which implies that,

$$\left(1-|a|^2\right)^3 \left|\bar{D}f(a)\right|^2 \le \frac{C}{R^3\left(1-R^2\right)^4 K\left(\frac{1-R}{R}\right)} \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx.$$

This completes the proof.

The importance of Theorem 2.2 is to give us a characterization for the quaternionic Bloch space by the help of integral norms of $\mathcal{Q}_K$ spaces of quaternion valued functions.

Also, with the same arguments used to prove the previous theorem, we can prove the following theorem for characterization of little hyperholomorphic Bloch space.

Theorem 2.3

Let $K: (0,\infty) \to [0,\infty)$, then $\mathcal{Q}_{K,0} = \mathcal{B}_0$ if and only if (6) holds.

Now we give a characterization for the quaternion $\mathcal{Q}_K$ spaces in terms of some different weighted functions in the unit ball of $\mathbb{R}^3$.

Define $I_{K,g}\big(f(a)\big) : \mathbb{B} \to [0,\infty)$ as

$$I_{K,\varphi}\left(f(a)\right) = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K(1-\left|\varphi_a(x)\right|^2) dx.$$

Theorem 2.4

For $K: (0,\infty) \to [0,\infty)$, let $f \in \mathcal{M}(\mathbb{B})$. Then,

$$f \in \mathcal{Q}_K \Leftrightarrow \sup_{a\in\mathbb{B}} I_{K,\varphi}\left(f(a)\right) < \infty. \quad (7)$$

Proof:

We consider the equivalence

$$I_{K,\varphi}\left(f(a)\right) \approx I_{K,g}\left(f(a)\right).$$

By the change of variable $x = \varphi_a(y)$ and Lemma 1.2, we have

$$I_{K,g}\left(f(a)\right) = \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(g(x,a)\right) dx = \int_{\mathbb{B}} \left|\Psi f_a(y)\right|^2 K\left(\frac{1-|y|}{|y|}\right) \frac{(1-|a|^2)^3}{|1-\bar{a}y|^2} dy$$

with $\Psi_{f_a}(y) = \frac{1-\bar{y}a}{|1-\bar{a}y|^3}\bar{D}f(\varphi_a(y))$, while

$$I_{K,\varphi}(f(a)) = \int_{\mathbb{B}} \left|\bar{D}f(x)\right|^2 K\left(1-\left|\varphi_a(x)\right|^2\right)dx$$

$$= \int_{\mathbb{B}} \left|\bar{D}f(\varphi_a(y))\right|^2 K\left(1-|y|^2\right) J^3(a,y)\,dy$$

$$= \int_{\mathbb{B}} \left|\Psi_{f_a}(y)\right|^2 K\left(1-|y|^2\right)\frac{(1-|a|^2)^3}{|1-\bar{a}y|^2}dy,$$

where $J(a,y) = \frac{1-|a|^2}{|1-\bar{a}y|^2}$the Jacobian determinant.

Then, we only need to show

$$K\left(\frac{1-|y|}{|y|}\right) \approx K\left(1-|y|^2\right), y \in \mathbb{B}.$$

This is obvious because of the assumptions for K, and the following obvious facts

- $\frac{3}{4} \leq 1-|y|^2 \leq 1 \leq \frac{1-|y|}{|y|}$, if $0 < |y| \leq \frac{1}{2}$
- $1-|y|^2 \leq \frac{1-|y|}{|y|} \leq 2(1-|y|^2)$, if $\frac{1}{2} \leq |y| < 1$.

The proof of Theorem 3.4 is completed.

3. Conclusion

Our aim in this paper lies at the interface of hyperholomorphic function spaces and operator theory. This paper is an attempt to synthesizethe achievements in the theory of hyperholomorphic function spaces. Many interesting and seemingly basic problems remain open. One of those open problems is the following question: What kind of operators act between the weighted hyperholomorphic function spaces like Bloch Q_p and Q_K spaces? In analytic case several authors have studied boundedness and compactness of composition and Toeplitz operators between some weighted classes of function spaces like BMOA (the space of analytic functions of bounded mean oscillation), Q_p and Q_K spaces (see [4,9,14] and others).

In quaternion sense the problem is that, $f(x)$ is hyperholomorphic, but $(f \circ \varphi)(x)$is not hyperholomorphic, where φ is a hyperholomorphic self-map of the unit ball B.

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