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Some Iterative Methods for Solving Nonlinear Equations

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Abstract In this paper, three iteration methods are introduced to solve nonlinear equations. The convergence criteria for these methods are also discussed. Several examples are presented and compared to other well-known methods, showing the accuracy and fast convergence of the proposed methods.

Keywords: *nonlinear equation, order of convergence, taylor series expansion, iterative methods*

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1. Introduction

One of the oldest and most basic problems in mathematics is that of solving an nonlinear equation $f(x)=0$. This problem has motivated many theoretical developments including the fact that solution formulas do not in general exist.

Thus, the development of algorithms for finding solution has historically been an important enterprise. Newton-Raphson method [11] is the most popular technique for solving nonlinear equations. Many topics related to Newton's method still attract attention from researchers. As is well known, a disadvantage of the method is that the initial approximation x_0, must be chosen sufficiently close to a true solution in order to guarantee their convergence.

Finding a criterion for choosing x_0 is quite difficult and difficult and therefore effective and globally convergent algorithms are needed. In recent years, several methods have been developed to solve the nonlinear equation $f(x)=0$, by Newton method and their modifications [2,4-11]. Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that convergent to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}}=\lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ. In general, a sequence with a high order of convergence more rapidly than a sequence with a lower order.

In this work, a cubic iterative methods based on Taylor's series expansion are introduced as follows:

Consider a nonlinear equation $f(x)=0$. The Taylor's series expansion around a given initial point $x=\gamma$, assuming γ being close enough to the simple root $x=\alpha$, is given as follows:

$$f(x)=f(\gamma)+f'(\gamma)(x-\gamma)+HOT, \tag{1}$$

where HOT denotes the higher order terms. Then the nonlinear equation becomes,

$$f(x)=f(\gamma)+f'(\gamma)(x-\gamma)+HOT=0, \tag{2}$$

when γ is close enough to α, equation (2) can be approximated as

$$f(\gamma)+f'(\gamma)(x-\gamma)\approx 0. \tag{3}$$

Thus we have

$$(x-\gamma)\approx\frac{f(\gamma)}{f'(\gamma)}. \tag{4}$$

By assuming $f'(\gamma)\neq 0$, which yields the one-step iteration method called Newton method [3] with second-order convergence

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{5}$$

2. New Iterative Methods

Depending on the relations (1)-(5), a new one-step iteration method can be constructed with third-order convergence. For this reason, we rewrite equation (2) as

$$f(x)=f(\gamma)+f'(\gamma)(x-\gamma)+\frac{f''(\gamma)}{2!}(x-\gamma)^2+HOT. \tag{6}$$

Equation (6) can be approximated as

$$f(\gamma)+f'(\gamma)(x-\gamma)+[(x-\gamma)]\frac{f''(\gamma)}{2!}(x-\gamma)\approx 0. \tag{7}$$

Substituting (4) into the bracket of equation (7),
We obtain

$$f(\gamma)+f'(\gamma)(x-\gamma)+[-\frac{f(\gamma)}{f'(\gamma)}]\frac{f''(\gamma)}{2!}(x-\gamma)\approx 0.$$

That is

$$f(\gamma)+(x-\gamma)[f'(\gamma)-\frac{f(\gamma)f''(\gamma)}{2f'(\gamma)}]\approx 0. \quad (8)$$

From equation (8),we obtain

$$(x-\gamma)\approx\frac{-2f(\gamma)f''(\gamma)}{2f'^2(\gamma)-f(\gamma)f''(\gamma)}. \quad (9)$$

Thus we can solve equation (9) by assuming

$$2f'^2(\gamma)-f(\gamma)f''(\gamma)\neq 0,$$

as

$$\alpha=\gamma-\frac{2f(\gamma)f''(\gamma)}{2f'^2(\gamma)-f(\gamma)f''(\gamma)}. \quad (10)$$

This suggests that following one-step iteration method:

Algorithm 1.1: For a given x_0, compute the approximate solution x_{n+1} by the one-step iteration scheme:

$$x_{n+1}=x_n-\frac{2f(x_n)f''(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}. \quad (11)$$

It will be shown that the proposed method (11) has third order convergence, and this will be done by applying the following Maple13 program:

$>restart;$

$>F:=x\to x-(2*f(x)*D(f)(x))/(2*(D(f)(x))\wedge 2-f(x)*(D@@2)(f)(x));$

$$F:=x\to x-\frac{2*f(x)D(f)(x)}{2D(f)(x)^2-f(x)D^{(2)}(f)(x)}$$

$>a\lg subs\ (f(r)=0,F(r));$

r

$>a\lg subs\ (f(r)=0,D(F)(r));$

0

$>a\lg subs\ (f(r)=0,(D@@2)(F)(r));$

0

$>a\lg subs\ (f(r)=0,(D@@3)(F)(r));$

$$\frac{-3D^{(2)}(f)(r)^2}{D(f)(r)}-\frac{4D^{(3)}(f)(r)}{D(f)(r)}+\frac{3}{2}\frac{3D^{(2)}(f)(r)^2+2D(f)(r)D^3(f)(r)}{D(f)(r)^2}.$$

To derive another iteration method, again, substituting (9) into the bracket of equation (7), we obtain

$$f(\gamma)+(x-\gamma)[f'(\gamma)+(\frac{-2f(\gamma)f'(\gamma)}{2f'^2(\gamma)-f(\gamma)f''(\gamma)})\frac{f''(\gamma)}{2!}]\approx 0.$$

That is

$$f(\gamma)+(x-\gamma)[\frac{2f'^3(\gamma)-2f(\gamma)f'(\gamma)f''(\gamma)}{2f'^2(\gamma)-f(\gamma)f''(\gamma)}]\approx 0.$$

From the above equation, we get

$$x-\gamma\approx -f(\gamma)[\frac{-2f(\gamma)f'^2(\gamma)+f^2(\gamma)f''(\gamma)}{2f'^3(\gamma)-2f(\gamma)f'(\gamma)f''(\gamma)}]. \quad (12)$$

Thus we can solve equation (12) by assuming

$$2f'^3(\gamma)-2f(\gamma)f'(\gamma)f''(\gamma)\neq 0,$$

as

$$x=\gamma-\frac{2f(\gamma)f'^2(\gamma)+f^2(\gamma)f''(\gamma)}{2f'^3(\gamma)-2f(\gamma)f'(\gamma)f''(\gamma)}. \quad (13)$$

This leads to the following new algorithm:

Algorithm 1.2: For given x_0, compute the approximate solution x_{n+1} by the one-step iteration scheme:

$$x_{n+1}=x_n-\frac{2f(x_n)f'^2(x_n)+f^2(x_n)f''(x_n)}{2f'^3(x_n)-2f(x_n)f'(x_n)f''(x_n)}; \quad (14)$$

for $n=0,1,\ldots$.

It will be shown that the proposed method (14) has third order convergence, and this will be done by applying the following Maple13 program:

$>restart;$

$>F:=x\to x-(f(x)*(D(f)(x))\wedge 2-(1/2)*((f)(x))\wedge 2*(D@@2)(f)(x))/((D(f)(x))\wedge 3=f(x)*D(f)(x)*D(D(f))(x));$

$$F:=x\to x-\frac{f(x)D(f)(x)^2-\frac{1}{2}f(x)^2D^{(2)}(f)(x)}{D(f)(x)^3-f(x)D(f)D(D(f))(x)};$$

$>a\lg subs\ (f(r)=0,F(r));$

r

$>a\lg subs\ (f(r)=0,D(F)(r));$

0

$>a\lg subs\ (f(r)=0,(D@@2)(F)(r));$

0

$>a\lg subs\ (f(r)=0,(D@@3)(F)(r));$

$$\frac{-9D(f)(r)D^{(2)}(f)(r)^2+4D(f)(r)^2D^{(3)}(f)(r)}{D(f)(r)^3}-\frac{3(3D(f)(r)D^{(2)}(f)(r)^2+D(f)(r)^2D^{(3)}(f)(r))}{D(f)(r)^3}.$$

Also, based on relation (4), another new one-step iteration method can be constructed with third-order convergence. For this reason, we rewrite equation (2) as follows:

$$f(x)=f(\gamma)+f'(\gamma)(x-\gamma)+\frac{f''(\gamma)}{2!}(x-\gamma)^2+\frac{f'''(\gamma)}{3!}(x-\gamma)^3+HOT. \quad (15)$$

Equation (15) can be approximate as

$$f(x)=f(\gamma)+(x-\gamma)\{f'(\gamma)+[(x-\gamma)]\frac{f''(\gamma)}{2!}+[(x-\gamma)^2]\frac{f'''(\gamma)}{3!}\}\approx 0. \quad (16)$$

Substituting (4) into the brackets of equation (16), we find

$$f(\gamma)+(x-\gamma)\{f'(\gamma)+[\frac{-f(\gamma)}{f'(\gamma)}]\frac{f''(\gamma)}{2!}+[\frac{-f(\gamma)}{f'(\gamma)}]^2\frac{f'''(\gamma)}{3!}\}\approx 0.$$

That is

$$(x-\gamma)\approx\frac{-6f(\gamma)f'^2(\gamma)}{6f'^3(\gamma)-3f'(\gamma)f''(\gamma)f(\gamma)+f^2(\gamma)f'''(\gamma)}.(17)$$

Using the above equation, we can suggest the following new one-step iteration method as follows:

Algorithm 1.3: For a given x_0, compute the approximate solution x_{n+1} by the one-step iteration scheme:

$$x_{n+1}=x_n-\frac{6f(x_n)f'^2(x_n)}{\left[\begin{array}{l}6f'^3(x_n)-3f'(x_n)f''(x_n)f(x_n)\\+f^2(x_n)f'''(x_n)\end{array}\right]}.$$

It will be shown that the proposed method (18) has third order convergence, and this will be done by applying the following Maple13 program:

$>restart;$

$>F:=x\to x-(6*(f)(x)*(D(f)(x))^\wedge 2/(6*(D(f)(x))^\wedge 3-3*D(f)(x)*(D@@2)(f)(x)*f(x)+(f(x))^\wedge 2*(D@@3)(f)(x));$

$$F:=x\to x-\frac{6f(x)D(f)(x)^2}{s};$$

where

$$s=6D(f)(x)^3-3D(f)(x)D^{(2)}(f)(x)f(x)+f(x)^2D^{(3)}(f)(x)$$

$>a\lg subs\ (f(r)=0,F(r));$

$$r$$

$>a\lg subs\ (f(r)=0,D(F)(r));$

$$0$$

$>a\lg subs\ (f(r)=0,(D@@2)(F)(r));$

$$0$$

$>a\lg subs\ (f(r)=0,(D@@3)(F)(r));$

$$\frac{-7D^{(3)}(f)(r)}{D(f)(r)}-\frac{12D^{(2)}(f)(r)^2}{D(f)(r)^2}+\frac{27D(f)(r)D^{(2)}(f)(r)^2+14D(f)(r)^2D^{(3)}(f)(r)}{2D(f)(r)^3}.$$

3. Numerical Examples

We present some examples to illustrate the efficiency of the new proposed methods in this paper. We compare the Newton's method (NM), the method of Saeed and Aziz [8](SA), the methods of Saeed and Khthr [9] (SK),the method of Saeed and Khthr [10] (SKh) and the method proposed in this paper by the algorithms 1.1-1.3. We use the following stopping criteria for computer program:

$$|x_{n+1}-x_n|<1.E-14$$

Displayed in Table 1 is the number of iterations (IT).

Table 1. Comparison of various iterative methods

functions[7]	x_0	Number of iterations by						
		Alg.1.1	Alg.1.2	Alg.1.3	NM	SA	SK	SKh
$f_1(x)=\sin^2 x-x^2+1$	3	5	6	5	6	5	5	4
$f_2(x)=xe^{x^2}-\sin^2 x+3\cos x+5$	-1	3	5	3	6	4	4	4
$f_3(x)=x^2-e^x-3x+2$	0.1	3	div..	div.	div.	4	7	3
$f_4(x)=x^3+4x-10$	1.5	3	4	3	4	3	3	3

4. Conclusion

We present a new method with three-order convergence for solving nonlinear equations. Analysis of efficiency and the number of iterations shows that the new algorithm is more efficient and it performs better than classical Newton's and similar or better that the methods proposed by [8,9,10]. Also, we see Algorithm 1.3 diverge for $f_3(x)$ because $f_3(x)$ is a symmetric function on the interval which contain x_0.

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