Factorization algebras in quantum field theory

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Contents

CHAPTER 1

Introduction and overview

1.1. Introduction

This book will provide the analog, in quantum field theory, of the deformation quantization approach to quantum mechanics. In this introduction, we will start by recalling how deformation quantization works in quantum mechanics.

The collection of observables in quantum mechanics form an associative algebra. The observables of a classical mechanical system form a Poisson algebra. In the deformation quantization approach to quantum mechanics, one starts with a Poisson algebra A^{cl}, and attempts to construct an associative algebra A^q, which is an algebra flat over the ring $\mathbb{C}[[\hbar]]$, together with an isomorphism of associative algebras $A^q/\hbar \cong A^{cl}$. In addition, if $a, b \in A^{cl}$, and $\widetilde{a}, \widetilde{b}$ are any lifts of a, b to A^q, then

$$\lim_{\hbar \to 0} \frac{1}{\hbar}[\widetilde{a}, \widetilde{b}] = \{a, b\} \in A^{cl}.$$

We will describe an analogous approach to studying perturbative quantum field theory. In order to do this, we need to explain the following.

- The structure present on the collection of observables of a *classical* field theory. This structure is the analog, in the world of field theory, of the commutative algebra which appears in classical mechanics. This structure we call a commutative factorization algebra (section 8).
- The structure present on the collection of observables of a *quantum* field theory. This structure is that of a factorization algebra (section 6.1). We view our definition of factorization algebra as a C^∞ analog of a definition introduced by Beilinson and Drinfeld. However, the definition we use is very closely related to other definitions in the literature, in particular to the Segal axioms.
- The extra structure on the commutative factorization algebra associated to a classical field theory which makes it "want" to quantize. This is the analog, in the world of field theory, of the Poisson bracket on the commutative algebra of observables.

- The quantization theorem we prove. This states that, provided certain obstruction groups vanish, the classical factorization algebra associated to a classical field theory admits a quantization. Further, the set of quantizations is parametrized (order by order in $\hbar$) by the space of deformations of the Lagrangian describing the classical theory.

This quantization theorem is proved using the physicists' techniques of perturbative renormalization, as developed mathematically in [**Cos11c**]. We claim that this theorem is a mathematical encoding of the perturbative methods developed by physicists.

This quantization theorem applies to many examples of physical interest, including pure Yang-Mills theory and σ-models. For pure Yang-Mills theory, it is shown in [**Cos11c**] that the relevant obstruction groups vanish, and that the deformation group is one-dimensional; so that there exists a one-parameter family of quantizations. A certain two-dimensional σ-model was constructed in this language in [**Cos10**, **Cos11a**]. Other examples are considered in [**GG11**] and [**CL11**].

Finally, we will explain how (under certain additional hypotheses) the factorization algebra associated to a perturbative quantum field theory encodes the correlation functions of the theory. This justifies the assertion that factorization algebras encode a large part of quantum field theory.

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1.2. The motivating example of quantum mechanics

The model problems of classical and quantum mechanics involve a particle moving in some Euclidean space $\mathbb{R}^n$ under the influence of some fixed field. Our main goal in this section is to describe these model problems in a way that makes the idea of a factorization algebra (section 6.1) emerge naturally, but we also hope to give mathematicians some feeling for the physical meaning of terms like "field" and "observable." We will not worry about making precise definitions, since that's what this book aims to do. As a narrative strategy, we describe a kind of cartoon of a physical experiment, and we ask that physicists accept this cartoon as a friendly caricature elucidating the features of physics we most want to emphasize.

1.2.1. A particle in a box. For the general framework we want to present, the details of the physical system under study are not so important. However, for concreteness, we will focus attention on a very simple system: that of a single particle confined to some region of space. We confine our particle inside some box and occasionally take measurements of this system. The set of possible trajectories of the particle around the box constitute all the imaginable behaviors of this particle; we might write this mathematically as $\mathrm{Maps}(I, box)$, where $I \subset \mathbb{R}$ denotes the time interval over which we conduct the experiment. In other words, the set of possible behaviors forms a space of *fields* on the timeline of the particle.

The behavior of our theory is governed by the action functional. The simplest case is the action of the massless free field theory, whose value on a function $f : I \to box$ is

$$S(f) = \int_I \langle f, \Delta f \rangle .$$

The aim of this section is to outline the structure one would expect the observables – that is, the possible measurements one can make – should satisfy.

1.2.2. Classical mechanics. Let us start by considering the much simpler case, where our particle is treated as a classical system. In that case, the trajectory of the particle is constrained to be in a solution to the Euler-Lagrange equations of our theory. For example, if the action functional governing our theory is that of the massless free theory, then a map $f : I \to box$ satisfies the Euler-Lagrange equation if it is a straight line.

We are interested in the observables for this classical field theory. Since the trajectory of our particle is constrained to be a solution to the Euler-Lagrange equation, the only measurements one can make are functions on the space of solutions to the Euler-Lagrange equation.

If $U \subset \mathbb{R}$ is an open subset, we will let $\mathrm{Fields(U)}$ denote the space of fields on U, that is, the space of maps $f : U \to box$. We will let

$$\mathrm{EL}(U) \subset \mathrm{Fields}(U)$$

denote the subspace consisting of those maps $f : U \to box$ which are solutions to the Euler-Lagrange equation. As U varies, $\mathrm{EL}(U)$ forms a sheaf of spaces on $\mathbb{R}$.

We will let $\mathrm{Obs}^{cl}(U)$ denote the space of functions on $\mathrm{EL}(U)$ (the precise class of functions we will consider will be discussed later). As U varies, the spaces $\mathrm{Obs}^{cl}(U)$ form a cosheaf of commutative algebras on $\mathbb{R}$. We will think of $\mathrm{Obs}^{cl}(U)$ as the space observables for our classical system which only consider the behavior of the particle on times contained in U.

Note that $\mathrm{Obs}^{cl}(U)$ is a cosheaf of commutative algebras on $\mathbb{R}$.

1.2.3. Measurements in quantum mechanics. The notion of measurement is fraught in quantum theory, but we will take a very concrete view. Taking a measurement means that we have physical measurement device (e.g., a camera) that we allow to interact with our system for a period of time. The measurement is then how our measurement device has changed due to the interaction. In other words, we *couple* the two physical systems, then decouple them and record how the measurement device has modified from its initial condition. (Of course, there is a symmetry in this situation: both systems are affected by their interaction, so a measurement inherently disturbs the system under study.)

The *observables* for a physical system are all the imaginable measurements we could take of the system. Instead of considering all possible observables, we might also consider those observables which occur within a specified time period. This period can be specified by an open interval $U \subset \mathbb{R}$.

Thus, we arrive at the following principle.

> **Principle 1**. For every open subset $U \subset \mathbb{R}$, we have a set $\mathrm{Obs}(U)$ of observables one can make on U.

The superposition principle tells us that quantum mechanics (and quantum field theory) is fundamentally linear. This leads to

> **Principle 2**. The set $\mathrm{Obs}(U)$ is a complex vector space.

We think of $\mathrm{Obs}(U)$ as being the space of ways of coupling a measurement device to our system on the region U. Thus, there is a natural map $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ if $U \subset V$ is an open subset. This means that the space $\mathrm{Obs}(U)$ forms a pre-cosheaf.

1.2.4. Combining observables. Measurements (and so observables) differ qualitatively in the classical and quantum settings. If we study a classical particle, the system is not noticeably disturbed by measurements, and so we can do multiple measurements at the same time. Hence, on each interval J we have a commutative multiplication map $Obs(J) \otimes Obs(J) \to Obs(J)$, as well as the maps that let us combine observables on disjoint intervals.

For a quantum particle, however, a measurement disturbs the system significantly. Taking two measurements simultaneously is incoherent, as the measurement devices are coupled to each other and thus also affect each other, so that we are no longer measuring just the particle. Quantum observables thus do not form a cosheaf of commutative algebras on the interval. However, there are no such problems with combining measurements occurring at different times. Thus, we find the following.

> **Principle 3**. If U, U' are disjoint open subsets of $\mathbb{R}$, and $U, U' \subset V$ where V is also open, then there is a map
>
> $$\star : \mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V).$$
>
> If $O \in \mathrm{Obs}(U)$ and $O' \in \mathrm{Obs}(U')$, then $O \star O'$ is defined by coupling our system to measuring device O for $t \in U$, and to device O' for $t \in U'$.
>
> Further, these maps are commutative, associative, and compatible with the maps $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ associated to inclusions $U \subset V$ of open subsets. (The precise meaning of these terms is detailed in section 3.1.)

1.2.5. Perturbative theory and the correspondence principle.

In the bulk of this book, we will be considering perturbative quantum theory. For us, this means that we work over the base ring $\mathbb{C}[[\hbar]]$, where at $\hbar = 0$ we find the classical theory. In perturbative theory, therefore, the space $\mathrm{Obs}(U)$ of observables on an open subset U is a $\mathbb{C}[[\hbar]]$-module, and the product maps are $\mathbb{C}[[\hbar]]$-linear.

The correspondence principle states that the quantum theory, in the $\hbar \to 0$ limit, must reproduce the classical theory. Applied to observables, this leads to the following principle.

> **Principle 4**. The vector space $\mathrm{Obs}^q(U)$ of quantum observables is a flat $\mathbb{C}[[\hbar]]$-module that, modulo $\hbar$, is the space $\mathrm{Obs}^{cl}(U)$ of classical observables.

These simple principles are at the heart of our approach to quantum field theory. They say, roughly, that the observables of a quantum field theory form a factorization algebra, which is a quantization of the factorization algebra associated to a classical field theory. The main theorem presented in this book is that one can use the techniques of perturbative renormalization to construct factorization algebras perturbatively quantizing a certain class of classical field theories (including many classical field theories of physical and mathematical interest).

1.2.6. Associative algebras in quantum mechanics.

The principles we have described so far indicate that the observables of a quantum mechanical system should assign, to every open subset $U \subset \mathbb{R}$, a vector space $\mathrm{Obs}(U)$, together with a product map

$$\mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V)$$

if U, U' are disjoint open subsets of an open subset V. This is the basic data of a factorization algebra (section 3.1).

It turns out that the factorization algebra produced by our quantization procedure applied to quantum mechanics has a special property: it is *locally constant* (section 6.2). This means that the map $\mathrm{Obs}((a,b)) \to \mathrm{Obs}(\mathbb{R})$ is an isomorphism for any interval (a,b). Let A be denote the vector space $\mathrm{Obs}(\mathbb{R})$; note that A is canonically isomorphic to $\mathrm{Obs}((a,b))$ for any interval (a,b).

The product map

$$\mathrm{Obs}((a,b)) \otimes \mathrm{Obs}((c,d)) \to \mathrm{Obs}((a,d))$$

(defined when $a < b < c < d$) becomes, when we perform this identification, a product map

$$m : A \otimes A \to A.$$

The axioms of a factorization algebra imply that this multiplication turns A into an associative algebra.

This should be familiar to topologists: associative algebras are algebras over the operad of little intervals in $\mathbb{R}$, and this is precisely what we have described. (As we will see later (section **??**), this associative algebra is the Weyl algebra one expects to find as the algebra of observables of quantum mechanics.)

One important point to take away from this discussion is that *associative algebras appear in quantum mechanics because associative algebras are connected with the geometry of* $\mathbb{R}$. There is no fundamental connection between associative algebras and any concept of "quantization": associative algebras only appear when one considers one-dimensional quantum field theories. As we will see later, when one considers quantum field theories on n-dimensional space times, one finds a structure reminiscent of an E_n-algebra instead of an E_1-algebra.

1.3. A preliminary definition of prefactorization algebras

Below (see section 3.1) we give a more formal definition, but here we provide the basic idea. Let M be a topological space (which, in practice, will be a smooth manifold).

1.3.0.1 Definition. *A prefactorization algebra $\mathcal{F}$ on M, taking values in cochain complexes, is a rule that assigns a cochain complex $\mathcal{F}(U)$ to each open set $U \subset M$ along with*

1. *a cochain map $\mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$;*
2. *a cochain map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for every finite collection of open sets where each $U_i \subset V$ and the U_i are disjoint;*
3. *the maps are compatible in a certain natural way. The simplest case of this compatibility is that if $U \subset V \subset W$ is a sequence of open sets, the map $\mathcal{F}(U) \to \mathcal{F}(W)$ agrees with the composition through $\mathcal{F}(V)$).*

Remark: A prefactorization algebra resembles a precosheaf, except that we tensor the cochain complexes rather than taking their direct sum.

The observables of a field theory (whether classical or quantum) form a prefactorization algebra on the spacetime manifold M. In fact, they satisfy a kind of local-to-global principle in the sense that the observables on a large open set are determined by the observables on small open sets. The notion of a factorization algebra (section 6.1) makes this local-to-global condition precise.

1.4. Prefactorization algebras in quantum field theory

The (pre)factorization algebras of interest in this book arise from perturbative quantum field theories. We have already discussed (section 1.2) how factorization algebras appear in quantum mechanics. In this section we will see that this picture extends in a very natural way to quantum field theory.

The manifold M on which the prefactorization algebra is defined is the space-time manifold of the quantum field theory. If $U \subset M$ is an open subset, we will interpret $\mathcal{F}(U)$ as the space of observables (or measurements) that we can make, which only depend on the behavior of the fields on U. Performing a measurement involves coupling a measuring device to the quantum system in the region U.

One can bear in mind the example of a particle accelerator. In that situation, one can imagine the space-time M as being of the form $M = A \times (0, t)$, where A is the interior of the accelerator and t is the duration of our experiment.

In this situation, performing a measurement on some open subset $U \subset M$ is something concrete. Let us take $U = V \times (\varepsilon, \delta)$, where $V \subset A$ is some small region in the accelerator, and (ε, δ) is a short time interval. Performing a measurement on U amounts to coupling a measuring device to our accelerator in the region V, starting at time ε and ending at time δ. For example, we could imagine that there is some piece of equipment in the region V of the accelerator, which is switched on at time ε and switched off at time δ.

1.4.1. Interpretation of the prefactorization algebra axioms.

Suppose that we have two different measuring devices, O_1 and O_2. We would like to set up our accelerator so that we measure both O_1 and O_2.

There are two ways we can do this. Either we insert O_1 and O_2 into disjoint regions V_1, V_2 of our accelerator. Then we can turn O_1 and O_2 on at any times we like, including for overlapping time intervals.

If the regions V_1, V_2 overlap, then we can not do this. After all, it doesn't make sense to have two different measuring devices at the same point in space at the same time.

However, we could imagine inserting O_1 into region V_1 during the time interval (a, b); and then removing O_1, and inserting O_2 into the overlapping region V_2 for the disjoint time interval (c, d).

These simple considerations immediately suggest that the possible measurements we can make of our physical system form a prefactorization algebra. Let $\mathrm{Obs}(U)$ denote the space of measurements we can make on an open subset $U \subset M$. Then, by combining measurements in the way outlined above, we would expect to have maps

$$\mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V)$$

whenever U, U' are disjoint open subsets of an open subset V. The associativity and commutativity properties of a prefactorization algebra are evident.

1.4.2. The cochain complex of observables.

In the approach to quantum field theory considered in this book, the factorization algebra of observables will be a factorization algebra of cochain complexes. One can ask for the physical meaning of the cochain complex $\mathrm{Obs}(U)$.

It turns out that the "physical" observables will be $H^0(\mathrm{Obs}(U))$. If $O \in \mathrm{Obs}^0(U)$ is an observable of cohomological degree 0, then the equation $\mathrm{d}O = 0$ can often be interpreted as saying that O is compatible with the gauge symmetries of the theory. Thus, only those observables $O \in \mathrm{Obs}^0(U)$ which are closed are physically meaningful.

The equivalence relation identifying $O \in \mathrm{Obs}^0(U)$ with $O + \mathrm{d}O'$, where $O' \in \mathrm{Obs}^{-1}(U)$, also has a physical interpretation, which will take a little more work to describe. Often, two observables on U are physically indistinguishable (that is, they can not be distinguished by any measurement one can perform). In the example of an accelerator outlined above, two measuring devices are equivalent if they always produce the same expectation values, no matter how we prepare our system, or no matter what boundary conditions we impose.

As another example, in the quantum mechanics of a free particle, the observable measuring the momentum of a particle at time t is equivalent to that measuring the momentum of a particle at another time t'. This is because, even at the quantum level, momentum is preserved (as the momentum operator commutes with the Hamiltonian).

From the cohomological point of view, if $O, O' \in \mathrm{Obs}^0(U)$ are observables which are in the kernel of d (and thus "physically meaningful"), then they are equivalent in the sense described above if they differ by an exact observable.

It is a little more difficult to provide a physical interpretation for the non-zero cohomology groups $H^n(\mathrm{Obs}(U))$. The first cohomology group $H^1(\mathrm{Obs}(U))$ contains anomalies (or obstructions) to lifting classical observables to the quantum level. For example, in a gauge theory, one might have a classical observable which respects gauge symmetry. However, it may not lift to a quantum observable respecting gauge symmetry; this happens if there is a non-zero anomaly in $H^1(\mathrm{Obs}(U))$.

The cohomology groups $H^n(\mathrm{Obs}(U))$, when $n < 0$, are best interpreted as symmetries, and higher symmetries, of observables. Indeed, we have seen that the physically meaningful observables are the closed degree 0 elements of $\mathrm{Obs}(U)$. One can construct a simplicial set, whose n-simplices are closed and degree 0 elements of $\mathrm{Obs}(U) \otimes \Omega^*(\Delta^n)$. The vertices of this simplicial set are observables, the edges are equivalences between observables, the faces are equivalences between equivalences, and so on.

The Dold-Kan correspondence tells us that the nth homotopy group of this simplicial set is $H^{-n}(\mathrm{Obs}(U))$. This allows us to interpret $H^{-1}(\mathrm{Obs}(U))$ as being the group of symmetries of the trivial observable $0 \in H^0(\mathrm{Obs}(U))$, and $H^{-2}(\mathrm{Obs}(U))$ as the symmetries of the identity symmetry of $0 \in H^0(\mathrm{Obs}(U))$, and so on.

Although the cohomology groups $H^n(\mathrm{Obs}(U))$ where $n > 1$ do not have such a clear physical interpretation, they are mathematically very natural objects and it is important not to discount them. For example, let us consider a gauge theory on a manifold M, and let D be a disc in M. Then it is often the case that elements of $H^1(\mathrm{Obs}(D))$ can be integrated over a circle in M to yield cohomological degree 0 observables (such as Wilson operators).

1.5. Classical field theory and factorization algebras

The main aim of this book is to present a deformation-quantization approach to quantum field theory. In this section we will outline how a classical field theory gives rise to the classical algebraic structure we consider.

We use the Lagrangian formulation throughout. Thus, classical field theory means the study of the critical locus of an action functional. In fact, we use the language of derived geometry, in which it becomes clear that functions on a derived critical locus (section 11.1) should form a P_0 algebra (section 8.3), that is, a commutative algebra with a Poisson bracket of cohomological degree 1. (For an overview of these ideas, see the section 1.8.)

Applying these ideas to infinite-dimensional spaces, such as the space of smooth functions on a manifold, one runs into analytic problems. Although there is no difficulty in constructing a commutative algebra Obs^{cl} of classical observabes, we find that the Poisson bracket on Obs^{cl} is not always well-defined. However, we show the following.

1.5.0.1 Theorem. *For a classical field theory (section 11.4) on a manifold M, there is a sub-commutative factorization algebra* $\widetilde{\mathrm{Obs}}^{c}l$ *of the commutative factorization algebra* Obs^{cl} *on which* *the Poisson bracket is defined, so that* $\widetilde{\mathrm{Obs}}^{cl}$ *forms a* P_0 *factorization algebra. Further, the inclusion* $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ *is a quasi-isomorphism of factorization algebras.*

Remark: Our approach to field theory involves both cochain complexes of infinite-dimensional vector spaces and families over manifolds (and dg manifolds). The infinite-dimensional vector spaces that appear are of the type studied in functional analysis: for example, spaces of smooth functions and of distributions. One approach to working with such vector spaces is to treat them as topological vector spaces. In this book, we will instead treat them as *differentiable* vector spaces. In particular, Obs^{cl} will be a factorization algebra valued in differentiable vector spaces. For a careful discussion of differential vector spaces, see Appendix B. The basic idea is as follows: a differentiable vector space is a vector space V with a smooth structure, meaning that we have a well-defined set of smooth maps from any manifold X into V; and further, we have enough structure to be able to differentiate any smooth map into V. These notions make it possible to efficiently study cochain complexes of vector spaces in families over manifolds.

1.5.1. A gloss of the main ideas. In the rest of this section, we will outline why one would expect that classical observables should form a P_0 algebra. More details are available in section 9.

The idea of the construction is very simple: if $U \subset M$ is an open subset, we will let $\mathcal{EL}(U)$ be the derived space of solutions to the Euler-Lagrange equation on U. Since we are dealing with perturbative field theory, we are interested in those solutions to the equations of motion which are infinitely close to a given solution.

The differential graded algebra $\mathrm{Obs}^{cl}(U)$ is defined to be the space of functions on $\mathcal{EL}(U)$. (Since $\mathcal{EL}(U)$ is an infinite dimensional space, it takes some work to define $\mathrm{Obs}^{cl}(U)$. Details will be presented later (Chapter **??**).

On a compact manifold M, the solutions to the Euler-Lagrange equations are the critical points of the action functional. If we work on an open subset $U \subset M$, this is no longer strictly true, because the integral of the action functional over U is not defined. However, fields on U have a natural foliation, where tangent vectors lying in the leaves of the foliation correspond to variations $\phi \to \phi + \delta\phi$, where $\delta\phi$ has compact support. In this case, the Euler-Lagrange equations are the critical points of a closed one-form $\mathrm{d}S$ defined along the leaves of this foliation.

Any derived scheme which arises as the derived critical locus (section 11.1) of a function acquires an extra structure: it's ring of functions is equipped with the structure of a P_0 algebra. The same holds for a derived scheme arising as the derived critical locus of a closed one-form define along some foliation. Thus, we would expect that $\mathrm{Obs}^{cl}(U)$ is equipped with a natural structure of P_0 algebra; and that, more generally, the commutative factorization algebra Obs^{cl} should be equipped with the structure of P_0 factorization algebra.

1.6. Quantum field theory and factorization algebras

Another aim of the book is to relate perturbative quantum field theory, as developed in [**Cos11c**], to factorization algebras. We give a natural definition of an *observable* of a quantum field theory, which leads to the following theorem.

1.6.0.1 Theorem. *For a classical field theory (section 11.4) and a choice of BV quantization (section 14.2), the quantum observables* Obs^q *form a factorization algebra over the ring* $\mathbb{R}[[\hbar]]$*. Moreover, the factorization algebra of classical observables* Obs^{cl} *is homotopy equivalent to* $\mathrm{Obs}^q \bmod \hbar$ *as a factorization algebra.*

Thus, the quantum observables form a factorization algebra and, in a very weak sense, are related to the classical observables. The quantization theorems will sharpen the relationship between classical and quantum observables.

The main result of [**Cos11c**] allows one to construct perturbative quantum field theories, term by term in $\hbar$, using cohomological methods. This theorem therefore gives a general method to quantize the factorization algebra associated to classical field theory.

1.7. The weak quantization theorem

We have explained how a classical field theory gives rise to P_0 factorization algebra Obs^{cl}, and how a quantum field theory (in the sense of [**Cos11c**]) gives rise to a factorization algebra Obs^q over $\mathbb{R}[[\hbar]]$, which specializes at $\hbar = 0$ to the factorization algebra Obs^{cl} of classical observables. In this section we will state our *weak quantization theorem*, which says that the Poisson bracket on Obs^{cl} is compatible, in a certain sense, with the quantization given by Obs^q.

This statement is the analog, in our setting, of a familiar statement in quantum-mechanical deformation quantization. Recall (section 1.1) that in that setting, we require that the associative product on the algebra A^q of quantum observables is related to the Poisson bracket on the Poisson algebra A^{cl} of classical observables by the formula

$$\{a, b\} = \lim_{\hbar \to 0} \hbar^{-1} [\widetilde{a}, \widetilde{b}]$$

where $\widetilde{a}, \widetilde{b}$ are any lifts of the elements $a, b \in A^{cl}$ to A^q.

One can make a similar definition in the world of P_0 algebras. If A^{cl} is any commutative differential graded algebra, and A^q is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$, then we can define a cochain map

$$\{-,-\}_{A^q} : A^{cl} \otimes A^{cl} \to A^{cl}$$

which measures the failure of the commutative product on A^{cl} to lift to a product on A^q, to first order in $\hbar$. (A precise definition is given in section 8.3).

Now, suppose that A^{cl} is a P_0 algebra. Let A^q be a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$. We say that A^q is a *weak quantization* of A^{cl} if the bracket $\{-,-\}_{A^q}$ on A^{cl}, induced by A^q, is homotopic to the given Poisson bracket on A^{cl}.

This is a very weak notion, because the bracket $\{-,-\}_{A^q}$ on A^{cl} need not be a Poisson bracket; it is simply a bilinear map. When we discuss the notion of strong quantization (section 1.8), we will explain how to put a certain operadic structure on A^q which guarantees that this induced bracket is a Poisson bracket.

1.7.1. The weak quantization theorem.

Now that we have the definition of weak quantization at hand, we can state our weak quantization theorem.

For every open subset $U \subset M$, $\mathrm{Obs}^{cl}(U)$ is a lax P_0 algebra. Given a BV quantization of our classical field theory, $\mathrm{Obs}^q(U)$ is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which coincides, modulo $\hbar$, with $\mathrm{Obs}^{cl}(U)$. Our definition of weak quantization makes sense with minor modifications for lax P_0 algebras as well as for ordinary P_0 algebras.

1.7.1.1 Theorem (The weak quantization theorem). *For every $U \subset M$, the cochain complex* $\mathrm{Obs}^q(U)$ *of classical observables on U is a weak quantization of the lax P_0 algebra* $\mathrm{Obs}^{cl}(U)$.

1.8. The strong quantization conjecture

We have seen (section 1.7) how the observables of a quantum field theory are a quantization, in a weak sense, of the lax P_0 algebra of observables of a quantum field theory. The definition of quantization appearing in this theorem is unsatisfactory, however, because the bracket on the classical observables arising from the quantum observables is not a Poisson observable.

In this section we will explain a stronger notion of quantization. We would like to show that the quantization of the classical observables of a field theory we construct lifts to a strong quantization. However, this is unfortunately is still a conjecture (except for the case of free fields).

1.8.0.2 Definition. *A* BD algebra *is a cochain complex A, flat over $\mathbb{C}[[\hbar]]$, equipped with a commutative product and a Poisson bracket of cohomological degree 1, satisfying the identity*

$$\mathrm{d}(a \cdot b) = a \cdot (\mathrm{d}b) \pm (\mathrm{d}a) \cdot b + \hbar\{a, b\}.$$

The BD operad is investigated in detail in section 8.4. Note that, modulo $\hbar$, a BD algebra is a P_0 algebra.

1.8.0.3 Definition. *A* quantization *of a P_0 algebra A^{cl} is a BD algebra A^q, flat over $\mathbb{C}[[\hbar]]$, together with an equivalence of P_0 algebras between $A^q/\hbar$ and A^{cl}.*

More generally, one can (using standard operadic techniques) define a concept of *homotopy BD algebra*. This leads to a definition of a homotopy quantization of a P_0 algebra.

Recall that the classical observables Obs^{cl} of a classical field theory have the structure of a P_0 factorization algebra on our space-time manifold M.

1.8.0.4 Definition. *Let $\mathcal{F}^{cl}$ be a P_0 factorization algebra on M. Then, a* strong quantization *of $\mathcal{F}^{cl}$ is a lift of $\mathcal{F}^{cl}$ to a homotopy BD factorization algebra $\mathcal{F}^q$, such that $\mathcal{F}^q(U)$ is a quantization (in the sense described above) of $\mathcal{F}^{cl}$.*

We conjecture that our construction of the factorization algebra of quantum observables associated to a quantum field theory has this structure. More precisely,

Conjecture. *Suppose we have a classical field theory on M, and a BV quantization of the theory. Then,* Obs^q *has the structure of a homotopy BD factorization algebra quantizing the P_0 factorization algebra* Obs^{cl}.

Part 1

Prefactorization algebras

CHAPTER 2

From Gaussian measures to factorization algebras

This chapter serves as a kind of second introduction. We will start by defining the observables of a free field theory, motivated by thinking about finite-dimensional Guassian integrals. We will see that for a free theory on a manifold M, there is a space of observables associated to any open subset $U \subset M$. We will see that that the operations we can perform on these spaces of observables give us the structure of a prefactorization algebra on M. This example will serve as further motivation for the idea that observables of a field theory are described by a prefactorization algebra.

2.1. Gaussian integrals

In physics, a free field theory is one where the action functional is purely quadratic. A basic example is the free scalar field theory on a manifold M, where the space of fields is the space $C^\infty(M)$ of smooth functions on M, and the action funtional is

$$S(\phi) = \tfrac{1}{2} \int_M \phi(\triangle + m^2)\phi.$$

Here $\triangle$ refers to the Laplacian with the convention that it's eigenvalues are non-negative. The positive real number m is the mass of the theory. The main quantities of interest in the free field theory are the correlation functions, defined by the heuristic expression

$$\langle \phi(x_1), \dots, \phi(x_n) \rangle = \int_\phi e^{-S(\phi)} \phi(x_1) \dots \phi(x_n)$$

where $x_1, \dots, x_n$ are points in M.

Our task is to explain how the language of prefactorization algebras provides a simple and natural way to make sense of this expression.

Like in most approaches to quantum field theory, we will motivate our definition of the prefactorization algebra of observables by studying finite dimensional Gaussian integrals. Thus, let A be an $n \times n$ symmetric positive-definite matrix, and consider Gaussian integrals of the form

$$\int_{x \in \mathbb{R}^n} \exp\left(-\tfrac{1}{2} \sum x_i A_{ij} x_j\right) f(x)$$

where f is a polynomial function on $\mathbb{R}^n$.

Most textbooks on quantum field theory would, at this point, explain Wick's lemma, which is a combinatorial expression for such integrals. Then, they would go on to define similar infinite-dimensional Gaussian integrals by the analogous combinatorial expression.

We will take a different approach, however. Instead of focusing on the combinatorial expression for the integral, we will focus on the divergence operator associated to the Gaussian measure.

Let $P(\mathbb{R}^n)$ denote the space of polynomial functions on $\mathbb{R}^n$. Let $\mathrm{Vect}(\mathbb{R}^n)$ denote the space of vector fields on $\mathbb{R}^n$ with polynomial coefficients. If $\mathrm{d}\mu$ denotes the Lebesgue measure on $\mathbb{R}^n$, let ω_A denote the measure

$$\omega_A = \exp\left(\tfrac{1}{2}\sum x_i A_{ij} x_j\right) \mathrm{d}\mu.$$

Then, the divergence operator Div_{ω_A} associated to this measure is a linear map

$$\mathrm{Div}_{\omega_A} : \mathrm{Vect}(\mathbb{R}^n) \to P(\mathbb{R}^n),$$

defined abstractly by saying that if $V \in \mathrm{Vect}(\mathbb{R}^n)$, then

$$\mathcal{L}_V \omega_A = (\mathrm{Div}_{\omega_A} V)\, \omega_A$$

where $\mathcal{L}_V$ refers to the Lie derivative. Thus, the divergence of V measures the infinitesimal change in volume that arises when one applies the infinitesimal diffeomorphism V.

In coordinates, the divergence is given by the formula

$$(\dagger) \qquad \mathrm{Div}_{\omega_A}\left(\sum f_i \frac{\partial}{\partial x_i}\right) = -\sum_{i,j} f_i x_j A_{ij} + \sum_i \frac{\partial f_i}{\partial x_i}.$$

By the definition of divergence, it is immediate that

$$\int \omega_A \,\mathrm{Div}_{\omega_A} V = 0$$

for all polynomial vector fields V. By changing basis of $\mathbb{R}^n$ into one where A is diagonal, one sees that the image of the divergence map is a codimension 1 linear subspace of the space $P(\mathbb{R}^n)$ of polynomials on $\mathbb{R}^n$. (This statement is true as long as A is non-degenerate, positive-definiteness is not required).

Let us identify $P(\mathbb{R}^n)/\operatorname{Im}\mathrm{Div}_{\omega_A}$ with $\mathbb{R}$ by taking the basis to be the image of the function 1. What we have shown so far can be summarized in the following lemma.

2.1.0.5 Lemma. *The quotient map*

$$P(\mathbb{R}^n) \to P(\mathbb{R}^n)/\operatorname{Im}\mathrm{Div}_{\omega_A} = \mathbb{R}$$

is the map that sends a function f to

$$\frac{\int_{\mathbb{R}^n} \omega_A f}{\int_{\mathbb{R}^n} \omega_A 1)}.$$

One nice feature of this approach to finite-dimensional Gaussian integrals is that it works over any ring det A is invertible in this ring (this follows from the explicit algebraic formula we wrote for the divergence of a polynomial vector field). This way of looking at finite-dimensional Gaussian integrals was further analyzed in [**?**] by the second author and Theo Johnson-Freyd, where it was shown that one can derive the Feynman rules for finite-dimensional Gaussian integration from such considerations.

2.2. Divergence in infinite dimensions

So far, we have seen that finite-dimensional Gaussian integrals are entirely encoded in thedivergence map from the Gaussian measure. In our approach to infinite-dimensional Gaussian integrals, the fundamental object we will define will be the divergence operator. We will recover the usual formulae for infinite-dimensional Gaussian integrals (in terms of the propagator) from our divergence operator. Further, we will see that analyzing the cokernel of the divergence operator will lead naturally to the notion of prefactorization algebra.

For concreteness, we will work with the free scalar field theory on a Riemannian manifold M which may or may not be compact. We will define a divergence operator for the putative Gaussian measure on $C^\infty(M)$ associated to the quadratic form $\frac{1}{2}\int_M \phi(\triangle + m^2)\phi$.

Before we define the divergence operator, we need to define spaces of polynomial functions and of polynomial vector fields. Let $U \subset M$ be an open subset. We will define polynomial functions and vector fields on the space $C^\infty(U)$.

The space of all continuous linear functionals on $C^\infty(U)$ is the space $\mathcal{D}_c(U)$ of compactly supported distributions on U. In order to define the divergence operator, we need to consider functionals with more regularity. The space of linear functionals we will consider will be $C_c^\infty(U)$, where every element $f \in C_c^\infty(U)$ defines a linear functional on $C^\infty(U)$ by the formula

$$\phi \mapsto \int_U f\phi \mathrm{d}\,\mathrm{Vol}$$

where $\mathrm{d}\,\mathrm{Vol}$ refers to the Riemannian volume form on U.

As a first approximation, we will define the space of polynomial functions on $C_c^\infty(U)$ to be the space

$$\widetilde{P}(C^\infty(U)) = \mathrm{Sym}^* C_c^\infty(U),$$

i.e. the symmetric algebra on $C_c^\infty(U)$. An element of $\widetilde{P}(C_c^\infty(U))$ which is homogeneous of degree n can be written as a finite sum of monomials $f_1 \dots f_n$ where the $f_i \in C_c^\infty(U)$. Such a monomial defines a function on the space $C^\infty(U)$ of fields by sending

$$\phi \mapsto \int_{x_1,\dots,x_n} f_1(x_1)\phi(x_1)\dots f_n(x_n)\phi(x_n).$$

Note that because $C_c^\infty(U)$ is a topological vector space, it might make more sense to use an appropriate completion of $\operatorname{Sym}^n C_c^\infty(U)$ instead of just the algebraic symmetric power. We will introduce the appropriate completion shortly, but for the moment we will just use the algebraic symmetric power. We use the notation $\widetilde{P}$ because this version of the algebra of polynomial functions is a little less natural than the completed version which we will call P.

We will define the space of polynomial vector fields in a similar way. Recall that if V is a finite-dimensional vector space, then the space of polynomial vector fields on V is $P(V) \otimes V$, where $P(V)$ is the space of polynomial functions on V.

In the same way, we let

$$\widetilde{\operatorname{Vect}}(C^\infty(U)) = \widetilde{P}(C^\infty(U)) \otimes C^\infty(U).$$

This is not, however, the vector fields we are really interested in. The space $C^\infty(U)$ has a foliation, coming from the linear subspace $C_c^\infty(U) \subset C^\infty(U)$. We are interested in vector fields along this foliation. Thus, we let

$$\widetilde{\operatorname{Vect}}_c(C^\infty(U)) = \widetilde{P}(C^\infty(U)) \otimes C_c^\infty(U).$$

Again, it is more natural to use a completion of this space which takes account of the topology on $C_c^\infty(U)$. We will discuss such completions shortly.

Any element of $\widetilde{\operatorname{Vect}}_c(C^\infty(U))$ can be written as an finite sum of monomials of the form

$$f_1 \dots f_n \frac{\partial}{\partial \phi}$$

for $f_i, \phi \in C_c^\infty(U)$. By $\frac{\partial}{\partial \phi}$ we mean the constant-coefficient vector field given by infinitesimal translation in the direction ϕ in $C^\infty(U)$.

Vector fields act on functions, in the usual way: the formula is

$$f_1 \dots f_n \frac{\partial}{\partial \phi}(g_1 \dots g_m) = f_1 \dots f_n \sum g_1 \dots \widehat{g_i} \dots g_m \int_U g_i(x)\phi(x) \mathrm{d}\operatorname{Vol}$$

where $\mathrm{d}\operatorname{Vol}$ is the Riemannian volume form on U.

2.2.0.6 Definition. *The divergence operator associated to the quadratic form $\int \phi(\triangle + m^2)\phi$ is the linear map*

$$\widetilde{\operatorname{Vect}}_c(C^\infty(U)) \to \widetilde{P}(C_c^\infty(U))$$

defined by

$$(\ddagger) \qquad \operatorname{Div}\left(f_1 \dots f_n \frac{\partial}{\partial \phi}\right) = -f_1 \dots f_n(\triangle + m^2)\phi + \sum_{i=1}^n f_1 \dots \widehat{f_i} \dots f_n \int_{x \in U} \phi(x) f_i(x).$$

Note that this formula is entirely parallel to the formula for divergence of a Gaussian measure in finite dimensions, given in formula (†).

2.2.1. As we mentioned above, it is probably more natural to use a completion of the spaces $\widetilde{P}(C^\infty(U))$ and $\mathrm{Vect}_c(C^\infty(U))$ of polynomial functions and polynomial vector fields.

Any element $F \in C_c^\infty(U^n)$ defines a polynomial function on $C^\infty(U)$ by

$$\phi \mapsto \int_{U^n} F(x_1, \dots, x_n)\phi(x_1)\dots\phi(x_n).$$

This functional doesn't change if we permute the arguments of F by an element of the symmetric group S_n, so that this only depends on the image of F in the coinvariants of $C_c^\infty(U^n)$ by the symmetric group action. This, of course, is the same as the symmetric group invariants.

Therefore we let

$$P(C^\infty(U)) = \oplus_{n\geq 0} C_c^\infty(U^n)_{S_n},$$

where the subscript indicates coinvariants. A dense subspace of $C_c^\infty(U^n)^{S_n}$ is given by the symmetric power $\mathrm{Sym}^n\, C_c^\infty(U)$. Thus, $\widetilde{P}(C^\infty(U))$ is a dense subspace of $P(C^\infty(U))$.

In a similar way, we can define $\mathrm{Vect}_c(C^\infty(U))$ by

$$\mathrm{Vect}_c(C^\infty(U)) = \oplus_n C_c^\infty(U^{n+1})_{S_n}$$

where the symmetric group acts on the first n factors. A dense subspace of $C_c^\infty(U^{n+1})^{S_n}$ is given by $\mathrm{Sym}^n\, C_c^\infty(U) \otimes C_c^\infty(U)$. Thus, $\widetilde{\mathrm{Vect}}_c(C^\infty(U))$ is a dense subspace of $\mathrm{Vect}_c(C^\infty(U))$.

2.2.1.1 Lemma. *The divergence map*

$$\mathrm{Div} : \widetilde{\mathrm{Vect}}_c(C^\infty(U)) \to \widetilde{P}(C^\infty(U))$$

extends continuously to a map

$$\mathrm{Vect}_c(C^\infty(U)) \to P(C^\infty(U)).$$

PROOF. Suppose that

$$F(x_1, \dots, x_{n+1}) \in C_c^\infty(U^{n+1})_{S_n} \subset \mathrm{Vect}_c(C^\infty(U)).$$

The divergence map in equation (‡) extends to a map which sends F to

$$-\triangle_{x_{n+1}} F(x_1, \dots, x_{n+1}) + \sum_i \int_{x_i \in U} F(x_1, \dots, x_i, \dots, x_n, x_i).$$

□

Now we can define the *quantum observables* of a free field theory.

2.2.1.2 Definition. *For an open subset $U \subset M$, we let*

$$H^0(\mathrm{Obs}^q(U)) = P(C^\infty(U)) / \operatorname{Im} \mathrm{Div} .$$

In other words, we let $H^0(\mathrm{Obs}^q(U))$ be the cokernel of the operator Div. Later we wil see that there is a cochain complex of quantum observables whose zeroth cohomology is what we just defined (this is why we write H^0).

Let us explain why we should interpret this space as a space of quantum observables. We expect that an observable in a field theory is a function on the space of fields. An observable on a field theory on an open subset $U \subset M$ is a function which only depends on the behaviour of the fields on U. Morally speaking, the expectation value of the observable is the integral of this function against the "functional measure" on the space of fields.

Our approach is that we will not try to define the functional measure, we will instead define the divergence operator. If we have some functional of $C^\infty(U)$ which is the divergence of an appropriate vector field, then the expectation value of the corresponding observable is zero. Thus, we would expect that the observable given by a divergence is not a physically measurable quantity, and so should be set to zero.

The appropriate vector fields on $C^\infty(U)$ – the ones for which our divergence operator makes sense – are vector fields along the foliation of $C^\infty(U)$ by compactly supported fields. Thus, the quotient of functions on $C^\infty(U)$ by the subspace of divergences of such vector fields gives a definition of observables.

2.3. The prefactorization structure on observables

Suppose that we have a Gaussian measure on $\mathbb{R}^n$. Then, every function on $\mathbb{R}^n$ with polynomial growth is integrable, and this space of functions forms an algebra. In our approach to quantum field theory, we defined the space of observables to be the quotient of a space of polynomial functions by the subspace of functions which are divergences. This subspace is *not* an ideal. Thus, we would not expect the space of observables to be a commutative algebra.

However, we will find that some shadow of this commutative algebra structure exists, which allows us to combine observables on disjoint subsets. This residual structure will give the spaces $H^0(\mathrm{Obs}^q(U))$ of observables associated to open subset $U \subset M$ the structure of a *prefactorization algebra*.

Let us make these statements precise. Note that $P(C^\infty(V))$ is a commutative algebra, as it is a space of polynomial functions on $C^\infty(V)$. Further, if $U \subset V$ there is a map of commutative algebras $P(C^\infty(U)) \to P(C^\infty(V))$, obtained by composing a polynomial map $C^\infty(U) \to \mathbb{R}$ with the restriction map $C^\infty(V) \to C^\infty(U)$. This map is injective. We will sometimes refer to an element of the subspace $P(C^\infty(U)) \subset P(C^\infty(V))$ as an element of $P(C^\infty(V))$ which is supported on U.

2.3.0.3 Lemma. *The product map*

$$P(C^\infty(V)) \otimes P(C^\infty(V)) \to P(C^\infty(V))$$

does not descend to a map

$$H^0(\mathrm{Obs}(V)) \otimes H^0(\mathrm{Obs}(V)) \to H^0(\mathrm{Obs}(V)).$$

If U_1, U_2 are disjoint open subset of M, both contained in V, then we have a map

$$P(U_1) \otimes P(U_2) \to P(V)$$

obtained by combining the inclusion maps $P(U_i) \hookrightarrow P(V)$ with the product map on $P(V)$. This map does descend to give a map

$$H^0(\mathrm{Obs}(U_1)) \otimes H^0(\mathrm{Obs}(U_2)) \to H^0(\mathrm{Obs}(V)).$$

In other words, although the product of general observables does not make sense, the product of observables with disjoint support does.

PROOF. Let U_1, U_2 be disjoint open subsets of M, both contained in an open$V$. Let us view the spaces $\mathrm{Vect}_c(C^\infty(U_i))$ and $P(C^\infty(U_i))$ as subspaces of $\mathrm{Vect}_c(C^\infty(V))$ and $P(C^\infty(V))$ respectively.

On an ordinary finite-dimensional manifold, the divergence with respect to any volume form has the following property: for a vector field X and a function f, we have

$$\mathrm{Div}(fX) - f \,\mathrm{Div}\, X = Xf,$$

where Xf is the action of X on f.

The same equation holds for the divergence operator we have defined in infinite dimensions. If $X \in \mathrm{Vect}_c(C^\infty(V))$ and $G \in P(C_c^\infty(V))$ then

$$\mathrm{Div}(GX) - G \,\mathrm{DivX} = X(G).$$

This tells us that the image of Div is not an ideal, as $X(G)$ is not necessarily in the image of Div.

However, suppose that G is in $P(C^\infty(U_1))$ and that X is in $\mathrm{Vect}_c(C^\infty(U_2))$. Then, $X(G) = 0$. This is clear from the fact that $\mathrm{Vect}_c(C^\infty(U_2))$ is the space of polynomial vector fields along the foliation of $C^\infty(U_2)$ by compactly supported functions. If $X \in \mathrm{Vect}_c(U_2)$ and $\phi \in C^\infty(U_2)$, let $X_\phi \in C_c^\infty(U_2)$ be the tangent vector associated to X at the point ϕ. The infinitesimal symmetry X sends ϕ to $\phi + \varepsilon X_\phi$. Since X_ϕ has compact support in U_2, ϕ and $\phi + \varepsilon X_\phi$ agree outside a compact set in U_2. (This is just rephrasing the fact that X

is a vector field on $C^\infty(U_2)$ along the foliation coming from compactly-supported smooth functions).

If $\phi \in C^\infty(V)$, then $G(\phi)$ only depends on the behaviour of ϕ on $C^\infty(U_1)$, whereas $\phi + \varepsilon X \phi$ is equal to ϕ outside a compact set in U_2, and in particular is equal to ϕ on U_1 which is disjoint from U_2. Therefore

$$G(\phi) + \varepsilon \left(XG\right)(\phi) = G(\phi + \varepsilon X_\phi) = G(\phi).$$

That is, $XG = 0$.

This shows that, if X and G have disjoint support,

$$\operatorname{Div}(GX) = G \operatorname{Div} X.$$

This is enough to show that the product map

$$P(U_1) \otimes P(U_2) \to P(V)$$

descends to a map

$$H^0(\mathrm{Obs}^q(U_1)) \otimes H^0(\mathrm{Obs}^q(U_2)) \to H^0(\mathrm{Obs}^q(V)).$$

Indeed, if $G_i \in P(U_2)$ and $X_i \in \mathrm{Vect}_c(U_i)$ then

$$G_2 \operatorname{Div}(X_1) = \operatorname{Div}(GX_2) \in \operatorname{Im} \operatorname{Div}$$

and similarly, $\operatorname{Div}(X_2) G_1 \in \operatorname{Im} \operatorname{Div}$. □

In a similar way, if $U_1, \ldots, U_n$ are disjoint opens all contained in V, then the map

$$P(U_1) \otimes \cdots \otimes P(U_n) \to P(V)$$

descends to a map

$$H^0(\mathrm{Obs}^q(U_1)) \otimes \cdots \otimes H^0(\mathrm{Obs}^q(U_n)) \to H^0(\mathrm{Obs}^q(V)).$$

Thus, we see that the spaces $H^0(\mathrm{Obs}^q(U))$ for open sets $U \subset M$ are naturally equipped with the structure maps necessary to define a prefactorization algebra. (See section 1.3 for a sketch of the definition of a factorization algebra, and section 3.1 for more details on the definition). It is straightforward to check that these structure maps satisfy the necessary compatibility conditions to define a prefactorization algebra.

2.4. From quantum to classical

Our general philosophy is that the quantum observables of a field theory are a factorization algebra which deform the classical observables. Classical observables are defined to be functions on the space of solutions to the equations of motion.

Let us see why this holds for a class of measures in finite dimensions. Let S be a polynomial function on $\mathbb{R}^n$. Let $\mathrm{d}\,\mathrm{Vol}$ denote the Lebesgue measure on $\mathbb{R}^n$, and consider the measure

$$\omega = e^{-S/\hbar}\mathrm{d}\,\mathrm{Vol}$$

where $\hbar$ is a small parameter. The divergence with respect to ω satisfies

$$\mathrm{Div}_{\omega}\left(\sum f_i \frac{\partial}{\partial x_i}\right) = \sum \frac{\partial f_i}{\partial x_i} - \frac{1}{\hbar}\sum f_i \frac{\partial S}{\partial x_i}.$$

Let $P(\mathbb{R}^n)$ denote the space of polynomial functions on $\mathbb{R}^n$ and let $\mathrm{Vect}(\mathbb{R}^n)$ denote the space of polynomial vector fields. We view the divergence operator Div_ω as a map $\mathrm{Vect}(\mathbb{R}^n) \to P(\mathbb{R}^n)$. Note that the image of Div_ω and of $\hbar\,\mathrm{Div}_\omega$ is the same as long as $\hbar \to 0$. As $\hbar \to 0$, the operator $\hbar\,\mathrm{Div}_\omega$ becomes the operator

$$\sum f_i \frac{\partial}{\partial x_i} \to -\sum f_i \frac{\partial S}{\partial x_i}.$$

Therefore, the $\hbar \to 0$ limit of the image of Div_ω is the ideal in $P(\mathbb{R}^n)$ which cuts out the critical locus of S.

Let us now check the analogous property for the observables of a free scalar field theory on a manifold M. We will consider the divergence for the putative Gaussian measure

$$\exp\left(-\tfrac{1}{\hbar}\int_M \phi(\triangle + m^2)\phi\right)\mathrm{d}\,\mathrm{Vol}$$

on $C^\infty(M)$. For any open subset $U \subset M$, this divergence operator gives us a map

$$\mathrm{Div}_\hbar : \mathrm{Vect}_c(C^\infty(U)) \to P(C^\infty(U))$$

which sends

$$f_1 \dots f_n \frac{\partial}{\partial \phi} \to \sum f_1 \dots \widehat{f_i} \int_M f\phi - \tfrac{1}{\hbar} f_1 \dots f_n (\triangle + m^2)\phi.$$

As before, in the $\hbar \to 0$ limit, the second term dominates; so that the $\hbar \to 0$ limit of the image of $\mathrm{Div}_\hbar$ is the closed subspace of $P(C^\infty(U))$ spanned by functionals of the form $f_1 \dots f_n(\triangle + m^2)\phi$, where f_i and ϕ are in $C_c^\infty(U)$). This is the topological ideal in $P(C^\infty(U))$ generated by linear functionals of the form $(\triangle + m^2)f$ where $f \in C_c^\infty(U)$. If $S(\phi) = \int \phi(\triangle + m^2)\phi$ is the action functional of our theory, then this is precisely the topological ideal generated by

$$\frac{\partial S}{\partial \phi}$$

for $\phi \in C_c^\infty(U)$. In other words, it is the ideal cut out by the Euler-Lagrangian equations. Let us call this ideal I_{EL}.

A more precise statement of what we have just sketched is the following. Define a prefactorization algebra $H^0(\mathrm{Obs}^{cl}(U))$ (the superscript cl stands for classical) which assigns to U the quotient of $P(C^\infty(U))$ by the ideal I_{EL}. Thus, $H^0(\mathrm{Obs}^{cl}(U))$ should be thought of as the space of polynomial functions on the space of solutions to the Euler-Lagrangian equations. Note that each constituent space $H^0(\mathrm{Obs}^{cl}(U))$ in this prefactorization algebra has the structure of a commutative algebra, and the structure maps are all maps of commutative algebras. This means that $H^0(\mathrm{Obs}^{cl})$ forms a *commutative prefactorization algebra*. Heuristically, this means that the product map defining the factorization structure is defined for all pairs of opens $U_1, U_2 \subset V$, and not just when U_1 and U_2 are disjoint.

2.4.0.4 Lemma. *There is a prefactorization algebra $H^0(\mathrm{Obs}^q_\hbar)$ over $\mathbb{C}[\hbar]$ which when specialized to $\hbar = 1$ is $H^0(\mathrm{Obs}^q)$ and to $\hbar = 0$ is $H^0(\mathrm{Obs}^{cl})$.*

This prefactorization algebra assigns to an open set the quotient of the map

$$\hbar \operatorname{Div}_\hbar : \mathrm{Vect}_c(C^\infty(M))[\hbar] \to P(C^\infty(M)j)[\hbar]$$

where $\operatorname{Div}_\hbar$ is the map discussed earlier.

We will see later that the $\mathbb{R}[\hbar]$-module $H^0(\mathrm{Obs}^q_\hbar(U))$ is free (although this is a special property of free theories and is not always true for an interacting theory.).

2.5. Correlation functions

We have seen that the observables of a free scalar field theory on a manifold M give rise to a factorization algebra. In this section, we will explain how the structure of a factorization algebra is enough to define correlation functions of observables. We will calculate certain correlation functions explicitly, and find the same answers physicists normally find.

Let us suppose that M is a compact Riemannian manifold, and as before, let us consider the observables of the free scalar field theory on M with mass $m > 0$. Then, we have the following.

2.5.0.5 Lemma. *If the mass m is positive, then $H^0(\mathrm{Obs}^q(M)) = \mathbb{R}$.*

We should compare this with the statement that if we have a Gaussian measure on $\mathbb{R}^n$, the image of the divergence map is of codimension 1 in the space of polynomial functions on $\mathbb{R}^n$. The assumption that the mass is positive is necessary to ensure that the quadratic form $\int_M \phi(\triangle + m^2)\phi$ is non-degenerate.

This lemma will follow from our more detailed analysis of free theories in chapter 4. The main point is that there is a family over $\mathbb{R}[\hbar]$ connecting $H^0(\mathrm{Obs}^q(M))$ and $H^0(\mathrm{Obs}^{cl}(M))$. Since the only solution to the equations of motion in the case that $m > 0$ is the function $\phi = 0$, the algebra $H^0(\mathrm{Obs}^{cl}(M))$ is $\mathbb{R}$. To conclude that $H^0(\mathrm{Obs}^q(M))$ is also

$\mathbb{R}$, we need to show that $H^0(\mathrm{Obs}^q_\hbar(M))$ is flat over $\mathbb{R}$, which will follow from a spectral sequence analysis we will perform later.

There is always a canonical observable $1 \in H^0(\mathrm{Obs}^q(U))$ for any open subset $U \subset M$. This is defined to be the image of the function $1 \in P(C^\infty(U))$. We identify $H^0(\mathrm{Obs}^q(M))$ with $\mathbb{R}$ by taking the observable 1 to be a basis.

2.5.0.6 Definition. *Let $U_1, \ldots, U_n \subset M$ be disjoint open subsets. The* correlator *is the prefactorization structure map*

$$\langle -, \ldots, - \rangle : H^0(\mathrm{Obs}^q(U_1)) \otimes \cdots \otimes H^0(\mathrm{Obs}^q(U_n)) \to H^0(\mathrm{Obs}^q(M)) = \mathbb{R}$$

We should compare this definition with what happens in finite dimensions. If we have a Gaussian measure on $\mathbb{R}^n$, then the space of polynomial functions modulo divergences is one-dimensional. If we take the image of a function 1 to be a basis of this space, then we get a map

$$P(\mathbb{R}^n) \to \mathbb{R}$$

from polynomial functions to $\mathbb{R}$. This map is the integral against the Gaussian measure, normalized so that the integral of the function 1 is 1.

In our infinite dimensional situation we are doing something very similar. Any reasonable definition of the correlation function of functions $O_1, \ldots, O_n$ with $O_i \in P(C^\infty(U_i))$ should only depend on the product function $O_1 \ldots O_n \in P(C^\infty(M))$. Thus, the correlation function map should be a linear map $P(C^\infty(M)) \to \mathbb{R}$. Further, it should send divergences to zero. We have seen that up to scale there is only one such map.

Next we will check explicitly that this correlation function map really matches up with what physicists expect. Let $f_i \in C_c^\infty(U_i)$ be compactly-supported smooth functions on the open sets $U_1, U_2 \subset M$. Let us view each f_i as a linear function on $C^\infty(U_i)$, and so as an element of $P(C^\infty(U_i))$.

Let $G \in \mathcal{D}(M \times M)$ be the unique distribution on $M \times M$ with the property that

$$(\triangle_x + m^2) G(x, y) = \delta_{\mathrm{Diag}}.$$

In other words, if we apply the operator $\triangle + m^2$ to the first factor of G, we find the delta function on the diagonal. Thus, G is the kernel for the operator $(\triangle + m^2)^{-1}$. In the physics literature, G is called the propagator, and in the math literature it is called the Green's function for the operator $\triangle + m^2$. Note that G is smooth away from the diagonal.

Then,

2.5.0.7 Lemma.

$$\langle f_1, f_2 \rangle = \int_{x,y \in M} f_1(x) G(x, y) f_2(y).$$

Note that this is exactly what a physicist would say.

PROOF. Later we will give a slicker and more general proof of this kind of statement. Here we will give a simple proof to illustrate how the Green's function arises from our homological approach to defining functional integrals.

Let

$$\phi = (\triangle + m^2)^{-1} f_2 \in C^\infty(M).$$

Thus,

$$\phi(x) = \int_{y \in M} G(x,y) f_2(y).$$

Consider the vector field

$$f_1 \frac{\partial}{\partial \phi} \in \mathrm{Vect}_c(C^\infty(M)).$$

Note that

$$\begin{aligned} \mathrm{Div}\left(f_1 \frac{\partial}{\partial \phi} \right) &= \int_{x \in M} f_1(x)\phi(x) - f_1\left((\triangle + m^2)\phi \right) \\ &= \left(\int_M f_1(x) G(x,y) f_2(y) \right) \cdot 1 - f_1 f_2. \end{aligned}$$

The element $f_1 f_2 \in P(C^\infty(M))$ represents the factorization product of the observables $f_i \in H^0(\mathrm{Obs}^q(U_i))$ in $H^0(\mathrm{Obs}^q(M))$. The displayed equation tells us that

$$f_1 f_2 = \left(\int_M f_1(x) G(x,y) f_2(y) \right) \cdot 1 \in H^0(\mathrm{Obs}^q(M)).$$

Since the observable 1 is chosen to be the basis element identifying $H^0(\mathrm{Obs}^q(M))$ with $\mathbb{R}$, the result follows. □

2.6. Further results on free field theories

In this chapter, we showed that, if we define the observables of a free field theory as a cokernel of a certain divergence operator, then these spaces of observables form a prefactorization algebra. We also showed that this prefactorization algebra contains enough information to allow us to define the correlation functions of observables, and that for linear observables we find the same formula that physicists would write.

In chapter 3 we will show that a certain class of factorization algebras on the real line are equivalent to associative algebras together with a derivation. The derivation comes from infinitesimal translation on the real line.

In chapter 4 we will analyze the factorization algebra of free field theories in more detail. We will show that if we consider the free field theory on $\mathbb{R}$, the factorization algebra

$H^0(\mathrm{Obs}^q)$ corresponds (under the relationship between factorization algebras on $\mathbb{R}$ and associative algebras) to the Weyl algebra. The Weyl algebra is generated by observables p, q corresponding to position and momentum satisfying $[p, q] = 1$. If we consider instead the family over $\mathbb{R}[\hbar]$ of factorization algebras $H^0(\mathrm{Obs}^q_\hbar)$ discussed above, then we find the commutation relation $[p, q] = \hbar$. This, of course, is what is traditionally called the algebra of observables of quantum mechanics. In this case, we will further see that the derivation of this algebra (corresponding to inner time translation) is an inner derivation, given by bracketing with the Hamiltonian

$$H = p^2 - m^2 q^2$$

which is the standard Hamiltonian for quantum mechanics.

More generally, we can consider a free scalar free theory on $N \times \mathbb{R}$ where N is a compact Riemannian manifold. This gives rise to a factorization algebra on $\mathbb{R}$ which assigns to an open subset $U \subset \mathbb{R}$ the space $H^0(\mathrm{Obs}^q(N \times U))$. We will see that there is a dense sub-factorization algebra of this factorization algebra on $\mathbb{R}$ which corresponds to an associative algebra. This associative algebra is a tensor product of Weyl algebras, with one Weyl algebra arising from each eigenspace of the operator $\triangle + m^2$ on $C^\infty(N)$. In other words, we find quantum mechanics on $\mathbb{R}$ with values in the infinite-dimensional space $C^\infty(N)$, where the λ eigenspace of the operator $\triangle + m^2$ is given mass $\sqrt{\lambda}$. This is entirely consistent with physics expectations.

2.7. Interacting theories

In any approach to quantum field theory, free field theories are easy to construct. The challenge is always to construct interacting theories. The core results of this book show how to construct the factorization algebra corresponding to interacting field theories, deforming the factorization algebra for free field theories discussed above.

Let us explain a little bit about the challenges we need to overcome in order to deal with interacting theories, and how we overcome these challenges.

Consider an interacting scalar field theory on a Riemannian manifold M. For instance, we could consider an action functional of the form

$$S(\phi) = -\tfrac{1}{2} \int_M \phi(\triangle + m^2)\phi + \int_M \phi^4.$$

In general, the action functional must be local: it must arise as the integral over M of some polynomial in ϕ and its derivatives.

We will let $I(\phi)$ denote the interacting term in our field theory, which consists of the cubic and higher terms in S. In the above example, $I(\phi) = \int \phi^4$. We will always assume that the quadratic term in S is of the form $-\frac{1}{2}\int_M \phi(\triangle + m^2)\phi$. (Of course, our techniques apply to a very general class of interacting theories, including gauge theories).

If $U \subset M$ is an open subset, we can consider, as before, the spaces $\mathrm{Vect}_c(C^\infty(M))$ and $P(C^\infty(M))$ of polynomial functions and vector fields on M. By analogy with the finite-dimensional situation, one can try to define the divergence for the putative measure $\exp S(\phi)/\hbar \mathrm{d}\mu$ (where $\mathrm{d}\mu$ refers to the "Lebesgue measure" on $C^\infty(M)$)) by the formula

$$\mathrm{Div}_\hbar \left(f_1 \dots f_n \frac{\partial}{\partial \phi} \right) = \tfrac{1}{\hbar} f_1 \dots f_n \frac{\partial S}{\partial \phi} + \sum f_1 \dots \widehat{f_i} \dots f_n \int_M f_i \phi.$$

This is the same formula we used earlier when S was quadratic.

Now we see that a problem arises. We defined $P(C^\infty(U))$ as the space of polynomial functions whose Taylor terms are given by integration against a smooth function on U^n. That is,

$$P(C^\infty(U)) = \oplus_n C_c^\infty(U^n)_{S_n}.$$

If $\phi \in C_c^\infty(U))$, then $\frac{\partial S}{\partial \phi}$ is not necessarily in this space of functions. For instance, if $I(\phi) = \int \phi^4$ is the interaction term in the example of the ϕ^4 theory, then

$$\frac{\partial I}{\partial \phi_0}(\phi) = \int_M \phi_0 \phi^3.$$

This is a cubic function on the space $C^\infty(U)$ but it is not given by integration against an element in $C_c^\infty(U^3)$. Instead it is given by integrating against a distribution on U^3, namely, the delta-distribution on the diagonal.

We can try to solve this by using a larger class of polynomial functions. Thus, we could let

$$\overline{P}(C^\infty(U)) = \oplus_n \mathcal{D}_c(U^n)_{S_n}$$

where $\mathcal{D}_c(U^n)$ is the space of compactly supported distributions on U. Similarly, we could let

$$\overline{\mathrm{Vect}}_c(U) = \oplus_n \mathcal{D}_c(U^{n+1})_{S_n}.$$

The spaces $P(C^\infty(U))$ and $\mathrm{Vect}_c(C^\infty(U))$ are dense subspaces of these spaces.

If $\phi_0 \in \mathcal{D}_c(U)$ is a compactly-supported distribution, then for any local functional S, $\frac{\partial S}{\partial \phi}$ is a well-defined element of $\overline{P}(C^\infty(U))$. Thus, it looks like this has solved our problem.

However, using this larger space of functions gives us an additional problem: the second term in the divergence operator now fails to be well-defined. For example, if $f, \phi \in \mathcal{D}_c(U)$, then we have

$$\mathrm{Div}_\hbar f\phi = \tfrac{1}{\hbar} f \frac{\partial S}{\partial \phi} + \int_M f\phi.$$

Now, $\int_M f\phi$ doesn't make sense, because it involves pairing the distribution $f \boxtimes \phi$ on the diagonal in M^2 with the delta-function on the diagonal.

If we take the $\hbar \to 0$ limit of $\hbar \operatorname{Div}_\hbar$ we find, however, a well-defined operator

$$\begin{aligned} \overline{\mathrm{Vect}}_c(C^\infty(U)) &\to \overline{P}(C^\infty(U)) \\ X &\to XS \end{aligned}$$

which sends a vector field X to its action on the local functional S.

The cokernel of this operator is the quotient of space $\overline{P}(C^\infty(U))$ by the ideal I_{EL} generated by the Euler-Lagrange equations. We thus let

$$H^0(\overline{\mathrm{Obs}}_S^{cl}(U)) = \overline{P}(C^\infty(U))/I_{EL}.$$

As U varies, this forms a factorization algebra on M, which we call the factorization algebra of classical observables associated to the action functional S.

2.7.0.8 Lemma. *If $S = -\frac{1}{2}\int \phi(\triangle + m^2)\phi$, then this definition of classical observables coincides with the one we discussed earlier:*

$$H^0(\overline{\mathrm{Obs}}_S^{cl}(U)) = H^0(\mathrm{Obs}^{cl}(U))$$

where $H^0(\mathrm{Obs}^{cl}(U))$ is defined, as earlier, to be the quotient of the space $P(C^\infty(M))$ by the ideal of Euler-Lagrange equations.

This result is a version of elliptic regularity.

Now we can see the challenge we have. If S is the action functional for the free field theory, then we have a factorization algebra of classical observables. This factorization algebra deforms in two ways: first, we can deform it into the factorization algebra of quantum observables for a free theory. Second, we can deform it into the factorization algebra of classical observables for an interacting field theory.

The challenge is to perform both of these deformations simultaneously.

The technique we use to construct the observables of an interacting field theory uses the renormalization technique of [**Cos11c**]. In [**Cos11c**], the first author gives a definition of a quantum field theory and a cohomological method for constructing field theories. A field theory as defined in [**Cos11c**] gives us (essentially from the definition) a family of divergence operators

$$\operatorname{Div}[L] : \overline{\mathrm{Vect}}_c(M) \to \overline{P}(C^\infty(M)),$$

one for every $L > 0$. These divergence operators, for varying L, can be conjugated to each other by linear continuous isomorphisms of $\overline{\mathrm{Vect}}_c(M))$ and $\overline{P}(C^\infty(M))$. These divergence operators do not map the space $\overline{\mathrm{Vect}}_c(U))$ to the space $\overline{P}(C^\infty(U))$, for an open subset $U \subset M$. However (roughly speaking) for L small the operator $\operatorname{Div}[L]$ only increases the support of an element of $\overline{\mathrm{Vect}}_c(U)$) by a small amount. This turns out to be enough to define the factorization algebra of quantum observables.

This construction of quantum observables for an interacting field theory is given in chapter 15, which is the most technically difficult chapter of the book. In the preceeding chapters, we will develop some aspects of the theory of factorization algebras in general; analyze in more detail the factorization algebra associated to a free theory; construct and analyze factorization algebras associated to vertex algebras such as the Kac-Moody vertex algebra; develop classical field theory using a homological approach arising from the BV formalism; and flesh out the description of the factorization algebra of classical observables we have sketched here.

CHAPTER 3

Prefactorization algebras and basic examples

In this chapter we will give a formal definition of the notion of prefactorization algebra. We also explain how to construct a factorization algebra from any sheaf of Lie algeras on a manifold M. This construction is called the *factorization envelope*, and is related to the universal enveloping algebra of a Lie algebra as well as to Beilinson-Drinfeld's [**BD04**] chiral envelope. Although the factorization envelope construction is very simple, it plays an important role in field theory. For example, the factorization algebra for any free theories is a factorization envelope, as is the factorization algebra corresponding to the Kac-Moody vertex algebra. More generally, factorization envelopes play an important role in our formulation of Noether's theorem for quantum field theories.

3.1. Prefactorization algebras

Let M be a topological space and let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We are particularly interested in the case where M is a smooth manifold and $\mathcal{C}$ is Vect or dgVect with the usual tensor product as the monoidal product. In this section we will give a formal definition of the notion of a prefactorization algebra.

3.1.1. The essential idea of a prefactorization algebra. A prefactorization algebra $\mathcal{F}$ on M, taking values in cochain complexes, is a rule that assigns a cochain complex $\mathcal{F}(U)$ to each open set $U \subset M$ along with

- a cochain map $\mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$;
- a cochain map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for every finite collection of open sets where each $U_i \subset V$ and the U_i are pairwise disjoint;
- the maps are compatible in the obvious way (e.g. if $U \subset V \subset W$ is a sequence of open sets, the map $\mathcal{F}(U) \to \mathcal{F}(W)$ agrees with the composition through $\mathcal{F}(V)$);

Thus $\mathcal{F}$ resembles a precosheaf, except that we tensor the cochain complexes rather than take their direct sum. These axioms imply that $\mathcal{F}(\emptyset)$ is a commutative algebra; we say that $\mathcal{F}$ is a unital prefactorization algebra if $\mathcal{F}(\emptyset)$ is a unital commutative algebra. In practise, $\mathcal{F}(\emptyset)$ is one of $\mathbb{C}, \mathbb{R}, \mathbb{C}[[\hbar]], \mathbb{R}[[\hbar]]$.

The crucial example to bear in mind is an associative algebra. Every associative algebra A defines a prefactorization algebra $\mathcal{F}_A$ on $\mathbb{R}$, as follows. To each open interval (a, b), we set $\mathcal{F}_A((a, b)) := A$. To any open set $U = \coprod_j I_j$, where each I_j is an open interval, we set $\mathcal{F}(U) := \bigotimes_j A$.[1] The structure maps simply arise from the multiplication map for A. Figure (1) displays the structure of $\mathcal{F}_A$. (Notice the resemblance to the notion of an E_1 or A_∞ algebra.)

FIGURE 1. Structure of $\mathcal{F}_A$

In the remainder of this section, we describe two other ways of phrasing this idea, but the reader who is content with this definition and eager to see examples should feel free to jump to section **??**, referring back as needed.

3.1.2. Prefactorization algebras in the style of algebras over an operad.

3.1.2.1 Definition. *Let* Fact$_M$ *denote the following multicategory associated to M.*

- *The objects consist of all connected open subsets of M.*
- *For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps* $\mathrm{Fact}_M(\{U_\alpha\}_{\alpha \in A}, V)$. *If the U_α are pairwise disjoint and all are contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty.*
- *The composition of maps is defined in the obvious way.*

Remark: By "multicategory" we mean what is also called a colored operad or a pseudotensor category. In [**Lei04**], there is an accessible discussion of multicategories; in Leinster's terminology, we work with "fat symmetric multicategories."

3.1.2.2 Definition. *Let $\mathcal{C}$ be a multicategory. A* prefactorization algebra *on M taking values in $\mathcal{C}$ is a functor (of multicategories) from* Fact$_M$ *to $\mathcal{C}$.*

Since symmetric monoidal categories are special kinds of multicategories, this definition makes sense for symmetric monoidal categories.

Remark: In other words, a prefactorization algebra *is* an algebra over the colored operad Fact$_M$.

Remark: If $\mathcal{C}$ is a symmetric monoidal category under coproduct, then a precosheaf on M with values in $\mathcal{C}$ defines a prefactorization algebra valued in $\mathcal{C}$. Hence, our definition

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[1] One can take infinite tensor products of unital algebras (see, for instance, exercise 23, chapter 2 [**AM69**]). The idea is simple. Given an infinite set I, consider the poset of finite subsets of I, ordered by inclusion. For each finite subset $J \subset I$, we can take the tensor product $A^J := \bigotimes_{j \in J} A$. For $J \hookrightarrow J'$, we define a map $A^J \to A^{J'}$ by tensoring with the identity $1 \in A$ for every $j \in J' \backslash J$. Then A^I is the colimit over this poset.

broadens the idea of "inclusion of open sets leads to inclusion of sections" by allowing more general monoidal structures to "combine" the sections on disjoint open sets.

Note that if $\mathcal{F}$ is any prefactorization algebra, then $\mathcal{F}(\emptyset)$ is a commutative algebra object of $\mathcal{C}$.

3.1.2.3 Definition. *We say a prefactorization algebra $\mathcal{F}$ is* unital *if the commutative algebra $\mathcal{F}(\emptyset)$ is unital.*

3.1.3. Prefactorization algebras in the style of precosheaves. Any multicategory $\mathscr{C}$ has an associated symmetric monoidal category $S\mathscr{C}$, which is defined to be the universal symmetric monoidal category equipped with a functor of multicategories $\mathscr{C} \to S\mathscr{C}$. Concretely, an object of $S\mathscr{C}$ is a formal tensor product $a_1 \otimes \cdots \otimes a_n$ of objects of $\mathscr{C}$. Morphisms in $S\mathscr{C}$ are characterized by the property that for any object b in $\mathcal{C}$, the set of maps $S\mathscr{C}(a_1 \otimes \cdots \otimes a_n, b)$ in the symmetric monoidal category is exactly the set of maps $\mathscr{C}(\{a_1, \ldots, a_n\}, b)$ in the multicategory category $\mathscr{C}$.

We can give an alternative definition of prefactorization algebra by working with the symmetric monoidal category $S\,\mathrm{Fact}_M$ rather than the multicategory Fact_M.

3.1.3.1 Definition. *Let $S\,\mathrm{Fact}_M$ denote the following symmetric monoidal category.*

- *The objects of $S\,\mathrm{Fact}_M$ consist of topological spaces U equipped with a map $U \to M$ which, on each connected component of U, is an open embedding embedding.*
- *A map from $U \to M$ to $V \to M$ is a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & V \arrow[dl] \\ M & \end{tikzcd}$$

  where the map i is an embedding.
- *The symmetric monoidal structure on $S\,\mathrm{Fact}_M$ is given by disjoint union.*

3.1.3.2 Lemma. *$S\,\mathrm{Fact}_M$ is the universal symmetric monoidal category containing the multicategory Fact_M.*

The alternative definition of prefactorization algebra is as follows.

3.1.3.3 Definition. *A* prefactorization algebra *with values in a symmetric monoidal category $\mathscr{C}$ is a symmetric monoidal functor*

$$S\,\mathrm{Fact}_M \to \mathscr{C}.$$

Remark: Although "algebra" appears in its name, a prefactorization algebra only allows one to "multiply" elements that live on disjoint open sets. The category of prefactorization algebras (taking values in some fixed target category) has a symmetric monoidal product, so we can study commutative algebra objects in that category. As an example, we will consider the observables for a classical field theory (chapter **??**).

3.2. Recovering associative algebras from prefactorization algebras on $\mathbb{R}$

We explained above how an associative algebra provides a prefactorization algebra on the real line. There are, however, prefactorization algebras on $\mathbb{R}$ that do *not* come from associative algebras. Here we will characterize those that do arise from associative algebras.

3.2.0.4 Definition. *Let $\mathcal{F}$ be a prefactorization algebra on $\mathbb{R}$ taking values in the category of vector spaces (without any grading). We say $\mathcal{F}$ is* locally constant *if the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism whenever the inclusion of opens $U \subset V$ is a homotopy equivalence.*

3.2.0.5 Lemma. *Let $\mathcal{F}$ be a locally constant, unital prefactorization algebra on $\mathbb{R}$ taking values in vector spaces. Let $A = \mathcal{F}(\mathbb{R})$. Then A has a natural structure of an associative algebra.*

Remark: Recall that $\mathcal{F}$ being unital means that the commutative algebra $\mathcal{F}(\emptyset)$ is equipped with a unit. We will find that A is an associative algebra over $\mathcal{F}(\emptyset)$.

PROOF. For any interval $(a,b) \subset \mathbb{R}$, the map

$$\mathcal{F}((a,b)) \to \mathcal{F}(\mathbb{R}) = A$$

is an isomorphism. Thus, we have a canonical isomorphism

$$A = \mathcal{F}((a,b))$$

for all intervals (a,b).

Notice that if $(a,b) \subset (c,d)$ then the diagram

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "\mathrm{Id}"'] & \mathcal{F}((a,b)) \arrow[d, "i_{(c,d)}^{(a,b)}"] \\ A \arrow[r, "\cong"] & \mathcal{F}((c,d)) \end{tikzcd}$$

commutes.

The product map $m : A \otimes A \to A$ is defined as follows. Let $a < b < c < d$. Then, the prefactorization structure on $\mathcal{F}$ gives a map

$$\mathcal{F}((a,b)) \otimes \mathcal{F}((c,d)) \to \mathcal{F}((a,d)),$$

and so, after identifying $\mathcal{F}((a,b))$, $\mathcal{F}((c,d))$ and $\mathcal{F}((a,d))$ with A, we get a map

$$A \otimes A \to A.$$

This is the multiplication in our algebra.

It remains to check the following.

(1) This multiplication doesn't depend on the intervals $(a,b) \amalg (c,d) \subset (a,d)$ we chose, as long as $(a,b) < (c,d)$.
(2) This multiplication is associative and unital.

This is an easy (and instructive) exercise. □

3.3. Comparisons with other axiom systems for field theories

Now that we have explained carefully what we mean by a prefactorization algebra, let us say a little about the history of this concept, and how it compares to other axiomatizations of quantum field theory.

3.3.1. Factorization algebras in the sense of Beilinson-Drinfeld. For us, one source of inspiration is Beilinson-Drinfeld's book [BD04]. These authors gave a geometric reformulation of the theory of vertex algebras in terms of an algebro-geometric version of the concept of factorization algebra. For Beilinson and Drinfeld, a factorization algebra on an algebraic curve X is a sheaf on a certain auxiliary space, the Ran space $\mathrm{Ran}(X)$ of X. The Ran space $\mathrm{Ran}(X)$ is the set of all finite non-empty subset of X, equipped with a certain natural structure of ind-scheme.

A factorization algebra is in particular a quasi-coherent sheaf $\mathfrak{F}$ on $\mathrm{Ran}(X)$. Thus, $\mathcal{F}$ is something which assigns to every finite set $I \subset X$ a vector space $\mathcal{F}_I$ (the fibre of $\mathcal{F}$ at I) which varies in an algebraic way with I. In addition, for $\mathcal{F}$ to be a factorization algebra, we must have natural isomorphisms

$$\mathcal{F}_{I \amalg J} \cong \mathcal{F}_I \otimes \mathcal{F}_J.$$

for disjoint finite sets $I, J \subset X$. These isomorphisms must vary nicely with the sets I and J and must satisfy natural associativity and commutativity properties similar to the ones we have considered.

Let us explain the heuristic dictionary between the definition used by Beilinson and Drinfeld and the one used here. We will sketch show how factorization algebras (in the sense used in this book) on a manifold M can be turned into Beilinson-Drinfeld style factorization algebras, and conversely.

Suppose we have a factorization algebra on M in the sense used in this book. At least heuristically, one gets an infinite-rank vector bundle on $\mathrm{Ran}(M)$ as follows. Given finite subset $I \subset M$, one can define $\mathcal{F}_I$ to be the costalk of $\mathcal{F}$ at I:

$$\mathcal{F}_I = \mathrm{holim}_{I \subset U} \mathcal{F}(U).$$

The symbol holim indicates the homotopy limit, which is necessary because the spaces $\mathcal{F}(U)$ are cochain complexes. Since we are only sketching a relationship between our picture and Beilinson-Drinfeld's, we will not go into details about homotopy limits.

Since we interpet $\mathcal{F}(U)$ as the observables of a quantum field theory on U, we should interpret $\mathcal{F}_I$ as the observables supported on the finite set $I \subset U$.

The factorization structure maps give us maps

$$\mathcal{F}_I \otimes \mathcal{F}_J \to \mathcal{F}_{I \amalg J}$$

if $I, J \subset M$ are disjoint finite subsets. The assumption that $\mathcal{F}$ is a factorization algebra and not just a prefactorization algebra will imply that these maps are isomorphisms.

From this presentation it's not entirely obvious that the spaces $\mathcal{F}_I$ vary nicely as I varies, especially as some points in I collide. This is necessary in Beilinson-Drinfeld's algebro-geometric picture to show that we have a quasi-coherent sheaf on the Ran space. In fact, in our main examples of factorization algebras (arising from field theories), we were unable to show that the spaces $\mathcal{F}_I$ vary nicely as the points in I collide, which is why we chose the particular axiom system for factorization algebras that we did.

A simpler relationship between factorization algebras as developed in this book and the Ran space is as follows. We will show how to construct a precosheaf on the Ran space of M from a factorization algebra on M.

If $U \subset M$ is an open subset, let $\mathrm{Ran}(U) \subset \mathrm{Ran}(M)$ be the open subset of $\mathrm{Ran}(M)$ consisting of finite subsets of U. More generally, if $U_1, \dots, U_n \subset M$ are disjoint open subsets, we have a map

$$\mathrm{Ran}(U_1) \times \cdots \times \mathrm{Ran}(U_n) \subset \mathrm{Ran}(M)$$

which sends a collection of finite subsets $I_i \subset U_i$ to their disjoint union. This map is injective and the image is an open subset. These sets form a basis for a certain natural topology on $\mathrm{Ran}(M)$ (which is the topology we will consider).

If we have a factorization algebra $\mathcal{F}$ on M, we define a precosheaf $\mathrm{Ran}(\mathcal{F})$ on $\mathrm{Ran}\, M$ in this topology by declaring that

$$\mathrm{Ran}(\mathcal{F})(\mathrm{Ran}(U_1) \times \cdots \times \mathrm{Ran}(U_n)) = \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n).$$

This precosheaf has a certain multiplicative property. We say two open subsets $W_1, W_2 \subset \mathrm{Ran}(M)$ are strongly disjoint if for every pair of finite sets $I_1, I_2 \subset M$ with $I_i \in W_i$, I_1 and I_2 are disjoint. If W_1, W_2 are strongly disjoint, we let $W_1 * W_2$ denote the open subset of the Ran space consisting of those finite sets $J \subset M$ which are a disjoint union $J = I_1 \amalg I_2$ where $I_1 \in W_1$ and $I_2 \in W_2$. The factorization structure on $\mathcal{F}$ induces a map

$$\mathrm{Ran}(\mathcal{F})(W_1) \otimes \mathrm{Ran}(\mathcal{F})(W_2) \to \mathrm{Ran}(\mathcal{F})(W_1 * W_2).$$

The factorization axiom will imply that this map is an isomorphism.

The codescent axiom we impose on factorization algebras is closely related to the codescent axiom for cosheaves on the Ran space.

3.3.2. Segal's axioms for quantum field theory. Segal has developed and studied some very natural axioms for quantum field theory []. These axioms were first studied in the world of topological field theory by Atiyah, Segal, Witten ... and in conformal field theory by Segal [].

According to Segal's philosophy, a d-dimensional quantum field theory (in Euclidean signature) is a symmetric functor from the category Cob_d^{Riem} of d-dimensional Riemannian cobordisms. An object of the category Cob_d^{Riem} is a compact $d-1$-manifold together with a germ of a d-dimensional Riemannian structure. A morphism is a d-dimensional Riemannian cobordism. The symmetric monoidal structure arises from disjoint union. As defined, this category does not have identity morphisms, but they can added in formally.

3.3.2.1 Definition. *A Segal field theory is a symmetric monoidal functor from* Cob_d^{Riem} *to the category of (topological) vector spaces.*

We won't get into details about what kind of topological vector spaces one should consider, because our aim is just to sketch a formal relationship between Segal's picture and our picture.

Segal's axioms admit obvious variants where the cobordisms are decorated with other geometric structures. The case relevant to our story is cobordisms embedded in an ambient manifold M. Thus, if X is a d-manifold, let $\mathrm{Cob}_d(X)$ denote the category whose objects are compact codimension 1 submanifolds of M, and whose morphisms are cobordisms embedded in X, that is, compact codimension 0 submanifolds of X with boundary. The symmetric monoidal structure on Cob_d^{Riem} is defined by disjoint union. However, disjoint union only makes sense for objects and morphisms of $\mathrm{Cob}_d(X)$ which are disjoint. So we can endow $\mathrm{Cob}_d(X)$ with a partially-defined symmetric monoidal structure by disjoint union of submanifolds which are disjoint.

In Segal's formalism, one could define a field theory on X to be a symmetric monoidal functor from the partially-defined symmetric monoidal category $\mathrm{Cob}_d(X)$ to topological vector spaces.

It turns out that a variant of this construction will give us something closely related to prefactorization algebras. Recall that Fact_X is the multicategory whose objects are open subsets in X, and where a multimorphism from $U_1, \dots, U_n \to V$ exists when the U_i are disjoint and contained in V.

Say an open subset $U \subset X$ is *nice* if it is the interior of a compact submanifold $M \subset X$ with boundary. Let us define a sub-multicategory $\mathrm{Fact}_X^{nice} \subset \mathrm{Fact}_X$ be whose objects are nice open sets $U \subset X$. Suppose $U_1, \dots, U_n, V$ are nice open subsets of X, and the U_i are disjoint and contained in V. Then, the corresponding multimorphism in Fact_X defines a multimorphism in Fact_X^{nice} if either

(1) $n = 1$ and $U_1 = V$.
(2) Or, the closures $\overline{U}_i$ of the sets U_i in X are disjoint and contained in V.

Note that 2 happens if the disjoint union of the $\overline{U}_i$ define a submanifold with boundary of $\overline{V}$, and the boundary of each $\overline{U}_i$ is disjoint from the boundary of $\overline{V}$.

3.3.2.2 Lemma. *There is a natural functor of multicategories*

$$\mathrm{Fact}_X^{nice} \to \mathrm{Cob}_d(X).$$

PROOF. The functor sends an object $U \in \mathrm{Fact}_X^{nice}$ to the boundary $\partial\overline{U}$ of its closure. By assumption, $\overline{U}$ is a submanifold with boundary, so this makes sense. The functor sends a non-trivial inclusion $U_1 \amalg \dots U_n \subset V$ to the cobordism

$$\overline{V} \setminus (U_1 \amalg \dots \amalg U_n)$$

which, by assumption, is a cobordism from the disjoint union of the submanifolds $\partial\overline{U}_i$ to $\partial\overline{V}$. □

The construction in this lemma identifies Fact_X^{nice} with the multicategory $\mathrm{Cob}_X^d/\emptyset$ whose objects are submanifolds of X equipped with a cobordism from the emptyset, or equivalently, of submanifolds which are presented as the boundary of codimension 0 manifold.

A factorization algebra by definition gives a functor of multicategories from $\mathrm{Fact}_X \to$ Vect. A Segal-style field theory on X gives a functor of multicategories from $\mathrm{Cob}_d(X) \to$ Vect. Both, therefore, restrict to give functors of multicategories $\mathrm{Fact}_X^{nice} \to$ Vect.

If we consider factorization algebras and not just prefactorization algebras, there is little difference between working with the subcategory Fact_X^{nice} and with Fact_X. The point is that the value of a factorization algebra on any open set is determined by its value on a sufficiently fine cover, and we can always make a sufficiently fine cover out of nice open subsets.

What this discussion shows is that the factorization algebras on X are essentially the same as solutions to Segal's axioms for a field theory on X where we only consider codimension 1 submanifolds $N \subset X$ which are presented as the boundary of an open region.

Further, the physical interpretation we give for factorization algebras is essentially the same as the physical interpretation of Segal's axioms. If $N \subset X$ is a codimension 1 submanifold, Segal tells us that the vector space associated to N in his axiom system is the

Hilbert space of the theory on N. Standard quantization yoga tells us that we should construct the Hilbert space as follows. The space of germs on N of solutions to the equations of motion is a symplectic manifold, which we can call $\mathrm{EOM}(N)$ (see Segal [] and Peierls). If N is the boundary of $M \subset X$, then the space $\mathrm{EOM}(M)$ of solutions to the equations of motion on M is a Lagrangian submanifold if $\mathrm{EOM}(N)$. The Hilbert space associated to N should be defined by geometric quantization. In general, if we have a symplectic manifold Y, then philosophy of geometric quantization tells us that the classical limit of a state in the Hilbert space is a Lagrangian submanifold $L \subset Y$.

FINISH THIS???

3.3.3. Locally constant factorization algebras.

A factorization algebra $\mathcal{F}$ on a manifold M, valued in cochain complexes, is called *locally constant* if, for any two discs $D_1 \subset D_2$ in M, the map $\mathcal{F}(D_1) \to \mathcal{F}(D_2)$ is a quasi-isomorphism. A theorem of Lurie [] shows that, given a locally constant factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$, the complex $\mathcal{F}(D)$ has the structure of an E_n algebra.

3.3.4. Algebraic quantum field theory.

FINISH THIS SUBSECTIO

3.4. A construction of the universal enveloping algebra

As an example of this relationship, we will present a construction of a factorization algebra on $\mathbb{R}$ from a Lie algebra whose corresponding associative algebra is the universal enveloping algebra of $\mathfrak{g}$. This construction is a basic special case of the "factorization envelope" construction, which appears in our formulation of Noether's theorem, and which produces many interesting examples of factorization algebras (such as the Kac-Moody vertex algebra). Another motivation for considering this example is that the argument we use to analyze this factorization algebra are a precursor to arguments we will see when we analyze the factorization algebra for quantum mechanics.

Let $\mathfrak{g}^{\mathbb{R}}$ denotes the cosheaf on $\mathbb{R}$ that assigns $(\Omega_c^*(U) \otimes \mathfrak{g}, \mathrm{d}_{dR})$ to each open U, with d_{dR} the exterior derivative. This is a cosheaf of cochain complexes, but it is only a precosheaf dg Lie algebras. This is because the cosheaf axiom involves the use of coproducts, and the coproduct in the category of dg Lie algebras is not given by direct sum.

Let $C_*\mathfrak{h}$ denote the Chevalley-Eilenberg complex for Lie algebra *homology*, written as a cochain complex. In other words, $C_*\mathfrak{h}$ is the graded vector space $\mathrm{Sym}(\mathfrak{h}[1])$ with a differential determined by the bracket of $\mathfrak{h}$.

Our main result shows how to construct the universal enveloping algebra $U\mathfrak{g}$ using $C_*\mathfrak{g}^{\mathbb{R}}$.

3.4.0.1 Proposition. *Let $\mathcal{H}$ denote the cohomology prefactorization algebra of $C_*\mathfrak{g}^{\mathbb{R}}$. That is, we take the cohomology of every open and every structure map, so*

$$\mathcal{H}(U) = H^*(C_*\mathfrak{g}^{\mathbb{R}}(U))$$

for any open U. Then $\mathcal{H}$ is locally constant, and the corresponding associative algebra is the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$.

PROOF. Local constancy of $\mathcal{H}$ is immediate from the fact that, if $I \subset J$ is an inclusion of intervals, the map of dg Lie algebras

$$\Omega_c^*(I) \otimes \mathfrak{g} \to \Omega_c^*(J) \otimes \mathfrak{g}$$

is a quasi-isomorphism. We let $A_{\mathfrak{g}}$ be the associative algebra constructed from $\mathcal{H}$ by lemma **??**.

The underlying vector space of $A_{\mathfrak{g}}$ is the space $\mathcal{H}(I)$ for any interval I. To be concrete, we will use the interval $\mathbb{R}$, so that we identify

$$A_g = \mathcal{H}(\mathbb{R}) = H^*(C_*(\Omega_c^*(\mathbb{R}) \otimes \mathfrak{g})).$$

Note that the dg Lie algebra $\Omega_c^*(\mathbb{R}) \otimes \mathfrak{g}$ is quasi-isomorphic to $H_c^*(\mathbb{R}) \otimes \mathfrak{g} = \mathfrak{g}[-1]$. This is Abelian because the cup product on $H_c^*(I)$ is zero.It follows that $C_*(\Omega_c^*(\mathbb{R}) \otimes \mathfrak{g})$ is quasi-isomorphic to chains of the Abelian Lie algebra $\mathfrak{g}[-1]$, which is simply $\mathrm{Sym}^* \mathfrak{g}$. Thus, as a vector space, $A_{\mathfrak{g}}$ is isomorphic to the symmetric algebra $\mathrm{Sym}^* \mathfrak{g}$.

There is a map

$$\Phi : \mathfrak{g} \to A_{\mathfrak{g}}$$

which sends an element $X \in \mathfrak{g}$ to $X\varepsilon$ where $\varepsilon \in H_c^1(I)$ is a basis for the compactly supported cohomology of the interval I whose integral is 1. It suffices to show that this is a map of Lie algebras where $A_{\mathfrak{g}}$ is given the Lie bracket coming from the associative structure.

Let us check this explicitly. Let $\delta > 0$ be small number, and let $f_0 \in C_c^\infty(-\delta, \delta)$ be a compactly supported smooth function with $\int f_0 \mathrm{d}x = 1$. Let $f_t(x) = f_0(x - t)$. Note that f_t is supported on the interval $(t - \delta, t + \delta)$. If $X \in \mathfrak{g}$, a cochain representative for $\Phi(X) \in A_{\mathfrak{g}}$ is provided by

$$X f_0 \mathrm{d}x \in \Omega_c^1((-\delta, \delta)) \otimes \mathfrak{g}.$$

Because the f_t for varying t are all cohomologous in $\Omega_c^1(\mathbb{R})$, the elements $X f_t$ are all cochain representatives of $\Phi(X)$.

Given elements $\alpha, \beta \in A_g$, the product $\alpha \cdot \beta$ is defined as follows.

(1) We choose intervals I, J with $I < J$.

(2) We regard α as an element of $\mathcal{H}(I)$ and β as an element of $\mathcal{H}(J)$ using the inverses to the isomorphisms $\mathcal{H}(I) \to \mathcal{H}(\mathbb{R})$ and $\mathcal{H}(J) \to \mathcal{H}(\mathbb{R})$ coming from the inclusions of I and J into $\mathbb{R}$.

(3) The product $\alpha \cdot \beta$ is defined by taking the image of $\alpha \otimes \beta$ under the factorization structure map
$$\mathcal{H}(I) \otimes \mathcal{H}(J) \to \mathcal{H}(\mathbb{R}) = A_g.$$

Let us see how this works in our example. The cohomology class $[X f_t \mathrm{d}x] \in \mathcal{H}(t-\delta, t+\delta)$ becomes $\Phi(X)$ under the natural map from $\mathrm{ch}(t-\delta, t+\delta)$ to $\mathrm{ch}(\mathbb{R})$. If we take δ to be sufficiently small, the intervals $(-\delta, \delta)$ and $(1-\delta, 1+\delta)$ are disjoint. It follows that the product $\Phi(X)\Phi(Y)$ is represented by the cocycle
$$(X f_0 \mathrm{d}x)(Y f_1 \mathrm{d}x) \in \mathrm{Sym}^2(\Omega^1_c(\mathbb{R}) \otimes \mathfrak{g}) \subset C_*(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g}).$$
Similarly, the commutator $[\Phi(X), \Phi(Y)]$ is represented by the expression
$$(X f_0 \mathrm{d}x)(Y f_1 \mathrm{d}x) - (X f_0 \mathrm{d}x)(Y f_{-1} \mathrm{d}x).$$
It suffices to show that this cocycle in $C_*(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g})$ is cohomologous to $\Phi([X, Y])$.

Note that the 1-form $f_1 \mathrm{d}x - f_{-1} \mathrm{d}x$ has integral 0. It follows that there exists a compactly supported function $h \in C^\infty_c(\mathbb{R})$ with
$$\mathrm{d}_{dR} h = f_{-1} \mathrm{d}x - f_1 \mathrm{d}x.$$
We can assume that h takes value 1 in the interval $(-\delta, \delta)$.

We will calculate the differential of the element
$$(X f_0 \mathrm{d}x)(Y h) \in C_*(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g}).$$
We have
$$\begin{aligned} \mathrm{d}\left((X f_0 \mathrm{d}x)(Y h)\right) &= (X f_0 \mathrm{d}x)(Y \mathrm{d}_{dR} h) + [X, Y] f_0 h \mathrm{d}x \\ &= (X f_0 \mathrm{d}x)(Y(f_{-1} - f_1)\mathrm{d}x) + [X, Y] f_0 h \mathrm{d}x. \end{aligned}$$
Since h takes value 1 on the interval $(-\delta, \delta)$, $f_0 h = f_0$. This equation tells us that a representative for $[\Phi(X), \Phi(Y)]$ is cohomologous to $\Phi([X, Y])$. □

3.5. Some functional analysis

Almost all of the examples of factorization algebras we will consider in this book will assign to an open subset $U \subset M$ a cochain complex built from vector spaces of analytical provenance: for example, smooth sections of a vector bundle, distributions on a manifold, etc. Such vector spaces are best viewed as being equipped with an extra structure (such as a topology) reflecting their analytical origin. In this section we will briefly sketch a flexible multicategory of vector spaces equipped with an extra "analytic" structure. Many more details are contained in the appendices **??**.

Probably the commonest way to encode the analytic structure on a vector space such as the space of smooth functions on a manifold is to endow it with a topology. Homological algebra with topological vector spaces is not easy, however (for instance, topological vector spaces do not form an abelian category). To get around this issue, we will work with *differentiable vector spaces*. Let us first define the slightly weaker notion of diffeological vector space.

3.5.0.2 Definition. *Let* Man *be the site of smooth manifolds, i.e. the category of smooth manifolds and smooth maps between them equipped with a structure of site where a cover is an open cover in the usual sense. Let* C^∞ *denote the sheaf of rings on* Man *which assigns to any manifold* M *the commutative algebra* $C^\infty(M)$*.*

A diffeological vector space is a sheaf of C^∞*-modules on the site* Man.

For example, if V is any topological vector space, then there is a natural notion of smooth map from any manifold M to V (see e.g. [KM97]). The space $C^\infty(M, V)$ of such smooth maps is a module over the algebra $C^\infty(M)$. Since smoothness is a local condition on M, sending M to $C^\infty(M, V)$ gives a sheaf of C^∞-modules on the site Man.

As an example of this construction, let us consider the case when V is the space of smooth functions on a manifold N (equipped with its natural Fréchet topology). One can show that $C^\infty(M, C^\infty(N))$ is naturally isomorphic to the space $C^\infty(M \times N)$ of smooth functions on $M \times N$.

As we will see shortly, we lose very little information when we view a topological vector space as a diffeological vector space.

Sheaves of C^∞-modules on Man which arise from topological vector spaces are endowed with an extra structure: we can always differentiate smooth maps from a manifold M to a topological vector space V. Differentiation can be viewed as an action of vector fields on M on the space $C^\infty(M, V)$, or dually, as coming from a connection

$$\nabla : C^\infty(M, V) \to \Omega^1(M, V)$$

where $\Omega^1(M, V)$ is defined to be the tensor product

$$\Omega^1(M, V) = \Omega^1(M) \otimes_{C^\infty(M)} C^\infty(M, V).$$

This tensor product is just the algebraic one, which is a reasonable thing to do because $\Omega^1(M)$ is a direct summand of a free $C^\infty(M)$-module of finite rank.

This connection is flat, in the usual sense that the curvature

$$F(\nabla) = \nabla \circ \nabla : C^\infty(M, V) \to \Omega^2(M, V).$$

This leads us to the following definition.

3.5.0.3 Definition. *Let Ω^1 denote the diffeological vector space which assigns $\Omega^1(M)$ to a manifold M.*

If F is a diffeological vector space, the diffeological vector space of k-forms valued in F is the space $\Omega^k(F)$ which assigns to a manifold M, the tensor product

$$\Omega^k(M, F) = \Omega^k(M) \otimes_{C^\infty(M)} F(M).$$

A connection on a diffeological vector space F is a map (of sheaves on the site Man*)*

$$\nabla : F \to \Omega^1(F),$$

which satisfies the Leibniz rule on every manifold M. A connection is flat if it is flat on every manifold M.

A differentiable vector space *is a diffeological vector space equipped with a flat connection.*

We say a sheaf $\mathcal{F}$ on the site of smooth manifolds is *concrete* if the natural map $\mathcal{F}(M) \to \operatorname{Hom}(M, \mathcal{F}(*))$ (where $\mathcal{F}(*)$ is the value of $\mathcal{F}$ on a point) is injective. Almost all of the differentiable vector spaces we will consider are concrete. For this reason, we will normally think of a differentiable vector space as being an ordinary vector space (the value on a point) together with an extra structure. We will refer to the value of the sheaf on a manifold M as the space of smooth maps to the value on a point. If V is a differentiable vector space, we often write $C^\infty(M, V)$ for the space of smooth maps from a manifold M.

As we have seen, every topological vector space gives rise to a differentiable vector space. For our purposes, differentiable vector spaces are much easier to use than topological vector spaces, because they are (essentially) sheaves on a site. Homological algebra for such objects is very well-developed.

3.5.0.4 Definition. *A differentiable cochain complex is a cochain complex in the category of differentiable vector spaces. A map $V \to W$ of differentiable cochain complexes is a quasi-isomorphism if the map $C^\infty(M, V) \to C^\infty(M, W)$ is a quasi-isomorphism for all manifolds M. This is equivalent to asking that the map be a quasi-isomorphism at the level of stalks.*

3.5.1. We have defined the notion of prefactorization algebra with values in any multicategory. In order to discuss factorization algebras valued in differentiable vector spaces, we need to define a multicategory structure on differentiable vector spaces. Let us first discuss the multicategory structure on the diffeological vector spaces.

3.5.1.1 Definition. *Let $V_1, \dots, V_n, W$ be differentiable vector spaces. A smooth multi-linear map*

$$\Phi : V_1 \times \dots V_n \to W$$

is a C^∞-multilinear map Φ of sheaves, which satisfies the following Leibniz rule with respect to the connections on the V_i and W. For every manifold M, and for every $v_i \in V_i(M)$, we require that

$$\nabla \Phi(v_1, \dots, v_n) = \sum i = 1^n \Phi(v_1, \dots, \nabla v_i, \dots, v_n) \in \Omega^1(M, W).$$

We let $\mathrm{Hom}_{DVS}(V_1 \times \dots \times V_n, W)$ denote this space of smooth multilinear maps.

In more down-to-earth terms, such a Φ gives a $C^\infty(M)$-multilinear map $V_1(M) \times \dots V_n(M) \to W(M)$ for every manifold M, in a way compatible with the connections and with the maps $V_i(M) \to V_i(N)$ associated to a map $f : N \to M$ of manifolds.

The category of differentiable cochain complexes acquires a multicategory structure from that on differentiable vector spaces, where the multi-maps are smooth multilinear maps which are compatible (in the usual way) with the differentials.

3.5.1.2 Definition. *A differentiable prefactorization algebra is a prefactorization algebra valued in the multicategory of differentiable cochain complexes.*

A differentiable prefactorization algebra $\mathcal{F}$ on a manifold X will asign a differeniable cochain complex $\mathcal{F}(U)$ to every open subset $U \subset X$, and smooth multi-linear maps (compatible with the differentials)

$$\mathcal{F}(U_1) \times \dots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

whenever $U_1, \dots, U_n$ are disjoint opens contained in V.

3.5.2. As we have seen, every topological vector space gives rise to a differentiable vector space. There is a rather beautiful theory developed in [KM97] concerning the precise relationship between topological vector spaces and differentiable vector spaces. These results are disucssed in much more detail in the appendix **??**: we will briefly summarize them now.

Let LCTVS denote the category of locally-convex Hausdorff topological vector spaces, and continuous linear maps. Let BVS denote the category with the same objects, but whose morphisms are *bounded* linear maps. Every continuous linear map is bounded, but not conversely. These categories have natural enrichments to multicategories, where the multi-maps are continuous (respectively, bounded) multi-linear maps. The category BVS is equivalent to a full subcategory of LCTVS whose objects are called bornological vector spaces.

Theorem. *The functor* LCTVS $\to$ DVS *extends to a functor* BVS $\to$ DVS, *which embeds* BVS *as a full sub multicategory of* DVS.

In other words: if V, W are topological vector spaces, and if $\Phi(V), \Phi(W)$ denote the corresponding differentiable vector spaces, then maps from $\Phi(V)$ to $\Phi(W)$ are the same as bounded linear maps from $V \to W$. More generally, if $V_1, \dots, V_n$ and W are topological vector spaces, bounded multi-linear maps $V_1 \times \cdots \times V_n \to W$ are the same as smooth multi-linear maps $\Phi(V_1) \times \cdots \times \Phi(V_n) \to \Phi(W)$.

This theorem tells us that we lose very little information if we think of a topological vector space as being a differentiable vector space.

So far, we have not, however, discussed how completeness of topological vector spaces appears in theory. We need a notion of completeness for a topological vector space which only depends on smooth maps to that vector space. The relevant concept was developed in [**?**]. We will view the category BVS as being a full subcategory of DVS.

3.5.2.1 Definition. *We say a topological vector space $V \in$ BVS is c^∞-complete, or convenient if every smooth map $\mathbb{R} \to V$ has an antiderivative.*

The category of convenient vector spaces, and bounded linear maps, will be called ConVS.

This completeness condition is a little weaker than the one normally studied for topological vector spaces. That is, every complete topological vector space is c^∞-complete.

Proposition. *The full subcategory* ConVS $\subset$ DVS *is closed under the formation of all limits and countable coproducts.*

We give ConVS the multicategory structure inherited from BVS (given by bounded multilinear maps). Since BVS is a full sub multicategory of DVS, so is ConVS.

Theorem. *The multicategory structure on* ConVS *is represented by a symmetric monoidal structure.*

This symmetric monoidal structure is called the completed bornological tensor product. If $E, F \in$ ConVS the bornological tensor product is written as $E \widehat{\otimes}_\beta F$. The statement that this represents the multicategory structure means that smooth (or equivalently bounded) bilinear maps $E_1 \times E_2 \to F$ are the same as bounded linear maps $E_1 \widehat{\otimes}_\beta E_2 \to F$, for objects E_1, E_2, F of ConVS.

If it will cause no confusion, we will often use the symbol $\otimes$ instead of $\widehat{\otimes}_\beta$ for this tensor product.

3.5.3. Let us now give some examples of differentiable vector spaces. These examples will include the basic building blocks for most of the factorization algebras we will consider.

Let E be a vector bundle on a manifold X, and let U be an open subset of X. We let $\mathscr{E}(U)$ denote the space of smooth sections of E on U, and we let $\mathscr{E}_c(U)$ denote the space of compactly supported sections of E on U.

Let us give these vector spaces the structure of differentiable vector spaces, as follows. If M is a manifold, we say a smooth map from M to $\mathscr{E}(U)$ is a smooth section of the bundle $\pi_X^* E$ on $M \times X$. We call this space $C^\infty(M, \mathscr{E}(U))$. Sending M to $C^\infty(M, \mathscr{E}(U))$ defines a sheaf of C^∞-modules on the site of smooth manifolds with a flat connection, and so a differentiable vector space.

Similarly, we say a smooth map from M to $\mathscr{E}_c(U)$ is a smooth section of the bundle $\pi_X^* E$ on $M \times X$, whose support maps properly to M. Let us denote this space by $C^\infty(M, \mathscr{E}_c(U))$; this defines, again, a sheaf of C^∞-modules on the site of smooth manifolds with a flat connection.

Theorem. *With this differentiable structure, the spaces $\mathscr{E}(U)$ and $\mathscr{E}_c(U)$ are in the full subcategory* ConVS *of convenient vector spaces. Further, this differentiable structure isthe same as the one that arises from the natural topologies on $\mathscr{E}(U)$ and $\mathscr{E}_c(U)$.*

The proof (like the proofs of all results in this section) are contained in the appendix, and based heavily on the book [KM97].

3.5.4. The category of differentiable vector spaces has internal Hom's, and internal multi-morphism spaces. If V, W are differentiable vector spaces, then a smooth map from a manifold M to the space $\mathrm{Hom}_{DVS}(V, W)$ is by definition an element of $\mathrm{Hom}_{DVS}(V, C^\infty(M, W))$. The space $C^\infty(M, W)$ is given the structure of differentiable vector space by saying that a smooth map from N to $C^\infty(M, W)$ is an element of $C^\infty(N \times M, W)$.

Theorem. *The category* ConVS *has internal Hom's and a Hom-tensor adjunction. That is, if E, F, G are objects of* ConVS*, then the object* $\mathrm{Hom}_{DVS}(E, F)$ *of* DVS *is actually in the full subcategory* ConVS*. Further, we have a natural isomorphism*

$$\mathrm{Hom}(E \widehat{\otimes}_\beta F, G) = \mathrm{Hom}(E, \mathrm{Hom}(F, G)).$$

3.5.5. Let E be a vector bundle on a manifold X, and let U be an open subset of X. Throughout this book, we will often use the notation $\overline{\mathscr{E}}(U)$ to denote the distributional sections on U, defined by

$$\overline{\mathscr{E}}(U) = \mathscr{E}(U) \otimes_{C^\infty(U)} \mathcal{D}(U),$$

where $\mathcal{D}(U)$ is the space of distributions on U. Similarly, let $\overline{\mathscr{E}}_c(U)$ denote the compactly supported distributional sections of E on U. There are natural inclusions

$$\mathscr{E}_c(U) \hookrightarrow \overline{\mathscr{E}}_c(U) \hookrightarrow \overline{\mathscr{E}}(U),$$

$$\mathscr{E}_c(U) \hookrightarrow \mathscr{E}(U) \hookrightarrow \overline{\mathscr{E}}(U),$$

by viewing smooth functions as distributions.

If, as above, E is a graded vector bundle on M, let $E^! = E^\vee \otimes \mathrm{Dens}_M$. We give $\mathscr{E}^!$ a differential that is the formal adjoint to Q on E. Let $\mathscr{E}^!(U)$, $\mathscr{E}_c^!(U)$ denote the cochain complexes of smooth and compactly supported sections of $E^!$, and let $\overline{\mathscr{E}}^!(U)$ and $\overline{\mathscr{E}}_c^!(U)$ denote the cochain complexes of distributional and compactly-supported distributional sections of $E^!$.

Note that $\overline{\mathscr{E}}_c(U)$ is the continuous dual to $\mathscr{E}^!(U)$, and that $\mathscr{E}_c(U)$ is the continuous dual to $\overline{\mathscr{E}}^!(U)$.

We need to give the spaces $\overline{\mathscr{E}}_c(U)$ and $\overline{\mathscr{E}}(U)$ differentiable structures. We will need the following proposition.

Proposition. *There is are natural identifications*

$$\begin{aligned}\mathrm{Hom}_{DVS}(\mathscr{E}_c^!(U), \mathbb{R}) &= \overline{\mathscr{E}}(U)\\ \mathrm{Hom}_{DVS}(\mathscr{E}^!(U), \mathbb{R}) &= \overline{\mathscr{E}}_c(U).\end{aligned}$$

This proposition, together with the fact that differentiable (and convenient) vector spaces have internal Hom's, give us a differentiable (and convenient) structure on the spaces $\overline{\mathscr{E}}(U)$ and $\overline{\mathscr{E}}_c(U)$. With these differentiable structures, a smooth map from a manifold M to $\overline{\mathscr{E}}(U)$ is a map $\mathscr{E}_c^!(U) \to C^\infty(M)$ of differentiable vector spaces, and similarly for $\overline{\mathscr{E}}(U)$.

This proposition should be compared with the fact that $\overline{\mathscr{E}}(U)$ and $\overline{\mathscr{E}}_c(U)$ are the continuous linear duals of $\mathscr{E}_c^!(U)$ and $\mathscr{E}^!(U)$. The content of the proposition is that every smooth linear functional on $\mathscr{E}_c^!(U)$ or $\mathscr{E}^!(U)$ is actually continuous. In the appendix we show that this differentiable structure on the spaces $\overline{\mathscr{E}}(U)$ and $\overline{\mathscr{E}}_c(U)$ coincides with the one coming from the standard topologies on these spaces.

3.5.6. The prefactorization algebras we will use for most of the book are built as algebras of functions or symmetric algebras of the convenient vector spaces $\mathscr{E}(U)$ or $\mathscr{E}_c(U)$. Recall that $\mathscr{E}(U)$ and $\overline{\mathscr{E}}_c(U)$ are both convenient vector spaces, and that, in the full subcategory ConVS $\subset$ DVS of convenient vector spaces, the multicategory is represented by a symmetric monoidal structure called the completed bornological tensur product and denoted by $\overline{\otimes}_\beta$.

Proposition. *Let X, Y be manifolds, and let E, F be vector bundles on X and Y respectively.*

$$\begin{aligned}C^\infty(X, E)\widehat{\otimes}_\beta C^\infty(Y, F) &= C^\infty(X \times Y, E \boxtimes F)\\ C_c^\infty(X, E)\widehat{\otimes}_\beta C_c^\infty(Y, F) &= C_c^\infty(X \times Y, E \boxtimes F).\end{aligned}$$

Remark: An alternative approach to the one we've taken is to use the category of nuclear topological vector spaces, with the completed projective tensor product, instead of the category of convenient (or differentiable) vector spaces. Using nuclear spaces raises a number of technical isssues, but one immediate one is the following: although it is true that $C^\infty(X)\widehat{\otimes}_\pi C^\infty(Y) = C^\infty(X \times Y)$ (where $\widehat{\otimes}_\pi$ refers to the completed projective tensor product, it is not true (or at least not obviously true) that the same statement holds if we use compactly supported smooth functions. The problem stems from the fact that the projective tensor product does not commute with colimits, whereas the bornological tensor product we use does.

We can define symmetric powers of convenient vector spaces using the symmetric monoidal structure we have described. If, as before, E is a vector bundle on X and U is an open subset of X, this proposition allows us to identify

$$\mathrm{Sym}^n(\mathscr{E}_c(U)) = C_c^\infty(U^n, E^{\boxtimes n}).$$

The symmetric algebra $\mathrm{Sym}^* \mathscr{E}_c(U)$ is defined as usual to be the direct sum of the symmetric powers. It is an algebra in the symmetric monoidal category of convenient vector spaces.

A related construction is the algebra of functions on a differentiable vector space. If V is a differentiable vector space, we can define, as we have seen, the space of linear functionals on V to be the space of maps $\mathrm{Hom}_{DVS}(V, \mathbb{R})$. Because the category DVS has internal Hom's, this is again a differentiable vector space. In a similar way, we can define the space of polynomial functions on V homogeneous of degree n to be the space

$$P_n(V) = \mathrm{Hom}_{DVS}(V \times \cdots \times V, \mathbb{R})_{S_n}.$$

In other words, we take smooth multilinar maps from n copies of V to $\mathbb{R}$, and then take the S_n coinvariants. This acquires the structure of differentiable vector space, by saying that a smoooth map from a manifold M to $P_n(V)$ is

$$C^\infty(M, P_n(B)) = \mathrm{Hom}_{DVS}(V \times \cdots \times V, C^\infty(M))_{S_n}.$$

One can then define the algebra of functions on V by

$$\mathscr{O}(V) = \prod_n P_n(V).$$

(We take the product rather than the sum, so that $\mathscr{O}(V)$ should be thought of as a space of formal power series on V). The space $\mathscr{O}(V)$ is a commutative algebra in a natural way.

This construction is a very general one, of course: one can define the algebra of functions on any object in any multicategory in the same way.

An important example is the following.

3.5.6.1 Lemma. *Let E be a vector bundle on a manifold X. Then, $P_n(C^\infty(X, E))$ is the S_n covariants of the space of compactly supported distributional sections of $E^!$ on X^n.*

PROOF. We know that the multicategory structure on the full subcategory ConVS $\subset$ DVS is represented by a symmetric monoidal category, and that in this symmetric monoidla category,

$$C^\infty(X, E)^{\widehat{\otimes}_\beta n} = C^\infty(X^n, E^{\boxtimes n}).$$

It follows form this that the space $P_n(C^\infty(X, E))$ is the S_n covariants of the space of smooth linear maps

$$C^\infty(X^n, E^{\boxtimes n}) \to \mathbb{R}.$$

We have seen that this space of smooth linear maps is (with its differentiable structure) the same as the space $\mathcal{D}_c(X^n, (E^!)^{\boxtimes n})$ of compactly supported distributional sections of the bundle $(E^!)^{\boxtimes n}$. $\square$

Note that $\mathscr{O}(\mathscr{E}(U)$ is naturally the same as $\mathrm{Hom}_{DVS}(\mathrm{Sym}^* \mathscr{E}(U), \mathbb{R})$, i.e. it's the dual of the symmetric algebra of $\mathscr{E}(U)$.

Remark: It is not true (at least, not obviously true) that $P_n(\mathscr{E}(U))$ is the n'th symmetric power of the space $P_1(\mathscr{E}(U)) = \overline{\mathscr{E}}_c^!(U)$.

3.6. The factorization envelope of a sheaf of Lie algebras

In this section, we will introduce an important class of examples of prefactorization algebras. We will show how to construct, for every sheaf of Lie algebras $\mathcal{L}$ on a manifold M, a prefactorization algebra which we call the *factorization envelope*. If $\mathcal{L}$ is a homotopy sheaf then the factorization envelope is a factorization algebra and not just a prefactorization algebra; we will often restrict attention to homotopy sheaves for this reason.

This construction is our version of Beilinson-Drinfeld's chiral envelope [**BD04**]. The construction can also be viewed as a natural generalization of the universal enveloping algebra of a Lie algebra. A special case of this construction yields the universal enveloping algebra of a Lie algebra (see 3.4).

The factorization envelope plays an important role in our story.

1. The factorization algebra associated to a free field theory is an example of a (twisted) factorization envelope.
2. In section 5.4, we will show (following Beilinson and Drinfeld) that the Kac-Moody vertex algebra arises as a (twisted) factorization envelope.
3. The most important appearance of factorization envelopes appears in our treatment of Noether's theorem at the quantum level. We will show in section 18.5 that, if a sheaf of Lie algebras $\mathcal{L}$ acts on a quantum field theory on a manifold M, then there is a homomorphism from a twisted factorization envelope of $\mathcal{L}$ to the quantum observables of the field theory. This construction is very useful. For example, we show in section **??** TK that this construction gives rise to our version

of the Segal-Sugawara construction: it allows us to construct a homomorphism from the factorization algebra for the Virasoro vertex algebra to the factorization algebra associated to a class of chiral conformal field theories.

3.6.1. Thus, let M be a manifold. Let $\mathcal{L}$ be a sheaf of dg Lie algebras on M. Let $\mathcal{L}_c$ denote the cosheaf of compactly supported sections of $\mathcal{L}$.

Remark: Note that, although $\mathcal{L}_c$ is a cosheaf of cochain complexes, and a precosheaf of dg Lie algebras, it is *not* a cosheaf of dg Lie algebras. This is because colimits of dg Lie algebras are not the same as colimits of cochain complexes.

We can view $\mathcal{L}_c$ as a prefactorization algebra valued in the category of dg Lie algebras with symmetric monoidal structure given by direct sum. Indeed, if U_i are disjoint opens in M contained in V, there is a natural map of dg Lie algebras

$$\oplus\mathcal{L}_c(U_i) = \mathcal{L}_c(\oplus U_i) \to \mathcal{L}_c(V)$$

giving the factorization product.

Taking Chevalley chains is a symmetric monoidal functor from dg Lie algebras, equipped with the direct sum monoidal structure, to cochain complexes.

3.6.1.1 Definition. *If $\mathcal{L}$ is a sheaf of dg Lie algebras on M, define the* factorization envelope *$U\mathcal{L}$ to be the prefactorization algebra obtained by applying the Chevalley-Eilenberg chain functor to $\mathcal{L}_c$, viewed as a factorization algebra valued in dg Lie algebras.*

Concretely, $U\mathcal{L}$ assigns to an open subset $V \subset M$ the complex

$$U(\mathcal{L})(V) = C_*(\mathcal{L}_c(V))$$

where C_* is the Chevalley-Eilenberg chain complex. The product maps are defined by applying the functor C_* to the dg Lie algebra map $\oplus\mathcal{L}_c(U)_i \to \mathcal{L}_c(V)$ associated to an inclusion of disjoint opens U_i into V.

We will see later **??** that – under the hypothesis that $\mathcal{L}$ is a homotopy sheaf – then this prefactorization algebra is a factorization algebra.

3.6.2. In practice, we will need an elaboration of this construction which involves a small amount of analysis.

3.6.2.1 Definition. *Let M be a manifold. A* local dg Lie algebra *on M consists of the following data.*

1. *A graded vector bundle L on M, whose sheaf of smooth sections will be denoted $\mathcal{L}$.*
2. *A differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$, of cohomological degree 1 and square 0.*

(3) An alternating bi-differential operator

$$[-,-] : \mathcal{L}^{\otimes 2} \to \mathcal{L}$$

which endows $\mathcal{L}$ with the structure of a sheaf of dg Lie algebras.

Remark: This definition will play an important role in our approach to classical field theory as detailed in Chapter **??**.

In section 3.5, we explained how spaces of sections of vector bundles on a manifold are, in a natural way, differentiable vector spaces. We also explained that they live in the full subcategory of convenient vector spaces, and that the multicategory structure on differentiable vector spaces is represented by a symmetric monoidal structure on the full subcategory of convenient vector spaces.

Therefore, $U \subset M$, the space $\mathcal{L}(U)$ is a convenient graded vector space. We would like to form, as above, the Chevalley-Eilenberg chain complex $C_*(\mathcal{L}_c(U))$. The underlying vector space of $C_*(\mathcal{L}_c(U)$ is the (graded) symmetric algebra on $\mathcal{L}_c(U)[1]$. As we explained in section 3.5, we need to take account of the differentiable structure on $\mathcal{L}_c(U)$ when we take the tensor powers of $\mathcal{L}_c(U)$. We define $(\mathcal{L}_c(U))^{\otimes n}$ to be the tensor power defined using the completed bornological tensor product on the convenient vector space $\mathcal{L}_c(U)$. Concretely, if $L^{\boxtimes n}$ denotes the vector bundle on M^n obtained as the external tensor product, then

$$(\mathcal{L}_c(U))^{\otimes n} = \Gamma_c(U^n, L^{\boxtimes n})$$

is the space of compactly supported smooth sections of $L^{\boxtimes n}$ on U^n. Symmetric (or exterior) powers of $\mathcal{L}_c(U)$ are defined by taking coinvariants of $\mathcal{L}_c(U)^{\otimes n}$ with respect to the action of the symmetric group S_n. The completed symmetric algebra on $\mathcal{L}_c(U)[-1]$ that is the underlying graded vector space of $C_*(\mathcal{L}_c(U))$ is defined using these completed symmetric powers. The Chevalley-Eilenberg differential is continuous, and therefore defines a differential on the completed symmetric algebra of $\mathcal{L}_c(U)[-1]$, giving us the cochain complex $C_*(\mathcal{L}_c(U))$.

3.6.2.2 Definition. *The factorization envelope of the local dg Lie algebra $\mathcal{L}$ is the factorization algebra which assigns to an open subset $U \subset M$ the chain complex $C_*(\mathcal{L}_c(U))$. This is a factorization algebra in the multicategory of differentiable vector spaces.*

Example: Let $\mathfrak{g}$ be a Lie algebra. There is a local Lie algebra on $\mathbb{R}$ given by the sheaf $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$. As we showed in section 3.4.0.1, the factorization envelope of $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ is a locally constant factorization algebra and so corresponds to an associative algebra. This associative algebra is the universal enveloping algebra of $\mathfrak{g}$.

In the same way, for any Lie algebra $\mathfrak{g}$ we can construct a factorization algebra on $\mathbb{R}^n$ as the factorization envelope of $\Omega^*_{\mathbb{R}^n} \otimes \mathfrak{g}$. The resulting factorization algebra is still locally constant: it has the property that the inclusion map from one disc to another is a quasi-isomorphism. A theorem of Lurie [**?**] tells us that locally-constant factorization algebras on $\mathbb{R}^n$ are the same as E_n algebras. The E_n algebra we have constructed is the E_n-enveloping algebra of $\mathfrak{g}$.

3.6.3. Many interesting factorization algebras (such as the Kac-Moody factorization algebra, and the factorization algebra associated to a free field theory) can be constructed from a variant of the factorization envelope construction, which we call the *twisted* factorization envelope.

3.6.3.1 Definition. *Let $\mathcal{L}$ be a local dg Lie algebra on a manifold M. A* local k-shifted central extension *of $\mathcal{L}$ is a dg Lie algebra structure on the precosheaf*

$$\widehat{\mathcal{L}}_c = \underline{\mathbb{C}}[k] \oplus \mathcal{L}_c$$

(where $\underline{\mathbb{C}}[k]$ is the constant precosheaf which assigns to every open set the vector space $\mathbb{C}$ in degree $-k$), such that

1. *$\mathbb{C}[k]$ is central, and the sequence*
$$0 \to \mathbb{C}[k] \to\to \widehat{\mathcal{L}}_c \to \mathcal{L} \to 0$$
is an exact sequence of dg Lie algebras.
2. *The differential and bracket maps from $\mathcal{L}_c(U) \to \mathbb{C}[k]$ and $\mathcal{L}_c(U)^{\otimes 2} \to \mathbb{C}[k]$ defining the central extension are* local, *meaning that they can be represented as compositions*
$$\mathcal{L}_c(U) \to \omega_c(U)[-k] \xrightarrow{\int} \mathbb{C}[k]$$
$$\mathcal{L}_c(U)^{\otimes 2} \to \omega_C(U)[-k] \xrightarrow{\int} \mathbb{C}[k]$$
where the map in the first line is a differential operator, and the second is a bidifferential operator.

In Chapter 10, subsection **??**, we will analyze the complex "local Lie algebra cochains" and see that the L_∞-version of local central extensions are classified by the cohomology of this cochain complex.

3.6.3.2 Definition. *In this situation, define the* twisted factorization envelope *to be the prefactorization algebra $U(\widetilde{\mathcal{L}})$ which sends an open set U to $C_*(\widetilde{\mathcal{L}}_c(U))$ (as above, we use the completed tensor product as above).*

The chain complex $C_*(\widetilde{\mathcal{L}}_c(U))$ is a module over chains on the Abelian Lie algebra $\mathbb{C}[k]$ for every open subset U. Thus, we will view the twisted factorization envelope as a prefactorization algebra in modules for $\mathbb{C}[c]$ where c has degree $-k-1$.

This is a factorization algebra over the base ring $\mathbb{C}[c]$. Of particular interest is the case when $k = -1$, so that the central parameter c is of degree 0.

Let us now introduce some important examples of this construction.

Example: Let $\mathfrak{g}$ be a simple Lie algebra, and let $\langle -, - \rangle_{\mathfrak{g}}$ denote an invariant pairing on $\mathfrak{g}$. Let us define the *Kac-Moody* factorization algebra as follows.

Let Σ be a Riemann surface, and consider the local Lie algebra $\Omega^{0,*} \otimes \mathfrak{g}$ on Σ, which sends an open subset U to the dg Lie algebra $\Omega^{0,*}(U) \otimes \mathfrak{g}$. There is a -1-shifted central extension of $\Omega^{0,*}_c \otimes \mathfrak{g}$ defined by the cocycle

$$\omega(\alpha, \beta) = \int_U \langle \alpha, \partial \beta \rangle_{\mathfrak{g}}$$

where $\alpha, \beta \in \Omega^{0,*}_c(U) \otimes \mathfrak{g}$ and $\partial : \Omega^{0,*} \to \Omega^{1,*}$ is the holomorphic de Rham operator. Note that this is a -1-shifted cocycle because $\omega(\alpha, \beta)$ is zero unless $\deg(\alpha) + \deg(\beta) = 1$.

The Kac-Moody factorization algebra on Σ is the twisted universal enveloping algebra $U_\omega(\Omega^{0,*} \otimes \mathfrak{g})$. We will analyze this example in more detail in chapter 5.

Example: In this example we will define a higher-dimensional analog of the Kac-Moody vertex algebra.

Let X be a complex manifold of dimension n. Let $\phi \in \Omega^{n-1,n-1}(X)$ be a closed form.

Then, given any Lie algebra equipped with an invariant pairing, we can construct a -1-shifted central extension of $\Omega^{0,*}_c \otimes \mathfrak{g}$, defined as above by the cocycle

$$\alpha \otimes \beta \mapsto \int_X \langle \alpha, \partial \beta \rangle_{\mathfrak{g}} \wedge \phi,$$

It is easy to verify that this is a cocycle. (The case of the Kac-Moody extension is when $n = 1$ and ϕ is a constant). The cohomology class of this cocycle is unchanged if we change ϕ to $\phi + \overline{\partial} \psi$ where $\psi \in \Omega^{n-1,n-2}(X)$ satisfies $\partial \psi = 0$.

Let $\mathcal{F}$ denote the twisted factorization envelope of this local dg Lie algebra. The factorization algebra $\mathcal{F}$ is closely related to the Kac-Moody algebra. For instance, if $X = \Sigma \times \mathbb{P}^{n-1}$ where Σ is a Riemann surface, and the form ϕ is the volume form on $\mathbb{P}^{n-1}$, then the push-forward of this factorization algebra to Σ is quasi-isomorphic to the Kac-Moody factorization algebra described above. (The push-forward is defined by

$$(p_* \mathcal{F})(U) = \mathcal{F}(p^{-1}(U))$$

for $U \subset \Sigma$).

An important special case of this construction is when $\dim_{\mathbb{C}}(X) = 2$, and the form ϕ is the curvature of a connection on the canonical bundle of X (so that ϕ represents $c_1(X)$). As we will show when we discuss Noether's theorem at the quantum level, if we have a field theory with an action of a local dg Lie algebra $\mathcal{L}$ then a twisted factorization envelopes of $\mathcal{L}$ will map to observables of the theory. One can show that the local dg Lie algebra

$\Omega^{0,*}_X \otimes \mathfrak{g}$ acts on a twisted $N = 1$ gauge theory with matter, and (following Johansen [**?**]) that the twisted factorization envelope – with central extension determined by $c_1(X)$ – maps to observables of this theory.

CHAPTER 4

Factorization algebras and free field theories

4.1. The divergence complex of a measure

We will start by motivating this construction of this cochain complex by considering Gaussian integrals in finite dimensions. Let M be a background and let ω_0 be a measure on M. Let f be a function on M. (For example, M could be a vector space, ω_0 the Lebesgue measure and f a quadratic form). The divergence operator for the measure $e^{f/\hbar}\omega_0$ is a map

$$\begin{aligned} \mathrm{Div}_\hbar : \mathrm{Vect}(M) &\to C^\infty(M) \\ X &\mapsto \hbar^{-1}(Xf) + \mathrm{Div}_{\omega_0} X. \end{aligned}$$

One way to define the divergence operator is to use the volume form $e^{f/\hbar}\omega_0$ to identify $\mathrm{Vect}(M)$ with $\Omega^{n-1}(M)$, and $C^\infty(M)$ with $\Omega^n(M)$ (where $n = \dim M$). Under this identification, the divergence operator is simply the de Rham operator from $\Omega^{n-1}(M)$ to $\Omega^n(M)$.

The de Rham operator, of course, is part of the de Rham complex. In a similar way, we can define the *divergence complex*, as follows. Let

$$\mathrm{PV}^i(M) = C^\infty(M, \wedge^i TM)$$

denote the space of polyvector fields on M. The divergence complex is the complex

$$\cdots \to \mathrm{PV}^i(M) \xrightarrow{\mathrm{Div}_\hbar} \mathrm{PV}^1(M) \xrightarrow{\mathrm{Div}_\hbar} \mathrm{PV}^0(M)$$

where the differential

$$\mathrm{Div}_\hbar : \mathrm{PV}^i(M) \to \mathrm{PV}^{i-1}(M)$$

is defined so that the diagram

$$\begin{tikzcd} \mathrm{PV}^i(M) \arrow[r, "\vee e^{f/\hbar}\omega_0"] \arrow[d, "\mathrm{Div}_\hbar"] & \Omega^{n-i}(M) \arrow[d, "\mathrm{d}_{dR}"] \\ \mathrm{PV}^{i-1}(M) \arrow[r, "\vee e^{f/\hbar}\omega_0"] & \Omega^{n-i+1}(M) \end{tikzcd}$$

commutes.

Thus, after contracting with the volume $e^{f/\hbar}\omega_0$, the divergence complex becomes the de Rham complex. It is easy to check that, as maps from $\mathrm{PV}^i(M)$ to $\mathrm{PV}^{i-1}(M)$, we have

$$\mathrm{Div}_\hbar = \vee \hbar^{-1} \mathrm{d} f + \mathrm{Div}_{\omega_0}$$

where $\vee \mathrm{d} f$ is the operator of contracting with $\mathrm{d} f$. In the $\hbar \to 0$ limit, the dominant term is $\vee \hbar^{-1} \mathrm{d} f$.

More precisely, there is a flat family of cochain complexes over the algebra $\mathbb{C}[\hbar]$ which at $\hbar \neq 0$ is isomorphic to the divergence complex, and at $\hbar = 0$ is the complex

$$\to \mathrm{PV}^2(M) \xrightarrow{\vee \mathrm{d} f} \mathrm{PV}^1(M) \xrightarrow{\vee \mathrm{d} f} \mathrm{PV}^0(M).$$

Note that this complex is a differential graded algebra (which is not the case for the divergence complex).

The image of the map $\vee \mathrm{d} f : \mathrm{PV}^1(M) \to \mathrm{PV}^0(M)$ is the ideal cutting out the critical locus. Indeed, this whole complex is the Koszul complex for the equations cutting out the critical locus. This leads to the following definition.

4.1.0.3 Definition. *The derived critical locus of f is the dg manifold whose functions are* $\mathrm{PV}^*(M)$ *with differential contracting with* $\mathrm{d} f$.

Remark: Since the purpose of this section is motivational, we will not go into any details on the theory of dg manifolds. Dg manifolds are one way to think about derived geometry: more details on derived geometry (from a different point of view) will be discussed in chapter **??**.

Let $\Gamma(\mathrm{d} f) \subset T^*M$ denote the graph of $\mathrm{d} f$. The ordinary critical locus of f is the intersection of $\Gamma(\mathrm{d} f)$ with the zero-section $M \subset T^*M$. The derived critical locus is defined to be the derived intersection. In derived geometry, functions on derived intersections are defined by derived tensor products:

$$C^\infty(\mathrm{Crit}^h(f)) = C^\infty(\Gamma(\mathrm{d} f)) \otimes_{C^\infty(T^*M)} C^\infty(M).$$

By using a Koszul resolution of $C^\infty(M)$ as a module for $C^\infty(T^*M)$, one sees finds a quasi-isomorphism of dg algebras between this derived intersection and the complex $\mathrm{PV}^*(M)$ with differential $\vee \mathrm{d} f$.

Thus, we find that the divergence complex has a $\hbar \to 0$ limit which is functions on the derived critical locus of f.

An important special case of this is when the function f is zero. In that case, the derived critical locus of f has functions the algebra $\mathrm{PV}^*(M)$ with zero differential. This can be viewed as the functions on the graded manifold $T^*[-1]M$. The derived critical locus for a general function f can be viewed as a deformation of $T^*[-1]M$ obtained by introducing a differential given by contracting with $\mathrm{d} f$.

4.1.1. We will define the factorization algebra of observables of a free scalar field theory as a divergence complex, just like we defined H^0 of observables to be given by functions modulo divergences in chapter 2. It turns out that there is a slick way to write this factorization algebra as a twisted factorization envelope of a certain sheaf of Heisenberg Lie algebras. We will explain this point in a finite-dimensional toy model, and then use the factorization envelope picture to define the factorization algebra of observables of the field theory.

Let V be a vector space, and let $q : V \to \mathbb{R}$ be a quadratic function on V. Let ω_0 be the Lebesgue measure on V. We want to understand the divergence complex for the measure $e^{q/\hbar}\omega_0$. The construction is quite general: we do not need to assume that q is non-degenerate.

The derived critical locus of the function q is a linear dg manifold. Linear dg manifolds are the same thing as cochain complexes: any cochain complex B^* gives rise to a dg manifold whose functions are the symmetric algebra on the dual of B^*.

The derived critical locus of q is described by the cochain complex

$$W = V \to V^*[-1]$$

where the differential sends $v \in V$ to the linear functional $\frac{\partial}{\partial q} v$.

Note that W is equipped with a graded anti-symmetric pairing of cohomological degree -1, defined by pairing V and V^*. In other words, W has a symplectic pairing of cohomological degree -1. We let

$$\mathcal{H}_W = \mathbb{C} \cdot \hbar[-1] \oplus W$$

where $\mathbb{C} \cdot \hbar$ indicates a one-dimensional vector space with basis $\hbar$. We give the cochain complex $\mathcal{H}_W$ a Lie bracket by saying that

$$[w, w'] = \hbar \left\langle w, w' \right\rangle$$

where $\langle -, - \rangle$ denotes the pairing on W. Thus, $\mathcal{H}_W$ is a shifted-symplectic version of the Heisenberg Lie algebra of an ordinary symplectic vector space.

Consider the Chevalley-Eilenberg chain complex $C_*\mathcal{H}_W$. This is defined to be the symmetric algebra of $\mathcal{H}_W[1] = W[1] \oplus \mathbb{C} \cdot \hbar$ equipped with a certain differential. Since the pairing on W identifies $W[1] = W^*$, we can identify

$$C_*(\mathcal{H}_W) = \mathrm{Sym}^* \left(W^*\right) [\hbar]$$

with a differential. Since, as a graded vector space, $W = V \oplus V^*[-1]$, we have a natural identification

$$C_*(\mathcal{H}_W) = \mathrm{PV}^*(V)[\hbar]$$

where $\mathrm{PV}^*(V)$ refers to polyvector fields on V witih polynomial coefficients, where as before we place $\mathrm{PV}^i(V)$ in degree $-i$.

4.1.1.1 Lemma. *The differential on $C_*(\mathcal{H}_W)$ is, under this identification, the operator*

$$\hbar\, \mathrm{Div}_{e^{q/\hbar}\omega_0} : \mathrm{PV}^i(V)[\hbar] \to \mathrm{PV}^{i-1}(V)[\hbar],$$

where ω_0 is the Lebesgue measure on V, and q is the quadratic function on V used to define the differential on the complex W.

PROOF. The proof is an explicit calculation, which we leave to the interested reader. The calculation is facilitated by choosing a basis of V in which we can explicitly write both the divergence operator and the differential on the Chevalley-Eilenberg complex $C_*(\mathcal{H}_W)$. □

In what follows, we will define the dg factorization algebra of observables of a free field theory as a Chevalley-Eilenberg chain complex of a certain Heisenberg Lie algebra, constructed as in this lemma.

4.2. The factorization algebra of classical observables of a free field theory

In this section, we will construct the prefactorization algebra associated to any free field theory. We will concentrate, however, on the free scalar field theory on a Riemannian manifold. We will show that, for one-dimensional manifolds, this prefactorization algebra recovers the familiar Weyl algebra, the algebra observables for quantum mechanics. In general, we will show how to construct correlation functions of observables of a free field theory and check that these agree with how physicists define correlation functions.

4.2.1. Defining the prefactorization algebra. We will start by defining the prefactorization algebra of classical observables of a free scalar field theory.

Let M be a Riemannian manifold, and so M is equipped with a natural density, arising from the metric. We will use this natural density to integrate functions, and also to provide an isomorphism between functions and densities that we use implicitly from hereon. The field theory we will discuss has as fields $\phi \in C^\infty(M)$ and has as action functional

$$S(\phi) = \tfrac{1}{2} \int_M \phi \triangle \phi,$$

where $\triangle$ is the Laplacian on M. (Normally we will reserve the symbol $\triangle$ for the Batalin-Vilkovisky Laplacian, but that's not necessary in this section.)

If $U \subset M$ is an open subset, then the space of solutions to the equation of motion on U is the space of harmonic functions on U.

In this book, we will always consider the *derived* space of solutions of the equation of motion. For more details about the derived philosophy, the reader should consult Chapter

10. In this simple situation, the derived space of solutions to the free field equations, on an open subset $U \subset M$, is the two-term complex

$$\mathscr{E}(U) = \left(C^\infty(U)^0 \xrightarrow{\triangle} C^\infty(U)^1 \right),$$

where the superscripts indicate the cohomological degree.

The classical observables of a field theory on an open subset $U \subset M$ are functions on the derived space of solutions to the equations of motion on U. As we explained in section 3.5, the space $\mathscr{E}(U)$ has the structure of differentiable cochain complex (essentially, it's a sheaf on the site of smooth manifolds). We define the space of polynomial functions homogeneous of degree n on $\mathscr{E}(U)$ to be the space

$$P_n(\mathscr{E}(U)) = \mathrm{Hom}_{DVS}(\mathscr{E}(U) \times \cdots \times \mathscr{E}(U), \mathbb{R})_{S_n}.$$

In other words, we take smooth multi-linear maps from n copies of $\mathscr{E}(U)$, and then take the S_n-coinvariants. The algebra of all polynomial functions on $\mathscr{E}(U)$ is the space $P(\mathscr{E}(U)) = \oplus_n P_n(\mathscr{E}(U))$.

As we discussed in section 3.5, we can identify

$$P_n(\mathscr{E}(U)) = \mathcal{D}_c(U^n, (E^!)^{\boxtimes n})_{S_n}$$

as the S_n-coinvariants of the space of compactly supported distributional sections of the bundle $(E^!)^{\boxtimes n}$ on U^n. In general, if $\mathscr{E}(U)$ is sections of a graded bundle E, then $E^!$ is $E^\vee \otimes \mathrm{Dens}$. In the case at hand, the bundle $E^!$ is two copies of the trivial bundle, one in degree -1 and one in degree 0.

For example, the space $P_1(\mathscr{E}(U)) = \mathscr{E}(U)^\vee$ of smooth linear functionals on $\mathscr{E}(U)$ is the space

$$\mathscr{E}(U)^\vee = \left(\mathcal{D}_c(U)^{-1} \xrightarrow{\triangle} \mathcal{D}_c(U)^0 \right),$$

where $\mathcal{D}_c(U)$ indicates the space of compactly supported distributions on U.

Using the space of all polynomial functionals will lead to difficulties defining the quantum observables. When we work with an interacting theory, these difficulties can only be surmounted using the techniques of renormalization. For a free field theory, though, there is a much simpler solution.

We have

$$\mathscr{E}_c^!(U) = \left(C_c^\infty(U)^{-1} \to C_c^\infty(U)^0 \right),$$

using our identification between densities and functions. Note that $\mathscr{E}(U)^\vee$ is the complex $\overline{\mathscr{E}}^!_c(U)$ of compactly supported distributional sections of the bundle $E^!$. There is therefore a natural cochain map $\mathscr{E}_c^!(U) \to \overline{\mathscr{E}}^!_c(U)$.

The observables we will work with is the space of "smeared observables", defined by

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^*(\mathscr{E}_c^!(U)) = \mathrm{Sym}^*(C_c^\infty(U)^{-1} \xrightarrow{\triangle} C_c^\infty(U)^0),$$

the symmetric algebra on $\mathscr{E}_c^!(U)$. As we explained in section 3.5, this symmetric algebra is defined using the natural symmetric monoidal structure on the full subcategory ConVS $\subset$ DVS of convenient vector spaces. Concretely, we can identify

$$\mathrm{Sym}^n(\mathscr{E}_c^!(U)) = C_c^\infty(U^n, (E^!)^{\otimes n})_{S_n}.$$

In other words, $\mathrm{Sym}^n\,\mathscr{E}_c^!(U)$ is the subspace of $P_n(\mathscr{E}(U))$ defined by taking all distributions to be smooth functions with compact support.

4.2.1.1 Lemma. *The map* $\mathrm{Obs}^{cl}(U) \to P(\mathscr{E}(U))$ *is a homotopy equivalence of cochain complexes of differentiable vector spaces.*

PROOF. It suffices to show that the map $\mathrm{Sym}^n\,\mathscr{E}_c^!(U) \to P_n(\mathscr{E}(U))$ is a smooth homotopy equivalence. For this, it suffices to show that the map

$$C_c^\infty(U, (E^!)^{\otimes n}) \to \mathcal{D}_c(U, (E^!)^{\otimes n})$$

is a smooth homotopy equivalence.

This is a special case of a general result proved in the appendix B.10, which says that the spaces of smooth and distributional sections of any elliptic complex are homotopy equivalent.

Note that by smooth homotopy equivalence we mean that there is a smooth inverse map $\mathscr{E}(U)^\vee \to \mathscr{E}_c^!(U)$ and smooth cochain homotopies between the two composed maps and the identity maps. Smooth means that all maps are in the category DVS of differentiable vector spaces; it suffices to construct a continuous homotopy equivalence. ☐

This lemma says that, since we are working homotopically, we can replace a distributional observable (given by integration against some distribution on U^n) by a smooth observable (given by integration against a smooth function on U^n). We will think of smooth linear observables as "smeared observables".

4.2.2. Let us describe the cochain complex $\mathrm{Obs}^{cl}(U)$ more explicitly, in order to clarify the relationship with what we discussed in chapter 2. The complex $\mathrm{Obs}^{cl}(U)$ looks like

$$\cdots \to \wedge^2 C_c^\infty(U) \otimes \mathrm{Sym}^*\, C_c^\infty(U) \to C_c^\infty(U) \otimes \mathrm{Sym}^*\, C_c^\infty(U) \to \mathrm{Sym}^*\, C_c^\infty(U).$$

All tensor products appearing in this expression are completed tensor products in the category of convenient vector spaces.

We should interpret $\mathrm{Sym}^* C_c^\infty(U)$ as being an algebra of polynomial functions on $C^\infty(U)$, using (as we explained above) the Riemannian volume form on U to identify $C_c^\infty(U)$ with a subspace of the dual of $C^\infty(U)$. There is a similar interpretation of the other terms in this complex using the geometry of the space $C^\infty(U)$ of fields. Let $T_c C^\infty(U)$ refer to the subbundle of the tangent bundle of $C^\infty(U)$ given by the subspace $C_c^\infty(U) \subset C^\infty(U)$. An element of a fibre of $T_c C^\infty(U)$ is a first-order variation of a field which is zero outside of a compact set. This subspace $T_c C^\infty(U)$ defines an integrable foliation on $C^\infty(U)$, and this foliation can be defined if we replace $C^\infty(U)$ by any sheaf of spaces.

Then, we can interpret $C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U)$ as a space of polynomial sections of $T_c C^\infty(U)$. Similarly, $\wedge^k C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U)$ should be interpreted as a space of polynomial sections of the bundle $\wedge^k T_c C^\infty(U)$.

That is, if $\mathrm{PV}_c(C^\infty(U))$ refers to polynomial polyvector fields on $C^\infty(U)$ along the foliation given by $T_c C^\infty(U)$, we have

$$\mathrm{Obs}^{cl}(U) = \mathrm{PV}_c(C^\infty(U)).$$

So far, this is just an identification of graded vector spaces. We need to explain how to identify the differential. Roughly speaking, the differential on $\mathrm{Obs}^{cl}(U)$ corresponds to the differential on $\mathrm{PV}_c(C^\infty(U))$ obtain this complex is given by contracting with the one-form $\mathrm{d}S$, where

$$S = \tfrac{1}{2} \int \phi \, \mathrm{D} \phi$$

is the action functional.

Let us explain the precise sense in which this is true. Note that the functional S is not well-defined for all fields ϕ (because the integral may not converge). However, the expression

$$\frac{\partial S}{\partial \phi_0}(\phi)$$

make sense for any $\phi \in C^\infty(U)$ and $\phi_0 \in C_c^\infty(U)$. This means that we can make sense of the one-form $\mathrm{d}S$, not as a section of the cotangent bundle of $C^\infty(U)$, but as a section of the space $T_c^* C^\infty(U)$, the dual of the subbundle $T_c C^\infty(U) \subset T C^\infty(U)$ describing vector fields along the leaves. This leafwise one-form is closed. Such one-forms are the kinds of things we can contract with elements of $\mathrm{PV}_c(C^\infty(U))$. The differential on $\mathrm{PV}_c(C^\infty(U))$ which matches the differential on Obs^{cl} is given by contracting with $\mathrm{d}S$.

4.2.3. General free field theories. Let's give an abstract definition of a free theory in general (although we will mostly focus on free scalar field theories in this chapter). More motivation for this general definition is presented in Chapter **??**, where we introduce the classical BV formalism from the point of view of derived geometry.

4.2.3.1 Definition. *Let M be a manifold. A* free field theory *on M is the following data:*

(1) *A graded vector bundle E on M, whose sheaf of sections will be denoted $\mathscr{E}$, and whose compactly supported sections will be denoted $\mathscr{E}_c$.*
(2) *A differential operator $\mathrm{d} : \mathscr{E} \to \mathscr{E}$, of cohomological degree 1 and square zero, making $\mathscr{E}$ into an elliptic complex.*
(3) *Let $E^! = E^\vee \otimes \mathrm{Dens}_M$, and let $\mathscr{E}^!$ be the sections of $E^!$. Let d be the differential on $\mathscr{E}^!$ which is the formal adjoint to the differential on $\mathscr{E}$. Note that there is a natural pairing between $\mathscr{E}_c(U)$ and $\mathscr{E}^!(U)$, and this pairing is compatible with differentials.*

    Then, we require an isomorphism of bundles $E \to E^![-1]$ compatible with differentials, in the sense that the induced map $\mathscr{E}(U) \to \mathscr{E}^!(U)[-1]$ is an isomorphism of cochain complexes. We further require that the induced pairing of cohomological degree -1 on each $\mathscr{E}_c(U)$ is graded anti-symmetric.

Note that the equations of motion for a free theory are always linear, so that the space of solutions is a vector space. Similarly, the derived space of solutions of the equations of motion of a free field theory is a cochain complex, which is a linear derived stack. The cochain complex $\mathscr{E}(U)$ should be thought of as the derived space of solutions to the equations of motion on an open subset U. As we will explain in Chapter **??**, the pairing on $\mathscr{E}_c(U)$ arises from the fact that the equations of motion of a field theory are not arbitrary differential equations, but describe the critical locus of an action functional.

For example, for the free scalar field theory on a manifold M with mass m, we have, as above,

$$\mathscr{E} = C^\infty_M \xrightarrow{\triangle + m^2} C^\infty_M[-1].$$

Our convention is that $\triangle$ is a non-negative operator, so that on $\mathbb{R}^n$, $\triangle = -\sum \frac{\partial}{\partial x_i}^2$. The pairing on $\mathscr{E}_c$ is defined by

$$\left\langle \phi^0, \phi^1 \right\rangle = \int_M \phi^0 \phi^1$$

for $\phi^i \in C^\infty(M)[-i]$.

As another example, let us describe Abelian Yang-Mills theory (with gauge group $\mathbb{R}$) in this language. Let M be a manifold of dimension 4. If $A \in \Omega^1(M)$ is a connection on the trivial $\mathbb{R}$-bundle on a manifold M, then the Yang-Mills action functional applied to A is

$$S_{YM}(A) = -\tfrac{1}{2} \int_M (\mathrm{d}A) * (\mathrm{d}A) = \tfrac{1}{2} \int_M A(\mathrm{d} * \mathrm{d})A.$$

The equations of motion are that $\mathrm{d} * \mathrm{d}A = 0$. There is also gauge symmetry, given by $X \in \Omega^0(M)$, which acts on A by $A \to A + \mathrm{d}X$. The complex $\mathscr{E}$ describing this theory is

$$\mathscr{E} = \Omega^0(M)[1] \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^3(M)[-1] \xrightarrow{\mathrm{d}} \Omega^4(M)[-2].$$

We will explain how to derived this statement in Chapter **??**. For now, note that $H^0(\mathscr{E})$ is the space of those $A \in \Omega^1(M)$ which satisfy the Yang-Mills equation $\mathrm{d} * \mathrm{d}A = 0$, modulo gauge symmetry.

For any free field theory with complex of fields $\mathscr{E}$, we can define the classical observables of the theory as

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^*(\mathscr{E}_c^!(U)) = \mathrm{Sym}^*(\mathscr{E}_c(U)[1]).$$

It is clear that classical observables form a prefactorization algebra. Indeed, $\mathrm{Obs}^{cl}(U)$ is a commutative differential graded algebra. If $U \subset V$, there is a natural algebra homomorphism

$$i_V^U : \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(V),$$

which on generators is just the natural map $C_c^\infty(U) \to C_c^\infty(V)$, extension by zero.

If $U_1, \ldots, U_n \subset V$ are disjoint open subsets, the prefactorization structure map is the continuous multilinear map

$$\begin{array}{ccc} \mathrm{Obs}^{cl}(U_1) \times \cdots \times \mathrm{Obs}^{cl}(U_n) & \to & \mathrm{Obs}^{cl}(V) \\ (\alpha_1, \ldots, \alpha_n) & \mapsto & \prod_{i=1}^n i_V^{U_i} \alpha_i. \end{array}$$

4.2.4. The one-dimensional case, in detail. Let us compute the space of classical observables for a free scalar theory in dimension 1 (i.e. for free quantum mechanics).

4.2.4.1 Lemma. *If $U = (a,b) \subset \mathbb{R}$ is an interval in $\mathbb{R}$, then the commutative algebra of classical observables for the free field with mass $m \geq 0$ has cohomology*

$$H^*(\mathrm{Obs}^{cl}((a,b))) = \mathbb{R}[p,q],$$

the polynomial algebra in two variables.

PROOF. The picture is the following. The equations of motion for free quantum mechanics on the interval (a,b) are that the field ϕ satisfies $(\mathrm{D} + m^2)\phi = 0$. This space is two dimensional, spanned by the functions $e^{\pm mx}$ (if $m > 0$) and by the functions $1, x$ if $m = 0$. Classical observables are functions on the space of solutions to the equations of motion. We would this expect that classical observables are a polynomial algebra in two generators.

We need to be a little more careful, however, because we used the *derived* version of the space of solutions to the equations of motion. We will show that the complex

$$\mathscr{E}_c^!((a,b)) = \left(C_c^\infty((a,b))^{-1} \xrightarrow{\triangle + m^2} C_c^\infty((a,b))^0 \right)$$

is smoothly homotopy equivalent to the complex $\mathbb{R}^2$ situated in degree 0. Since the algebra $\mathrm{Obs}^{cl}((a,b))$ of observables is defined to be the symmetric algebra on this complex, this will imply the result. Without loss of generality, we can take $a = -1$ and $b = 1$.

First, let us introduce some notation. If $m = 0$, let $\phi_q = 1$ and $\phi_p = x$. If $m > 0$, let

$$\begin{aligned}\phi_q &= \tfrac{1}{2}\left(e^{mx} + e^{-mx}\right)\\ \phi_p &= \tfrac{1}{2m}\left(e^{mx} - e^{-mx}\right).\end{aligned}$$

For any value of m, the functions ϕ_p and ϕ_q are annhilated by the operator $-\partial_x^2 + m^2$, and they form a basis for the kernel of this operator. Further, $\phi_p(0) = 1$ and $\phi_q(0) = 0$, whereas $\phi_p'(0) = 0$ and $\phi_q'(0) = 1$. Finally, $\phi_q = \phi_p'$.

Define a map

$$\pi : \mathscr{E}_c^!((-1,1)) \to \mathbb{R}\{p,q\}$$

to the vector space spanned by p, q, by sending

$$g \mapsto \pi(g) = q\int g(x)\phi_q \mathrm{d}x + p\int g(x)\phi_p \mathrm{d}x.$$

This is a cochain map, because if $g = (\mathrm{D} + m^2)f$, where f has compact support, then $\pi(g) = 0$. This map is easily seen to be surjective.

We need to construct a contracting homotopy on the kernel of π. That is, if $\operatorname{Ker}\pi^0 \subset C_c^\infty((-1,1))$ refers to the kernel of π in cohomological degree 0, we need to construct an inverse to the differential

$$C_c^\infty((-1,1)) = \operatorname{Ker}\pi^{-1} \xrightarrow{\mathrm{D}+m^2} \operatorname{Ker}\pi^0.$$

This is defined as follows. Let $G(x) \in C^0(\mathbb{R})$ be the Green's function for the operator $\mathrm{D} + m^2$. Explicitily, we have

$$G(x) = \begin{cases} \frac{m}{2}e^{-m|x|} & \text{if } m > 0 \\ -\frac{1}{2}\,|x| & \text{if } m = 0. \end{cases}$$

Then $(\mathrm{D} + m^2)G$ is the delta function at 0. The inverse map sends a function

$$f \in \operatorname{Ker}\pi^0 \subset C_c^\infty((-1,1))$$

to

$$G \star f = \int_y G(x-y)f(y)\mathrm{d}y.$$

The fact that $\int f\phi_q = 0$ and $\int f\phi_p = 0$ implies that $G \star f$ has compact support. The fat that G is the Green's function implies that this operator is the inverse to $\mathrm{D} + m^2$. It is clear that the operator of convolution with G is smooth (and even continuous), so the result follows.

□

4.2.5. The Poisson bracket.

We now return to the general case.

Suppose we have any free field theory on a manifold M, with complex of fields $\mathscr{E}$. Classical observables are the symmetric algebra $\operatorname{Sym}\mathscr{E}_c(U)[1]$. Recall that the complex $\mathscr{E}_c(U)$ is equipped with an antisymmetric pairing of cohomological degree -1. Thus, $\mathscr{E}_c(U)[1]$ is equipped with a symmetric pairing of degree 1.

4.2.5.1 Lemma. *There is a unique smooth Poisson bracket on* $\mathrm{Obs}^{cl}(U)$ *of cohomological degree* 1, *with the property that for* $\alpha, \beta \in \mathscr{E}_c(U)[1]$, *we have*

$$\{\alpha, \beta\} = \langle \alpha, \beta \rangle .$$

Recall that "smooth" means that the Poisson bracket is a smooth bilinear map

$$\{-,-\} : \mathrm{Obs}^{cl}(U) \times \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

as defined in section 3.5.

PROOF. The argument we will give is very general, and applies in any reasonable symmetric monoidal category. Recall that, as stated in section 3.5, the category of convenient vector spaces is a symmetric monoidal category with internal Hom's and a Hom-tensor adjunction.

If A is any commutative algebra object in the category ConVS^* of convenient cochain complexes, and M is an A-modules, then we can define $\mathrm{Der}(A, M)$ to be the space of algebra homomorphisms $A \to A \oplus M$ which are trivial modulo the ideal M. (Here $A \oplus M$ is given the square-zero algebra structure, where the product of any two elements in M is zero and the product of an element in A with one in M is the module structure.)

Since the category of convenient cochain complexes has internal Hom's, the cochain complex $\mathrm{Der}(A, M)$ is again an A-module in ConVS^*.

The commutative algebra

$$\mathrm{Obs}^{cl}(U) = \operatorname{Sym}^* \mathscr{E}_c^!(U)$$

is the universal commutative algebra in the category ConVS^* of convenient cochain complexes equipped with a smooth linear cochain map $\mathscr{E}_c^!(U) \to \mathrm{Obs}^{cl}(U)$. It follows from this that, for any module M over $\operatorname{Sym}^* \mathscr{E}_c^!(U)$,

$$\mathrm{Der}(\operatorname{Sym}^* \mathscr{E}_c^!(U), M) = \mathrm{Hom}(\mathscr{E}_c^!(U), M)$$

A Poisson bracket on $\operatorname{Sym}^* \mathscr{E}_c^!(U)$ is in particular a biderivation. A biderivation is something that assigns to an element of $\operatorname{Sym}^* \mathscr{E}_c^!(U)$ a derivation of the algebra $\operatorname{Sym}^* \mathscr{E}_c^!(U)$. Thus, the space of biderivations is the space

$$\mathrm{Der}\left(\operatorname{Sym}^* \mathscr{E}_c^!(U), \mathrm{Der}\left(\operatorname{Sym}^* \mathscr{E}_c^!(U), \operatorname{Sym}^* \mathscr{E}_c^!(U)\right)\right).$$

What we have said so far identifies this space of biderivations with

$$\operatorname{Hom}(\mathscr{E}_c^!(U), \operatorname{Hom}(\mathscr{E}_c^!(U), \operatorname{Sym}^* \mathscr{E}_c^!(U))) = \operatorname{Hom}(\mathscr{E}_c^!(U) \otimes \mathscr{E}_c^!(U), \operatorname{Sym}^* \mathscr{E}_c^!(U)).$$

In this line we have used the Hom-tensor adjunction in the category ConVS.

The Poisson bracket we are constructing corresponds to the biderivation which is the pairing on $\mathscr{E}_c^!(U)$ viewed as a map

$$\mathscr{E}_c^!(U) \otimes \mathscr{E}_c^!(U) \to \mathbb{R} = \operatorname{Sym}^0 \mathscr{E}_c^!(U).$$

This biderivation is antisymmetric and satisfies the Jacobi identity. Since Poisson brackets are a subspace of biderivations, we have proved both the existence and uniqueness clauses.

$\square$

Note that for U_1, U_2 disjoint open subsets of V and for observables $\alpha_i \in \operatorname{Obs}^{cl}(U_i)$, we have

$$\{i_V^{U_1}\alpha_1, i_V^{U_2}\alpha_2\} = 0.$$

That is, observables coming from disjoint open subsets commute with respect to the Poisson bracket. This means that $\operatorname{Obs}^{cl}(U)$ defines a P_0 prefactorization algebra. (We will see later in section 6.3 that this prefactorization algebra is actually a factorization algebra.)

In the case of classical observables of the free scalar field theory, we can think of $\operatorname{Obs}^{cl}(U)$ as a certain space of polyvector fields on $C^\infty(U)$ along the foliation of $C^\infty(U)$ defined by the subspace $C_c^\infty(U) \subset C^\infty(U)$. The Poisson bracket we have just defined is the Schouten bracket on polyvector fields.

4.2.6. Quantum observables. In Chapter 2, we construct a prefactorization algebra which we called $H^0(\operatorname{Obs}^q)$ of quantum observables of a free scalar field theory on a manifold. This space is defined as a space of functions on the space of fields, modulo the image of a certain divergence operator. The aim of this section is to lift this vector space to a cochain complex. As we explained in section 4.1, this complex will be the analog of the divergence complex of a measure in finite dimensions.

In section 4.1, we explained that for a quadratic function q on a vector space V, the divergence complex for the measure $e^{q/\hbar}\omega_0$ on V (where ω_0 is the Lebesgue measure) can be realized as the Chevalley-Eilenberg chain complex of a certain Heisenberg Lie algebra.

The prefactorization algebra $\operatorname{Obs}^q(U)$ is an example of a twisted prefactorization envelope of a sheaf of Lie algebras. Let

$$\widehat{\mathscr{E}_c}(U) = \mathscr{E}_c(U) \oplus \mathbb{R} \cdot \hbar$$

where $\mathbb{R}\hbar$ is situated in degree 1. We give $\widehat{\mathscr{E}}_c(U)$ a Lie bracket by saying that, for $\alpha, \beta \in \mathscr{E}_c(U)$,

$$[\alpha, \beta] = \hbar \langle \alpha, \beta \rangle .$$

Thus, $\widehat{\mathscr{E}}_c(U)$ is a graded version of a Heisenberg algebra, centrally extending the abelian dg Lie algebra $\mathscr{E}_c(U)$.

Let

$$\mathrm{Obs}^q(U) = C_*(\widehat{\mathscr{E}}_c(U)),$$

where C_* denotes the Chevalley-Eilenberg complex for the Lie algebra homology of $\widehat{\mathscr{E}}_c(U)$, defined using the completed tensor product on the category of convenient vector spaces (as discussed in section 3.5). Thus,

$$\begin{aligned}\mathrm{Obs}^q(U) &= \left(\mathrm{Sym}^* \left(\widehat{\mathscr{E}}_c(U)[1]\right), \mathrm{d}\right) \\ &= \left(\mathrm{Obs}^{cl}(U)[\hbar], \mathrm{d}\right)\end{aligned}$$

where the differential arises from the Lie bracket and differential on $\widehat{\mathscr{E}}_c(U)$. The symbol $[1]$ indicates a shift of degree down by one. (Note that we always work with cochain complexes, so our grading convention of C_* is the negative of one popular convention.)

As we discussed in section 4.2, we can view classical observables on an open set U for the free scalar theory as a certain complex of polyvector fields on the space $C^\infty(U)$:

$$\mathrm{Obs}^{cl}(U) = (\mathrm{PV}_c(C^\infty(U)), \vee \mathrm{d}S) .$$

By $\mathrm{PV}_c(C^\infty(U))$ we mean polylvector fields defined using the foliation $T_c C^\infty(U) \subset TC^\infty(U)$ of compactly-supported variants of a field. Concretely,

$$\mathrm{Obs}^{cl}(U) = \ldots \xrightarrow{\vee \mathrm{d}S} \wedge^2 C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U) \xrightarrow{\vee \mathrm{d}S} C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U) \xrightarrow{\vee \mathrm{d}S} \xrightarrow{\vee \mathrm{d}S} \mathrm{Sym}^* C_c^\infty(U)$$

where all tensor products appearing in this expression are completed bornological tensor products.

In a similar way, quantum observables look like

$$\mathrm{Obs}^q(U) = \ldots \xrightarrow{\vee \mathrm{d}S + \hbar\, \mathrm{Div}} \wedge^2 C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U)[\hbar] \xrightarrow{\vee \mathrm{d}S + \hbar\, \mathrm{Div}} C_c^\infty(U) \otimes \mathrm{Sym}^* C_c^\infty(U) \xrightarrow{\vee \mathrm{d}S} \xrightarrow{\vee \mathrm{d}S + \hbar\, \mathrm{Div}} \mathrm{Sym}^* C_c^\infty($$

The operator Div is the extension to all polyvector fields of the operator we defined in Chapter 2 as a map from polynomial vector fields on $C^\infty(U)$ to polynomial functions. Thus, $H^0(\mathrm{Obs}^q(U))$ is the same vector space (and same prefactorization algebra) that we defined in Chapter 2.

Since this is an example of the general construction we discussed in section 3.6 we see that $\mathrm{Obs}^q(U)$ has the structure of a prefactorization algebra.

4.2.7. As we explained in section **??**, our philosophy is that we should take a P_0 prefactorization algebra and deform it into a BD prefactorization algebra. In the situation we are considering in this section, we will construct a prefactorization algebra of quantum observables Obs^q with the property that, as a vector space,

$$\mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)[\hbar],$$

but with a differential d such that

(1) modulo $\hbar$, d coincides with the differential on $\mathrm{Obs}^{cl}(U)$, and
(2) the equation
$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot b + \hbar\{a, b\}$$
holds.

Here, $\cdot$ indicates the commutative product on $\mathrm{Obs}^{cl}(U)$. These properties imply that Obs^q defines a BD prefactorization algebra quantizing the P_0 prefactorization algebra Obs^{cl}.

4.2.8. Let us identify $\mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)[\hbar]$ as above. Then, $\mathrm{Obs}^q(U)$ acquires a product and Poisson bracket from that on $\mathrm{Obs}^{cl}(U)$. Further, the differential on $\mathrm{Obs}^q(U)$ satisfies the BD-algebra axiom

$$\mathrm{d}(ab) = (\mathrm{d}a)b \pm a(\mathrm{d}b) + \hbar\{a, b\}.$$

Thus, Obs^q defines a prefactorization BD algebra quantizing Obs^{cl}.

It follows from the fact that Obs^{cl} is a factorization algebra (which we prove in section 6.3) that Obs^q is a factorization algebra over $\mathbb{R}[\hbar]$.

Remark: Those readers who are operadically inclined might notice that the Lie algebra chain complex of a Lie algebra $\mathfrak{g}$ is the E_0 version of the universal enveloping algebra of a Lie algebra. Thus, our construction is an E_0 version of the familiar construction of the Weyl algebra as a universal enveloping algebra of a Heisenberg algebra.

4.3. Quantum mechanics and the Weyl algebra

We will now show that our construction of the free scalar field on $\mathbb{R}$ recovers the Weyl algebra, which is the associative algebra of observables in quantum mechanics.

First, we must check that this prefactorization algebra is locally constant, and so gives us an associative algebra.

4.3.0.1 Lemma. *The prefactorization algebra on* $\mathbb{R}$ *constructed from the free scalar field theory with mass* m *is locally constant.*

PROOF. Let Obs^q denote this prefactorization algebra. Recall that $\mathrm{Obs}^q(U)$ is the Lie algebra chains on the Heisenberg Lie algebra $\mathcal{H}(U)$ built as a central extension of $C_c^\infty(U) \xrightarrow{\triangle+m^2} C_c^\infty(U)[-1]$. Let us filter $\mathrm{Obs}^q(U)$ by saying that

$$F^{\leq i}\,\mathrm{Obs}^q(U) = \mathrm{Sym}^{\leq i}(\mathcal{H}(U)[1]).$$

The associated graded for this filtration is $\mathrm{Obs}^{cl}(U)[\hbar]$. Thus, to show that $H^*\,\mathrm{Obs}^q$ is locally constant, it suffices (by considering the spectral sequence associated to this filtration) to show that $H^*\,\mathrm{Obs}^{cl}$ is locally constant. We have already seen (in lemma 4.2.4.1) that $H^*(\mathrm{Obs}^{cl}(a,b)) = \mathbb{R}[p,q]$ for any interval (a,b), and that the inclusion maps $(a,b) \to (a',b')$ induces isomorphisms. Thus, the cohomology of Obs^{cl} is locally constant, as desired. □

It follows that the cohomology of this prefactorization algebra is an associative algebra, which we call A_m. We will show that A_m is the Weyl algebra.

In fact, we will show a little more. The prefactorization algebra for the free scalar field theory on $\mathbb{R}$ is built as Chevalley chains of the Heisenberg algebra based on $C_c^\infty(U) \xrightarrow{\triangle+m^2} C_c^\infty(U)[-1]$. The operator $\frac{\partial}{\partial x}$ act on $C_c^\infty(U)$ and commute with the operator $\triangle + m^2$. It also preserve the cocycle defining the central extension, and therefore acts naturally on the Chevalley chains of the Heisenberg algebra. One can check that this operator is a derivation for the prefactorization product. That is, the operator $\frac{\partial}{\partial x}$ commutes with inclusions of one open subset into another, and if U, V are disjoint and $\alpha \in \mathrm{Obs}^q(U)$, $\beta \in \mathrm{Obs}^q(V)$, we have

$$\frac{\partial}{\partial x}(\alpha \cdot \beta) = \frac{\partial}{\partial x}(\alpha) \cdot \beta + \alpha \cdot \frac{\partial}{\partial x}(\beta) \in \mathrm{Obs}^q(U \amalg V).$$

(Derivations are discussed in more detail in section 4.7).

It follows immediately that $\frac{\partial}{\partial x}$ defines a derivation of the associative algebra A_m coming from the cohomology of $\mathrm{Obs}^q((0,1))$.

4.3.0.2 Definition. *The* Hamiltonian H *is the derivation of the associative* A_m *arising from the derivation* $-\frac{\partial}{\partial x}$ *of the prefactorization algebra* Obs^q *of observables of the free scalar field theory with mass* m.

4.3.0.3 Proposition. *The associative algebra* A_m *coming from the free scalar field theory with mass* m *is the Weyl algebra, generated by* $p, q, \hbar$ *with the relation* $[p,q] = \hbar$ *and all other commutators being zero. The Hamiltonian* H *is the derivation*

$$H(a) = \tfrac{1}{2\hbar}[p^2 - m^2 q^2, a].$$

PROOF. We will start by writing down elements of Obs^q corresponding to position and momentum. Recall that

$$\mathrm{Obs}^q((a,b)) = \mathrm{Sym}^*\left(C_c^\infty((a,b))^{-1} \oplus C_c^\infty((a,b))^0\right)[\hbar]$$

with a certain differential.

We let $\phi_q, \phi_p \in C^\infty(\mathbb{R})$ be the functions introduced in the proof of lemma 4.2.4.1. Explicitly, if $m = 0$, then $\phi_q = 1$ and $\phi_p = x$, whereas if $m \neq 0$ we have

$$\begin{aligned}\phi_q &= \tfrac{1}{2}\left(e^{mx} + e^{-mx}\right)\\ \phi_p &= \tfrac{1}{2m}\left(e^{mx} - e^{-mx}\right).\end{aligned}$$

Thus, ϕ_q, ϕ_p are both in the kernel of the operator $-\partial^2 + m^2$, with the properties that $\partial\phi_p = \phi_q$ and that ϕ_q is symmetric under $x \mapsto -x$, whereas ϕ_p is antisymmetric.

Choose a function $f_0 \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ which is symmetric under $x \mapsto -x$ and has the property that

$$\int_{-\infty}^{\infty} f_0(x)\phi_q(x)\mathrm{d}x = 1.$$

The symmetry of f_0 implies that the integral of f_0 against ϕ_p is zero.

Let $f_t \in C_c^\infty((t - \frac{1}{2}, t + \frac{1}{2}))$ be $f_t(x) = f_0(x - t)$. We define observables P_t, Q_t by

$$\begin{aligned}Q_t &= f_t\\ P_t &= -f_t'\end{aligned}$$

The observables Q_t, P_t are in the space $C_c^\infty(I)[1] \oplus C_c^\infty(I)$ of linear observables, where $I = ((t - \frac{1}{2}, t + \frac{1}{2})$. They are also of cohomological degree zero. If we think of observables as functionals of a field in $C^\infty(I) \oplus C^\infty(I)[-1]$, then these are linear observables given by integrating f_t or $-f_t'$ against the field ϕ.

Thus, Q_t and P_t represent average measurements of positions and momenta of the field ϕ in a neighborhood of t.

Because the cohomology classes $[P_0], [Q_0]$ generate the commutative algebra $H^*(\mathrm{Obs}^{cl}(\mathbb{R}))$, it is automatic that they still generate the associative algebra $H^0\,\mathrm{Obs}^q(\mathbb{R})$. We thus need to show that they satisfy the Heisenberg commutation relation

$$[[P_0], [Q_0]] = \hbar$$

for the associative product on $H^0\,\mathrm{Obs}^q(\mathbb{R})$, which is an associative algebra by virtue of the fact that $H^*\,\mathrm{Obs}^q$ is locally constant. (The other commutators automatically vanish).

The Hamiltonian acting on $[P_0]$ gives the t-derivative of $[P_t]$ at $t = 0$, and similarly for Q_0. Note that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}P_t = -\tfrac{\mathrm{d}}{\mathrm{d}t}f_0'(x - t) = f_0''(x - t).$$

Since, in cohomology, the image of $-\partial_x^2 + m^2$ is zero, we see that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}[P_t] = m^2[f_0(x-t)] = m^2[Q_t].$$

In particular, when $m = 0$, $[P_t]$ is independent of t: this is conservation of momentum. Similarly, $\frac{\mathrm{d}}{\mathrm{d}t} Q_t = P_t$.

Thus, the Hamiltonian satisfies

$$\begin{aligned} H([P_0]) &= m^2[Q_0] \\ H([Q_0]) &= [P_0]. \end{aligned}$$

If we assume that the commutation relation $[[P_0],[Q_0]] = \hbar$ holds, this implies that

$$H(a) = \tfrac{1}{2\hbar}[[P_0]^2 - m^2[Q_0]^2, a]$$

as desired.

Therefore, to prove the proposition, it suffices to prove the commutation relation between $[P_0]$ and $[Q_0]$.

The first lemma we need is the following.

4.3.0.4 Lemma. *Let $a(t)$ be any function which satisfies $a''(t) = m^2 a(t)$. Let Q_t^{-1} denote the observable in cohomological degree -1 given by $f_t \in C_c^\infty((t-\frac{1}{2}, t+\frac{1}{2})[1]$. Then,*

$$\frac{\partial}{\partial t}\left(a(t)P_t - a'(t)Q_t\right) = -\mathrm{d}(a(t)Q_t^{-1})$$

where d *is the differential on observables.*

PROOF. Let us prove this equation explicitly.

$$\begin{aligned} \frac{\partial}{\partial t}\left(a(t)P_t - a'(t)Q_t\right) &= -a'(t)\frac{\partial}{\partial x}f_t(x) - a(t)\frac{\partial}{\partial t}\frac{\partial}{\partial x}f_t(x) - a''(t)f_t(x) - a'(t)\frac{\partial}{\partial t}f_t(x) \\ &= a(t)\frac{\partial}{\partial x}^2 f_t(x) - a''(t)f_t(x) \\ &= a(t)\frac{\partial}{\partial x}^2 f_t(x) - m^2 a(t)f_t(x) \\ &= -\mathrm{d}(a(t)Q_t^{-1}). \end{aligned}$$

□

We will define modified observables $\mathscr{P}_t$, $\mathscr{Q}_t$ which are independent of t at the cohomological level. We let

$$\begin{aligned} \mathscr{P}_t &= \phi_q(t)P_t - \phi_q'(t)Q_t \\ \mathscr{Q}_t &= \phi_p(t)P_t - \phi_p'(t)Q_t. \end{aligned}$$

Since ϕ_p and ϕ_q are in the kernel of the operator $-\partial^2 + m^2$, the observables $\mathscr{P}_t$, $\mathscr{Q}_t$ are independent of t at the level of cohomology. In the case $m = 0$, then $\phi_q(t) = 1$ so that $\mathscr{P}_t = P_t$. The statement that $\mathscr{P}_t$ is independent of t corresponds, in this case, to conservation of momentum.

In general, $\mathscr{P}_0 = P_0$ and $\mathscr{Q}_0 = Q_0$. We also have

$$\frac{\partial}{\partial t}\mathscr{P}_t = -\mathrm{d}(\phi_q(t) Q_t^{-1})$$

and similarly for $\mathscr{Q}_t$.

It follows from the lemma that if we define a linear degree -1 observable $h_{s,t}$ by

$$h_{s,t} = \int_{u=s}^{t} \phi_q(u) Q_u^{-1}(x)\mathrm{d}u,$$

then

$$\mathrm{d}h_{s,t} = \mathscr{P}_s - \mathscr{P}_t.$$

Note that if $|t| > 1$, the observables $\mathscr{P}_t, \mathscr{Q}_t$ and $\mathscr{P}_0, \mathscr{Q}_0$ have disjoint support. This means that we can use the prefactorization structure map

$$\mathrm{Obs}^q((-\tfrac{1}{2}, \tfrac{1}{2})) \otimes \mathrm{Obs}^q((t - \tfrac{1}{2}, t + \tfrac{1}{2})) \to \mathrm{Obs}^q(\mathbb{R})$$

to define a product observable

$$Q_0 \cdot \mathscr{P}_t \in \mathrm{Obs}^q(\mathbb{R}).$$

We will let $\star$ denote the associative multiplication on $H^0\,\mathrm{Obs}^q(\mathbb{R})$. We defined this multiplication as follows. If $\alpha, \beta \in H^0(\mathrm{Obs}^q(\mathbb{R}))$, we represent α by an element of $H^0(\mathrm{Obs}^q(I))$ and β by an element of $H^0(\mathrm{Obs}^q(J))$, where I, J are disjoint intervals with $I < J$. The map

$$H^0(\mathrm{Obs}^q(I)) \to H^0(\mathrm{Obs}^q(\mathbb{R}))$$

(and similarly for J) is an isomorphism, which allows us to choose such representations of α and β.

A representative of $[P_0]$ which is supported in the interval $(t - \frac{1}{2}, t + \frac{1}{2})$ is given by $\mathscr{P}_t$. This follows from the fact that the cohomology class of $\mathscr{P}_t$ is independent of t, and that $\mathscr{P}_0 = P_0$.

Thus, the products between $[P_0]$ and $[Q_0]$ are defined by

$$[Q_0] \star [P_0] = [Q_0 \cdot \mathscr{P}_t] \text{ if } t > 1$$
$$[P_0] \star [Q_0] = [Q_0 \cdot \mathscr{P}_t] \text{ if } t < -1.$$

Thus, it remains to show that, if $t > 1$,

$$[Q_0 \cdot \mathscr{P}_t] - [Q_0 \cdot \mathscr{P}_{-t}] = \hbar.$$

We will construct an observable whose differential is the difference between the left and right hand sides. Consider the observable of cohomological degree 1 defined by

$$S = f_0(x)h_{-t,t}(y) \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R})[1],$$

where the functions f and $h_{-t,t}$ were defined above. We view $h_{-t,t}$ as being of cohomological degree -1, and f as being of cohomological degree 0.

Recall that the differential on $\mathrm{Obs}^q(\mathbb{R})$ has two terms: one coming from the Laplacian $\triangle + m^2$ mapping $C_c^\infty(\mathbb{R})^{-1}$ to $C_c^\infty(\mathbb{R})^0$, and one arising from the bracket of the Heisenberg Lie algebra. The second term maps

$$\mathrm{Sym}^2\left(C_c^\infty(\mathbb{R})^{-1} \oplus C_c^\infty(\mathbb{R})^0\right) \to \mathbb{R}\hbar.$$

Applying this differential to the observable S, we find that

$$\begin{aligned}(\mathrm{d}S) &= f_0(x)(-\partial^2 + m^2)h_{-t,t}(y) + \hbar \int_{\mathbb{R}} h_{-t,t}(x) f_0(x)\mathrm{d}x \\ &= Q_0 \cdot (\mathscr{P}_{-t} - \mathscr{P}_t) + \hbar \int f_0(x)h_{-t,t}(x).\end{aligned}$$

Therefore

$$[Q_0\mathscr{P}_t] - [Q_0\mathscr{P}_{-t}] = \hbar \int f_0(x)h_{-t,t}(x).$$

It remains to compute the integral. This integral can be rewritten as

$$\int_{u=-t}^{t} \int_{x=-\infty}^{\infty} f_0(x)f_0(x-u)\phi_q(u)\mathrm{d}u.$$

Note that the answer is automatically independent of t for t sufficiently large, because $f(x)$ is supported near the origin so that $f_0(x)f_0(x-u) = 0$ for u sufficiently large. Thus, we can sent $t \to \infty$.

Since f_0 is also symmetric under $x \to -x$, we can replace $f_0(x-u)$ by $f_0(u-x)$. We can perform the u-integral by changing coordinates $u \to u - x$, leaving the integrand as $f_0(x)f_0(u)\phi_q(u+x)$. Note that

$$\phi_q(u+x) = \tfrac{1}{2}\left(e^{m(x+u)} + e^{-m(x+u)}\right).$$

Now, by assumption on f_0,

$$\int f_0(x)e^{mx} = \int f_0(x)e^{-mx} = 1.$$

It follows that

$$\int_{u=-\infty}^{\infty} \phi_q(u+x)f_0(u)\mathrm{d}u = \phi_q(x).$$

We can then perform the remaining x integral $\int \phi_q(x)f_0(x)\mathrm{d}x$, which gives 1, as desired.

Thus, we have proven

$$[[Q_0],[P_0]] = \hbar,$$

as desired. □

4.4. Free field theories and canonical quantization

Consider the free scalar field theory on a manifold of the form $N \times \mathbb{R}$, with the product metric. We assume for simplicity that N is compact. Let Obs^q denote the prefactorization algebra of observables of the free scalar field theory with mass m on $N \times \mathbb{R}$. Let $\pi : N \times \mathbb{R} \to \mathbb{R}$ be the projection map. There is a push-forward prefactorization algebra $\pi_* \mathrm{Obs}^q$, defined by

$$(\pi_* \mathrm{Obs}^q)(U) = \mathrm{Obs}^q(\pi^{-1}(U)).$$

In this section, we will explain how to relate this prefactorization algebra on $\mathbb{R}$ to an infinite tensor product of the prefactorization algebras associated to quantum mechanics on $\mathbb{R}$.

Let e_i be an ortho-normal basis of eigenvectors of the operator $\triangle + m^2$ on $C^\infty(N)$, with eigenvalue λ_i. The space $\oplus \mathbb{R} \cdot e_i$ is a dense subspace of $C^\infty(N)$.

For $m \in \mathbb{R}$, let A_m denote the cohomology of the prefactorization algebra associated to the free one-dimensional scalar field theory with mass m. Thus, A_m is the Weyl algebra $\mathbb{R}[p,q,\hbar]$ with commutator $[p,q] = \hbar$. The dependence on m is only through the Hamiltonian.

We can form the tensor product

$$\otimes_{\mathbb{R}[\hbar]} A_{\sqrt{\lambda_i}} = A_{\sqrt{\lambda_1}} \otimes_{\mathbb{R}[\hbar]} A_{\sqrt{\lambda_2}} \otimes_{\mathbb{R}[\hbar]} \cdots$$

(The infinite tensor product is defined to be the colimit of the finite tensor products, where the maps in the colimit come from the unit in each algebra).

4.4.0.5 Proposition. *There is a dense sub-factorization algebra of* $\pi_* \mathrm{Obs}^q$ *which is locally constant, and has cohomology* $\otimes_{\mathbb{R}[\hbar]} A_{\sqrt{\lambda_i}}$.

Remark: The prefactorization algebra $\pi_* \mathrm{Obs}^q$ has a derivation, the Hamiltonian, coming from infinitesimal translation in $\mathbb{R}$. The prefactorization algebra $A_{\sqrt{\lambda_i}}$ also has a Hamiltonian, given by bracketing with $\frac{1}{2\hbar}[p^2 - \lambda_i q^2, -]$. The map from $\otimes A_{\sqrt{\lambda_i}}$ to $H^*(\pi_* \mathrm{Obs}^q)$ intertwines these derivations.

PROOF. The prefactorization algebra $\pi_* \mathrm{Obs}^q$ on $\mathbb{R}$ assigns to an open subset $U \subset \mathbb{R}$ the Chevalley chains of a Heisenberg Lie algebra given by a central extension of

$$C_c^\infty(U \times N) \xrightarrow{\triangle + m^2} C^\infty(U \times N)[-1].$$

A dense subcomplex of this is

$$(\dagger) \qquad \oplus_i \left(C_c^\infty(U) \cdot e_i \xrightarrow{\triangle_{\mathbb{R}} + \lambda_i} \oplus C_c^\infty(U) \cdot e_i[-1] \right).$$

Let $\mathcal{F}_i$ be the prefactorization algebra on $\mathbb{R}$ associated to quantum mechanics with mass $\sqrt{\lambda_i}$. This is the prefactorization algebra associated to Heisenberg central extension of

$$C_c^\infty(U) \cdot e_i \xrightarrow{\triangle_{\mathbb{R}} + \lambda_i} \oplus_i C_c^\infty(U) \cdot e_i[-1]$$

Note that $\mathcal{F}_i$ is a prefactorization algebra over $\mathbb{R}[\hbar]$. We can define the tensor product prefactorization algebra

$$\otimes_{\mathbb{R}[\hbar]} \mathcal{F}_i = \mathcal{F}_1 \otimes_{\mathbb{R}[\hbar]} \mathcal{F}_2 \otimes_{\mathbb{R}[\hbar]} \cdots$$

to be the colimit of the finite tensor products of the $\mathcal{F}_i$ under the inclusion maps coming from the unit $1 \in \mathcal{F}_i(U)$ for any open subset.

We can, equivalently, view this tensor product as being associated to the Heisenberg central extension of the complex (†) above.

Because the complex (†) is a dense subspace of the complex whose Heisenberg extension defines $\pi_* \operatorname{Obs}^q$, we see that there's a map of prefactorization algebras with dense image

$$\otimes_{\mathbb{R}[\hbar]} \mathcal{F}_i \to \pi_* \operatorname{Obs}^q$$

Passing to cohomology, we have a map

$$\otimes_{\mathbb{R}[\hbar]} H^*(\mathcal{F}_i) \to H^*(\pi_* \operatorname{Obs}^q).$$

The prefactorization algebra $H^*(\mathcal{F}_i)$ is $A_{\sqrt{\lambda_i}}$. □

4.5. Quantizing classical observables

We have given an abstract definition of the prefactorization algebra of quantum observables of a free field theory, as the prefactorization envelope of a certain Heisenberg dg Lie algebra. The prefactorization algebra of quantum observables Obs^q, viewed as a graded prefactorization algebra with no differential, coincides with $\operatorname{Obs}^{cl}[\hbar]$. The only difference between Obs^q and $\operatorname{Obs}^{cl}[\hbar]$ is in the differential.

In this section we will give an alternative, but equivalent, description of Obs^q. We will construct an isomorphism of precosheaves $\operatorname{Obs}^q \cong \operatorname{Obs}^{cl}[\hbar]$ which is compatible with differentials. This isomorphism is not, however, compatible with the prefactorization product. Thus, this isomorphism induces a deformed prefactorization product on $\operatorname{Obs}^{cl}[\hbar]$ corresponding to the prefactorization product on Obs^q.

In other words, instead of viewing Obs^q as being obtained from Obs^{cl} by keeping the prefactorization product fixed but deforming the differential, we will show that it can be obtained from Obs^{cl} by keeping the differential fixed but deforming the product.

One advantage of this alternative description is that it is easier to construct correlation functions and vacua in this language.

4.5.1. The isomorphism we will construct between the cochain complexes of quantum and classical observables relies on a Green's function for the Laplacian.

4.5.1.1 Definition. *A* Green's function *is a distribution G on $M \times M$ which is symmetric, and which satisfies*

$$(\triangle \otimes 1)G = \delta_{\triangle}$$

where $\delta_{\triangle}$ is the δ-distribution on the diagonal.

A Green's function for the Laplacian with mass satisfies the equation

$$(\triangle \otimes 1)G + m^2 G = \delta_{\triangle}.$$

(The convention is that $\triangle$ has positive eigenvalues, so that on $\mathbb{R}^n$, $\triangle = -\sum \frac{\partial}{\partial x_i}^2$.)

If M is compact, then there is no Green's function for the Laplacian. Instead, there is a unique function $\widetilde{G}$ satisfying

$$(\triangle \otimes 1)\widetilde{G} = \delta_{\triangle} - \pi$$

where π is the kernel for the operator of projection on to harmonic functions.

However, if we introduce a non-zero mass term, then the operator $\triangle + m^2$ on $C^\infty(M)$ is an isomorphism, so that there is a unique Green's function.

If M is non-compact, then there can be a Green's function for the Laplacian without mass term. For example, if M is $\mathbb{R}^n$, then a choice of Green's function is

$$G(x,y) = \begin{cases} \frac{1}{4\pi^{d/2}}\Gamma(d/2-1)\,|x-y|^{2-n} & \text{if } n \neq 2 \\ -\frac{1}{2\pi}\log|x-y| & \text{if } n = 2. \end{cases}$$

Let us now turn to the construction of the isomorphism of graded vector spaces between $\mathrm{Obs}^{cl}(U)$ and $\mathrm{Obs}^q(U)$ in the presence of a Green's function.

The underlying graded vector space of $\mathrm{Obs}^q(U)$ is $\mathrm{Sym}^*(C_c^\infty(U)^{-1} \oplus C_c^\infty(U))[\hbar]$. In general, for any vector space V, any element $P \in (V^\vee)^{\otimes 2}$ defines a differential operator ∂_P of order two on $\mathrm{Sym}^* V$, uniquely characterized by the condition that it is zero on $\mathrm{Sym}^{\leq 1} V$ and that on $\mathrm{Sym}^2 V$ it is given by pairing with P. The same holds when we define the symmetric algebra using the completed tensor product.

In the same way, for every distribution P on $U \times U$, we can define a continuous order 2 differential operator on $\mathrm{Sym}^*(C_c^\infty(U)^{-1} \oplus C_c^\infty(U)^0)$ which is uniquely characterized by the properties that on $\mathrm{Sym}^{\leq 1}$ it is zero, that on $\mathrm{Sym}^2(C_c^\infty(U)^{-1} \oplus C_c^\infty(U)^0)$ it is zero on elements of cohomological degree less than zero, and that for $\phi, \psi \in C_c^\infty(U)^0$ we have

$$\partial_P(\phi\psi) = \int P(x,y)\phi(x)\psi(y).$$

Let us choose a Green's function G for the Laplacian on M. Then, G restricts to a Green's function for the Laplacian on any open subset U of M.

Therefore, we can define an order two differential operator ∂_G on $\mathrm{Sym}^*(C_c^\infty(U)^{-1} \oplus C_c^\infty(U)^0)$. Since we have an identification (as graded vector spaces)

$$\mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)[\hbar] = \mathrm{Sym}^*(C_c^\infty(U)^{-1} \oplus C_c^\infty(U)^0)[\hbar]$$

we can extend this by $\mathbb{R}[\hbar]$-linearity to an operator on the graded vector space $\mathrm{Obs}^q(U)$.

Now, the differential on $\mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)[\hbar]$ can be written as $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$, where d_1 is the differential on classical observables (which cohomologically imposes the equations of motion), and d_2 is the quantum correction which corresponds to divergence. Note that d_1 is a first-order differential operator and that d_2 is a second-order operator. The operator d_1 is the derivation arising from the differential on the complex $C_c^\infty(U) \xrightarrow{\triangle} C_c^\infty(U)$. The operator d_2 is the term arising from the Lie bracket on the Heisenberg dg Lie algebra; it is a continuous $\hbar$-linear order two differential operator uniquely characterized by the property that $\mathrm{d}_2(\phi^{-1}\phi^0) = \hbar \int \phi^{-1}\phi^0$ for ϕ^i in the copy of $C_c^\infty(U)$ in degree i, sitting inside of $\mathrm{Obs}^q(U)$.

The Green's function G satisfies

$$((\triangle + m^2) \otimes 1)G = \delta_\triangle$$

where $\delta_\triangle$ is the Green's function on the diagonal.

It follows that

$$[\hbar\partial_G, \mathrm{d}_1] = \mathrm{d}_2.$$

Indeed, both sides of this equation are order two differential operators, so to check the equation, it suffices to calculate how the act on an element of $\mathrm{Sym}^2(C_c^\infty(\mathbb{R}^n)[1] \oplus C_c^\infty(\mathbb{R}^n))$.

If ϕ^i for $i = -1, 0$ are elements of the copy of $C_c^\infty(\mathbb{R}^n)$ in degree i, we have

$$\begin{aligned}\hbar\partial_G \mathrm{d}_1(\phi^0\phi^{-1}) &= \hbar\partial_G(\phi^0(\triangle + m^2)\phi^{-1}) \\ &= \int G(x,y)\phi^0(x)((\triangle + m^2)\phi^{-1})(y) \\ &= \int ((\triangle_y + m^2)G(x,y))\phi^0(x)\phi^{-1}(y) \\ &= \int \phi^0(x)\phi^{-1}(x) \qquad\qquad = \mathrm{d}_2(\phi^0\phi^{-1}).\end{aligned}$$

On the fourth line, we have used the fact that $\triangle_y + m^2$ applied to $G(x,y)$ is the delta-distribution on the diagonal.

It is also immediate that ∂_G commutes with d_2. Thus, if we let

$$W(\alpha) = e^{\hbar\partial_G}(\alpha)$$
$$(\mathrm{d}_1 + \mathrm{d}_2)W(\alpha) = W(\mathrm{d}_1\alpha).$$

In other words, W is a cochain map from the complex $\mathrm{Obs}^q(U)$ with differential d_1 to the same graded vector space with differential $\mathrm{d}_1 + \mathrm{d}_2$. Since the differential on $\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^*(C_c^\infty(M)[1] \oplus C_c^\infty(U))$ is d_1, we see that, as desired, W gives an $\mathbb{R}[\hbar]$-linear cochain isomorphism

$$W : \mathrm{Obs}^{cl}(U)[\hbar] \to \mathrm{Obs}^q(U).$$

Note that W is *not* a map of prefactorization algebras. Thus, W induces a prefactorization product on classical observables which quantizes the original prefactorization product. Let us denote this quantum product by $\star_\hbar$, whereas the original product on classical observables will be denoted by a dot. Then, if U, V are disjoint open subsets of M, and $\alpha \in \mathrm{Obs}^{cl}(U)$, $\beta \in \mathrm{Obs}^{cl}(V)$, we have

$$\alpha \star_\hbar \beta = e^{-\hbar\partial_G}\left\{\left(e^{\hbar\partial_G}\alpha\right) \cdot \left(e^{\hbar\partial_G}\beta\right)\right\}$$

This can be compared to the Moyal formula for the product on the Weyl algebra.

We have described this construction for the case of a free scalar field theory. This construction can be readily generalized to the case of an arbitrary free theory. Suppose we have such a theory on a manifold M, with space of fields $\mathscr{E}(M)$ and differential d. Instead of a Green's function, we require a symmetric and continuous linear operator $G : (\mathscr{E}(M)[1])^{\otimes 2} \to \mathbb{R}$ such that

$$G\mathrm{d}(e_1 \otimes e_2) = \langle e_1, e_2 \rangle$$

where $\langle -, - \rangle$ is the pairing on $\mathscr{E}(M)$ which is part of the data of a free field theory. In the case that M is compact and $H^*(\mathscr{E}(M)) = 0$, the propagator of the theory satisfies this property. If $M = \mathbb{R}^n$, then we can generally construct such a G from the Green's function for the Laplacian. In this context, the operator $e^{\hbar\partial_G}$ is, in the terminology of [Cos11c], the renormalization group flow operator from scale zero to scale ∞.

4.6. Correlation functions

Let us suppose we have a free field theory on a compact manifold M, with the property that $H^*(\mathscr{E}(M)) = 0$. As an example, consider the massive scalar field theory on M where

$$\mathscr{E}(M) = C^\infty(M) \xrightarrow{\triangle + m^2} C^\infty(M)[-1].$$

Our conventions are such that the eigenvalues of $\triangle$ are non-negative. Adding a non-zero mass term m^2 gives an operator with no zero eigenvalues, so that this complex has no cohomology.

As above, let $\widehat{\mathscr{E}}(M)$ be the Heisenberg dg Lie algebra whose underlying cochain complex is $\mathscr{E}(M) \oplus \mathbb{R} \cdot \hbar[-1]$, where the central element $\hbar$ is in degree 1. The inclusion $\mathbb{R} \cdot \hbar[-1] \to \widehat{\mathscr{E}}(M)$ is a quasi-isomorphism of dg Lie algebras. It follows that there is an isomorphism of $\mathbb{R}[\hbar]$-modules

$$H^*(\mathrm{Obs}^q(M)) \cong \mathbb{R}[\hbar].$$

Let us normalize this isomorphism by asking that the element 1 in

$$\mathrm{Sym}^0(\widehat{\mathscr{E}}[1]) = \mathbb{R} \subset C_*(\widehat{\mathscr{E}}[1])$$

gets send to $1 \in \mathbb{R}[\hbar]$.

4.6.0.2 Definition. *In this situation, define the* correlation functions *of the free theory as follows. If $U_1, \dots, U_n \subset M$ are disjoint opens, and $O_i \in \mathrm{Obs}^q(U_i)$ are closed elements, then we define*

$$\langle O_1, \dots, O_n \rangle = [O_1 \dots, O_n] \in H^*(\mathrm{Obs}^q(M)) = \mathbb{R}[\hbar].$$

Here $O_1 \dots O_n \in \mathrm{Obs}^q(M)$ is defined by the product structure of the prefactorization algebra.

The map W constructed in the previous section allows us to calculate correlation functions. Since we have a non-zero mass term and M is compact, there is a unique Green's function for the operator $\triangle + m^2$.

4.6.0.3 Lemma (Wick's lemma). *Let*

$$\alpha_i \in \mathrm{Obs}^{cl}(U_i) = \mathrm{Sym}^*(C_c^\infty(U)_i[1] \oplus C_c^\infty(U_i))$$

be classical observables, and let

$$W(\alpha_i) = e^{\hbar \partial_G} \alpha_i \in \mathrm{Obs}^q(U_i)$$

be the corresponding quantum observables under the isomorphism of cochain complexes W from classical to quantum observables. Then,

$$\langle W(\alpha_1), \dots, W(\alpha_n) \rangle = W^{-1} \left(W(\alpha_1) \cdots W(\alpha_n) \right) (0) \in \mathbb{R}[\hbar]$$

On the right hand side $W(\alpha_1) \cdot W(\alpha_2)$ indicates the product in the algebra $\mathrm{Sym}^*(C^\infty(M)[1] \oplus C^\infty(M))[\hbar]$. *The symbol (0) indicates evaluating a function on $C^\infty(M)[1] \oplus C^\infty(M)[\hbar]$ at zero, that is, taking the term in* Sym^0.

PROOF. The map W gives an isomorphism of cochain complexes between $\mathrm{Obs}^{cl}(U)[\hbar]$ and $\mathrm{Obs}^q(U)$ for every open subset U of M. As above, let $\star_\hbar$ denote the prefactorization product on $\mathrm{Obs}^{cl}[\hbar]$ corresponding, under the isomorphism W, to the prefactorization product on Obs^q. Then,

$$\alpha_1 \star_\hbar \cdots \star_\hbar \alpha_n = e^{-\hbar \partial_G} \left((e^{\hbar \partial_G} \alpha_1) \cdots (e^{\hbar \partial_G} \alpha_n) \right).$$

Since W gives an isomorphism of prefactorization algebras between $(\mathrm{Obs}^{cl}[\hbar], \star_\hbar)$ and Obs^q, the correlation functions of the observables $W(\alpha_i)$ is the cohomology class of $\alpha_1 \star_\hbar \cdots \star_\hbar \alpha_n$ in $\mathrm{Obs}^{cl}(M)[\hbar]$. The map $\mathrm{Obs}^{cl}(M)[\hbar] \to \mathbb{R}[\hbar]$ sending an observable α to $\alpha(0)$ is an $\mathbb{R}[\hbar]$-linear cochain map inducing an isomorphism on cohomology, and it sends 1 to 1. There is a unique (up to cochain homotopy) such map. Therefore,

$$\langle W(\alpha_1), \ldots, W(\alpha_n) \rangle = (\alpha_1 \star_\hbar \cdots \star_\hbar \alpha_n)(0).$$

□

This lemma gives us an explicit formula in terms of the Green's functions for the correlation functions of any observable.

As an example, let us suppose that we have two linear observables α_1, α_2 of cohomological degree 0, defined on open sets U, V. Thus, $\alpha_1 \in C_c^\infty(U)$ and $\alpha_2 \in C_c^\infty(V)$. Then, $W(\alpha_1) = \alpha_1$, but

$$W^{-1}(\alpha_1 \alpha_2) = \alpha_1 \alpha_2 - \hbar \partial_G(\alpha_1 \alpha_2).$$

Further,

$$\partial_G(\alpha_1 \alpha_2) = \int_{M \times M} G(x,y) \alpha_1(x) \alpha_2(y).$$

It follows that

$$\langle W(\alpha_1), W(\alpha_2) \rangle = -\hbar \int_{M \times M} G(x,y) \alpha_1(x) \alpha_2(y)$$

which is up to sign what a physicist would write down to the expectation value of two linear observables.

Remark: We have set things up so that we are computing the functional integral against the measure which is $e^{S/\hbar} \mathrm{d}\mu$ where S is the action functional and $\mathrm{d}\mu$ is the (non-existent) "Lebesgue measure" on the space of fields. Physicists often use the convention where we use $e^{-S/\hbar}$ or $e^{iS/\hbar}$. One goes between these different conventions by a change of coordinates $\hbar \to -\hbar$ or $\hbar \to i\hbar$.

4.7. Translation-invariant prefactorization algebras

In this section we will analyze in detail the notion of *translation-invariant* prefactorization algebras on $\mathbb{R}^n$. Most theories from physics possess this property. For one-dimensional field theories, one often uses the phrase "time-independent Hamiltonian" to indicate this property, and in this section we will explain, in examples, how to relate the Hamiltonian formalism of quantum mechanics to our approach.

4.7.1. We now turn to the definition of a translation-invariant prefactorization algebra. If $U \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, let

$$T_x(U) := \{y \ : \ y - x \in U\}$$

denote the translate of U by x.

4.7.1.1 Definition. *A prefactorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ is* discretely translation-invariant *if we have isomorphisms*

$$T_x : \mathcal{F}(U) \cong \mathcal{F}(T_x(U))$$

for all $x \in \mathbb{R}^n$ and all open subsets $U \subset \mathbb{R}^n$. These isomorphisms must satisfy a few conditions. First, we require that $T_x \circ T_y = T_{x+y}$ for every $x, y \in \mathbb{R}^n$. Second, for all disjoint open subsets $U_1, \ldots, U_k$ in V, the diagram

$$\begin{tikzcd}
\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \arrow[r, "T_x"] \arrow[d] & \mathcal{F}(T_x U_1) \otimes \cdots \otimes \mathcal{F}(T_x U_k) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "T_x"] & \mathcal{F}(T_x V)
\end{tikzcd}$$

commutes. (Here the vertical arrows are the structure maps of the prefactorization algebra.)

Example: Consider the prefactorization algebra of quantum observables of the free scalar field theory on $\mathbb{R}^n$, as defined in section 4.2.6. This has complex of fields

$$\mathscr{E} = \left\{ C^\infty \xrightarrow{\triangle} C^\infty[-1] \right\},$$

where the superscript indicates cohomological degree.

By definition, $\mathrm{Obs}^q(U)$ is the Chevalley-Eilenberg chains of a -1-shifted central extension $\widehat{\mathscr{E}_c}(U)$ of $\mathscr{E}_c(U)$, with cocycle defined by $\int \phi^0 \phi^1$ where $\phi^i \in C_c^\infty(U)^i$.

This Heisenberg algebra is defined only using the Riemannian structure on $\mathbb{R}^n$, and is therefore automatically invariant under all isometries of $\mathbb{R}^n$. In particular, the resulting prefactorization algebra is discretely translation-invariant.

We are interested in a refined version of this notion, where the structure maps of the prefactorization algebra depend smoothly on the position of the open sets. It is a bit subtle to talk about "smoothly varying an open set," and in order to do this, we introduce some notation.

Firstly, we need to introduce the notion of a *derivation* of a prefactorization algebra on a manifold M. We will construct a differential graded Lie algebra of derivations of any prefactorization algebra.

4.7.1.2 Definition. *A* degree k derivation *of a prefactorization algebra $\mathcal{F}$ is a collection of maps $D_U : \mathcal{F}(U) \to \mathcal{F}(U)$ of cohomological degree k for each open subset $U \subset M$, with the property that, if $U_1, \dots, U_n \subset V$ are disjoint, and $\alpha_i \in \mathcal{F}(U_i)$, then*

$$D_V m_V^{U_1,\dots,U_n}(\alpha_1, \dots, \alpha_n) = \pm \sum m_V^{U_1,\dots,U_n}(\alpha_1, \dots, D_{U_i}\alpha_i, \dots, \alpha_n),$$

where ± indicates the usual Koszul rule of signs.

Example: Let us consider, again, the prefactorization algebra of the free scalar field on $\mathbb{R}^n$. Observables on U are Chevalley-Eilenberg chains of the Heisenberg algebra $\widehat{\mathscr{E}_c}(U)$. Suppose that X is a Killing vector field on $\mathbb{R}^n$ (i.e. X is an infinitesimal isometry). For example, we could take X to be a translation vector field $\frac{\partial}{\partial x^j}$. The Heisenberg dg Lie algebra has a derivation which sends $\phi^i \to X\phi^i$, for $i = 0, 1$, and is zero on the central element $\hbar$. (Recall that ϕ^i is notation for an element of the copy of $C_c^\infty(U)$ situated in degree i). The fact that X is a Killing vector field on $\mathbb{R}^n$ implies that it commutes with the differential on the Heisenberg algebra, and the equality

$$\int (X\phi^0)\phi^1 + \int \phi^0(X\phi^1)$$

implies that X is a derivation of dg Lie algebras.

By naturality, X extends to an endomorphism of $\mathrm{Obs}^q(U) = C_*(\widehat{\mathscr{E}_c}(U))$. This endomorphism defines a derivation of the prefactorization algebra Obs^q of observables of the free scalar field theory.

Let $\mathrm{Der}^k(\mathcal{F})$ denote the derivations of degree k; it is easy to verify that $\mathrm{Der}^*(\mathcal{F})$ forms a differential graded Lie algebra. The differential is defined by $(\mathrm{d}D)_U = [\mathrm{d}_U, D_U]$, where d_U is the differential on $\mathcal{F}(U)$. The Lie bracket is defined by

$$[D, D']_U = [D_U, D'_U].$$

The concept of derivation allows us to talk about the action of a dg Lie algebra on a prefactorization algebra $\mathcal{F}$. Such an action is simply a homomorphism of differential graded Lie algebras

$$\mathfrak{g} \to \mathrm{Der}^*(\mathcal{F}).$$

Next, let us introduce some notation which will help us describe the smoothness conditions for a discretely translation-invariant prefactorization algebra.

Let $U_1, \dots, U_k \subset V$ be disjoint open subsets. Let $W \subset (\mathbb{R}^n)^k$ be the set of those $x_1, \dots, x_k$ such that the sets $T_{x_1}(U_1), \dots, T_{x_k}(U_k)$ are all disjoint and contained in V. It parametrizes the way we can move the open sets without causing overlaps. Let us assume that W has non-empty interior, which happens when the closure of the U_i are disjoint and contained in V.

Let $\mathcal{F}$ be any discretely translation-invariant prefactorization algebra. Then, for each $(x_1, \dots, x_k) \in W$, we have a multilinear map obtained as a composition

$$m_{x_1,\dots,x_k} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_k) \to \mathcal{F}(T_{x_1}U_1) \times \cdots \times \mathcal{F}(T_{x_k}U_k) \to \mathcal{F}(V),$$

where the second map arises from the inclusion

$$T_{x_1}U_1 \amalg \cdots \amalg T_{x_k}U_k \hookrightarrow V.$$

4.7.1.3 Definition. *A discretely translation-invariant prefactorization algebra $\mathcal{F}$ is* smoothly translation-invariant *if the following conditions hold.*

1. *The map $m_{x_1,\dots,x_k}$ above depends smoothly on $(x_1, \dots, x_k) \in W$.*
2. *The prefactorization algebra $\mathcal{F}$ is equipped with an action of the Abelian Lie algebra $\mathbb{R}^n$ of translations. If $v \in \mathbb{R}^n$, we will denote the corresponding action maps by*

$$\frac{\mathrm{d}}{\mathrm{d}v} : \mathcal{F}(U) \to \mathcal{F}(U).$$

We view this Lie algebra action as an infinitesimal version of the global translation invariance.

3. *The infinitesimal action is compatible with the global translation invariance in the following sense. If $v \in \mathbb{R}^n$, let $v_i \in (\mathbb{R}^n)^k$ denote the vector with v placed in the ith position and 0 in the other $k-1$ slots. If $\alpha_i \in \mathcal{F}(U_i)$, then we require that*

$$\frac{\mathrm{d}}{\mathrm{d}v_i} m_{x_1,\dots,x_k}(\alpha_1, \dots, \alpha_k) = m_{x_1,\dots,x_k}\left(\alpha_1, \dots, \frac{\mathrm{d}}{\mathrm{d}v}\alpha_i, \dots, \alpha_k\right).$$

When we refer to a translation-invariant prefactorization algebra without further qualification, we will always mean a smoothly translation-invariant prefactorization algebra.

Example: We have already seen that the prefactorization algebra of the free scalar field theory on $\mathbb{R}^n$ is discretely translation invariant, and is equipped with an action of the Abelian Lie algebra $\mathbb{R}^n$ by derivations. It is easy to verify that this prefactorization algebra is smoothly translation-invariant.

Example: Suppose that $\mathcal{F}$ is a locally-constant, smoothly translation invariant prefactorization algebra on $\mathbb{R}$, valued in vector spaces. Then, $A = \mathcal{F}((0,1))$ has the structure of an associative algebra.

For any two intervals $(0,1)$ and $(t, t+1)$, there is an isomorphism $\mathcal{F}((0,1)) \cong \mathcal{F}((t, t+1))$ coming from the isomorphism $\mathcal{F}((a,b)) \to \mathcal{F}(\mathbb{R})$ associated to inclusion of an interval into $\mathbb{R}$.

The fact that $\mathcal{F}$ is translation invariant means that there is *another* isomorphism $\mathcal{F}((0,1)) \to \mathcal{F}((t,t+1))$ for any $t \in \mathbb{R}$. Composing these two isomorphism yields an action of the group $\mathbb{R}$ on $A = \mathcal{F}((0,1))$. One can check (excercise!) that this is an action on $\mathbb{R}$ by associative algebras.

The fact that $\mathcal{F}$ is smoothly translation-invariant means that the action of $\mathbb{R}$ on A is smooth, and differentiates to an infinitesimal action of the Lie algebra $\mathbb{R}$ on A by derivations. The basis element $\frac{\partial}{\partial x}$ of $\mathbb{R}$ becomes a derivation H of A, called the *Hamiltonian*.

In the case that $\mathcal{F}$ is the cohomology of the prefactorization algebra of observables of the free scalar field theory on $\mathbb{R}$ with mass m, we have seen in section 4.3 that the algebra A is the Weyl algebra, generated by $p, q, \hbar$ with commutation relation $[p,q] = \hbar$. The Hamiltonian is given by

$$H(a) = \tfrac{1}{2\hbar}[p^2 - m^2 q^2, a].$$

Remark: As always, we work with prefactorization algebras taking values in the category of differentiable cochain complexes. Generalities about differentiable cochain complexes are developed in appendix B. There we explain what it means for a smooth multilinear map between differentiable cochain complexes to depend smoothly on some parameters.

4.7.2. Next, we will explain how to think of the structure of a translation-invariant prefactorization algebra on $\mathbb{R}^n$ in more operadic terms. This description has a lot in common with the E_n algebras familiar from topology.

Let $r_1, \dots, r_k, s \in \mathbb{R}_{>0}$. Let

$$\mathrm{Discs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{R}^n)^k$$

be the (possibly empty) open subset consisting of $x_1, \dots, x_k \in \mathbb{R}^n$ with the property that the closures of the balls $B_{r_i}(x_i)$ are all disjoint and contained in $B_s(0)$ (where $B_r(x)$ denotes the open ball of radius r around x).

4.7.2.1 Definition. *Let* Discs_n *be the* $\mathbb{R}_{>0}$*-colored operad in the category of smooth manifolds whose space of k-ary morphisms is the space* $\mathrm{Discs}_n(r_1, \dots, r_k \mid s)$ *between* $r_i, s \in \mathbb{R}_{>0}$ *described above.*

Note that a colored operad is the same thing as a multicategory (recall remark 3.1.2). An $\mathbb{R}_{>0}$-colored operad is thus a multicategory whose set of objects is $\mathbb{R}_{>0}$.

The essential data of the colored operad structure on the spaces $\mathrm{Discs}_n(r_1, \dots, r_k \mid s)$ is the following. We have maps

$$\begin{aligned} \circ_i : \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{Discs}_n(t_1, \dots, t_m \mid s) \\ \to \mathrm{Discs}_n(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s). \end{aligned}$$

This map is defined by inserting the outgoing ball (of radius t_i) of a configuration $x \in \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i)$ into the ith incoming ball of a point $y \in \mathrm{Discs}_n(t_1, \dots, t_k \mid s)$.

These maps satisfy the natural associativity and commutativity properties of a multicategory.

4.7.3. Next, let $\mathcal{F}$ be a translation-invariant prefactorization algebra on $\mathbb{R}^n$. Let

$$\mathcal{F}_r = \mathcal{F}(B_r(0))$$

denote the cochain complex $\mathcal{F}$ that assigns to a ball of radius r. This notation is reasonable because translation invariance gives us an isomorphism between $\mathcal{F}(B_r(0))$ and $\mathcal{F}(B_r(x))$ for any $x \in \mathbb{R}^n$.

The structure maps for a translation-invariant prefactorization algebra yield, for each $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$, multiplication operations

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

The map $m[p]$ is a smooth multilinear map of differentiable spaces; and furthermore, this map depends smoothly on p.

These operations make the complexes $\mathcal{F}_r$ into an algebra over the $\mathbb{R}_{>0}$-colored operad $\mathrm{Discs}_n(r_1, \dots, r_k \mid s)$, valued in the multicategory of differentiable cochain complexes. In addition, the complexes $\mathcal{F}_r$ are endowed with an action of the Abelian Lie algebra $\mathbb{R}^n$. This action is by derivations of the Discs_n-algebra $\mathcal{F}$ compatible with the action of translation on Discs_n, as described above.

4.7.4. Now, let us unravel explicitly what it means to be such a Discs_n algebra.

The first property is that, for each $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$, the map $m[p]$ is a multilinear map, of cohomological degree 0, compatible with differentials.

Second, let N be a manifold and let $f_i : N \to \mathcal{F}_{r_i}^{d_i}$ be smooth maps into the space $\mathcal{F}_{r_i}^{d_i}$ of elements of degree d_i. The smoothness properties of the map $m[p]$ mean that the map

$$\begin{aligned} N \times \mathrm{Discs}_n(r_1, \dots, r_k \mid s) &\to \mathcal{F}_s \\ (x, p) &\mapsto m[p](f_1(x), \dots, f_k(x)) \end{aligned}$$

is smooth.

Next, note that a permutation $\sigma \in S_k$ gives an isomorphism

$$\sigma : \mathrm{Discs}_n(r_1, \dots, r_k \mid s) \to \mathrm{Discs}_n(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s).$$

We require that, for each $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$ and each $\alpha_i \in \mathcal{F}_{r_i}$,

$$m[\sigma(p)](\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}) = m[p](\alpha_1, \dots, \alpha_k).$$

Finally, we require that the maps $m[p]$ are compatible with composition, in the following sense. For $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i)$, $q \in \mathrm{Discs}_n(t_1, \dots, t_l \mid s)$, $\alpha_i \in F_{r_i}$, and $\beta_j \in \mathcal{F}_{t_j}$, we require that

$$m[q]\,(\beta_1, \dots, \beta_{i-1}, m[p](\alpha_1, \dots, \alpha_k), \beta_{i+1}, \dots, \beta_l) \\ = m[q \circ_i p](\beta_1, \dots, \beta_{i-1}, \alpha_1, \dots, \alpha_k, \beta_{i+1}, \dots, \beta_l).$$

In addition, the action of $\mathbb{R}^n$ on each $\mathcal{F}^r$ is compatible with these multiplication maps, in the way described above.

4.7.5. Let us give one more equivalent way of rewriting these axioms, which will be useful when we discuss the holomorphic context. These alternative axioms will say that the spaces $C^\infty(\mathrm{Discs}_n(r_1, \dots, r_k \mid s))$ form an $\mathbb{R}_{>0}$-colored co-operad when we use the appropriate completed tensor product (we will use the completed tensor product on the category of convenient vector spaces). Since we know how to tensor a differentiable vector space with the space of smooth functions on a manifold, it makes sense to talk about an algebra over this colored co-operad in the category of differentiable cochain complexes.

The smoothness axiom for the product map

$$m[p] : \mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_k} \to \mathcal{F}_s,$$

where $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$, can be rephrased as follows. For any differentiable vector space V and smooth manifold M, we use the notation $V \otimes C^\infty(M)$ interchangeably with the notation $C^\infty(M, V)$; both indicate the differentiable vector space of smooth maps $M \to V$. The smoothness axiom states that the map above extends to a smooth map of differentiable spaces

$$\mu(r_1, \dots, r_k \mid s) : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s \otimes C^\infty(\mathrm{Discs}_n(r_1, \dots, r_k \mid s)).$$

In general, if $V_1, \dots, V_k, W$ are differentiable vector spaces and if X is a smooth manifold, let

$$C^\infty(X, \mathrm{Hom}(V_1, \dots, V_k \mid W))$$

denote the space of smooth multilinear maps

$$V_1 \times \cdots \times V_k \to C^\infty(X, W).$$

Note that there is a natural gluing map

$$\circ_i : C^\infty(X, \mathrm{Hom}(V_1, \dots, V_k \mid W_i)) \times C^\infty(Y, \mathrm{Hom}(W_1, \dots, W_l \mid T)) \\ \to C^\infty(XxY, \mathrm{Hom}(W_1, \dots, W_{i-1}, V_1, \dots, V_k, W_{i+1}, \dots, W_l \mid T)).$$

With this notation in hand, there are elements

$$\mu(r_1, \dots, r_k \mid s) \in C^\infty\left(\mathrm{Discs}(r_1, \dots r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)\right)$$

with the following properties.

(1) $\mu(r_1, \dots, r_k \mid s)$ is closed under the natural differential, arising from the differentials on the cochain complexes $\mathcal{F}_{r_i}$.
(2) If $\sigma \in S_k$, then

$$\sigma_* \mu(r_1, \dots, r_k \mid s) = \mu(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s)$$

where

$$\sigma_* : C^\infty(\mathrm{Discs}(r_1, \dots r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)) \\ \to C^\infty(\mathrm{Discs}(r_{\sigma(1)}, \dots r_{\sigma(k)} \mid s), \mathrm{Hom}(\mathcal{F}_{r_{\sigma(1)}}, \dots, \mathcal{F}_{r_{\sigma(k)}} \mid \mathcal{F}_s))$$

is the natural isomorphism.
(3) As before, let

$$\circ_i : \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{Discs}_n(t_1, \dots, t_m \mid s) \\ \to \mathrm{Discs}_n(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s).$$

denote the gluing map. Then, we require that

$$\circ_i^* \mu(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_l) = \mu(r_1, \dots, r_k \mid t_i) \circ_i \mu(t_1, \dots, t_l \mid s).$$

These elements equip the $\mathcal{F}_r$ with the structure of an algebra over the colored co-operad, as stated earlier.

4.7.6. Let us write down explicit formula for these product maps in the case of the free massless scalar field theory. Let G be the Green's function for the Laplacian on $\mathbb{R}^n$. Thus, $G = \left(4\pi^{d/2}\right)^{-1} \Gamma(d/2 - 1) \left|x - y\right|^{2-n}$ for $n \neq 2$, and $G = -(2\pi)^{-1} \log |x - y|$ if $n = 2$.

We have seen in section 4.5 that the choice of a Green's function G leads to an isomorphism of cochain complexes $W : \mathrm{Obs}^{cl}(U)[\hbar] \to \mathrm{Obs}^q(U)[\hbar]$, for every open subset U. This allows us to transfer the product in the prefactorization algebra $\mathrm{Obs}^q(U)$ to a deformed product $\star_\hbar$ in the prefactorization algebra $\mathrm{Obs}^{cl}(U)[\hbar]$, defined by

$$\alpha \star_\hbar \beta = W^{-1}(W(\alpha) \cdot W(\beta))$$

where on the right hand side $\cdot$ indicates the product in the prefactorization algebra Obs^q.

This leads to a completely explicit description of the product maps

$$\mu^s_{r_1, \dots, r_k} : \mathcal{F}_{r_1} \otimes \dots \otimes \mathcal{F}_{r_k} \to C^\infty(P(r_1, \dots, r_k \mid s), \mathcal{F}_s)$$

discussed above, in the case that $\mathcal{F}$ arises from the prefactorization algebra of quantum obseravbles of a free scalar field theory, or equivalently from the prefactorization algebra $(\mathrm{Obs}^{cl}[\hbar], \star_\hbar)$.

For example, on $\mathbb{R}^2$, let α_1, α_2 be compactly supported smooth functions on discs $D(0, r_i)$ of radii r_i around 0. Let us view each α as a a cohomological degree 0 element of

$$\mathcal{F}_{r_i} = \mathrm{Obs}^{cl}(D(0, r_i))[\hbar] = \mathrm{Sym}^*(C_c^\infty(D(0, r_i))[1] \oplus C_c^\infty(D(0, r_i)))[\hbar].$$

Let $T_x \alpha_i$ denote the translate of α_i to an element of $C_c^\infty(D(x, r_i))$.

Then, for $x_1, x_2 \in \mathbb{R}^2$ which are such that $D(x_i, r_i)$ are disjoint and contained in $D(0, s)$, we have

$$\begin{aligned} \mu^s_{r_1, r_2}(\alpha_1, \alpha_2) &= T_{x_1}\alpha_1 \star_\hbar T_{x_2}\alpha_2 \\ &= (T_{x_1}\alpha_1) \cdot (T_{x_2}\alpha_2) - \hbar \int_{u_1, u_2 \in \mathbb{R}^2} \alpha_1(u_1 + x_1)\alpha_2(u_2 + x_2) \log |u_1 - u_2| \, . \end{aligned}$$

4.7.7. Vacua.

4.7.7.1 Definition. *Let $\mathcal{F}$ be a smoothly translation-invariant prefactorization algebra on $\mathbb{R}^n$, over a ring R (which in practice is $\mathbb{R}, \mathbb{C}$ or $\mathbb{R}[[\hbar]], \mathbb{C}[[\hbar]]$).*

A state *for $\mathcal{F}$ is a smooth linear map*

$$\langle - \rangle : H^0(\mathcal{F}(\mathbb{R}^n)) \to \mathbb{R}$$

(or to $\mathbb{C}$ if we work with complex coefficients).

The smooth means the following. Because $\mathcal{F}(\mathbb{R}^n)$ is a differentiable vector space, it is a sheaf on the site of smooth manifolds, so that $H^*(\mathcal{F}(\mathbb{R}^n))$ is a presheaf on the site of smooth manifolds: that is, if M is a smooth manifold, and $\alpha \in H^*(C^\infty(M, \mathcal{F}(\mathbb{R}^n))$, then we require that $\langle \alpha \rangle$ is a smooth map from M to R.

A state $\langle - \rangle$ is translation-invariant if it commutes with the action of both the group $\mathbb{R}^n$ and of the infinitesimal action of the Lie algebra $\mathbb{R}^n$, where $\mathbb{R}^n$ acts trivially on $\mathbb{R}$

A state $\langle - \rangle$ allows us to define correlation functions of observables. If $O_i \in \mathcal{F}(D(0, r_i))$ are cohomologically closed observables, then we can construct $T_{x_i} O_i \in \mathcal{F}(D(x_i, r_i)$. If x_i are such that the discs $D(x_i, r_i)$ are disjoint – so that $(x_1, \dots, x_k) \in P(r_1, \dots, r_k \mid \infty)$– then we can define the correlation function

$$\langle O_1(x_1), \dots, O_n(x_n) \rangle \in R.$$

by applying $\langle - \rangle$ to the cohomology class of the product observable $T_{x_1} O_1 \dots T_{x_n} O_n \in \mathcal{F}(\mathbb{R}^n)$. The fact that we have a smoothly translation invariant prefactorization algebra, and that the vacuum is assumed to be a smooth map, implies that $\langle O_1(x_1), \dots, O_n(x_n) \rangle$ is a smooth function of the x_i. Further, this function is invariant under simultaneous translation of all the points x_i.

4.7.7.2 Definition. *A translation-invariant state* $\langle - \rangle$ *is a* vacuum *if it satisfies the* cluster decomposition principle, *which states that, in the situation above, for all cohomology classes* O_1, O_2 *where* $O_i \in H^*(\mathcal{F}(D(0,r_i)))$ *we have*

$$\langle O_1(0), O_2(x) \rangle - \langle O_1 \rangle \langle O_2 \rangle \to 0 \text{ as } x \to \infty.$$

A vacuum is massive *if* $\langle O_1(0), O_2(x) \rangle - \langle O_1 \rangle \langle O_2 \rangle$ *tends to zero exponentially fast.*

Example: Consider the free scalar field theory on $\mathbb{R}^n$ with mass m. We have seen that the choice of a Green's function G for the operator $\triangle + m^2$ leads to, for every open subsets $U \subset \mathbb{R}^n$, an isomorphism of cochain complexes $\mathrm{Obs}^{cl}(U)[\hbar] \cong \mathrm{Obs}^q(U)$; this becomes an isomorphism of prefactorization algebras if we endow $\mathrm{Obs}^{cl}[\hbar]$ with a deformed prefactorization product $\star_\hbar$ defined using the Green's function.

If $m > 0$, there is a unique Green's function G of the form $G = f(x-y)$ where f is a distribution on $\mathbb{R}^n$ which is smooth away from the origin, and which tends to zero exponentially fast at infinity. For example, if $n = 1$, the function f is

$$f(x) = \frac{1}{2m} e^{-m|x|}.$$

If $m = 0$, there is a canonically-defined Green's function which, if $n \neq 2$, is $G(x,y) = \frac{1}{4\pi^{d/2}} \Gamma(d/2 - 1) |x-y|^{2-n}$ and is $-(2\pi)^{-1} \log |x-y|$ if $n = 2$.

Because $\mathrm{Obs}^{cl}(\mathbb{R}^n)$ is, as a cochain complex, the symmetric algebra on the complex $C_c^\infty(\mathbb{R}^n)[1] \xrightarrow{\triangle + m^2} C_c^\infty(\mathbb{R}^n)$, there is a map from $\mathrm{Obs}^{cl}(\mathbb{R}^n)$ to $\mathbb{R}$ which is the identity on Sym^0 and sends $\mathrm{Sym}^{>0}$ to 0. This map extends to an $\mathbb{R}[\hbar]$-linear cochain map

$$\langle - \rangle : \mathrm{Obs}^{cl}(\mathbb{R}^n)[\hbar] \to \mathbb{R}[\hbar].$$

Clearly, this map is translation invariant and smooth. Thus, because we have a cochain isomorphism between $\mathrm{Obs}^{cl}(\mathbb{R}^n)[\hbar]$ and $\mathrm{Obs}^q(\mathbb{R}^n)$, we have produced a translation-invariant state.

4.7.7.3 Lemma. *If* $m > 0$, *this state is a massive vacuum. If* $m = 0$, *this state is a vacuum if* $n > 2$, *otherwise it does not satisfy the cluster decomposition principle.*

PROOF. Let $F_1 \in C_c^\infty(D(0,r_1))^{\otimes k_1}$ and $F_2 \in C_c^\infty(D(0,r_2))^{\otimes k_2}$. We will view F_1, F_2 as observables in $\mathrm{Obs}^{cl}(D(0,r_i))$ by using the natural map from $C_c^\infty(U)^{\otimes k}$ to the coinvariants $\mathrm{Sym}^k C_c^\infty(U)$. Let $T_c F_2$ be the translation of F_2 by c, where c is sufficiently large so that the discs $D(0,r_1)$ and $D(c,r_2)$ are disjoint. Explicitly, $T_c F_2$ is represented by the function $F_2(y_1 - c, \ldots, y_{k_2} - c)$. We are interested in computing the expectation value $\langle F_1, T_c F_2 \rangle$.

We gave, above, an explicit formula for the quantum prefactorization product $\star_\hbar$ on $\mathrm{Obs}^{cl}[\hbar]$. In this case, it leads to the formula

$$F_1 \star_\hbar T_c F_2 = \sum_{r=0}^{\min(k_1,k_2)} \hbar^r \sum_{\substack{1\leq i_1<\dots<i_r\leq k_1\\ 1\leq j_1<\dots<j_r\leq k_2}} \int_{\substack{x_{i_1},\dots,x_{i_r}\in\mathbb{R}^n\\ y_{i_1},\dots,y_{i_r}\in\mathbb{R}^n}}$$
$$F_1(x_1,\dots,x_{k_1})F_2(y_1-c,\dots,y_{k_2}-c)G(x_{i_1},y_{j_1})\dots G(x_{i_r},y_{j_r})\mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_r}\mathrm{d}y_{i_1}\dots\mathrm{d}y_{i_r}.$$

Note that after performing the integral, we are left with a function of the k_1+k_2-2r copies of $\mathbb{R}^n$ we have not integrated over. This function is then viewed as an observable in $\mathrm{Sym}^{k_1+k_2-2r} C_c^\infty(\mathbb{R}^n)$.

If $k_1,k_2>0$, then $\langle F_1\rangle = 0$ and $\langle F_2\rangle = 0$. Further, $\langle F_1, T_c F_2\rangle$ selects the constant term in the expression for $F_1 \star_\hbar T_c F_2$. There is only a non-zero constant term if $k_1=k_2=k$. In that case, the constant term is (up to a combinatorial factor)

$$\langle F_1, T_c F_2\rangle = \hbar^k \int_{x_i,y_i\in\mathbb{R}^n} F_1(x_1,\dots,x_k)F_2(y_1-c,\dots,y_k-c)G(x_1,y_1)\dots G(x_k,y_k)\in\mathbb{R}[\hbar].$$

To check whether the cluster decomposition principle holds, we need to check whether or not

$$\langle F_1, T_c F_2\rangle = \langle F_1, T_c F_2\rangle - \langle F_1\rangle\langle F_2\rangle$$

tends to zero as $c\to\infty$. If $m>0$, we know that $G(x,y)$ tends to zero exponentially fast as $x-y\to\infty$. This implies immediately that we have a massive vacuum in this case.

If $m=0$, the Green's function $G(x,y)$ tends to zero like the inverse of a polynomial as long as $n>2$. In this case, we have a vacuum. If $n=1$ or $n=2$, then $G(x,y)$ does not tend to zero, so we don't have a vacuum.

□

Example: We can give a more abstract construction for the vacuum of associated to a massive scalar field theory on $\mathbb{R}^n$. Let $\mathscr{E}_c = C_c^\infty \xrightarrow{\triangle+m^2} C_c^\infty$ be, as before, the complex of fields. Let $\widehat{\mathscr{E}_c}$ be the Heisenberg central extension $\widehat{\mathscr{E}_c} = \mathscr{E}_c\oplus\mathbb{R}\cdot\hbar$ where $\hbar$ has degree 1. We defined the complex of observables on U as the Chevalley-Eilenberg chain complex of $\widehat{\mathscr{E}_c}(U)$.

Let $\mathscr{E}_{\mathcal{S}}(\mathbb{R}^n)$ be the complex $\mathcal{S}(\mathbb{R}^n)\xrightarrow{\triangle+m^2}\mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions on $\mathbb{R}^n$. (Recall that a smooth function is Schwartz if it and all its derivatives tends to zero at ∞ faster than the reciprocal of any polynomial).

The Heisenberg dg Lie algebra can be defined using $\mathscr{E}_{\mathcal{S}}(\mathbb{R}^n)$ instead of $\mathscr{E}_c(\mathbb{R}^n)$. We let $\widehat{\mathscr{E}_{\mathcal{S}}}(\mathbb{R}^n)$ be $\mathscr{E}_{\mathcal{S}}(\mathbb{R}^n)\oplus\mathbb{R}\cdot\hbar[-1]$, with bracket defined by $[\phi^0,\phi^1]=\hbar\int\phi^0\phi^1$. Here ϕ^i are

Schwartz functions of degrees 0 and 1. This makes sense, because the product of any two Schwartz function is Schwartz and Schwartz functions are integrable.

Schwartz functions have a natural topology, so we will view them as being a convenient vector space. Since the topology is nuclear Fréchet, a result discussed in appendix C tells us that the tensor product in the category of convenient vector spaces coincides with that in the category of nuclear Fréchet spaces. A result of Grothendieck [Gro52] tells us that

$$\mathcal{S}(\mathbb{R}^n)\widehat{\otimes}\mathcal{S}(\mathbb{R}^m) = \mathcal{S}(\mathbb{R}^{n+m}),$$

and similarly for Schwartz sections of vector bundles.

This allows us to define the Chevalley chain complex

$$\mathrm{Obs}^q_{\mathcal{S}}(\mathbb{R}^n) \overset{\text{def}}{=} C_*(\widehat{\mathscr{E}}_{\mathcal{S}}(\mathbb{R}^n))$$

of the Heisenberg algebra based on Schwartz functions; as usual, we use the completed tensor product on the category of convenient vector spaces when defining the symmetric algebra.

There's a map

$$\mathrm{Obs}^q(\mathbb{R}^n) \to \mathrm{Obs}^q_{\mathcal{S}}(\mathbb{R}^n)$$

given by viewing a compactly supported function as a Schwartz function.

Now,

4.7.7.4 Lemma. *The cohomology of* $\mathrm{Obs}^q_{\mathcal{S}}(\mathbb{R}^n)$ *is* $\mathbb{R}[\hbar]$.

PROOF. The complex

$$\mathscr{E}_{\mathcal{S}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) \xrightarrow{\triangle+m^2} \mathcal{S}(\mathbb{R}^n)$$

has no cohomology. We can see this using Fourier duality: the Fourier transform is an isomorphism on the space of Schwartz functions, and the Fourier dual of the operator $\triangle + m^2$ is the operator $p^2 + m^2$, where $p^2 = \sum p_i^2$ and p_i are coordinates on the Fourier dual $\mathbb{R}^n$. (Note that our convention is that the Laplacian is $-\sum \frac{\partial}{\partial x_i}^2$). The operator of multiplication by $p^2 + m^2$ is invertible on the space of Schwartz functions, just because if f is a Schwartz function then so is $(p^2 + m^2)^{-1} f$. (At this point we need to use our assumption that $m \neq 0$). The inverse is a smooth linear map, so that this complex is smoothly homotopy equivalent to the zero complex.

This implies immediately that the Chevalley chain complex of $\widehat{\mathscr{E}}_{\mathcal{S}}(\mathbb{R}^n)$ has cohomology the same as that of the Abelian Lie algebra $\mathbb{R} \cdot \hbar$, i.e. $\mathbb{R}[\hbar]$ as desired. □

Thus, we have a translation-invariant state

$$H^*(\mathrm{Obs}^q(\mathbb{R}^n)) \to H^*(\mathrm{Obs}^q_{\mathcal{S}}(\mathbb{R}^n)) = \mathbb{R}[\hbar].$$

This state is characterized uniquely by the fact that it is defined on Schwartz observables. Since the state constructed more explicitly above also has this property, we see that these two states coincide, so that this state is a massive vacuum.

CHAPTER 5

Holomorphic field theories and vertex algebras

This chapter serves two purposes. On the one hand, we develop several examples that exhibit how to understand the observables of a two-dimensional theory from the point of view of factorization algebras and how this approach recovers standard examples of vertex algebras. On the other hand, we provide a precise definition of a factorization algebra on $\mathbb{C}^n$ whose structure maps vary holomorphically, much as we defined translation-invariant factorization algebras in section 4.7. We then give a proof that when $n = 1$ and the factorization algebra possesses a $U(1)$-action, we can extract a vertex operator algebra. For $n > 1$, the structure we find is a higher-dimensional analog of a vertex algebra. Such higher-dimensional vertex algebras appear, for example, as the factorization algebra of observables of partial twists of supersymmetric gauge theories.

5.0.8. Reminder on vertex algebras. In mathematics, the notion of a vertex algebras is a standard formalization of the observables of a chiral conformal field theory (a theory on the complex plane $\mathbb{C}$). Before embarking on our own approach, we recall the definition of a vertex algebra and various properties as given in [FBZ04].

5.0.8.1 Definition. *Let V be a vector space. An element $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}$ in* End $V[[z, z^{-1}]]$ *is a* field *if, for each $v \in V$, there is some N such that $a_j v = 0$ for all $j > N$.*

Remark: The usage of the term "field" in the theory of vertex operators often provokes confusion. In this book, the term *field* is used to refer to a configuration in a classical field theory: for example, in a scalar field theory on a manifold M, a field is an element of $C^\infty(M)$. The term "field" as used in the theory of vertex algebras is *not* related to this usage.

5.0.8.2 Definition (Definition 1.3.1, [FBZ04]). *A* vertex algebra *is the following data:*

- *a vector space V over $\mathbb{C}$ (the* state space*);*
- *a nonzero vector $|0\rangle \in V$ (the* vacuum vector*);*
- *a linear map $T : V \to V$ (the* shift operator*);*
- *a linear map $Y(-, z) : V \to$* End $V[[z, z^{-1}]]$ *sending every vector v to a field (the* vertex operation*);*

subject to the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = \mathbb{1}_V$ *and* $Y(v,z)|0\rangle \in v + zV[[z]]$ *for all* $v \in V$;
- (translation axiom) $[T, Y(v,z)] = \partial_z Y(v,z)$ *for every* $v \in V$ *and* $T|0\rangle = 0$;
- (locality axiom) *for any pair of vectors* $v, v' \in V$*, there exists a nonnegative integer* N *such that* $(z-w)^N [Y(v,z), Y(v',w)] = 0$ *as an element of* $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$.

The vertex operation is best understood in terms of the following intuition. The vector space V represents the set of pointwise measurements one can make of the fields, and one should imagine labeling each point $z \in \mathbb{C}$ by a copy of V, which we'll denote V_z. Moreover, in a disk D containing the point z, the measurements at z are a dense subspace of the measurements one can make in D. We'll denote the observables on D by V_D. The vertex operation is a way of combining pointwise measurements. Let D be a disk centered on the origin. For $z \neq 0$, we can multiply observables to get a map

$$Y_z : V_0 \otimes V_z \to V_D,$$

and it should vary holomorphically in $z \in D \setminus \{0\}$. In other words, we should get something with properties like the formal definition $Y(-,z)$ above. (This picture clearly resembles the "pair of pants" product from two-dimensional topological field theories.)

An appealing aspect of our approach to observables is that this intuition becomes explicit and rigorous. Our procedure describes the observables on every disk and gives the structure maps in a coordinate-free way. By choosing a coordinate z on $\mathbb{C}$, we recover the usual formulas for vertex algebras.

This is seen in the main theorem in this chapter, which we will now state. The theorem connects a certain class of factorization algebras on $\mathbb{C}$ with vertex algebras. The factorization algebras of interest are holomorphically translation invariant factorization algebra. We will give a precise definition of what this means shortly; but here is a rough definition. Let $\mathcal{F}_r$ denote the cochain complex $\mathcal{F}(D(0,r))$ that $\mathcal{F}$ assigns to a disc of radius r. Recall, as we explained in section 4.7 of chapter 4, if $\mathcal{F}$ is smoothly translation invariant then we have a operator product map

$$\mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_n} \to C^\infty(\mathrm{Discs}(r_1, \dots, r_k \mid s), \mathcal{F}_s)$$

where $\mathrm{Discs}(r_1, \dots, r_k \mid s)$ refers to the open subset of $\mathbb{C}^k$ consisting of points $z_1, \dots, z_k$ such that the discs of radius r_i around z_i are all disjoint and contained in the disc of radius s around the origin. If $\mathcal{F}$ is holomorphically translation invariant, then this lifts to a cochain map

$$\mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_n} \to \Omega^{0,*}(\mathrm{Discs}(r_1, \dots, r_k \mid s), \mathcal{F}_s)$$

where on the right hand side we have used the Dolbeault complex of the complex manifold $\mathrm{Discs}(r_1, \dots, r_k \mid s)$. We also require some compatibility of these lifts with compositions, which we will detail later.

In other words, being holomorphically translation invariant means that the operator product map is holomorphic (up to homotopy) in the location of the discs.

5.0.8.3 Theorem. *Let us suppose that $\mathcal{F}$ is a holomorphically translation invariant factorization algebra on $\mathbb{C}$. Let us suppose that $\mathcal{F}$ is also invariant under the action of S^1 on $\mathbb{C}$ by rotation. Let $\mathcal{F}_r^k$ denote the weight k eigenspace of the S^1 action on the complex $\mathcal{F}_r$. Let us assume that the maps $\mathcal{F}_r^k \to \mathcal{F}_s^k$ (for $r < s$) associated to the inclusion $D(0,r) \subset D(0,s)$ are quasi-isomorphisms.*

Then, the space

$$V = \oplus_k H^*(\mathcal{F}_r^k)$$

has the structure of a vertex algebra. The vertex algebra structure map

$$V \otimes V \to V[[z, z^{-1}]]$$

is the Laurent expansion of operator product map

$$H^*(\mathcal{F}_{r_1}^{k_1}) \otimes H^*(\mathcal{F}_{r_2}^{k_2}) \to \mathrm{Hol}(\mathrm{Discs}(r_1, r_2 \mid s), H^*(\mathcal{F}_s)).$$

On the right hand side, Hol *denotes the space of holomorphic maps.*

In other words, the intuition that the vertex algebra structure map is the operator product expansion is made precise in our formalism.

This result should be compared with the classic result of Huang [], who relates chiral conformal field theories at genus 0 in the sense used by Segal with vertex algebras. As we have seen in section 3.3 of chapter 3.3, the axioms for factorization algebras are very closely related to Segal's axioms. Our axioms for a holomorphically translation invariant field theory is similarly related to Segal's axioms for a two-dimensional chiral field theory. Although our result is closely related to Huang's, it is a little different because of the technical differences between a factorization algebra and a Segal-style chiral conformal field theory.

One nice feature of our definition of holomorphically translation-invariant factorization algebra is that it makes sense in any complex dimension. The structure present on the cohomology of a higher-dimensional holomorphically translation-invariant factorization algebra is a higher dimensional version of the axioms of a vertex algebra. This structure was discussed briefly in [**?**].

Another nice feature of our approach is that our general construction of field theories allows one to construct vertex algebras by perturbation theory, starting with a Lagrangian. This should lead to the construction of many interesting vertex algebras.

5.0.9. Before we turn to a more detailed discussion of the main theorems in this chapter, we will recollect some further properties of vertex algebras.

Although vertex algebras are not (typically) associative algebras, they possess an important "associativity" property:

5.0.9.1 Proposition. *Let V be a vertex algebra. For any $v_1, v_2, v_3 \in V$, we have the following equality in $V((w))((z-w))$:*

$$Y(v_1, z)Y(v_2, w)v_3 = Y(Y(v_1, z-w)v_2, w)v_3.$$

Finally, there is a powerful "reconstruction" theorem that provides simple criteria to uniquely construct a vertex algebra given "generators and relations." We will exploit this theorem to verify that we have indeed recovered the standard vertex algebras in our examples.

5.0.9.2 Theorem (Reconstruction, Theorem 4.4.1, [FBZ04]). *Let V be a complex vector space equipped with a nonzero vector $|0\rangle$, an endomorphism T, a countable ordered set $\{a^\alpha\}_{\alpha \in S}$ of vectors, and fields*

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]$$

such that

(1) *for all α, $a^\alpha(z)|0\rangle = a^\alpha + O(z)$;*
(2) *$T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all α;*
(3) *all fields $a^\alpha(z)$ are mutually local;*
(4) *V is spanned by the vectors*

$$a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle$$

with the $j_i < 0$.

Then, using the formula

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle, z) := \frac{1}{(-j_1-1)! \cdots (-j_m-1)!} : \partial_z^{-j_1-1} a^{\alpha_1}(z) \cdots \partial_z^{-j_m-1} a^{\alpha_m}(z) :$$

to define a vertex operation, we obtain a well-defined and unique vertex algebra $(V, |0\rangle, T, Y)$ satisfying conditions (1)-(4) and $Y(a^\alpha, z) = a^\alpha(z)$.

Here $: a(z)b(w) :$ denotes the *normally ordered product* of fields, defined as

$$: a(z)b(w) := a(z)_+ b(w) + b(w)a(z)_-$$

where

$$a(z)_+ := \sum_{n \geq 0} a_n z^n \text{ and } a(z)_- := \sum_{n<0} a_n z^n.$$

Normal ordering eliminates various "divergences" that appear in naively taking products of fields.

5.0.10. Organization of this chapter. We will start this chapter by stating and proving our main theorem. The first thing we define, in section 5.1, is the notion of a holomorphically translation invariant factorization algebra on $\mathbb{C}^n$ for every $n \geq 1$. Holomorphic translation invariance guarantees that the operator products are all holomorphic. We will then show to construct from such an object in dimension 1 a vertex algebra.

The rest of this chapter is devoted to analyzing examples. In section 5.3, we discuss the factorization algebra associated to a very simple two-dimensional chiral conformal field theory: the free $\beta\gamma$ system. We will show that the vertex algebra associated to the factorization algebra of observables of this theory is an object called the $\beta\gamma$ vertex algebra in the literature. Then, in section 5.4, we will construct a factorization algebra encoding the affine Kac-Moody algebra. This factorization algebra again encodes a vertex algebra, which is the standard Kac-Moody vertex algebra.

5.1. Holomorphically translation-invariant factorization algebras

In this section we will analyze in detail the notion of translation-invariant prefactorization algebras on $\mathbb{C}^n$. On $\mathbb{C}^n$ we can ask for a translation-invariant prefactorization algebra to have a *holomorphic* structure; this implies that all structure maps of the prefactorization algebra are (in a sense we will explain shortly) holomorphic. There are many natural field theories where the corresponding prefactorization algebra is holomorphic: for instance, chiral conformal field theories in complex dimension 1, and minimal twists [**Cos11c**] of supersymmetric field theories in complex dimension 2.

5.1.1. We now explain what it means for a (smoothly) translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ to be *holomorphically translation-invariant*. For this definition to make sense, we require that $\mathcal{F}$ is defined over $\mathbb{C}$: that is, the vector spaces $\mathcal{F}(U)$ are complex vector spaces and all structure maps are complex linear.

Recall that such a factorization algebra has, as part of its structure, an action of the real Lie algebra $\mathbb{R}^{2n} = \mathbb{C}^n$ by derivations. This action is as a real Lie algebra; since $\mathcal{F}$ is defined over $\mathbb{C}$, the action extends to an action of the complexified translation Lie algebra $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. We will denote the action maps by

$$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j} : \mathcal{F}(U) \to \mathcal{F}(U).$$

5.1.1.1 Definition. *A translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ is* holomorphically translation-invariant *if it is equipped with derivations $\eta_i : \mathcal{F} \to \mathcal{F}$ of cohomological degree -1, for $i = 1 \dots n$, with the property that*

$$\mathrm{d}\eta_i = \frac{\partial}{\partial \overline{z}_i} \in \mathrm{Der}(\mathcal{F}).$$

Here, d *refers to the differential on the dg Lie algebra* $\operatorname{Der}(\mathcal{F})$*.*

We should understand this definition as saying that the vector fields $\frac{\partial}{\partial \overline{z}_i}$ act homotopically trivially on $\mathcal{F}$. In physics terminology, the operators $\frac{\partial}{\partial x_i}$ (where x_i are real coordinates on $\mathbb{C}^n = \mathbb{R}^{2n}$) are related to the energy-momentum tensor. We are asking that the components of the energy momentum tensor in the $\overline{z}_i$ directions are exact for the differential on observables (in physics, this might be called "BRST exact"). This is rather similar to a phenomenon that sometimes appears in the study of topological field theory [Wit88], where a topological theory depends on a metric, but the variation of the metric is exact for the BRST differential.

5.1.2. Now we will interpret holomorphically translation-invariant prefactorization algebras in the language of $\mathbb{R}_{>0}$-colored operads. When we work in complex geometry, it is better to use polydiscs instead of balls, as is standard in complex analysis.

Thus, if $z \in \mathbb{C}^n$, let

$$PD_r(z) = \{w \in \mathbb{C}^n \mid |w_i - z_i| < r\}$$

be the polydisc of radius r around z. Let

$$\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s) \subset (\mathbb{C}^n)^k$$

be the space of $z_1, \ldots, z_k \in \mathbb{C}^n$ with the property that the closures of the polydiscs $PD_{r_i}(z_i)$ are disjoint and contained in the polydisc $PD_s(0)$.

It is clear that the spaces $\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)$ form a $\mathbb{R}_{>0}$-colored operad in the category of complex manifolds.

Now, let $\mathcal{F}$ be a holomorphically translation-invariant prefactorization algebra on $\mathbb{C}^n$. Let $\mathcal{F}_r$ denote the differentiable cochain complex $\mathcal{F}(PD_r(0))$ associated to the polydisc of radius r.

Then, as above, for each $p \in \mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)$ we have a map

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

This map is smooth, multilinear, and compatible with the differential. Further, this map varies smoothly with p.

The fact that $\mathcal{F}$ is a holomorphically translation-invariant prefactorization algebra means that these maps are equipped with extra structure. We have derivations η_j of $\mathcal{F}$ which make the derivations $\frac{\partial}{\partial \overline{z}_j}$ homotopically trivial.

For $i = 1, \dots, k$ and $j = 1, \dots, n$, let $z_{ij}, \overline{z}_{ij}$ refer to coordinates on $(\mathbb{C}^n)^k$, and so on the open subset

$$\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{C}^n)^k.$$

Thus, we have operations

$$\frac{\partial}{\partial \overline{z}_{ij}} m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

obtained by differentiating the operation $m[p]$, which depends smoothly on p, in the direction $\overline{z}_{ij}$.

Let $m[p] \circ_i \eta_j$ denote the operation

$$\begin{aligned} m[p] \circ_i \eta_j : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} &\to \mathcal{F}_s \\ \alpha_1 \times \cdots \times \alpha_k &\mapsto \pm m[p](\alpha_1, \dots, \eta_j \alpha_i, \dots, \alpha_k) \end{aligned}$$

(where ± indicates the usual Koszul rule of signs).

The axioms of a (smoothly) translation invariant factorization algebra tell us that

$$\frac{\partial}{\partial \overline{z}_{ij}} m[p] = m[p] \circ_i \frac{\partial}{\partial \overline{z}_j}$$

where $\frac{\partial}{\partial \overline{z}_j}$ is the derivation of the factorization algebra $\mathcal{F}$.

This, together with the fact that $[\mathrm{d}, \eta_i] = \frac{\partial}{\partial \overline{z}_j}$, tells us that

$$\frac{\partial p}{\partial \overline{z}_{ij}} m[p] = \left[\mathrm{d}, m[p] \circ_i \eta_j\right]$$

holds. This tells us that the product map $m[p]$ is holomorphic in p, up to a homotopy given by η_i.

5.1.3. In the smooth case, we saw that we could describe the structure as that of an algebra over a $\mathbb{R}_{>0}$-colored co-operad built from smooth functions on the spaces $\mathrm{Discs}_n(r_1, \dots, r_k \mid s)$. In this section we will see that there is an analogous story in the complex world, where we use the Dolbeault complex of the spaces $\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)$.

Let us first introduce some notation. For any complex manifold X, and any collection $V_1, \dots, V_k, W$ of differentiable cochain complexes over $\mathbb{C}$, let

$$\Omega^{0,*}(X, \mathrm{Hom}(V_1, \dots, V_k \mid W_i))$$

denote the cochain complex of smooth multilinear maps

$$V_1 \times \cdots \times V_k \to \Omega^{0,*}(X, W).$$

Recall that

$$\Omega^{0,*}(X, W) = C^\infty(X, W) \otimes_{C^\infty(X)} \Omega^{0,*}(X);$$

as we see in the appendix, a differentiable vector space W has enough structure to define the $\overline{\partial}$ operator on $\Omega^{0,*}(X, W)$. The differential on $\Omega^{0,*}(X, \mathrm{Hom}(V_1, \dots, V_k \mid W_i))$ is a combination of the Dolbeault differential on X with the differentials on the differentiable cochain complexes V_i, W.

Since the spaces $\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)$ form an colored operad in the category of complex manifolds, it is automatic that the Dolbeault complexes of these spaces form a colored cooperad in the category of convenient cochain complexes. This is because the contravariant functor sending a complex manifold to its Dolbeault complex, viewed as a convenient vector space, is symmetric monoidal:

$$\Omega^{0,*}(X \times Y) = \Omega^{0,*}(X) \otimes \Omega^{0,*}(Y)$$

where $\otimes$ denotes the symmetric monoidal structure on the category of convenient vector spaces.

Explicitly, the colored co-operad structure is given as follows. The operad structure on the complex manifolds $\mathrm{PDiscs}_n(r_1, \dots, r_k \mid t_i)$ is given by maps

$$\begin{aligned} \circ_i : \mathrm{PDiscs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{PDiscs}_n(t_1, \dots, t_m \mid s) \\ \to \mathrm{PDiscs}_n(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s). \end{aligned}$$

We let

$$\begin{aligned} \circ_i^* : \Omega^{0,*}(\mathrm{PDiscs}_n(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s)) \\ \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{PDiscs}_n(t_1, \dots, t_m \mid s) \end{aligned}$$

be the corresponding pullback map on Dolbeault complexes.

The factorization algebras we are interested in take values in the category of differentiable vector spaces. We want to say that if $\mathcal{F}$ is a holomorphically translation invariant factorization algebra on $\mathbb{C}^n$, then it defines a coalgebra over the colored co-operad given by the Dolbeault complex of the spaces PDiscs_n. *A priori*, this doesn't make sense, because the category of differentiable vector spaces is not a symmetric monoidal category; only its full subcategory of convenient vector spaces has a tensor structure.

However, in order to make this definition, all we need to be able to do is to tensor differentiable vector spaces with the Dolbeault complexes of complex manifolds. We know how to do this: of X is a complex manifold and V a differentiable vector space, we can view $\Omega^{0,*}(X, V)$ as a tensor product of $\Omega^{0,*}(X)$ with V. This tensor product is functorial for maps $\Omega^{0,*}(X) \to \Omega^{0,*}(Y)$ which arise from holomorphic maps $Y \to X$, and for arbitrary smooth maps between differentiable vector spaces. Since the co-operad structure maps in our colored dg cooperad $\Omega^{0,*}(\mathrm{PDiscs}_n)$ arise from maps of complex manifolds, it makes sense to ask for a coalgebra over this co-operad in the category of differentiable vector spaces.

5.1.3.1 Proposition. *Let $\mathcal{F}$ be a holomorphically translation-invariant factorization algebra on $\mathbb{C}^n$. Then, $\mathcal{F}$ defines a coalgebra over the dg co-operad $\Omega^{0,*}(\mathrm{PDiscs}_n)$. More precisely, the product maps*

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s$$

for $p \in \mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)$ lift to smooth multilinear maps

$$\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s) \in \mathcal{F}_{r_1} \times \ldots \mathcal{F}_{r_k} \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s), \mathcal{F}_s))$$

which are compatible with differentials, and which satisfy the usual properties needed to define a coalgebra over a cooperad.

Explicitly, these properties are follows.

(1) Let $\mathrm{d}_{\mathcal{F}}$ denote the differential on the cochain complexes $\mathcal{F}_r$. Let f_i denote elements of $\mathcal{F}_{r_i}$. Then,

$$\sum \pm \mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s)(f_1, \ldots, \mathrm{d}_{\mathcal{F}} f_i, \ldots, f_k) = (\mathrm{d}_{\mathcal{F}} + \overline{\partial})\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s)(f_1, \ldots, f_k).$$

(2) Let $\sigma \in S_k$. Then, as in the smooth case,

$$\mu^{\overline{\partial}}(r_{\sigma(1)}, \ldots, r_{\sigma(k)} \mid s)(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) = (\sigma^{-1})^* \mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s)(f_1, \ldots, f_k)$$

where σ^{-1} is the isomorphism

$$\sigma^{-1} : \mathrm{PDiscs}(r_1, \ldots, r_k \mid s) \to \mathrm{PDiscs}(r_{\sigma(1)}, \ldots, r_{\sigma(k)} \mid s).$$

(3) The usual associativity rule relating composition in the cooperad $\Omega^{0,*}(\mathrm{PDiscs}_n)$ and of the multilinear maps $\mu^{\overline{\partial}}$ holds. For example,

$$\circ_2^* \mu^{\overline{\partial}}(t_1, r_1, r_2 \mid s)(f, g_1, g_2) = \mu^{\overline{\partial}}(t_1, t_2 \mid s) \left(f, \mu^{\overline{\partial}}(r_1, r_2 \mid t_2)(g_1, g_2) \right)$$
$$\in \Omega^{0,*}\left(\mathrm{PDiscs}_n(t_1, t_2 \mid s) \times \mathrm{PDiscs}_n(r_1, r_2 \mid t_2) \right).$$

The interested reader should consult, for example, [Get94] and [Cos07] for similar axiom systems in the context of topological field theory, and [Seg04] for a related system of axioms for chiral conformal field theories. This construction is also closely related to the construction of "descendents" that appears in the study of topological field theory in the physics literature (see for example [Wit88, Wit91, ?]). Suppose one has a field theory which depends on a metric, but for where the variation with respect to the metric is exact for what the physicists call the BRST operator (which corresponds to the differential on observables in our language). The metric-dependent functional which makes the variation of the original action functional BRST exact can be viewed as a one-form on some moduli space of metrics. Higher homotopies yield forms on the moduli space of metrics, or, in two dimensions, on the moduli of conformal classes of metrics; i.e. the moduli of Riemann surfaces.

In our approach, because we are working with holomorphic rather than topological theories, we find elements of the Dolbeault complex of appropriate moduli spaces of complex manifolds, which in our simple case are the spaces PDiscs_n.

PROOF OF THE PROPOSITION. Giving a smooth multilinear map

$$\mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s), \mathcal{F}_s)$$

which is compatible with differentials is equivalent to giving an element of

$$\Omega^{0,*}\left(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)\right)$$

which is closed with respect to the differential $\overline{\partial} + \mathrm{d}_{\mathcal{F}}$, where $\mathrm{d}_{\mathcal{F}}$ refers to the natural differential on the differentiable cochain complex $\mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)$ of smooth multilinear maps

$$\mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

We will produce the desired element

$$\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s) \in \Omega^{0,*}\left(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s), \mathrm{Him}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid \mathcal{F}_s\right)$$

starting from the operations

$$\mu^0(r_1, \ldots, r_k \mid s) \in C^\infty(\mathrm{PDiscs}(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s))$$

that we already have because $\mathcal{F}$ is a smoothly translation-invariant factorization algebra.

First, we need to introduce some notation. Recall that

$$\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s) \subset (\mathbb{C}^n)^k$$

is an open (possibly empty) subset. Thus,

$$\Omega^{0,*}\left(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)\right) = \Omega^{0,0}\left(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)\right) \otimes \mathbb{C}[\mathrm{d}\overline{z}_{ij}]$$

where the $\mathrm{d}\overline{z}_{ij}$ are commuting variables of cohomological degree 1, with $i = 1, \ldots, k$ and $j = 1, \ldots, n$. We let $\frac{\partial}{\partial(\mathrm{d}\overline{z}_{ij})}$ denote the graded derivation which removes $\mathrm{d}\overline{z}_{ij}$.

As before, let $\eta_j : \mathcal{F}_r \to \mathcal{F}_r$ denote the derivation which cobounds the derivation $\frac{\partial}{\partial \overline{z}_j}$. We can compose any element

$$(\dagger) \qquad \alpha \in \Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s))$$

with η_j acting on $\mathcal{F}_{r_i}$, to get

$$\alpha \circ_i \eta_j \in \Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s)).$$

We use the shorthand notations:

$$\begin{aligned} l_{ij}(\alpha) &= \alpha \circ_i \eta_j \\ r_{ij}(\alpha) &= \alpha \circ_i \frac{\partial}{\partial \overline{z}_j} \end{aligned}$$

where $\frac{\partial}{\partial \overline{z}_j}$ refers to the derivation acting on $\mathcal{F}_{r_j}$.

Note that

$$[\mathrm{d}_{\mathcal{F}}, l_{ij}] = r_{ij},$$

if $\mathrm{d}_{\mathcal{F}}$ is the differential on the graded vector space (†) above arising from the differentials on the spaces $\mathcal{F}_{r_i}$, $\mathcal{F}_s$.

Then, the cochains $\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s)$ are defined by

$$\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s) = \exp(-\sum \mathrm{d}\overline{z}_{ij} l_{ij}) \mu^0(r_1, \ldots, r_k \mid s).$$

Here $\mathrm{d}\overline{z}_{ij} l_{ij}$ denotes the composition of wedging with $\mathrm{d}\overline{z}_{ij}$ with the operator l_{ij}. Note that these two operators graded commute.

In what follows, we will write simply $\mu^{\overline{\partial}}$ instead of $\mu^{\overline{\partial}}(r_1, \ldots, r_k \mid s)$, and similarly for μ^0.

Next, we need to verify that $(\overline{\partial} + \mathrm{d}_{\mathcal{F}})\mu^{\overline{\partial}} = 0$, where $\overline{\partial} + \mathrm{d}_{\mathcal{F}}$ is the differential on graded vector space (†) above which arises by combining the $\overline{\partial}$ operator with the differential arising from the complexes $\mathcal{F}_{r_i}$, $\mathcal{F}_s$.

We need the following identities:

$$\begin{aligned} [\mathrm{d}_{\mathcal{F}}, \sum \mathrm{d}\overline{z}_{ij} l_{ij}] &= \sum \mathrm{d}\overline{z}_{ij} r_{ij} \\ r_{ij} \mu^0 &= \frac{\partial}{\partial \overline{z}_{ij}} \mu^0 \\ \mathrm{d}_{\mathcal{F}} \mu^0 &= 0. \end{aligned}$$

The second and third identities are part of the axioms for a smoothly translation-invariant factorization algebra.

These identities allows us to calculate that

$$\begin{aligned} (\overline{\partial} + \mathrm{d}_{\mathcal{F}})\mu^{\overline{\partial}} &= (\overline{\partial} + \mathrm{d}_{\mathcal{F}}) \exp(-\sum \mathrm{d}\overline{z}_{ij} l_{ij}) \mu^0 \\ &= -\left(\exp(-\sum \mathrm{d}\overline{z}_{ij} l_{ij})\right) \left(\sum \mathrm{d}\overline{z}_{ij} r_{ij}\right) \mu^0 + \left(\exp(-\sum \mathrm{d}\overline{z}_{ij} l_{ij})\right) (\overline{\partial} + \mathrm{d}_{\mathcal{F}}) \mu^0 \\ &= \left(\exp(-\sum \mathrm{d}\overline{z}_{ij} l_{ij})\right) \left(-\sum \mathrm{d}\overline{z}_{ij} \frac{\partial}{\partial \overline{z} ij} + \overline{\partial}\right) \mu^0 \\ &= 0. \end{aligned}$$

Thus shows that $\mu^{\overline{\partial}}$ is closed.

It is straightforward to verify that the elements $\mu^{\overline{\partial}}$ are compatible with composition and with the symmetric group actions. ☐

5.2. A general method for constructing vertex algebras

In this section we will prove that the cohomology of a holomorphically translation invariant prefactorization algebra on $\mathbb{C}$ with a compatible circle action gives rise to a vertex algebra. Together with the main result of this book, which allows one to construct factorization algebras by obstruction theory starting from the Lagrangian of a classical field theory, this gives a general method to construct vertex algebras.

5.2.1. Equivariant factorization algebras.

In section 4.7 we gave a definition of smoothly translation invariant prefactorization algebra on $\mathbb{R}^n$. In the course of constructing the vertex algebra associated to a factorization algebra, we will need to discuss factorization algebras invariant under the action of a more general Lie group.

5.2.1.1 Definition. *Let M be a manifold with an action of a group G. Let $\mathcal{F}$ be a factorization algebra on M. We say $\mathcal{F}$ is G-*equivariant*, if for all $g \in G$ and all open subset $U \subset M$ we are given isomorphisms*

$$\sigma_g : \mathcal{F}(U) \cong \mathcal{F}(g(U))$$

such that

1. *$\sigma_g \circ \sigma_h = \sigma_{gh} : \mathcal{F}(U) \to \mathcal{F}(gh(U))$.*
2. *T_g respects the factorization product. If $U_1, \dots, U_k$ are disjoint opens contained in V, then the diagram*

$$\begin{tikzcd}
\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \arrow[r] \arrow[d] & \mathcal{F}(g(U_1)) \otimes \cdots \otimes \mathcal{F}(g(U_n)) \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \mathcal{F}(g(V))
\end{tikzcd}$$

commutes.

We will now define what it meanas for a factorization algebra $\mathcal{F}$ to be smoothly equivariant under the action of a Lie group G on a manifold M. We need to introduce some notation before making this definition. Let $U_1, \dots, U_k$ be disjoint subsets of V. Let $W \subset G^k$ be the open subset consisting of those $g_1, \dots, g_k$ where $g_1(U_1), \dots, g_k(U_k)$ continue to be disjoint and contained in W. If $(g_1, \dots, g_k) \in W$, we have a map

$$m_{g_1,\dots,g_k} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V)$$

defined by the composition

$$\otimes F(U_i) \xrightarrow{\otimes \sigma_{g_i}} \otimes \mathcal{F}(g_i(U_i)) \to \mathcal{F}(V)$$

where the second map is the prefactorization multiplication.

5.2.1.2 Definition. *$\mathcal{F}$ is a* smoothly G-equivariant prefactorization algebra *if the following conditions hold.*

1. *$m_{g_1,\dots,g_k}$ depends smoothly on $(g_1,\dots,g_k) \in W \subset G^k$.*
2. *There's an action of the Lie algebra $\mathfrak{g}$ of G on $\mathcal{F}$ by derivations.*
3. *This action is compatible with the action of G, in the following sense. For all $X \in \mathfrak{g}$, all k and all i with $1 \le i \le k$, we require that*

   $$\frac{\partial}{\partial X_i} m_{g_1,\dots,g_k}(\alpha_1,\dots,\alpha_k) = m_{g_1,\dots,g_k}(\alpha_1,\dots,X(\alpha_i),\dots,\alpha_k).$$

   On the left hand side $\frac{\partial}{\partial X_i}$ indicates the action of the left-invariant vector field associated to X on the i^{th} factor of G^k.

The case of interest is the action of the group $S^1 \ltimes \mathbb{C}$ of isometries of $\mathbb{C}$ on $\mathbb{C}$.

5.2.1.3 Definition. *A* holomorphically translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}$ with a compatible S^1 action *is a smoothly $S^1 \times \mathbb{R}^2$-invariant factorization algebra $\mathcal{F}$, defined over the base field of complex numbers, together with an extension of the action of the complex Lie algebra*

$$\mathrm{Lie}_{\mathbb{C}}(S^1 \ltimes \mathbb{R}^2) = \mathbb{C}\{\partial_\theta, \partial_z, \partial_{\bar{z}}\}$$

(where ∂_θ is a basis of $\mathrm{Lie}_{\mathbb{C}}(S^1)$*) to an action of the dg Lie algebra*

$$\mathbb{C}\{\partial_\theta, \partial_z, \partial_{\bar{z}}\} \oplus \mathbb{C}\{\eta\}$$

where η is of cohomological degree -1, and the differential is

$$\mathrm{d}\eta = \partial_{\bar{z}}.$$

In this dg Lie algebra, all commutators involving η vanish except for

$$[\partial_\theta, \eta] = -\eta.$$

Note that, in particular, $\mathcal{F}$ is a holomorphically translation invariant factorization algebra on $\mathbb{C}$.

5.2.2. Let us now turn to the theorem relating S^1-equivariant holomorphically-translation invariant factorization algebras with vertex algebras. Before stating the theorem, we need a few technical remarks.

Every differentiable vector space is a sheaf on the site of smooth manifolds. If E is a differentiable vector space, we denote the sections of this sheaf on a manifold M by $C^\infty(M, E)$: we think of this as the space of smooth maps from M to E. If E is a complex of differentiable vector spaces, then sending an open subset U of M to $C^\infty(U, E)$ defines a sheaf of cochain complexes on M. The cohomology of a sheaf of cochain complexes is a presheaf of graded vector spaces, which we can sheafify to get a sheaf of graded vector spaces. We let $C^\infty(U, H^*(E))$ denote the sections on U of these cohomology sheaves.

We are allowed to differentiate smooth maps to a differentiable vector spaces. Thus, if M is a complex manifold and E is a cochain complex of differentiable vector spaces over $\mathbb{C}$, we have, for every open $U \subset M$, a cochain map

$$\overline{\partial} : C^\infty(U, E) \to \Omega^{0,1}(U, E).$$

These maps form a map of sheaves of cochain complexes on M. Taking cohomology, we find a map of sheaves of graded vector spaces

$$\overline{\partial} : C^\infty(U, H^*(E)) \to \Omega^{0,1}(U, H^*(E)).$$

The kernel of this map defines a sheaf on M, whose sections we denote by $\mathrm{Hol}(U, H^*(E))$.

The theorem on vertex algebras will require an extra hypothesis regarding the S^1 action on the factorization algebra. Note that for any compact Lie group G, the space $\mathcal{D}(G)$ of distributions on G is an algebra under convolution. Since spaces of distributions are naturally differentiable vector spaces, and the product map $\mathcal{D}(G) \times \mathcal{D}(G) \to \mathcal{D}(G)$ is smooth, this forms an algebra in the category of differentiable vector spaces. There is a map

$$G \to \mathcal{D}(G)$$

sending an element g to the δ-distribution at g. This is a smooth map, and is also a homomorphism of monoids.

Given a complex E of differentiable vector spaces with a G action, we can ask that it extends to a smooth action of the algebra $\mathcal{D}(G)$. That is, we can ask for a smooth bilinear map, compatible with differentials,

$$\mathcal{D}(G) \times E \to E$$

which extends the smooth map $G \times E \to E$, and which defines an action of the algebra $\mathcal{D}(G)$.

Let us now specialize to the case when $G = S^1$, which is the case which is relevant for the theorem on vertex algebras. The irreducible representation ρ_k of S^1 where $\lambda \in S^1 \subset \mathbb{C}^\times$ acts by λ^k lifts to a representation ρ_k of the algebra $\mathcal{D}(S^1)$, defined by

$$\rho_k(\phi) = \left\langle \phi, \lambda^k \right\rangle$$

where $\langle -, - \rangle$ indicates the pairing between distributions and functions.

Let $E_k \subset E$ denote the subspace on which $\mathcal{D}(S^1)$ acts by ρ_k. We call this the weight k eigenspace for the S^1-action on E.

In the algebra $\mathcal{D}(S^1)$, the element λ^k (viewed as a distribution on S^1) is an idempotent. If we denote the action of $\mathcal{D}(S^1)$ on E by $*$, then the map

$$\lambda^{-k} * : E \to E$$

defines a projection from E onto E_k. We will denote this projection by π_k.

(Of course, all this holds for a general compact Lie group where instead of the λ^k we use the characters of irreducible representations).

Now we can state the main theorem of this section.

5.2.2.1 Theorem. *Let $\mathcal{F}$ be a unital S^1-equivariant holomorphically translation invariant prefactorization algebra on $\mathbb{C}$ valued in differentiable vector spaces. Assume that, for each disc $D(0,r)$ around the origin, the action of S^1 on $\mathcal{F}(D(0,r)$ extends to a smooth action of the algebra $\mathcal{D}(S^1)$ of distributions on S^1.*

Let $\mathcal{F}_k(D(0,r))$ be the k^{th} eigenspace of the S^1 action on $\mathcal{F}(D(0,r))$. Let us make the following additional assumptions.

1. *Assume that, for $r < r'$, the map*
$$\mathcal{F}_k(D(0,r)) \to \mathcal{F}_k(D(0,r'))$$
is a quasi-isomorphism.
2. *The vector space $H^*(\mathcal{F}_k(D(0,r))$ is zero for $k \gg 0$.*
3. *For each k and r, we require that $H^*(\mathcal{F}_k(D(0,r))$ is isomorphic, as a sheaf on the site of smooth manifolds, to a countable sequential colimit of finite-dimensional graded vector spaces.*

Let $V_k = H^*(\mathcal{F}_k(D(0,r))$, and let $V = \oplus V_k$. This space is independent of r by assumption.

Then, V has the structure of a vertex algebra.

Remark: (1) If V is not concentrated in cohomological degree 0, then it will have the structure of a vertex algebra valued in the symmetric monoidal category of graded vector spaces, that is, the Koszul rule of signs will apear in the axioms.

(2) We will often deal with factorization algebras $\mathcal{F}$ equipped with a complete decreasing filtration $F^i\mathcal{F}$, so that $\mathcal{F} = \lim \mathcal{F}/F^i\mathcal{F}$. In this situation, to construct the vertex algebra we need that the properties listed in the theorem hold on each graded piece $\mathrm{Gr}^l \mathcal{F}$. This implies, by a spectral sequence, that they hold on each $\mathcal{F}/F^i\mathcal{F}$, allowing us to construct an inverse system of vertex algebras associated to the prefactorization algebra $\mathcal{F}/F^i\mathcal{F}$. The inverse limit of this system of vertex algebras is the factorization algebra associated to $\mathcal{F}$.

$\Diamond$

The conditions of the theorem are always satisfied in practise by factorization algebras arising from quantizing a holomorphically translation invariant classical field theory.

Because our factorization algebra $\mathcal{F}$ is translation invariant, the cochain complex $\mathcal{F}(D(z,r))$ associated to a disc of radius r is independent of z. We can therefore use the notation $\mathcal{F}(r)$ for $\mathcal{F}(D(z,r))$. We will use this notation when $r = \infty$ to denote $\mathcal{F}(\mathbb{C})$.

Let $\mathcal{F}_k(r)$ denote the k^{th} weight space of the S^1 action on $\mathcal{F}$. The S^1 action on each $\mathcal{F}(r)$ extends to an action of $\mathcal{D}(S^1)$. If λ denotes an element of $S^1 \subset \mathbb{C}^\times$, and $*$ denotes this action, then $\lambda^{-k}*$ gives a projection map

$$\mathcal{F}(r) \to \mathcal{F}_k(r).$$

At the level of cohomology, it gives a map

$$\pi_k : H^*(\mathcal{F}(r)) \to H^*(\mathcal{F}_k(r)) = V_k$$

of differentiable vector spaces, splitting the natural inclusion.

Further, the map $H^*(\mathcal{F}(r)) \to H^*(\mathcal{F}(r'))$ associated to the inclusion $D(0,r) \hookrightarrow D(0,r')$ is the identity on V.

By assumption, the natural differentiable vector space structure on $V_k = H^*(\mathcal{F}_k(r))$ has the property that a smooth map from a manifold M to V_k is a map which locally on M is given by a smooth (in the ordinary sense) map to a finite dimensional subspace of V_k. Thus, V_k is the colimit in the category of differentiable vector spaces of all of its finite-dimensional subspaces.

Let

$$\overline{V} = \prod V_k$$

where the product is taken in the category of differentiable vector spaces.

There is a natural map

$$H^*(\mathcal{F}(r)) \to \overline{V}$$

given by the product of all the projection maps.

The strategy of the proof is as follows. We will analyze the structure on V given to us by the axioms of a translation-invariant factorization algebra. The factorization product will become the operator product expansion or state-field map. The locality axiom of a vertex algebra will follow from the associativity axioms of the factorization algebra.

Let us start by analyzing the structure on V given by the factorization (or operator) product. Let $r_1, \dots, r_k, s \in \mathbb{R}_{>0}$. Recall that we have a complex manifold

$$\mathrm{Discs}(r_1, \dots, r_k \mid s) = \{z_1, \dots, z_k \in \mathbb{C} \mid D(z_1, r_1) \amalg \cdots \amalg D(z_k, r_k) \subset D(0,s)\} \subset \mathbb{C}^k.$$

Note that $\mathbb{C}$ acts on $\mathrm{Discs}(r_1,\dots,r_k \mid \infty)$ by translation.

As we have seen, the factorization (or operator) product is a multilinear map, compatible with differentials,

$$\mu^{\overline{\partial}}(r_1,\dots,r_k \mid \infty) : \mathcal{F}(r_1)\times\cdots\times\mathcal{F}(r_k) \to \Omega^{0,*}(\mathrm{Discs}(r_1,\dots,r_k \mid \infty),\mathcal{F}(\infty)).$$

Note that for any complex manifold X there is a map

$$H^*\left(\Omega^{0,*}(X,\mathcal{F}(\infty))\right) \to \mathrm{Hol}(X, H^*(\mathcal{F}(\infty))).$$

If $\alpha \in \Omega^{0,*}(X,\mathcal{F}(\infty))$ is a closed element, this map is defined by first extracting the component α^0 of α which is in $\Omega^{0,0}(X,\mathcal{F}(\infty))$. The fact that α is closed means that α^0 is closed for the differential $D_{\mathcal{F}}$ on $\mathcal{F}(\infty)$ and that $\overline{\partial}\alpha^0$ is exact. Thus, the cohomology class $[\alpha^0]$ is a smooth map from X to $\mathcal{F}(\infty)$, and $\overline{\partial}[\alpha^0] = 0$ so that $[\alpha^0]$ is holomorphic.

This means that the operator product map, at the level of cohomology, gives a smooth multilinear map

$$m_{z_1,\dots,z_k} : H^*(\mathcal{F}(r_1))\times\cdots\times H^*(\mathcal{F}(r_k)) \to \mathrm{Hol}(\mathrm{Discs}(r_1,\dots,r_k \mid \infty), H^*(\mathcal{F}(\infty))).$$

Here z_i indicate the positions of the centers of the discs in $\mathrm{Discs}(r_1,\dots,r_k \mid \infty)$.

If $r_i' < r_i$, then

$$\mathrm{Discs}(r_1,\dots,r_k \mid \infty) \subset \mathrm{Discs}(r_1',\dots,r_k' \mid \infty).$$

There is also a natural map $\mathcal{F}(r') \to \mathcal{F}(r)$, given by including $D(0,r') \hookrightarrow D(0,r)$. The following diagram commutes:

$$\begin{tikzcd}
H^*(\mathcal{F}(r_1'))\otimes\cdots\otimes H^*(\mathcal{F}(r_k')) \arrow[r] \arrow[d] & \mathrm{Hol}(\mathrm{Discs}(r_1',\dots,r_k' \mid \infty), H^*(\mathcal{F}(\infty))) \arrow[d] \\
H^*(\mathcal{F}(r_1))\otimes\cdots\otimes H^*(\mathcal{F}(r_k)) \arrow[r] & \mathrm{Hol}(\mathrm{Discs}(r_1,\dots,r_k \mid \infty), H^*(\mathcal{F}(\infty))).
\end{tikzcd}$$

Note that V maps to $\lim_{r\to 0} H^*(\mathcal{F}(r))$. Therefore, the operator product, when restricted to V, gives a map

$$m^{H^*(\mathcal{F}(\infty))}_{z_1,\dots,z_k} : V^{\otimes k} \to \lim_{r\to 0}\mathrm{Hol}(\mathrm{Discs}(r,\dots,r \mid \infty), H^*(\mathcal{F}(\infty))) = \mathrm{Hol}(\mathrm{Conf}_k(\mathbb{C}), H^*(\mathcal{F}(\infty)))$$

where $\mathrm{Conf}_k(\mathbb{C})$ is the configuration space of k ordered distinct points in $\mathbb{C}$.

If all the z_i lie in a disc $D(0,r)$, this lifts to a map

$$m^{H*(\mathcal{F}(r))}_{z_1,\dots,z_k} : V^{\otimes k} \to \mathrm{Hol}(\mathrm{Conf}_k(D(0,r)), H^*(\mathcal{F}(r))).$$

Composing the factorization product map $m^{H^*(\mathcal{F}(\infty))}_{z_1,\dots,z_k}$ with the map

$$\prod \pi_k : H^*(\mathcal{F}(\infty)) \to \overline{V} = \prod V_k$$

gives a map

$$m_{z_1,\ldots,z_k} : V^{\otimes k} \to \mathrm{Hol}(\mathrm{Conf}_k(\mathbb{C}), \overline{V}) = \prod_l \mathrm{Hol}(\mathrm{Conf}_k(\mathbb{C}), V_l).$$

This map does not involve the spaces $\mathcal{F}(r)$ anymore, only the space V and its natural completion $\overline{V}$. If there is potential confusion, we will refer to this version of the operator product map by $m^V_{z_1,\ldots,z_k}$ instead of just $m_{z_1,\ldots,z_k}$.

The operator product will, of course, be constructed from the maps $m_{z_1,\ldots,z_k}$. We can consider the map

$$m_{z,0} : V \otimes V \to \prod_l \mathrm{Hol}(\mathbb{C}^\times, V_l)$$

where we have restricted the map $m^V_{z,w}$ to the locus where $w = 0$. Since each space V_l is a discrete vector space, that is, a colimit of finite dimensional vector spaces, we can form the ordinary Laurent expansion of an element in $\mathrm{Hol}(\mathbb{C}^\times, V_l)$ to get a map

$$\mathcal{L}_z m^V_{z,0} : V \otimes V \to \overline{V}[[z, z^{-1}]].$$

5.2.2.2 Lemma. *The image of $\mathcal{L}_z m^V_{z,0}$ is in the subspace $V((z))$.*

Proof. The map $m^V_{z,0}$ is S^1-equivariant, where S^1 acts on V and $\mathbb{C}^\times$ in the evident way. Therefore so is $\mathcal{L}_z m^V_{z,0}$. Since every element in $V \otimes V$ is in a finite sum of the S^1-eigenspaces, the image of $\mathcal{L}_z m^V_{z,0}$ is in the subspace of $\overline{V}[[z, z^{-1}]]$ spanned by finite sums of eigenvectors. An element of $\overline{V}[[z, z^{-1}]]$ is in the kth eigenspace of the S^1 action if it is of the form

$$\sum z^{k-l} v_l$$

where $v_l \in V_l$. Since $V_l = 0$ for $l \gg 0$, every such element is in $V((z))$. □

Let us now define the structures on V which will correspond to the vertex algebra structure.

(1) **The vacuum element $\mathbb{1} \in V$**: By assumption, $\mathcal{F}$ is a unital prefactorization algebra. Therefore, the commutative algebra $\mathcal{F}(\emptyset)$ has a unit element $\mathbb{1}$. The prefactorization structure map $\mathcal{F}(\emptyset) \to \mathcal{F}(D(0, r))$ for any r gives an element $\mathbb{1} \in \mathcal{F}(r)$. This element is automatically S^1-invariant, and therefore in V_0.

(2) **The translation map $T : V \to V$**: The structure of holomorphically translation factorization algebra on $\mathcal{F}$ includes a derivation $\frac{\partial}{\partial z}$ corresponding to infinitesimal translation in the (complex) direction z. The fact that $\mathcal{F}$ has a compatible S^1 action means that, for all r, the map $\frac{\partial}{\partial z}$ maps

$$\mathcal{F}_k(r) \to \mathcal{F}_{k-1}(r).$$

Therefore, passing to cohomology, it becomes a map $\frac{\partial}{\partial z} : V_k \to V_{k-1}$. We let T be the map $V \to V$ which on V_k is $\frac{\partial}{\partial z}$.

(3) **The state-field map** $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$: We let

$$Y(v, z)(v') = \mathcal{L}_z m_{z,0}(v, v') \in V((z)).$$

Note that $Y(v, z)$ is a field in the sense used in the axioms of a vertex algebra, because $Y(v, z)(v')$ has only finitely many negative powers of z.

It remains to check the axioms of a vertex algebra. We need to verify the

(1) **Vacuum axiom**: $Y(\mathbb{1}, z)(v) = v$.

(2) **Translation axiom**:

$$Y(Tv_1, z)(v_2) = \frac{\partial}{\partial z} Y(v_1, z)(v_2).$$

and $T\mathbb{1} = 0$.

(3) **Locality axiom**:

$$(z_1 - z_2)^N [Y(v_1, z_1), Y(v_2, z_2)] = 0$$

for N sufficiently large.

The vacuum axiom follows from the fact that the unit $\mathbb{1} \in \mathcal{F}(\emptyset)$, viewed as an element of $\mathcal{F}(D(0, r))$, is a unit for the factorization product.

The translation axiom follows immediately from the corresponding axiom of factorization algebras, which is built into our definition of holomorphic translation invariance: namely,

$$\frac{\partial}{\partial z} m_{z,0}(v_1, v_2) = m_{z,0}(\partial_z v_1, v_2).$$

The fact that $T\mathbb{1} = 0$ follows from the fact that the derivation ∂_z of $\mathcal{F}$ gives a derivation of the commutative algebra $\mathcal{F}(\emptyset)$, which must therefore send the unit to zero.

It remains to prove the locality axiom. This will follow from the associativity property of factorization algebras. The rest of this section will be devoted to the proof.

5.2.3. Proof of the locality axiom. Let us write down some useful properties of the operator product maps $m_{z_1,\dots,z_k}$. Firstly, the map $m_{z_1,\dots,z_k}$ is S^1-covariant, where we use the diagonal S^1 action on $V^{\otimes k}$, and the S^1 action on the right hand side comes from the natural S^1 action on $\mathrm{Conf}_k(\mathbb{C})$ by rotation coupled to the S^1 action on $\overline{V}$ coming from the structure of S^1-invariant factorization algebra. It is also S_k-covariant, where S_k acts on $V^{\otimes k}$ and on

$\mathrm{Conf}_k(\mathbb{C})$ in the evident way. It is also invariant under translation, in the sense that for arbitrary $v_i \in V$,

$$m_{z_1+\lambda,\dots,z_k+\lambda}(v_1,\dots,v_k) = \rho_\lambda\left(m_{z_1,\dots,z_k}(v_1,\dots,v_k)\right) \in \overline{V}$$

where ρ_λ denotes the action of $\mathbb{C}$ on $\overline{V}$ which integrates the translation action of the Lie algebra $\mathbb{C}$.

Let us write

$$\mathcal{L}_z m_{z,0}(v_1,v_2) = \sum_k z^k m^k(v_1,v_2)$$

where $m^k(v_1,v_2) \in V$. As we have seen, $m^k(v_1,v_2)$ is zero for $k \ll 0$.

The key proposition which will allow us to prove the locality axiom is the following.

5.2.3.1 Proposition. *Let $U_{ij} \subset \mathrm{Conf}_k(\mathbb{C})$ be the open subset where*

$$\left|z_j - z_i\right| < \left|z_j - z_l\right|$$

for $l \neq i,j$.

Then, for $(z_1,\dots,z_k) \in U_{ij}$, we have the following identity:

$$m_{z_1,\dots,z_k}(v_1,\dots,v_k) = \sum (z_i - z_j)^n m_{z_1,\dots,\widehat{z_j},\dots,z_k}(v_1,\dots,v_{i-1},m^n(v_i,v_j),\dots,\widehat{v_j},\dots,v_k) \in \mathrm{Hol}(U_{ij},\overline{V}).$$

The sum on the right hand side converges.

In this expression, $\widehat{z_j}$ and $\widehat{v_j}$ indicate that we skip these entries.

When $n = 3$, this identity allows us to expand m_{z_1,z_2,z_3} in two different ways. We find that

$$m_{z_1,z_2,z_3}(v_1,v_2,v_3) = \begin{cases} \sum (z_2 - z_3)^k m_{z_1,z_3}(m^k(v_1,v_2),v_3) & \text{if } |z_2 - z_3| < |z_1 - z_3| \\ \sum (z_1 - z_3)^k m_{z_2,z_3}(v_2, m^k(v_1,v_3)) & \text{if } |z_2 - z_3| > |z_1 - z_3|. \end{cases}$$

This should be compared to the associativity axiom in the theory of vertex algebras.

PROOF. By symmetry, we can reduce to the case when $i = 1$ and $j = 2$.

If all of the z_i lie in a particular disc $D(0,r) \subset \mathbb{C}$, then the map $m_{z_1,\dots,z_k}$ lifts to a map

$$m^{H^*(\mathcal{F}(r))}_{z_1,\dots,z_k} : \mathrm{Hol}(\mathrm{Conf}_k(D(0,r)), H^*(\mathcal{F}(r))).$$

Further, if we take some $\varepsilon > 0$ and consider the open subset

$$U \subset \mathrm{Conf}_k(\mathbb{C})$$

where the z_j for $j \neq 1$ are disjoint from the disc of radius ε around z_1, then the map $m_{z_1,\ldots,z_k}$ extends to a map

$$H^*(\mathcal{F}(\varepsilon)) \otimes V \cdots \otimes V \to \mathrm{Hol}(U, \overline{V}).$$

The axioms of a prefactorization algebra tell us that the following associativity condition holds:

$$(\dagger) \qquad m_{z_1,z_2,\ldots,z_k}(v_1,\ldots,v_k) = m_{z_2,z_3,\ldots,z_k}(m_{z_1,z_2}(v_1,v_2),v_3,\ldots,v_k)$$

in the following sense. First, we must choose $\varepsilon > 0$ and restrict the z_i to lie in the open set where the discs the discs $D(z_i,\varepsilon)$ are all disjoint. Further, we ask that $D(z_2,\delta)$ (where $\delta > \varepsilon$) contains $D(z_1,\varepsilon)$ and is disjoint from each $D(z_i,\varepsilon)$ for $i > 2$. (Here ε and δ can be chosen arbitrarily small). Then, we are viewing $m_{z_1,z_2}(v_1,v_2)$ as an element of $H^*(\mathcal{F}(D(z_2,\delta)))$, depending holomorphically on z_1, z_2 which lie in the open set where $D(z_1,\varepsilon)$ is disjoint from $D(z_2,\varepsilon)$ and contained in $D(z_2,\delta)$. By translating z_2 to 0, we identify $H^*(\mathcal{F}(D(z_2,\delta)))$ with $H^*(\mathcal{F}(\delta))$. This associativity property is an immediate consequence of the axioms of a prefactorization algebra.

By taking $\varepsilon \to 0$, we see that this identity holds when the z_i are restricted to lie in the set U_{12}^{δ} where $|z_2 - z_1| < \delta$ and $|z_2 - z_i| > \delta$ for $i > 2$. Note that the union of the sets U_{12}^{δ} as δ varies is U_{12}.

We can, without loss of generality, assume that each v_i is homogeneous of weight $|v_i|$ under the S^1 action on V.

Let $\pi_k : H^*(\mathcal{F}(\delta)) \to V_k$ be the projection onto the eigenspace V_k for the S^1 action. Recall that $\mathcal{D}(S^1)$ acts on $\mathcal{F}(\delta)$ and so on $H^*(\mathcal{F}(\delta))$. If this action is denoted by $*$, and if $\lambda \in S^1 \subset \mathbb{C}^\times$ denotes an element of the circle viewed as a complex number, then

$$\pi_k(f) = \lambda^{-k} * f.$$

Note that S^1-equivariance of the operator product map means that we can write the Laurent expansion of $m_{z,0}(v_1,v_2)$ as

$$m_{z,0}(v_1,v_2) = \sum z^k m^k(v_1,v_2) = \sum \pi_{|v_1|+|v_2|-k} m_{z,0}(v_1,v_2).$$

That is, $z^k m^k(v_1,v_2)$ is the projection of $m_{z,0}(v_1,v_2)$ onto the $|v_1+v_2|-k$ eigenspace of $\overline{V}$.

We want to show that

$$\sum (z_1-z_2)^n m_{z_2,z_3,\ldots,z_k}(m^n(v_1,v_2),\ldots,v_k) = m_{z_1,\ldots,z_k}(v_1,\ldots,v_k)$$

where the z_i lie in U_{ij}^{δ}. This is equivalent to showing that

$$\sum_n m_{z_2,z_3,\ldots,z_k}(\pi_n m_{z_1,z_2}(v_1,v_2),\ldots,v_k) = m_{z_1,\ldots,z_k}(v_1,\ldots,v_k).$$

Indeed, the Laurent expansion of $m_{z_1,z_2}(v_1,v_2)$ and the expansion in terms of eigenspaces of the S^1 action on $\overline{V}$ differ only by a reordering of the sum.

Fix $v_1, \ldots, v_n \in V$. Define a map

$$\begin{aligned}\Phi : \mathcal{D}(S^1) &\to \mathrm{Hol}(U_{ij}^{\delta}, \overline{V})\\ \Phi(\alpha) &= m_{z_2,z_3,\ldots,z_k}(\alpha * m_{z_1,z_2}(v_1, v_2), v_3, \ldots, v_n).\end{aligned}$$

Here $\alpha*$ refers to the action of $\mathcal{D}(S^1)$ on $H^*(\mathcal{F}(\delta))$.

Note that Φ is a smooth map. Note also that the associativity identity (†) of factorization algebras implies that

$$\Phi(\delta_1) = m_{z_1,\ldots,z_k}(v_1, \ldots, v_k).$$

The point is that δ_1* is the identity on $H^*(\mathcal{F}(\delta))$.

To prove the proposition, it now suffices to prove that

$$\Phi\left(\sum_{n\in\mathbb{Z}} \lambda^n\right) = \Phi(\delta_1).$$

In the space $\mathcal{D}(S^1)$ with its natural topology the sum $\sum_{n\in\mathbb{Z}} \lambda^n$ converges to δ_1 (this is simply the Fourier expansion of the delta-function). So, to prove the proposition, it suffices to prove that Φ is continuous, where the spaces $\mathcal{D}(S^1)$ and $\mathrm{Hol}(U_{ij}^{\delta}, \overline{V})$ are endowed with their natural topologies. (In the topology on $\mathrm{Hol}(U_{ij}^{\delta}, \overline{V})$, a sequence converges if its projection to each V_k converges uniformly, with all derivatives, on compact sets of U_{ij}^{δ})).

We know that Φ is smooth. The spaces $\mathcal{D}(S^1)$ and $\mathrm{Hol}(U_{ij}^{\delta}, \overline{V})$ both lie in the essential image of the functor from locally-convex topological vector spaces to differentiable vector spaces. A result of [KM97] tells us that smooth linear maps between topological vector spaces are bounded. Therefore, the map Φ is a bounded linear map of topological vector spaces.

In lemma C.2.0.8 in Appendix C we show (using results of [KM97]) that the space of compactly supported distributions on any manifold has the *bornological* property, meaning that a bounded linear map from it to any topological vector space is the same as a continuous linear map. It follows that Φ is continuous, thus completing the proof. ☐

As a corollary, we find the following.

5.2.3.2 Corollary. *For $v_1, \ldots, v_k \in V$, $m_{z_1,\ldots,z_k}(v_1, \ldots, v_k)$ has finite order poles on every diagonal in* $\mathrm{Conf}_k(\mathbb{C})$. *That is, for some N sufficiently large,*

$$\prod_{i,j}(z_i - z_j)^N m_{z_1,\ldots,z_k}(v_1, \ldots, v_k)$$

extends to an element of $\mathrm{Hol}(\mathbb{C}^k, \overline{F})$.

PROOF. This is an immediate corollary of the previous proposition. ☐

Now we are ready to prove the locality axiom.

5.2.3.3 Proposition. *The locality axiom holds:*

$$(z_1 - z_2)^N [Y(v_1, z_1), Y(v_2, z_2)] = 0$$

for $N \gg 0$.

PROOF. For any holomorphic function $F(z_1, \ldots, z_k)$ of variables $z_1, \ldots, z_k \in \mathbb{C}^\times$, we let $\mathcal{L}_{z_i} F$ denote the Laurent expansion of F in the variable z_i. This is an expansion that converges when $|z_i| < |z_j|$ for all j. It can be defined by fixing the values of z_j with $j \neq i$, then viewing F as a function of z_i on the punctured disc where $0 < |z_i| < |z_j|$, and taking the usual Laurent expansion. We can define iterated Laurent expansions. For example, if F is a function of $z_1, z_2 \in \mathbb{C}^\times$, we can define

$$\mathcal{L}_{z_2} \mathcal{L}_{z_1} \in \mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$$

by first taking the Laurent expansion with respect to z_1, yielding a series in z_1 whose coefficients are holomorphic functions of $z_2 \in \mathbb{C}^\times$; and then applying the Laurent expansion with respect to z_2 to each of the coefficient functions of the expansion with respect to z_1.

Recall that we define

$$Y(v_1, z)(v_2) = \mathcal{L}_z m_{z,0}(v_1, v_2) \in V((z)).$$

We define $m_k(v_1, v_2)$ so that

$$\mathcal{L}_z m_{z,0}(v_1, v_2) = \sum z^k m_k(v_1, v_2).$$

Note that, by definition,

$$\begin{aligned} Y(v_1, z_1) Y(v_2, z_2)(v_3) &= \mathcal{L}_{z_1} m_{z_1,0} \left(v_1, \mathcal{L}_{z_2} m_{z_2,0}(v_2, v_3)\right) \\ &= \sum \mathcal{L}_{z_1} \sum z_2^n m_{z_1,0} \left(v_1, m_n(v_2, v_3)\right) \in V[[z_1^{\pm 1}, v_2^{\pm 1}]]. \end{aligned}$$

Proposition 5.2.3.3 tells us that

$$z_2^n m_{z_1,0} \left(v_1, \sum m_n(v_2, v_3)\right) = m_{z_1,z_2,0}(v_1, v_2, v_3)$$

as long as $|z_2| < |z_1|$. Thus,

$$Y(v_1, z_1) Y(v_2, z_2)(v_3) = \mathcal{L}_{z_1} \mathcal{L}_{z_2} m_{z_1,z_2,0}(v_1, v_2, v_3).$$

Similarly,

$$\begin{aligned} Y(v_2, z_2) Y(v_1, z_1)(v_3) &= \mathcal{L}_{z_2} \mathcal{L}_{z_1} m_{z_2,z_2,0}(v_2, v_1, v_3) \\ &= \mathcal{L}_{z_2} \mathcal{L}_{z_1} m_{z_1,z_2,0}(v_1, v_2, v_3). \end{aligned}$$

Therefore,

$$[Y(v_1, z_1), Y(v_2, z_2)](v_3) = \left(\mathcal{L}_{z_1} \mathcal{L}_{z_2} - \mathcal{L}_{z_2} \mathcal{L}_{z_1}\right) m_{z_1,z_2,0}(v_1, v_2, v_3).$$

Since $\mathcal{L}_{z_1}$ and $\mathcal{L}_{z_2}$ are maps of $\mathbb{C}[z_1, z_2]$ modules, we have

$$(z_1 - z_2)^N \left(\mathcal{L}_{z_1}\mathcal{L}_{z_2} - \mathcal{L}_{z_2}\mathcal{L}_{z_1}\right) m_{z_1,z_2,0}(v_1, v_2, v_3) \\ = \left(\mathcal{L}_{z_1}\mathcal{L}_{z_2} - \mathcal{L}_{z_2}\mathcal{L}_{z_1}\right) (z_1 - z_2)^N m_{z_1,z_2,0}(v_1, v_2, v_3)$$

Finally, we know that for N sufficiently large, $(z_1 - z_2)^N z_1^N z_2^N m_{z_1,z_2,0}$ has no poles, and so extends to a function on $\mathbb{C}^2$. It follows from the fact that partial derivatives commute that

$$\left(\mathcal{L}_{z_1}\mathcal{L}_{z_2} - \mathcal{L}_{z_2}\mathcal{L}_{z_1}\right) (z_1 - z_2)^N z_1^N z_2^N m_{z_1,z_2,0}(v_1, v_2, v_3) = 0.$$

Since Laurent expansion is a map of $\mathbb{C}[z_1, z_2]$-modules, and since z_1, z_2 act invertibly on $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$, the result follows. $\square$

Remark: In the vertex algebra literature, the idea that $Y(v_1, z_1)Y(v_2, z_2)(v_3)$ and $Y(v_2, z_2)Y(v_1, z_1)(v_3)$ arise as expansions of a holomorphic function of $z_1, z_2 \in \mathbb{C}^\times$ in the regions when $|z_1| < |z_2|$ and $|z_2| < |z_1|$ is often cited as a heuristic justification of the locality axiom for a vertex algebra. Our approach makes this idea rigorous.

5.3. The $\beta\gamma$ system and vertex algebras

This section focuses on one of the simplest holomorphic field theories, the free $\beta\gamma$ system. Our goal is to study it just as we studied the free particle in section **??**. Following the methods developed there, we will construct the factorization algebra for this theory, show that it is holomorphically translation-invariant, and finally show that the associated vertex algebra is what is known in the vertex algebra literature as the $\beta\gamma$ system. Along the way, we will compute the simplest operator product expansions for the theory using purely homological methods.

5.3.1. The $\beta\gamma$ system. Let $M = \mathbb{C}$ and let $\mathscr{E} = \left(\Omega_M^{0,*} \oplus \Omega_M^{1,*}, \overline{\partial}\right)$ be the Dolbeault complex resolving holomorphic functions and holomorphic 1-forms as a sheaf on M. Following the convention of physicists, we denote by γ an element of $\Omega^{0,*}$ and by β an element of $\Omega^{1,*}$. The pairing $\langle -, - \rangle$ is

$$\begin{aligned} \langle -, - \rangle : \mathscr{E}_c \otimes \mathscr{E}_c &\to \mathbb{C}, \\ (\gamma_0 + \beta_0) \otimes (\gamma_1 + \beta_1) &\mapsto \int_{\mathbb{C}} \gamma_0 \wedge \beta_1 + \beta_0 \wedge \gamma_1. \end{aligned}$$

Thus we have the data of a free BV theory. The action functional for the theory is

$$S(\gamma, \beta) = \langle \gamma + \beta, \overline{\partial}(\gamma + \beta) \rangle = 2 \int_M \beta \wedge \overline{\partial}\gamma$$

The Euler-Lagrange equation is simply $\overline{\partial}\gamma = 0 = \overline{\partial}\beta$. One should think of $\mathscr{E}$ as the "derived space of holomorphic functions and 1-forms on M." Note that this theory is well-defined on any Riemann surface, and one can study how it varies over the moduli space of curves.

Remark: One can add d copies of $\mathscr{E}$ (equivalently, tensor $\mathscr{E}$ with $\mathbb{C}^d$) and let S_d be the d-fold sum of the action S on each copy. The Euler-Lagrange equations for S_d picks out "holomorphic maps γ from M to $\mathbb{C}^d$ and holomorphic sections β of $\Omega^1_M(\gamma^* T_{\mathbb{C}^d})$."

5.3.2. The quantum observables of the $\beta\gamma$ system.

To construct the quantum observables, following section **??**, we start by defining a certain graded Heisenberg Lie algebra and then take its Chevalley-Eilenberg complex for Lie algebra homology.

For each open $U \subset \mathbb{C}$, we set

$$\mathcal{H}(U) = \Omega^{0,*}_c(U) \oplus \Omega^{1,*}_c(U) \oplus (\mathbb{C}\hbar)^1,$$

where $\mathbb{C}\hbar$ is situated in cohomological degree 1. The Lie bracket is simply

$$[\mu, \nu] = \hbar \int_U \mu \wedge \nu,$$

so $\mathcal{H}$ is a central extension of the abelian dg Lie algebra given by all the Dolbeault forms (with $\overline{\partial}$ as differential).

The factorization algebra Obs^q of quantum observables assigns to each open $U \subset \mathbb{C}$, the cochain complex $C_*(\mathcal{H}(U))$, which we will write as

$$\mathrm{Obs}^q(U) := \left(\mathrm{Sym}\left(\Omega^{1,*}_c(U)[1] \oplus \Omega^{0,*}_c(U)[1]\right)[\hbar], \overline{\partial} + \hbar\Delta\right).$$

The differential has a component $\overline{\partial}$ arising from the underlying cochain complex of $\mathcal{H}$ and a component arising from the Lie bracket, which we'll denote Δ. It is the *BV Laplacian* for this theory.

Below we unpack what information Obs^q encodes by examining some simple open sets and the cohomology $H^* \mathrm{Obs}^q$ on those open sets. As usual, the meaning of a complex is easiest to garner through its cohomology. First, though, we discuss how this example is holomorphically translation invariant (as defined in section 5.1).

Everything in this construction is manifestly translation-invariant, so it remains to verify that the action of $\frac{\partial}{\partial \overline{z}}$ is homotopically trivial. Consider the operator

$$\eta = \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}\overline{z})},$$

which acts on the space of fields $\mathscr{E}$. The operator η maps $\Omega^{1,1}$ to $\Omega^{1,0}$ and $\Omega^{0,1}$ to $\Omega^{0,0}$. Then we see that

$$[\overline{\partial} + \hbar\Delta, \eta] = \frac{\mathrm{d}}{\mathrm{d}\overline{z}},$$

so that the action of $\mathrm{d}/\mathrm{d}\overline{z}$ is homotopically trivial, as desired.

We would like to apply the result of theorem 5.2.2.1, which shows that a holomorphically translation invariant factorization algebra with certain additional conditions gives rise to a vertex algebra. We need to check the conditions for the example of interest. The conditions are the following.

(1) The factorization algebra must have an action of S^1 covering the action on $\mathbb{C}$ by rotation.
(2) For every disc $D(0,r) \subset \mathbb{C}$ (including $r = \infty$), the S^1 action on $\mathrm{Obs}^q(D(0,r))$ must extend to an action of the algebra $\mathcal{D}(S^1)$ of distributions on S^1.
(3) If $\mathrm{Obs}^q_k(D(0,r))$ denotes the kth eigenspace of the S^1 action, then we require that the map
$$H^*(\mathrm{Obs}^q_k(D(0,r))) \to H^*(\mathrm{Obs}^q_k(D(0,s)))$$
is an isomorphism for $r < s$.
(4) Finally, we require that the space $H^*(\mathrm{Obs}^q_k(D(0,r)))$ is a discrete vector space, that is, a colimit of finite dimensional vector spaces.

The first condition is obvious in our example: the S^1 action arises from the natural action of S^1 on $\Omega^{0,*}_c(\mathbb{C})$ and $\Omega^{1,*}_c(\mathbb{C})$. The second condition is also easy to check: if $f \in C^\infty_c(D(0,r))$ then the expression
$$z \mapsto \int_{\lambda \in S^1} \phi(\lambda) f(\lambda z)$$
makes sense for any distribution $\phi \in \mathcal{D}(S^1)$, and defines a continuous and hence smooth map
$$\mathcal{D}(S^1) \times C^\infty_c(D(0,r)) \to C^\infty_c(D(0,r)).$$
To check the remaining two conditions, we need to analyze $H^*(\mathrm{Obs}^q(D(0,r)))$ more explicitly.

5.3.3. Analytic preliminaries. In Chapter **??**, we showed that for any free field theory for which there exists a Green's function for the differential defining the elliptic complex of fields, then there is an isomorphism of differentiable cochain complexes
$$\mathrm{Obs}^{cl}(U)[\hbar] \cong \mathrm{Obs}^q(U)$$
for any open subset U in space time. In our example, we want to understand $H^*(\mathrm{Obs}^q(D(0,r)))$ as a differentiable cochain complex, and in particular, understand its decomposition into eigenspaces for the action of S^1. This remark shows that it suffices to understand the cohomology of the corresponding complex of classical observables. We will do this in this subsection.

We remind the reader of some facts from the theory of several complex variables (references for this material are [GR65], [For91], and [Ser53]). We then use these facts to describe the cohomology of the observables.

5.3.3.1 Proposition. *Every open set $U \subset \mathbb{C}$ is Stein* [**For91**]. *As the product of Stein manifolds is Stein, every product $U^n \subset \mathbb{C}^n$ is Stein.*

Remark: Behnke and Stein [**BS49**] proved that every noncompact Riemann surface is Stein, so the arguments we develop here extend farther than we exploit them.

We need a particular instance of Cartan's theorem B about coherent analytic sheaves [**GR65**].

5.3.3.2 Theorem (Cartan's Theorem B). *Let X be a Stein manifold and let E be a holomorphic vector bundle on X. Then,*

$$H^k(\Omega^{0,*}(X,E),\overline{\partial}) = \begin{cases} 0, & k \neq 0 \\ \mathrm{Hol}(X,E), & k = 0, \end{cases}$$

where $\mathrm{Hol}(X,E)$ *denotes the holomorphic sections of E on X.*

We use a corollary first noted by Serre [**Ser53**]; it is a special case of the Serre duality theorem. (Nowadays, people normally talk about the Serre duality theorem for compact complex manifolds, but in Serre's original paper he proved it for noncompact manifolds too, under some additional hypothesis that will be satisfied on Stein manifolds).

Note that we use the Fréchet topology on $\mathrm{Hol}(X,E)$, obtained as a closed subspace of $C^\infty(X,E)$. We let $E^!$ be the holomorphic vector bundle $E^\vee \otimes K_X$ where K_X is the canonical bundle of X.

5.3.3.3 Corollary. *For X a Stein manifold of complex dimension n, the* compactly-supported *Dolbeault cohomology is*

$$H^k(\Omega_c^{0,*}(X,E),\overline{\partial}) = \begin{cases} 0, & k \neq n \\ (\mathrm{Hol}(X,E^!)^\vee, & k = n, \end{cases}$$

where $(\mathrm{Hol}(X,E^!)^\vee$ *denotes the continuous linear dual to* $\mathrm{Hol}(X,E^!)$.

PROOF. The Atiyah-Bott lemma (see lemma B.10) shows that the inclusion

$$(\Omega_c^{0,*}(X,E),\overline{\partial}) \hookrightarrow (\overline{\Omega_c^{0,*}(X,E)},\overline{\partial})$$

is a chain homotopy equivalence. (Recall that the bar denotes "distributional sections.") As $\overline{\Omega_c^{0,k}}(X,E)$ is the continuous linear dual of $\Omega^{0,n-k}(X,E^!)$, it suffices to prove the desired result for the continuous linear dual complex.

Consider the acyclic complex

$$0 \to \mathrm{Hol}(X,E) \overset{i}{\hookrightarrow} C^\infty(X,E) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X,E) \to \cdots \to \Omega^{0,n}(X) \to 0.$$

Our aim is to show that the linear dual of this complex is also acyclic. Note that this is a compex of Fréchet spaces. The result follows from the following lemma. □

5.3.3.4 Lemma. *If V^* is an acyclic cochain complex of Fréchet spaces, then the dual complex $(V^*)^\vee$ is also acyclic.*

PROOF. Let $\mathrm{d}_i : V^i \to V^{i+1}$ denote the differential. We need to show that the sequence

$$(V^{i+1})^\vee \to (V^i)^\vee \to (V^{i-1})^\vee$$

is exact in the middle. That is, we need to show that if $\alpha : V^i \to \mathbb{C}$ is a continuous linear map, and if $\alpha \circ \mathrm{d}_{i-1} = 0$, then there exists some $\beta : V^{i+1} \to \mathbb{C}$ such that $\alpha = \beta \circ \mathrm{d}_i$.

Note that α is zero on $\operatorname{Im} \mathrm{d}_{i-1} = \operatorname{Ker} \mathrm{d}_i$, so that α descends to a linear map

$$V^i / \operatorname{Ker} \mathrm{d}_i = \operatorname{Im} \mathrm{d}_i \to \mathbb{C}.$$

Since the complex is acyclic, $\operatorname{Im} \mathrm{d}_i = \operatorname{Ker} \mathrm{d}_{i+1}$ as vector spaces. However, it is *not* automatically true that they are the same as *topological* vector spaces, where we view $\operatorname{Im} \mathrm{d}_i$ as a quotient of V^i and $\operatorname{Ker} \mathrm{d}_{i+1}$ as a subspae of V^{i+1}. This is where we use the Fréchet hypothesis: the open mapping theorem holds for Fréchet spaces, and tells us that any surjective map between Fréchet spaces is open. Since $\operatorname{Ker} \mathrm{d}_{i+1}$ is a closed subspae of V^{i+1}, it is a Fréchet space. The map $V^i \to \operatorname{Ker} \mathrm{d}_{i+1}$ is surjective, and therefore open. It follows that $\operatorname{Im} \mathrm{d}_i = \operatorname{Ker} \mathrm{d}_{i+1}$ as topological vector spaces.

From this, we see that our $\alpha : V^i \to \mathbb{C}$ descends to a continuous linear functional on $\operatorname{Ker} \mathrm{d}_{i+1}$. Since this is a closed subspace of V^{i+1}, the Hahn-Banach theorem tells us that it extends to a continuous linear functional on V^{i+1}. ☐

These lemmas allow us to understand the cohomology of classical observables just as a vector space. Since we treat classical observables as a differentiable vector space, i.e. a sheaf of D-modules on the site of smooth manifolds, we are really interested in its cohomology as a sheaf on the site of smooth manifolds. It turns out (perhaps surprisingly) that the isomorphism

$$H^n(\Omega_c^{0,*}(X, E)) = \operatorname{Hol}(X, E^!)^\vee$$

(where X is a Stein manifold of dimension n) is *not* an isomorphism of sheaves o n the smooth site, where we define a smooth map from a manifold M to $\operatorname{Hol}(X, E^!)^\vee$ to be a continuous linear map $\operatorname{Hol}(X, E) \to C^\infty(M)$. However, we have a different interpretation of compactly-supported Dolbeault cohomology that will give a description of it as a sheaf on the site of smooth manifolds.

We will present this description for polydiscs (although it works more generally). If $0 < r_1, \dots, r_n \leq \infty$, let $D_r \subset \mathbb{C}$ be the disc of radius r, and let

$$D_{r_1,\dots,r_n} = D_{r_1} \times \cdots \times D_{r_k} \subset \mathbb{C}^n$$

be the corresponding polydisc.

We can view D_r as an open subset in $\mathbb{P}^1$. Let $\mathscr{O}(-1)$ denote the holomorphic line bundle on $\mathbb{P}^1$ consisting of functions vanishing at ∞.

In general, if X is a complex manifold and $C \subset X$ is a closed subset, we can define

$$\operatorname{Hol}(C) = \operatorname*{colim}_{C \subset U} \operatorname{Hol}(U)$$

to be the germs of holomorphic functions on C. The space $\operatorname{Hol}(C)$ has a natural structure of differentiable vector space, where we view $\operatorname{Hol}(U)$ as a differentiable vector space and take the colimit in the category of differentiable vector spaces. The same definition holds for the space $\operatorname{Hol}(C, E)$ of germs on C of holomorphic sections of a holomorphic vector bundle on E.

We have the following theorem, describing compactly supported Dolbeault cohomology as a differentiable vector space.

5.3.3.5 Theorem. *For a polydisc $D_{r_1,\dots,r_n} \subset \mathbb{C}^n$, we have a natural isomorphism of differentiable vector spaces*

$$H^n(\Omega_c^{0,*}(D_{r_1,\dots,r_n})) \cong \operatorname{Hol}((\mathbb{P}^1 \setminus D_{r_1}) \times \dots (\mathbb{P}^1 \times D_{r_n}), \mathscr{O}(-1)^{\boxtimes n}).$$

Further, all other cohomology groups of $\Omega_c^{0,*}(D_{r_1,\dots,r_n})$ are zero as differentiable vector spaces.

Note that $\mathscr{O}(-1)^{\boxtimes n}$ denotes the line bundle on $(\mathbb{P}^1)^n$ consisting of functions which vanish at infinity in each variable. This isomorphism is invariant under holomorphic symmetries of the polydisc $D_{r_1,\dots,r_k}$, and in particular, under the actions of S^1 by rotation in each coordinate, and under the action of the symmetric group (if the r_i are all the same).

Before we prove this theorem, we need a technical result.

5.3.3.6 Proposition. *For any complex manifold X and holomorphic vector bundle E on X, and any manifold M, the complex $C^\infty(M, \Omega^{0,*}(X, E))$ is a fine resolution of the sheaf on $M \times X$ consisting of smooth sections of the bundle $\pi_X^* E$ which are holomorphic in X.*

Further, if we assume that $H^i(\Omega^{0,*}(X, E)) = 0$ for $i > 0$, then

$$H^i(C^\infty(M, \Omega^{0,*}(X, E))) = \begin{cases} C^\infty(M, \operatorname{Hol}(X, E)) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

PROOF. Note that the first statement follows from the second statement. As, locally on X, the Dolbeault-Grothendieck lemma tells us that the sheaf $\Omega^{0,*}(X, E)$ has no higher cohomology, and the sheaves $C^\infty(M, \Omega^{0,i}(X, E))$ are certainly fine.

To prove the second statement, consider the exact sequence

$$0 \to \operatorname{Hol}(X, E) \to \Omega^{0,0}(X, E) \to \Omega^{0,1}(X, E) \cdots \to \Omega^{0,n}(X, E) \to 0.$$

This is an exact sequence of Fréchet spaces. Theorem A1.6 of [**?**] tells us that the completed projetive tensor product of nuclear Fréchet spaces is an exact functor, that is, takes exact sequences to exact sequences. A result of Grothendieck [Gro52] tells us that for any complete locally convex topological vector space F, $C^\infty(M,F)$ is naturally isomorphic to $C^\infty(M)\widehat{\otimes}_\pi F$, where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. The result follows. ☐

Proof of the theorem. Let M be a smooth manifold. We need to produce an isomorphism, natural in M,

$$C^\infty(M, H^n(\Omega_c^{0,*}(D_{r_1,\dots,r_n}))) \cong C^\infty(M, \mathrm{Hol}((\mathbb{P}^1 \setminus D_{r_1}) \times \dots (\mathbb{P}^1 \times D_{r_n}), \mathscr{O}(-1)^{\boxtimes n})).$$

We also need to show that $C^\infty(M, H^i(\Omega_c^{0,*}(D_{r_1,\dots,r_n})))$ are zero for $i < n$. Note that we are taking the cohomology sheaves, i.e. the sheafification of the presheaf on M which sends U to

$$H^i(C^\infty(U, \Omega_c^{0,*}(D_{r_1,\dots,r_n}))).$$

The first thing to observe is that, for any complex manifold X and holomorphic vector bundle E on X, and any open subset $U \subset X$, there is an exact sequence of sheaves of cochain complex on M

$$0 \to C^\infty(M, \Omega_c^{0,*}(U,E)) \to C^\infty(M, \Omega^{0,*}(X,E)) \to C^\infty(M, \Omega^{0,*}(X \setminus U, E)) \to 0.$$

Indeed, if the K_i are an exhausting family of compact subsets of U, we define $\Omega_{K_i}^{0,*}(U,E)$ to be the kernel of the map

$$\Omega^{0,*}(U,E) \to \Omega^{0,*}(U \setminus K_i, E).$$

This coincides with $\Omega_{K_i}^{0,*}(X,E)$. The sequence

$$0 \to C^\infty(M, \Omega_{K_i}^{0,*}(X,E)) \to C^\infty(M, \Omega^{0,*}(X,E)) \to C^\infty(M, \Omega^{0,*}(X \setminus K_i, E))$$

is therefore exact. It is not necessarily exact on the right.

Now, let us take the colimit of this sequence as $i \to \infty$, where the colimit is taken in the category of sheaves on M. We find an exact sequence

$$0 \to C^\infty(M, \Omega_c^{0,*}(U,E)) \to C^\infty(M, \Omega^{0,*}(X,E)) \to C^\infty(M, \Omega^{0,*}(X \setminus U, E)).$$

Now, we claim that this sequence is exact on the right, as a sequence of sheaves on M. The point is that, locally on M, every smooth function on $M \times (X \setminus U)$ extends to a smooth function on some $M \times (X \setminus K_i)$ and by applying a bump function which is 1 on a neighbourhood of $M \times (X \setminus U)$ and zero on the interior of K_i we can extend it to a smooth function on $M \times X$.

Let's apply this to the case when $X = (\mathbb{P}^1)^n$, $U = D_{r_1,\dots,r_n}$ and $E = \mathscr{O}(-1)^{\boxtimes n}$. Note that in this case E is trivialized on U, so that we find, on taking cohomology, an exact sequence of sheaves on M

$$\cdots \to C^\infty(M, H^i(\Omega_c^{0,*}(D_{r_1,\dots,r_n})) \to C^\infty(M, H^i(\Omega^{0,*}((\mathbb{P}^1)^n, (\text{Ø}(-1))^{\boxtimes n})) \to C^\infty(M, H^i(\Omega^{0,*}((\mathbb{P}^1)^n \setminus D_{r_1,\dots,r_n}, \mathscr{O}(-1$$

Note that the Dolbeault cohomology of $(P^1)^n$ with coefficients in $\mathscr{O}(-1)^{\boxtimes n}$ vanishes. This implies, using the previous proposition, that the middle term in this exact sequence vanishes, so that we get an isomorphism

$$C^\infty(M, H^i(\Omega_c^{0,*}(D_{r_1,\dots,r_n}))) = C^\infty(M, H^{i-1}(\Omega^{0,*}((\mathbb{P}^1)^n \setminus D_{r_1,\dots,r_n}, \mathscr{O}(-1)^{\boxtimes n}))).$$

We need to compute the right hand side of this. We can view the complex appearing on the right hand side as the colimit, as $\varepsilon \to 0$, of

$$C^\infty(M, \Omega^{0,*}((\mathbb{P}^1)^n \setminus \overline{D}_{r_1-\varepsilon,\dots,r_n-\varepsilon}, \mathscr{O}(-1)^{\boxtimes n})).$$

Here $\overline{D}$ indicates the closed disc.

We have a sheaf on $M \times \left((\mathbb{P}^1)^n \setminus \overline{D}_{r_1-\varepsilon,\dots,r_n-\varepsilon}\right)$, which sends an open set $U \times V$ to $C^\infty(U, \Omega^{0,*}(V, \mathscr{O}(-1)^{\boxtimes n}))$. This is a cochain complex of fine sheaves. We are interested in the cohomology of global sections. We can compute this sheaf cohomology using the local-to-global spectral sequence associated to a Čech cover of $(\mathbb{P}^1)^n \setminus \overline{D}_{r_1-\varepsilon,\dots,r_n-\varepsilon}$. We use the Čech cover is given by setting

$$U_i = \mathbb{P}^1 \times \dots (\mathbb{P}^1 \setminus \overline{D}_{r_i-\varepsilon}) \cdots \times \mathbb{P}^1.$$

We take the corresponding cover of $M \times \left((\mathbb{P}^1)^n \setminus \overline{D}_{r_1-\varepsilon,\dots,r_n-\varepsilon}\right)$ given by the opens $M \times U_i$.

Note that, for $k < n$, we have

$$H^*(\Omega^{0,*}(U_{i_1,\dots,i_k}, \mathscr{O}(-1)^{\boxtimes n})) = 0.$$

The previous proposition implies that we also have

$$H^*(C^\infty(M, \Omega^{0,*}(U_{i_1,\dots,i_k}, \mathscr{O}(-1)^{\boxtimes n}))) = 0.$$

The local-to-global spectral sequence then tells us that we have a natural isomorphism

$$H^*(C^\infty(M, \Omega^{0,*}(\mathbb{P}^1 \setminus \overline{D}_{r_1-\varepsilon,\dots,r_n-\varepsilon}, \mathscr{O}(-1)^{\boxtimes n}))) = C^\infty(M, \mathrm{Hol}(U_{1,\dots,n}, \mathscr{O}(-1)^{\boxtimes n}))[-n].$$

That is, all cohomology groups on the left hand side of this equation are zero, except for the top cohomology, which is equal to the vector space on the right.

Note that

$$U_{1,\dots,n} = (\mathbb{P}^1 \setminus \overline{D}_{r_1-\varepsilon}) \times \cdots \times (\mathbb{P}^1 \setminus \overline{D}_{r_n-\varepsilon}).$$

Taking the colimit as $\varepsilon \to 0$, combined with our previous calculations, gives the desired result. Note that this colimit must be taken in the category of sheaves on M. We are also using the fact that sequential colimits commute with formation of cohomology. □

5.3.4. A description of observables. This analytic discussion will allow us to understand both classical and quantum observables of the theory we are considering. Let us first define some basic classical observables.

5.3.4.1 Definition. *On any disk $D(x,r)$ centered at the point x, let $c_n(x)$ denote the linear classical observable*

$$c_n(x) : \gamma \in \Omega^{0,0}(D(x,r)) \mapsto \frac{1}{n!}(\partial_z^n \gamma)(x).$$

Likewise, for $n > 0$, let $b_n(x)$ denote the linear functional

$$b_n(x) : \beta\, dz \in \Omega^{1,0}(D(x,r)) \mapsto \frac{1}{(n-1)!}(\partial_z^{n-1}\beta)(x).$$

These observables descent to elements of $H^0(\mathrm{Obs}^{cl}(D(0,r))$.

Let us introduce some notation to deal with products of these. If $K = (k_1, \dots, k_n)$ or $L = (l_1, \dots, l_m)$ is a multi-index, we let

$$\begin{aligned} b_K(x) &= b_{k_1}(x) \dots b_{k_n}(x) \in H^0(\mathrm{Obs}^{cl}(D(x,r)) \\ c_L(x) &= c_{l_1}(x) \dots c_{l_m}(x). \end{aligned}$$

Of course this makes sense only if $k_i > 0$ for all i. Note that under the natural S^1 action on $H^0(\mathrm{Obs}^{cl}(D(x,r))$, $b_K(x)$ and $c_L(x)$ are of weights $-|K|$ and $-|L|$, where $|K| = \sum k_i$ and similarly for L.

5.3.4.2 Lemma. *(1) The cohomology groups $H^i(\mathrm{Obs}^{cl}(D(x,r))$ vanish (as differentiable vector spaces) unless $i = 0$.*

(2) The monomials $b_K(x)c_L(x)$ form a basis for the weight space $H^0_{|K|+|L|}(\mathrm{Obs}^{cl}(D(x,r))$. Further, they form a basis in the sense of differentiable vector spaces, meaning that any smooth map

$$f : M \to H^0_n(\mathrm{Obs}^{cl}(D(x,r))$$

can be expressed uniquely as a sum

$$f = \sum_{|K|+|L|=n} f_{KL} b_K(x) c_L(x)$$

where the $f_{KL} \in C^\infty(M)$ and locally on M all but finitely many of the f_{KL} are zero.

(3) More generally, any smooth map

$$f : M \to H^0(\mathrm{Obs}^{cl}(D(x,r))$$

can be expressed uniquely as a sum (convergent in a natural topology on $C^\infty(M, H^0(\mathrm{Obs}^{cl}(D(x,r))))$

$$f = \sum f_{KL} b_k(x) c_L(x)$$

where the coefficient functions f_{KL} have the following properties:

(a) Locally on M, $f_{k_1,\dots,k_n,l_1,\dots,l_m} = 0$ for $n + m \gg 0$.

(b) For fixed n and m, the sum

$$\sum f_{k_1,\dots,k_n,l_1,\dots,l_m} z_1^{-k_1} \dots z_n^{-k_n} w_1^{-l_1-1} \dots w_1^{-l_m-1}$$

is absolutely convergent when $|z_i| \geq r$ *and* $|w_j| \geq r$*, in the natural topology on* $C^\infty(M)$*.*

This lemma then gives an explicit description of the spaces $C^\infty(M, H^(\operatorname{Obs}^{cl}(D(x,r))))$.

PROOF. We will set $x = 0$. By definition, $\operatorname{Obs}^{cl}(D(0,r))$ is a direct sum, as differentiable cochain complexes,

$$\operatorname{Obs}^{cl}(U) = \oplus_n \left(\Omega_c^{0,*}(D(0,r)^n, E^{\boxtimes n})[n]\right)_{S_n}$$

where E is the holomorphic vector bundle $\mathscr{O} \oplus K$ on $\mathbb{C}$.

It follows from theorem 5.3.3.5 that, as differentiable vector spaces,

$$H^i(\Omega^{0,*}(D(0,r)^n, E^{\boxtimes n}) = 0$$

unless $i = n$, and that

$$H^n(\Omega^{0,*}(D(0,r)^n, E^{\boxtimes n}) = \operatorname{Hol}((\mathbb{P}^1 \setminus D(0,r)), (\mathscr{O}(-1) \oplus \mathscr{O}(-1)\mathrm{d}z)^{\boxtimes n}).$$

Theorem 5.3.3.5 was stated only for the trivial bundle; but the bundle E is trivial on $D(x,r)$. It is not, however, S^1-equivariantly trivial. The notation $\mathscr{O}(-1)\mathrm{d}z$ indicates that we change the S^1-action on $\mathscr{O}(-1)$.

It follows that we have an isomorphism, of S^1-equivariant differentiable vector spaces,

$$H^0(\operatorname{Obs}^{cl}(D(0,r)) \cong \oplus_n H^0((\mathbb{P}^1 \setminus D(0,r))^n, (\mathscr{O}(-1) \oplus \mathscr{O}(-1)\mathrm{d}z)^{\boxtimes n})_{S_n}.$$

Under this isomorphism, the observable $b_K(0)c_L(0)$ goes to the function

$$\frac{1}{(2\pi i)^{n+m}} z_1^{-k_1} \mathrm{d}z_1 \dots z_n^{-k_n} \mathrm{d}z_n w_1^{-l_1-1} \dots w_m^{-l_m-1}$$

Everything in the statement is now immediate; the topology we use on $H^0(\operatorname{Obs}^{cl}(D(0,r))$ is the one arising as the colimit of the topologies on holomorphic functions on $(\mathbb{P}^1 \setminus \overline{D}(0, r-\varepsilon))^n$ as $\varepsilon \to 0$.

☐

5.3.4.3 Lemma. *We have*

$$H^*(\operatorname{Obs}^q(U)) = H^*(\operatorname{Obs}^{cl}(U))[\hbar]$$

as S^1*-equivariant differentiable vector spaces. This isomorphism is compatible with maps induced from inclusions* $U \hookrightarrow V$ *of open subsets of* $\mathbb{C}$ *(but* not *compatible with the factorization product map).*

PROOF. This follows, as explained in Chapter 4 from the existence of a Green's function for the $\overline{\partial}$ operator, namely

$$G(z_1, z_2) = \frac{\mathrm{d}z_1 - \mathrm{d}z_2}{z_1 - z_2} \in \mathscr{E}(\mathbb{C}) \otimes \mathscr{E}(\mathbb{C})$$

where $\mathscr{E}$ denotes the complex of fields of the $\beta - \gamma$ system. We will make this more explicit later. ☐

We will use the notation $b_K(x)c_L(x)$ for the quantum observables on $D(x,r)$ which arise from the classical observables discussed above, using the isomorphism given by this lemma.

5.3.4.4 Corollary. *The properties listed in lemma 5.3.4.2 also hold for quantum observables. As a result, all the conditions of theorem 5.2.2.1 are satisfied, so that the structure of factorization algebra leads to a* $\mathbb{C}[\hbar]$*-linear vertex algebra structure on the space*

$$V = \oplus_k H^*_k(\mathrm{Obs}^q(D(0,r))$$

where $H^*_l(\mathrm{Obs}^q(D(0,r))$ *indicates the kth eigenspace of the* S^1 *action.*

PROOF. The only conditions we have not so far checked are that the inclusion maps

$$H^*_k(\mathrm{Obs}^q(D(0,r))) \to H^*_k(\mathrm{Obs}^q(D(0,s)))$$

for $r < s$ are quasi-isomorphisms, and that the differentiable vector spaces $V_k = H^*_k(\mathrm{Obs}^q(D(0,r)))$ are countable colimits of finite-dimensional vector spaces, in the category of differentiable vector spaces. Both of these conditions follow immediately from the analog of lemma 5.3.4.2 that applies to quantum observables. ☐

5.3.5. An isomorphism of vertex algebras. Our goal in this subsection is to demonstrate that the vertex algebra constructed by theorme 5.2.2.1 is from the quantum observables of the $\beta - \gamma$-system is isomorphic to a vertex algebra considered in the physics literature called the $\beta - \gamma$ vertex algebra.

We will first describe the $\beta - \gamma$ vertex algebra. We follow [**FBZ04**], notably chapters 11 and 12, to make the dictionary clear.

Let W denote the space of polynomials $\mathbb{C}[a_n, a^*_m]$ where generated by variables a_n, a^*_m $n < 0$ and $m \leq 0$.

5.3.5.1 Definition. *The* $\beta\gamma$ vertex algebra *has state space* W, *vacuum vector* 1, *translation operator* T *the map sending*

$$a_i \to -ia_{i-1}$$
$$a^*_i \to -(i-1)a^*_{i-1},$$

and the vertex operator satisfies

$$Y(a_{-1}, z) = \sum_{n<0} a_n z^{-1-n} + \sum_{n\geq 0} \frac{\partial}{\partial a^*_{-n}} z^{-1-n}$$

and

$$Y(a^*_0, z) = \sum_{n\leq 0} a^*_n z^{-n} - \sum_{n>0} \frac{\partial}{\partial a_{-n}} z^{-n}$$

By the reconstruction theorem 5.0.9.2, these determine the vertex algebra.

The main theorem of this section is the following.

5.3.5.2 Theorem. *Let $V_{\hbar=2\pi i}$ denote the vertex algebra constructed from quantum observables of the $\beta - \gamma$ system, specialized to $\hbar = 2\pi i$. Then, there is an S^1-equivariant isomorphism of vertex algebras*

$$V_{\hbar=2\pi i} \cong W.$$

The circle S^1 acts on W by giving a_i, a^*_j weights i, j respetively.

PROOF. Note that $V_{\hbar=2\pi i}$ is the polynomial algebra on the generators b_n, c_m where $n \geq 1$ and $m \geq 0$. Also b_n, c_m have weights $-n, -m$ respectively, under the S^1-action. We define an isomorphism $V_{\hbar=2\pi i}$ to W by sending b_n to a_{-n} and c_m to a^*_{-m}, and extending it to be an isomorphism of commutative algebras. By the reconstruction theorem, it suffices to calculate $Y(b_1, z)$ and $Y(c_0, z)$.

By the way we defined the vertex algebra associated to the factorization algebra of quantum observables in theorem 5.2.2.1, we have

$$Y(b_1, z)(\alpha) = \mathcal{L}_z m_{z,0}(b_1, \alpha) \in V_{\hbar=2\pi i}((z))$$

where $\mathcal{L}_z$ is denotes Laurent expansion, and

$$m_{z,0} : V_{\hbar=2\pi i} \otimes V_{\hbar=2\pi i} \to \overline{V}_{\hbar=2\pi i}$$

is the map associated to the factorization product coming from the inclusion of the disjoint discs $D(z, r)$ and $D(0, r)$ into $D(0, \infty)$ (where r can be taken to be arbitrarily small).

We are using the Green's function for the $\overline{\partial}$ operator to identify classical and quantum observables. Let us recall how the Green's function leads to an explicit formula for the factorization product.

Let $\mathscr{E}$ denote the sheaf on $\mathbb{C}$ of fields of our theory, so that

$$\mathscr{E}(U) = \Omega^{0,*}(U, \mathscr{O} \oplus K).$$

This is sections of a graded bundle E on $\mathbb{C}$. Let $\mathscr{E}^! = \mathscr{E}[1]$ denote sections of $E^\vee \otimes \omega$ where ω is the bundle of 2-forms on $\mathbb{C}$. We let $\overline{\mathscr{E}}$ denote the sheaf of distributional sections, and $\mathscr{E}_c$ denote compactly-supported sections.

The propagator (or Green's function) is

$$\begin{aligned} P &= \frac{\mathrm{d}z_1 \otimes 1 - 1 \otimes \mathrm{d}z_2}{2\pi i(z_1 - z_2)} \\ &\in \overline{\mathscr{E}}(\mathbb{C}) \widehat{\otimes}_\pi \overline{\mathscr{E}}(\mathbb{C}) \\ &= \mathcal{D}(\mathbb{C}^2, E \boxtimes E) \end{aligned}$$

where $\mathcal{D}$ denotes the space of distributional sections.

We can also view at as being a symmetric and smooth linear map

$$(\dagger) \qquad P : \mathscr{E}_c^!(\mathbb{C}) \otimes \mathscr{E}_c^!(\mathbb{C}) = C_C^\infty(\mathbb{C}^2, (E^!)^{\boxtimes 2}) \to \mathbb{C}.$$

Here $\otimes$ denotes the completed bornological tensor product on the category of convenient vector spaces.

Recall that we identify

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^* \mathscr{E}_c^!(U) = \mathrm{Sym}^* \mathscr{E}_c(U)[1]$$

where the symmetric algebra is defined using the completed tensor product on the category of convenient vector spaces.

We have an order two differential operator

$$\partial_P : \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

for every $U \subset \mathbb{C}$, characterized by the fact that it is a smooth (or, equivalently, continuous) order two differential operator, which is zero on $\mathrm{Sym}^{\leq 1}$ and on Sym^2 is given by applying the map in (†).

For example, if $x, y \in U$, then

$$\partial_P(b_i(x)c_j(y)) = \frac{1}{(i-1)!j!} \frac{\partial^{i-1}}{\partial^{i-1}x} \frac{\partial^j}{\partial^j y} \frac{1}{2\pi i(x-y)}$$

We can identify $\mathrm{Obs}^q(U)$ as a graded vector space with $\mathrm{Obs}^{cl}(U)[\hbar]$, with differential $\mathrm{d} = \mathrm{d}_1 + \hbar \mathrm{d}_2$ a sum of two terms, where d_1 is the differential on $\mathrm{Obs}^{cl}(U)$ and d_2 is the differential arising from the Lie bracket in the shifted Heisenberg Lie algebra whose Chevalley chain complex defines $\mathrm{Obs}^q(U)$.

It is easy to check that

$$[\hbar \partial_P, \mathrm{d}_1] = \mathrm{d}_2.$$

This follows immediately from the fact that $(1,1)$-current $\overline{\partial} P$ on $\mathbb{C}^2$ is the delta-current on the diagonal.

As we explained in section **??**, we get an isomorphism of cochain complexes

$$\begin{aligned} W : \mathrm{Obs}^{cl}(U)[\hbar] &\mapsto \mathrm{Obs}^{q}(U) \\ \alpha &\mapsto e^{\hbar\partial_P}\alpha. \end{aligned}$$

Further, the factorization product map

$$\star_\hbar : \mathrm{Obs}^{cl}(U_1)[\hbar] \times \mathrm{Obs}^{cl}(U_2)[\hbar] \to \mathrm{Obs}^{cl}(V)[\hbar]$$

(if U_1, U_2 are disjoint and in V) which arises from that on Obs^q the identification W is given by the formula

$$\alpha \star_\hbar \beta = e^{-\hbar\partial_P}\left\{ \left(e^{\hbar\partial_P}\alpha\right) \cdot \left(e^{\hbar\partial_P}\beta\right) \right\}.$$

Here $\cdot$ refers to the commutative product on classical observables.

Let us apply this formula to $\alpha = b_1(z)$ and β in the algebra generated by $b_i(0)$ and $c_j(0)$. First note that since $b_1(z)$ is linear, $\hbar\partial_P b_1(z) = 0$. Note also that $[\partial_P, b_1(z)]$ commutes with ∂_P. Thus, we find that

$$\begin{aligned} b_1(z) \star_\hbar \beta &= e^{-\hbar\partial_P}\left(b_1(z) e^{\hbar\partial_P}\beta\right) \\ &= b_1(z)\beta - [\hbar\partial_P, b_1(z)]\beta. \end{aligned}$$

Note that $[\partial_P, b_1(z)]$ is an order one operator, and so a derivation. So it suffices to calculate what it does on generators. We find that

$$\begin{aligned} [\partial_P, b_1(z)]c_j(0) &= \frac{1}{2\pi i}\frac{1}{j!}\frac{\partial^j}{\partial^j w}\frac{1}{z-w} \text{ evaluated at } w = 0 \\ &= \frac{1}{2\pi i} z^{-j-1}. \\ [\partial_P, b_1(z)]b_j(0) &= 0. \end{aligned}$$

In other words,

$$[\partial_P, b_1(z)] = \frac{1}{2\pi i}\sum z^{-j-1}\frac{\partial}{\partial c_j(0)}.$$

Note also that we can expand the cohomology class $b_1(z) \in H^0(\mathrm{Obs}^{cl}(D(0,r))$ (for $|z| < r$) as a sum

$$b_1(z) = \sum_{n=0}^{\infty} b_{n+1}(0) z^n.$$

Indeed, for a classical field $\gamma \in \Omega^1_{hol}(D(0,r))$ solving the equations of motion,

$$\begin{aligned} b_1(z)(\gamma) &= \gamma(z) \\ &= \sum z^n \frac{1}{n!}\gamma^{(n)}(0) \\ &= \sum z^n b_{n_1}(0)(\gamma). \end{aligned}$$

Putting all this together, we find that, for β in the algebra generated by $c_j(0), b_i(0)$, we have

$$b_1(z) \star_\hbar \beta = \left(\sum_{n=0}^{\infty} b_{n+1}(0) z^n + \frac{\hbar}{2\pi i} \sum_{m=0}^{\infty} \frac{\partial}{\partial c_m(0)} z^{-m-1} \right) \beta \in H^0(\mathrm{Obs}^{cl}(D(0,r))[\hbar].$$

Thus, if we set $\hbar = 2\pi i$, we see that the operator product on the space $V_{\hbar=2\pi i}$ matches the one on W if we sent $b_n(0)$ to a_{-n} and $c_n(0)$ to a^*_{-n}. A similar calculation of the operator product of $c_0(0)$ completes the proof. □

5.4. Affine Kac-Moody algebras and factorization algebras

In this section, we will construct a holomorphically translation invariant factorization algebra whose associated vertex algebra is the affine Kac-Moody vertex algebra. This construction is an example of the twisted factorization envelope construction, which also produces the factorization algebras for free field theories (see section 3.6). This construction is our version of Beilinson-Drinfeld's [BD04] chiral envelope construction. We will things up in somewhat greater generality then needed for this theorem.

The input data is the following:

- a Riemann surface Σ ;
- a Lie algebra $\mathfrak{g}$ (for simplicity, we stick to ordinary Lie algebras like $\mathfrak{sl}_2$);
- a $\mathfrak{g}$-invariant symmetric pairing $\kappa : \mathfrak{g}^{\otimes 2} \to \mathbb{C}$.

From this data, we obtain a cosheaf on Σ,

$$\mathfrak{g}^\Sigma : U \mapsto (\Omega_c^{0,*}(U) \otimes \mathfrak{g}, \bar{\partial}),$$

where U denotes an open in Σ. Note that $\mathfrak{g}^\Sigma$ is a cosheaf of dg vector spaces and merely a precosheaf of dg Lie algebras. When κ is nontrivial (though not necessarily nondegenerate), we obtain an interesting -1-shifted central extension on each open:

$$\mathfrak{g}^\Sigma_\kappa : U \mapsto (\Omega_c^{0,*}(U) \otimes \mathfrak{g}, \bar{\partial}) \oplus \underline{\mathbb{C}} \cdot c,$$

where $\underline{\mathbb{C}}$ denotes the locally constant cosheaf on Σ and c is a central element of cohomological degree 1. The bracket is defined by

$$[\alpha \otimes X, \beta \otimes Y]_\kappa := \alpha \wedge \beta \otimes [X, Y] - \frac{1}{2\pi i} \left(\int_U \partial\alpha \wedge \beta \right) \kappa(X, Y) c,$$

with $\alpha, \beta \in \Omega^{0,*}(U)$ and $X, Y \in \mathfrak{g}$. (These constants are chosen to match with the use of κ for the affine Kac-Moody algebra below.)

Remark: As discussed in section 10.1, every dg Lie algebra $\mathfrak{g}$ has a geometric interpretation as a *formal moduli space* $B\mathfrak{g}$. The dg Lie algebra $\mathfrak{g}^\Sigma(U)$ in fact possesses a natural geometric interpretation: it describes "deformations *with compact support in* U of the trivial G-bundle on Σ." Equivalently, it describes the moduli space of holomorphic G-bundles on U which are trivialized outside of a compact set. For U a disc, it is closely related to the affine Grassmannian of G. The affine Grassmannian is defined to be the space of algebraic bundles on a formal disc trivialized away from a point, whereas our formal moduli space describes G-bundles on an actual disc trivialized outside a compact set.

The choice of κ has the interpretation of a line bundle on the formal moduli problem $B\mathfrak{g}^{\Sigma}(U)$ for each U. In general, -1-shifted central extensions of a dg Lie algebra $\mathfrak{g}$ are the same as L_∞-maps $\mathfrak{g} \to \mathbb{C}$, that is, as rank-one representations. Rank-one representations of a group are line bundles on the classifying space of the group. In the same way, rank-one representations of a Lie algebra are line bundles on the formal moduli problem $B\mathfrak{g}$. ☐

As explained in section **??**, we can form the twisted factorization envelope of $\Omega_c^{0,*} \otimes \mathfrak{g}$. Concretely, this factorization algebra assigns to an open subset $U \subset \Sigma$, the complex

$$\mathcal{F}^\kappa : U \mapsto C_*(\mathfrak{g}_\kappa^\Sigma(U)) = (\mathrm{Sym}(\Omega_c^{0,*}(U) \otimes \mathfrak{g}[1])[c], \overline{\partial} + d_{CE}),$$

where c now has cohomological degree 0 in the Lie algebra homology complex. This is a factorization algebra in modules for the ring $\mathbb{C}[c]$ generated by the central parameter. We should therefore think of it as a family of factorization algebras depending on the central parameter c.

Remark: Given a dg Lie algebra $(\mathfrak{g}, d)$, we interpret $C_*\mathfrak{g}$ as the "distributions with support on the closed point of the formal space $B\mathfrak{g}$." Hence, our factorization algebras $\mathcal{F}^\kappa(U)$ describes the κ-twisted distributions supported at the point in $Bun_G(U)$ given by the trivial bundle on U.

This description is easier to understand in its global form, particularly when Σ is a closed Riemann surface. Each point of $P \in Bun_G(\Sigma)$ has an associated dg Lie algebra $\mathfrak{g}_P$ describing the formal neighborhood of P. This dg Lie algebra, in the case of the trivial bundle, is precisely the global sections over Σ of $\mathfrak{g}^\Sigma$. For a nontrivial bundle P, the Lie algebra $\mathfrak{g}_P$ is also global sections of a natural cosheaf, and we can apply the enveloping construction to this cosheaf to obtain a factorization algebras. By studying families of such bundles, we recognize that our construction $C_*\mathfrak{g}_P$ should recover differential operators on $Bun_G(\Sigma)$. When we include a twist κ, we should recover κ-twisted differential operators. When the twist is integral, the twist corresponds to a line bundle on $Bun_G(\Sigma)$ and the twisted differential operators are precisely differential operators for that line bundle.

It is nontrivial to properly define differential operators on the stack $Bun_G(\Sigma)$, and much work continues on porting the full machinery of D-modules onto this stack. At the formal neighbourhood of a point, however, there are no difficulties and our statements are rigorous.

5.4.1. The main result. Note that if we take our Riemann surface to be $\mathbb{C}$, the factorization algebra $\mathcal{F}^\kappa$ is holomorphically translation invariant. This follows from the fact that, on the Lie algebra $\mathfrak{g}_\kappa(U)$ for any open subset U in $\mathbb{C}$, the derivation $\frac{\partial}{\partial \bar{z}}$ is homotopically trivial, where the homotopy is given by $\frac{\partial}{\partial \mathrm{d}\bar{z}}$. It follows that we are in a situation where we can (if certain other properties hold) apply theorem 5.2.2.1. The main result of this section is the following.

Theorem. *The holomorphically translation invariant factorization algebra $\mathcal{F}^\kappa$ on $\mathbb{C}$ satisfies the conditions of theorem 5.2.2.1, and so defines a vertex algebra. This vertex algebra is isomorphic to the affine Kac-Moody vertex algebra.*

Before we can prove this statement, we of course need to describe the affine Kac-Moody vertex aigebra.

Recall that the Kac-Moody Lie algebra is the central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$,

$$0 \to \mathbb{C} \cdot c \to \widehat{\mathfrak{g}}_\kappa \to L\mathfrak{g} \to 0.$$

As vector spaces, we have $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot c$, and the Lie bracket is given by the formula

$$[f(t) \otimes X, g(t) \otimes Y]_\kappa := f(t)g(t) \otimes [X, Y] + \left(\oint f \partial g \right) \kappa(X, Y)$$

for $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$. Here c has cohomological degree 0 and is central. Note, that $\oint t^n \partial t^m$ is $2\pi i m \delta_{m+n,0}$.

Note that $\mathfrak{g}[t]$ is a sub Lie algebra of the Kac-Moody algebra. The *vacuum module* W for the Kac-Moody algebra is the induced representation from the trivial rank one representation of $\mathfrak{g}[t]$. That there is a natural map $\mathbb{C} \to W$ of $\mathfrak{g}[t]$-modules, where $\mathbb{C}$ is the trivial representation. We let $|\emptyset\rangle \in W$ be the image of $1 \in \mathbb{C}$.

As a vector space, we can identify the vacuum representation canonically as

$$W = U(\mathbb{C} \cdot c \oplus t^{-1}\mathfrak{g}[t^{-1}]).$$

As, $\mathbb{C} \cdot c \oplus t^{-1}\mathfrak{g}[t^{-1}]$ is a sub-Lie algebra of $\widehat{\mathfrak{g}}_\kappa$. Thus, the universal enveloping algebra of this sub-algebra acts on W; the action on the vacuum element $|\emptyset\rangle \in W$ gives rise to this isomorphism.

The vacuum module W is a $\mathbb{C}[c]$ module in a natural way, because $\mathbb{C}[c]$ is inside the universal enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$.

5.4.1.1 Definition. *The Kac-Moody vertex algebra is defined as follows. It is a vertex algebra structure over the base ring $\mathbb{C}[c]$ on the vector space W. (Working over the base ring $\mathbb{C}[c]$ simply means all maps are $\mathbb{C}[c]$-linear). By the reconstruction theorem 5.0.9.2, to specify the vertex algebra structure it suffices to specify the state-field map on a subset of elements of W which generate all* *of W (in the sense of the reconstruction theorem). The following state-field operations define the vertex algebra structure on W.*

1. *(1) The vacuum element $|\emptyset\rangle \in W$ is the unit for the vertex algebra, that is, $Y(|\emptyset\rangle, z)$ is the identity.*
2. *(2) If $X \in \mathfrak{g} \subset \widehat{\mathfrak{g}}_\kappa$, we have an element $X|\emptyset\rangle \in W$. We declare that*

$$Y(X|\emptyset\rangle, z) = \sum X_n z^{-1-n}$$

where $X_n = t^n X \in \widehat{\mathfrak{g}}_\kappa$, and we are viewing elements of $\widehat{\mathfrak{g}}_\kappa$ as endomorphisms of W.

5.4.2. Verification of the conditions to define a vertex algebra. We need to verify that $\mathcal{F}^\kappa$ satisfies the conditions listed in theorem **??** guaranteeing that we can construct a vertex algebra. This is entirely parallel to the corresponding statements for the $\beta - \gamma$ system, so we will be brief. The first thing to check is that the natural S^1 action on $\mathcal{F}^\kappa(D(0,r))$ extends to an action of the algebra $\mathcal{D}(S^1)$ of distributions on the circle. This is easy to see.

Then, we need to check that, if $\mathcal{F}^\kappa_l(D(0,r))$ denotes the eigenspace for the S^1-action, then the following properties hold.

1. (1) The inclusion $\mathcal{F}^\kappa_l(D(0,r)) \to \mathcal{F}^\kappa_l(D(0,s))$ for $r < s$ is a quasi-isomorphism.
2. (2) The cohomology $H^*(\mathcal{F}^\kappa_l(D(0,r))$ vanishes (as a sheaf on the site of smooth manifolds) for $l \gg 0$.
3. (3) The differentiable vector spaces $H^*(\mathcal{F}^\kappa_l(D(0,r))$ are countable sequential colimits (in the category of differentiable vector spaces) of finite-dimensional vector spaces.

Note that

$$\mathcal{F}^\kappa(D(0,r)) = \mathrm{Sym}^*\left(\Omega^{0,*}_c(D(0,r),\mathfrak{g})[1] \oplus \mathbb{C}\cdot c\right)$$

with differential a sum $\overline{\partial} + \mathrm{d}_{CE}$ where d_{CE} is the Chevalley-Eilenberg differential. Give $\mathcal{F}^\kappa(D(0,r)$ an increasing filtration, by degree of the symmetric power. This filtration is compatible with the action of S^1 and of $\mathcal{D}(S^1)$. In the associated graded, the differential is just that from the $\overline{\partial}$ differential on $\Omega^{0,*}_c(D(0,r))$.

It follows that there is a spectral sequence (in the category of sheaves on the site of smooth manifolds)

$$H^*\left(\mathrm{Gr}^*\,\mathcal{F}^\kappa_l(D(0,r))\right) \Rightarrow H^*\left(\mathcal{F}^\kappa_l(D(0,r))\right).$$

The analytic results we proved in section **??** concerning compactly supported Dolbeault cohomology immediately imply that $H^*\left(\mathrm{Gr}^*\,\mathcal{F}^\kappa_l(D(0,r))\right)$ satisfy properties $1-3$ above. It follows that the same holds for $H^*(\mathcal{F}^\kappa_l(D(0,r))$.

5.4.3. Proof of the theorem. Let us now prove that the vertex algebra associated to $\mathcal{F}_\kappa$ is isomorphic to the Kac-Moody vertex algebra. The proof will be a little different than the proof of the corresponding result for the $\beta - \gamma$ system.

We first prove a statement concerning the behaviour of the factorization algebra $\mathcal{F}_\kappa$ on annuli. Consider the radial projection map

$$\rho : \mathbb{C}^\times \to \mathbb{R}_{>0}.$$

We can define a factorization algebra $\rho_* \mathcal{F}_\kappa$ on $\mathbb{R}_{>0}$ which assigs to any open subset $U \subset \mathbb{R}_{>0}$ the cochain complex $\mathcal{F}_\kappa(\rho^{-1}(U))$. In aparticular, this factorization algebra assigns to an interval (a, b) the space $\mathcal{F}_\kappa(A(a, b))$ where $A(a, b)$ indicates the annulus of those z with $a < |z| < b$. The operator product map associated to the inclusion of two disjoint intervals in a larger one arises from the operator product map from the inclusion of two disjoint annuli in a larger one.

Recall **??** that any associative algebra A gives rise to a factorization algebra on $\mathbb{R}$ whcih assigns to the interval (a, b) the algebra A, and where the operator product map is the multiplication in A. We let A^{fact} denote the factorization algebra on $\mathbb{R}$ associated to A. The first result we will show is the following.

5.4.3.1 Theorem. *There is an injective map of $\mathbb{C}[c]$-linear factorization algebras on $\mathbb{R}_{>0}$*

$$U(\widehat{\mathfrak{g}}_\kappa)^{fact} \to H^*(\rho_* \mathcal{F}^\kappa)$$

whose image is a dense subspace.

This map is characterized by the following property. Observe that, for every open subset $U \subset \mathbb{C}$, the space

$$\Omega_c^{0,*}(U, \mathfrak{g})[1] \oplus \mathbb{C} \cdot c \subset C_*(\Omega_c^{0,*}(U, \mathfrak{g}) \oplus \mathbb{C} \cdot c[1]) = \mathcal{F}^\kappa(U)$$

of linear elements of $\mathcal{F}^\kappa(U)$ is in fact a subcomplex. By applying this to $U = \rho^{-1}(I)$ for an interval $I \subset \mathbb{R}_{>0}$ and taking cohomology, we obtain a natural cochain map

$$H^1(\Omega_c^{0,*}(\rho^{-1}(I))) \otimes \mathfrak{g}) \oplus \mathbb{C} \cdot c \to H^0(\mathcal{F}^\kappa(I)).$$

We have the natural identification

$$H^1(\Omega_c^{0,*}(\rho^{-1}(I))) = \Omega^1_{hol}(\rho^{-1}(I))^\vee.$$

where $\vee$ indicates continuous linear dual. If $n \in \mathbb{Z}$, then performing a contour integral against z^n defines a linear function on $\Omega^1_{hol}(\rho^{-1}(I))$, and so an element of $H^1(\Omega_c^{0,*}(\rho^{-1}(I)))$ which we call $\phi(z^n)$.

The map

$$\widehat{\mathfrak{g}}_\kappa \to H^*(\mathcal{F}^\kappa(\rho^{-1}(I)))$$

constructed by the theorem factors through the map

$$\begin{aligned} \widehat{\mathfrak{g}}_\kappa &\to H^1(\Omega_c^{0,*}(\rho^{-1}(I))) \otimes \mathfrak{g} \oplus \mathbb{C} \cdot c \\ z^n X &\mapsto \phi(z^n) X \\ c &\mapsto c \end{aligned}$$

PROOF. In Chapter 3 section 3.4, we showed how the universal enveloping algebra of any Lie algebra $\mathfrak{a}$ arises as a factorization envelope. Let $U^{fact}(\mathfrak{a})$ denote the factorization algebra on $\mathbb{R}$ which assigns to U the complex

$$U^{fact}(\mathfrak{a})(U) = C_*(\Omega_c^*(U, a)).$$

Then, we showed that the cohomology of $U^{fact}(\mathfrak{a})$ is locally constant and corresponds to the ordinary universal enveloping algebra $U\mathfrak{a}$.

Let us apply this construction to $\mathfrak{a} = \widehat{\mathfrak{g}}_\kappa$. To prove the theorem, we need to produce a map of factorization algebras on $\mathbb{R}_{>0}$

$$U^{fact}(\widehat{\mathfrak{g}}_\kappa) \to \rho_* \mathcal{F}^\kappa.$$

Since both sides are defined as Chevalley chains of certain Lie algebras, it suffices to produce such a map at the level of dg Lie algebras. We will produce such a map in the homotopical sense.

Let us introduce some notation to describe the Lie algebras we consider. We let $\mathcal{L}_1$ be the precosheaf of dg Lie algebras on $\mathbb{R}$ which assigns to U the dg Lie algebra

$$\mathcal{L}_1(U) = \Omega_c^* \otimes \left(\mathfrak{g}[z, z^{-1}]\right) \oplus \mathbb{C} \cdot c[-1].$$

Thus, $\mathcal{L}_1$ is a central extension of $\Omega_c^* \otimes \mathfrak{g}[z, z^{-1}]$. The cocycle defining the central extension on the open set U is

$$\alpha z^n X \otimes \beta z^m Y \mapsto \left(\int_U \alpha \wedge \beta\right) \kappa(X, Y) \left(\oint z^n \partial_z z^m\right)$$

where $\alpha, \beta \in \Omega_c^*(U)$.

We let

$$\mathcal{L}_2 = \rho_*(\Omega_c^{0,*}(U) \otimes \mathfrak{g} \oplus \mathbb{C} \cdot c[-1])$$

be the precosheaf of dg Lie algebras which assigns to an open subset U the dg Lie algebra

$$\mathcal{L}_2(U) = \Omega_c^{0,*}(\rho^{-1}(U)) \otimes \mathfrak{g} \oplus \mathbb{C} \cdot c[-1]$$

with central extension the one discussed earlier.

Note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are prefactorization algebras valued in the category of dg Lie algebras equipped with the direct sum symmetric monoidal structure. A precosheaf $\mathcal{L}$ of dg Lie algebras is such a prefactorization algebra if it has the property that that, if

U, V are disjoint opens in W, then elements in $\mathcal{L}(W)$ which are in the image of the map from $\mathcal{L}(U)$ commute with those coming from $\mathcal{L}(V)$.

Note also that there is a natural map of factorization dg Lie algebras

$$\Omega_c^* \otimes \widehat{\mathfrak{g}}_\kappa \to \mathcal{L}_1$$

which is the identity on $\Omega_c^* \otimes \mathfrak{g}[z, z^{-1}]$ and sends

$$\alpha c \mapsto \left(\int_U \alpha \right) c$$

for $\alpha \in \Omega_c^*(U)$.

This map is clearly a quasi-isomorphism when U is an interval. It follows immediately that the map

$$U^{fact}(\widehat{\mathfrak{g}}_\kappa) = C_*(\Omega_c^* \otimes \widehat{\mathfrak{g}}_\kappa) \to C_* \mathcal{L}_1$$

is a map of prefactorization algebras which is a quasi-isomorphism on intervals. Therefore, the cohomology prefactorization algebra of $C_*\mathcal{L}_1$ assigns to an interval $U(\widehat{\mathfrak{g}}_\kappa)$, and the factorization product is just the associative product on this algebra.

It suffices to produce a map of precosheaves of dg Lie algebras

$$\mathcal{L}_1 \to \mathcal{L}_2.$$

We do this as follows. First, we define $\mathcal{L}_1'$ to be, like $\mathcal{L}_1$, a central extension of $\Omega_c^* \otimes \mathfrak{g}[z, z^{-1}]$, but where the cocycle defining the central extension is

$$\alpha z^n X \otimes \beta z^n Y \mapsto \left(\int_U \alpha \wedge \beta \right) \kappa(X, Y) \left(\oint z^n \partial_z z^m \right) + \pi \kappa(X, Y) \delta_{n+m,0} \left(\int_U \alpha r \frac{\partial}{\partial r} \beta \right)$$

where the vector field $r \frac{\partial}{\partial r}$ acts by Lie derivative on the form $\beta \in \Omega_c^*(U)$.

It is easy to verify that this cochain is closed, and so defines a central extension. In fact, this central extension, as well as the ones defining $\mathcal{L}_1$ and $\mathcal{L}_2$, are local central extensions of local dg Lie algebras in the sense of definition 3.6.3.1 of Chapter 3. This concept is studied in more detail in subsection **??** of Chapter 10.

To prove the result, we will do the following.

(1) Prove that $\mathcal{L}_1$ and $\mathcal{L}_1'$ are homotopy equivalent prefactorization dg Lie algebras.
(2) Construct a map of prefactorization dg Lie algebras $\mathcal{L}_1' \to \mathcal{L}_2$.

For the first point, note that the extra term $\pi \kappa(X, Y) \delta_{n+m,0} \int_U \alpha r \frac{\partial}{\partial r} \beta$ in the coycle for the central extension of $\Omega_c^* \otimes L\mathfrak{g}$ defining $\mathcal{L}_1'$ is an exact cocycle. It is cobounded (in the sense of dg Lie algebras) by the cochain

$$\pi \kappa(X, Y) \delta_{n+m,0} \int_U \alpha r \iota_{\frac{\partial}{\partial r}} \beta$$

where $\iota_{\frac{\partial}{\partial r}}$ indicates contraction. The fact that this expression cobounds follows from the Cartan homotopy formula. Since the cobounding cochain is also local, $\mathcal{L}_1$ and $\mathcal{L}_1'$ are homotopy equivalent as prefactorization dg Lie algebras.

Now we will produce the desired map $\mathcal{L}_1' \to \mathcal{L}_2$. We use the following notation: r will denote the coordinate on $\mathbb{R}_{>0}$ and the radial coordinate in $\mathbb{C}^\times$, θ the angular coordinate on $\mathbb{C}^\times$ and X will denote an element of $\mathfrak{g}$. We will view $\widehat{\mathfrak{g}}_\kappa$ as $\mathfrak{g}[z, z^{-1}] \oplus \mathbb{C} \cdot c$.

If $U \subset \mathbb{R}_{>0}$ is open, we send

$$\begin{aligned} f(r) z^n X &\mapsto f(r) z^n X \in \mathcal{L}_2 \text{ if } f \in \Omega^0_c(U). \\ f(r) z^n \mathrm{d}r &\mapsto \tfrac{1}{2} e^{i\theta} f(r) z^n X \mathrm{d}\overline{z}. \\ c &\mapsto c. \end{aligned}$$

We need to verify that this is a map of dg Lie algebras (it is obviously a map of precosheaves). Compatibility with the differential follows from the formula for $\frac{\partial}{\partial \overline{z}}$ in polar coordinates:

$$\frac{\partial}{\partial \overline{z}} = \tfrac{1}{2} e^{i\theta} \left(\frac{\partial}{\partial r} - \tfrac{1}{ir} \frac{\partial}{\partial \theta} \right)$$

The only possible issue that can arise when checking compatibility with the Lie bracket is from the central extension. Let us denote the map we have constructed by Φ. The fact that the central extension terms match up follows from the following simple identity:

$$\begin{aligned} & c\kappa(X, Y) \int_{\rho^{-1}(U)} f(r) z^n \partial (z^m g(r) \tfrac{1}{2} e^{i\theta}) \mathrm{d}\overline{z} \\ & \qquad = c\kappa(X, Y) \left\{ 2\pi i m \delta_{n+m,0} \left(\int_U f(r) g(r) \mathrm{d}r \right) + \pi \delta_{n+m,0} \int_U f(r) r \frac{\partial}{\partial r} g(r) \mathrm{d}r \right\} \end{aligned}$$

where $U \subset \mathbb{R}_{>0}$ is an interval. The expression on the right hand side gives the central extension term in the Lie bracket on $\mathcal{L}_1'$, whereas that on the left is the central extension term in the bracket on $\mathcal{L}_2$ applied to the elements $\Phi(f(r) z^n X)$ and $\Phi(g(r) \mathrm{d}r z^m Y)$.

Applying Chevalley chains, we get a map of prefactorization algebras

$$C_* \mathcal{L}_1' \to C^* \mathcal{L}_2 = \rho_* \mathcal{F}_\kappa.$$

Since $C_* \mathcal{L}_1'$ is equivalent to $C_* \mathcal{L}_1$, we get the desired map of factorization algebras on $\mathbb{R}_{>0}$

$$U \widehat{\mathfrak{g}}_\kappa \to \rho_* \mathcal{F}_\kappa.$$

The analytical results (concerning compactly supported Dolbeault cohomology) presented in section 5.3.3 imply immediately this map has dense image.

Finally we will check that this map has the properties stated in the discussion following the statement of the theorem. Suppose that $U \subset \mathbb{R}_{>0}$ is an interval. Then, under the isomorphism

$$U(\widehat{\mathfrak{g}}_\kappa) \cong H^*(C_*(\mathcal{L}_1'(U))$$

the element $z^n X$ (where $X \in \mathfrak{g}$) is represented by the element

$$f(r) z^n X \mathrm{d}r \in \mathcal{L}_1'(U)[1]$$

where f is of compact support and is chosen so that $\int f(r)\mathrm{d}r = 1$.

To check the desired properties, we need to verify that if α is a holomorphic 1-form on $\rho^{-1}(U)$, then

$$\int_{\rho^{-1}(U)} \alpha f(r) z^n \tfrac{1}{2} e^{i\theta} \mathrm{d}\overline{z} = \oint_{|z|=1} \alpha z^n.$$

This follows immediately from Stoke's theorem and the identity

$$f(r) \tfrac{1}{2} e^{i\theta} \mathrm{d}\overline{z} = \overline{\partial}(h(r))$$

where, as above, $h(r) = \int_\infty^r f(r)\mathrm{d}r$. ☐

This theorem shows how to relate observables on an annulus to the universal enveloping algebra of the affine Kac-Moody algebra. Recall that the Kac-Moody vertex algebra is the structure of vertex algebra on the vacuum representation. The next result will show that the vertex algebra associated to the factorization algebra $\mathcal{F}^\kappa$ is also a vertex algebra structure on the vacuum representation.

More precisely, we will show the following.

5.4.3.2 Proposition. *Let*

$$V = \oplus H^*(\mathcal{F}_l^\kappa(D(0,\varepsilon))$$

be the cohomology of the direct sum of the weight spaces of the S^1 *action on* $\mathcal{F}_\kappa(D(0,\varepsilon))$*.*

Then, the map

$$U(\widehat{\mathfrak{g}}_\kappa) \to H^*(\mathcal{F}^\kappa(A(r,r'))$$

constructed in the previous theorem (where $A(r,r')$ *is the annulus) induces an action of* $U(\widehat{\mathfrak{g}}_\kappa)$ *on* V*.*

There is a unique isomorphism of $U(\widehat{\mathfrak{g}}_\kappa)$*-modules from* V *to the vacuum module* W*, which sends the unit observable* $1 \in V$ *to the vacuum element* $|\emptyset\rangle \in W$*.*

Proof. If $\varepsilon < r < r'$, then the factorization product gives a map

$$\mathcal{F}^\kappa(D(0,\varepsilon) \times \mathcal{F}^\kappa(A(r,r')) \to \mathcal{F}^\kappa(D(0,r'))$$

of cochain complexes. Passing to cohomology, and using the relationship between $U(\widehat{\mathfrak{g}}_\kappa)$ and $\mathcal{F}^\kappa(A(r,r'))$, we get a map

$$V \otimes U(\widehat{\mathfrak{g}}_\kappa) \to H^*(\mathcal{F}^\kappa(D(0,r')).$$

Note that on the left hand side, every element is a finite sum of elements in S^1-eigenspaces. This map is S^1-equivariant. Therefore, it lands in the subspace of $H^*(\mathcal{F}(D(0,r'))$ which consists of finite sums of S^1-eigenvectors. This subspace is V. We therefore find a map

$$(\dagger) \qquad U(\widehat{\mathfrak{g}}_\kappa) \otimes V \to V.$$

The fact that the map $U(\widehat{\mathfrak{g}}_\kappa) \to H^*(\mathcal{F}^\kappa(A(r,r'))$ takes the asosciative product on $U(\widehat{\mathfrak{g}}_\kappa)$ to the factorization product map

$$H^*(\mathcal{F}^\kappa(A(r,r')) \otimes H^*(\mathcal{F}^\kappa(A(s,s')) \to H^*(\mathcal{F}^\kappa(A(r,s'))$$

(for $r < r' < s < s'$), combined with the fact that the following diagram

$$\begin{tikzcd}
\mathcal{F}^\kappa(D(0,\varepsilon)) \otimes \mathcal{F}^\kappa(A(r,r')) \otimes \mathcal{F}^\kappa(A(s,s')) \arrow[r] \arrow[d] & \mathcal{F}^\kappa(D(0,\varepsilon)) \otimes \mathcal{F}^\kappa(A(r,s')) \arrow[d] \\
\mathcal{F}^\kappa(D(0,r')) \otimes \mathcal{F}^\kappa(A(s,s')) \arrow[r] & \mathcal{F}^\kappa(D(0,s'))
\end{tikzcd}$$

(whose arrows are the factorization product maps) commutes, implies that the map in equation (†) defines an action of the algebra $U(\widehat{\mathfrak{g}}_\kappa)$ on V.

We need to show that this action identifies V with the vacuum representation W. The putative map $W \to V$ sends the vacuum element $|\emptyset\rangle$ to the unit observable $1 \in V$. To show that this map is well-defined, we need to show that $1 \in V$ is annihilated by the elements $z^n X \in \widehat{\mathfrak{g}}_\kappa$ for $n \geq 0$.

The unit axiom for factorization algebras implies that the following diagram commutes:

$$\begin{tikzcd}
U(\widehat{\mathfrak{g}}_\kappa) \arrow[r] \arrow[d] & H^*(\mathcal{F}^\kappa(A(s,s')) \arrow[d] \\
V \arrow[r] & H^*(\mathcal{F}^\kappa(D(0,s'))
\end{tikzcd}$$

where the left vertical arrow is given by the action of $U(\widehat{\mathfrak{g}}_\kappa)$ on the unit element $1 \in V$, and the right vertical arrow is the map arising from the inclusion $A(r,r') \subset D(0,r')$. The bottom right arrow is the inclusion onto the direct sum of S^1-eigenspaces, which is injective.

The proof of theorem 5.4.3.1 gives an explicit representative for the element $z^n X \in \widehat{\mathfrak{g}}_\kappa$ in $F^\kappa(A(s,s'))$. Namely, let $f(r)$ be a function which is supported in the interval (s,s') and whose integral $\int f(r)\mathrm{d}r$ is one. Then, $z^n X$ is represented by

$$\tfrac{1}{2} e^{i\theta} f(r) \mathrm{d}\bar{z} z^n X \in \Omega^{0,1}_c(A(s,s')).$$

We need to show that this is exact when viewed as an element of $\Omega^{0,1}_c(D(0,s'))$. If we let

$$h(r) = \int_\infty^r f(t)\mathrm{d}t$$

then $h(r) = 0$ for $r \gg s'$ and $h(r) = 1$ for $r < s$. Also, $\frac{\partial}{\partial r} h = f$. The polar-coordinate representation of $\frac{\partial}{\partial \overline{z}}$ tells us that

$$\overline{\partial} h(r) z^n = \tfrac{1}{2} e^{i\theta} f(r) \mathrm{d}\overline{z} z^n X.$$

Thus, we have shown that elements $z^n X \in \widehat{\mathfrak{g}}_\kappa$, where $n \geq 0$, act by zero on the element $1 \in V$. We thus have a unique map of $U(\widehat{\mathfrak{g}}_\kappa$-modules $W \to V$ sending $|\emptyset\rangle \to 1$.

It remains to show that this is an isomorphism. Note that every object we are discussing is filtered. The universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa$ is filtered by saying that $F^i U(\widehat{\mathfrak{g}}_\kappa)$ is the subspace spanned by products of $\leq i$ elements of $\widehat{\mathfrak{g}}_\kappa$. Similarly, the space

$$\mathcal{F}^\kappa(U) = C_*(\Omega^{0,*}_c(U, \mathfrak{g}) \oplus \mathbb{C} \cdot c[-1])$$

is filtered by saying that $F^i \mathcal{F}^\kappa(U)$ is the subcomplex $C_{\leq i}$. All the maps we have been discussing are compatible with these increasing filtrations.

The associated graded of $U(\widehat{\mathfrak{g}}_\kappa)$ is the symmetric algebra of $\widehat{\mathfrak{g}}_\kappa$, and the associated graded of $\mathcal{F}^\kappa(U)$ is the appropriate completed symmetric algebra on $\Omega^{0,*}_c(U, \mathfrak{g})[1] \oplus \mathbb{C} \cdot c$. Upon taking associated graded, the maps

$$\begin{aligned} \operatorname{Gr} U(\widehat{\mathfrak{g}}_\kappa) &\to \operatorname{Gr} H^* \mathcal{F}^\kappa(A(s, s')) \\ \operatorname{Gr} H^*(\mathcal{F}^\kappa(A(s, s')) &\to \operatorname{Gr} H^* \mathcal{F}^\kappa(D(0, s')) \end{aligned}$$

are maps of commutative algebras. It follows that the map

$$\operatorname{Gr} U(\widehat{\mathfrak{g}}_\kappa) \to \operatorname{Gr} V \subset H^* \operatorname{Sym}^* \left(\Omega^{0,*}_c(D(0, s), \mathfrak{g})[1] \oplus \mathbb{C} \cdot c \right)$$

is a map of commutative algebras. Now, $\operatorname{Gr} V$ is the direct sum of the S^1-eigenspaces in the space on the right of this equation. The direct sum of the S^1-eigenspaces in $H^1(\Omega^{0,1}(D(0, s))$ is naturally identified with $z^{-1}\mathbb{C}[z^{-1}]$. Thus, we find the associated graded of the map $U(\widehat{\mathfrak{g}}_\kappa)$ to V is a map of commutative algebras

$$\left(\operatorname{Sym}^* \mathfrak{g}[z, z^{-1}] \right) [c] \to \left(\operatorname{Sym}^* z^{-1} \mathfrak{g}[z^{-1}] \right) [c].$$

We have already calculated that on the generators of the commutative algebra, it arises from the natural projection map

$$\mathfrak{g}[z, z^{-1}] \oplus \mathbb{C} \cdot c \to z^{-1}\mathfrak{g}[z^{-1}] \oplus \mathbb{C} \cdot c.$$

It follows immediately that the map $\operatorname{Gr} W \to \operatorname{Gr} V$ is an isomorphism, as desired. ☐

In order to complete the proof that the vertex algebra associated to the factorization algebra $\mathcal{F}^\kappa$ is isomorphic to the Kac-Moody vertex algebra, we need to identify the operator product expansion map

$$V \otimes V \to V((z)).$$

Recall that this map is defined, in terms of the factorization algebra $\mathcal{F}^\kappa$, as follows. Consider the map

$$m_{z,0} : V \otimes V \to H^*(\mathcal{F}^\kappa(D(0,\infty))$$

defined by restricting the factorization product map

$$H^*(\mathcal{F}^\kappa(D(z,\varepsilon)) \times H^*(\mathcal{F}^\kappa(D(0,\varepsilon)) \to H^*(\mathcal{F}^\kappa(D(0,\infty))$$

to the subspace

$$V \subset H^*(\mathcal{F}^\kappa(D(z,\varepsilon)) = H^*(\mathcal{F}^\kappa(D(0,\varepsilon)).$$

Composing the map $m_{z,0}$ with the map

$$H^*(\mathcal{F}^\kappa(D(0,\infty)) \to \overline{V} = \prod_k V_k$$

where the V_k are the S^1 eigenspaces and the map is the product of the projection maps, we get a map

$$m_{z,0} : V \otimes V \to \overline{V}.$$

This depends holomorphically on z. The operator product map is obtained as the Laurent expansion of $m_{z,0}$.

Our aim is to calculate the operator product map and identify it with the vertex operator map in the Kac-Moody vertex algebra. We use the following notation. If $X \in \mathfrak{g}$, let $X_i = z^i X \in \widehat{\mathfrak{g}}_\kappa$. We denote the action of $\widehat{\mathfrak{g}}_\kappa$ on V by the symbol $\cdot$. Then we have the following.

5.4.3.3 Proposition. *For all $v \in V$,*

$$m_{z,0}(X_{-1} \cdot 1, v) = \sum_{i \in \mathbb{Z}} z^{-i-1}(X_i \cdot v) \in \overline{V}$$

where the sum on the right hand side converges.

Before we prove this proposition, let us observe that it proves our main result:

5.4.3.4 Corollary. *This isomorphism $W \to V$ of $U(\widehat{\mathfrak{g}}_\kappa)$ from the vacuum representation W to V is an isomorphism of vertex algebras, where V is given the vertex algebra structure arising from the factorization algebra $\mathcal{F}^\kappa$, and W is given the Kac-Moody vertex algebra structure defined in 5.4.1.1.*

PROOF. This follows immediately from the reconstruction theorem 5.0.9.2. ☐

PROOF OF THE PROPOSITION. As before, we let $X_i \in \widehat{\mathfrak{g}}_\kappa$ denote Xz^i if $X \in \mathfrak{g}$. As we explained in the discussion following theorem 5.4.3.1, we can view the element

$$X_{-1} \cdot 1 \in V$$

as being represented by X times the linear functional

$$\Omega^1_{hol}(D(0,s)) \to \mathbb{C}$$

which sends

$$\alpha \mapsto \oint z^{-1}\alpha.$$

Cauchy's theorem tells us that if this linear functional sends $h(z)\mathrm{d}z$ (where h is holomorphic) to $2\pi i h(0)$.

Let us fix $z_0 \in A(s,s')$, and let

$$\iota_{z_0} : V \to H^*(\mathcal{F}^\kappa(A(s,s')))$$

denote the map arising from the restriction to V of the map

$$H^*(\mathcal{F}^\kappa(D(0,\varepsilon))) = H^*(\mathcal{F}^\kappa(D(z_0,\varepsilon))) \to H^*(\mathcal{F}^\kappa(A(s,s')))$$

arising from the inclusion of the disc $D(z_0,\varepsilon)$ into the annulus $A(s,s')$. It is clear from the definition of $m_{z_0,0}$ and the axioms of a prefactorization algebra that the following diagram commutes:

$$\begin{tikzcd}
V \otimes V \otimes \mathrm{Id} \arrow[r, "m_{z_0,0}"] \arrow[d, "\iota_{z_0}"'] & \overline{V} \\
H^*(\mathcal{F}^\kappa(A(s,s')) \otimes V \arrow[r] & H^*(\mathcal{F}^\kappa(D(0,\infty))) \arrow[u]
\end{tikzcd}$$

where the bottom right arrow is the restriction to V of the factorization structure map

$$H^*(\mathcal{F}^\kappa(A(s,s')) \otimes H^*(\mathcal{F}^\kappa(D(0,\varepsilon)) \to H^*(\mathcal{F}^\kappa(D(0,\infty)).$$

It therefore suffices to show that

$$\iota_{z_0}(X_{-1}\cdot 1) = \sum_{i\in\mathbb{Z}} X_i z_0^{-i-i} \in H^*(\mathcal{F}^\kappa(A(s,s')))$$

where we view $X_i \in \widehat{\mathfrak{g}}_\kappa$ as elements of $H^0(\mathcal{F}^\kappa(A(s,s')))$ via the map

$$U(\widehat{\mathfrak{g}}^\kappa) \to H^0(\mathcal{F}^\kappa(A(s,s')))$$

constructed in theorem 5.4.3.1.

It is clear from how we construct the factorization algebra $\mathcal{F}^\kappa$ that $\iota_{z_0}(X_{-1}\cdot 1)$ is in the image of the natural map

$$\Omega^{0,*}_c(A(s,s')) \otimes \mathfrak{g} \to \mathcal{F}^\kappa(A(s,s')).$$

Recall that the cohomology of $\Omega^{0,*}_c(A(s,s'))$ is the linear dual of the space of holomorphic 1-forms on the annulus $A(s,s')$. The element $\iota_{z_0}(X_{-1}\cdot 1)$ can thus be represented by a continuous linear map

$$\Omega^1_{hol}(A(s,s')) \to \mathfrak{g}.$$

The map is the one that sends

$$h(z)\mathrm{d}z \mapsto 2\pi i h(z_0) X.$$

Similarly, the elements $X_i \in H^*(\mathcal{F}^\kappa(A(s,s')))$ are represented by the linear maps which send

$$h(z)\mathrm{d}z \mapsto \left(\oint z^i h(z)\mathrm{d}z \right) X.$$

It remains to show that, for all holomorphic functions $h(z)$ on the annulus $A(s,s')$, we have

$$2\pi i h(z_0) = \sum_{i \in \mathbb{Z}} z_0^i \left(\oint_{|z|=s+\varepsilon} z^{-i-1} h(z)\mathrm{d}z \right).$$

This can be proved by a simple argument using Cauchy's theorem. ?As,

$$2\pi i h(z_0) = \oint_{|z|=s+\varepsilon} \frac{h(z)}{z-z_0}\mathrm{d}z - \oint_{|z|=s'-\varepsilon} \frac{h(z)}{z-z_0}\mathrm{d}z.$$

Expanding $(z-z_0)^{-1}$ in the regions when $|z| < |z_0|$ (relevant for the first integral) and when $|z| > |z_0|$ (relevant for the second integral) gives the desired expression.

□

Part 2

Factorization algebras

CHAPTER 6

Factorization algebras: definitions and constructions

Our definition of a prefactorization algebra is closely related to that of a precosheaf or of a presheaf. Mathematicians have found it useful to refine the axioms of a presheaf to those of a sheaf: a sheaf is a presheaf whose value on a large open set is determined, in a precise way, by values on arbitrarily small subsets. In this chapter we describe a similar "descent" axiom for prefactorization algebras. We call a prefactorization algebra satisfying this axiom a *factorization algebra*.

After defining this axiom, our next task is to verify that the examples we have constructed so far, such as the observables of a free field theory, satisfy it. This we do in sections 6.3 and 6.4.

Philosophically, our descent axiom for factorization algebras is important: a prefactorization algebra satisfying descent (i.e., a factorization algebra) is built from local data, in a way that a general prefactorization algebra need not be. However, for many practical purposes, such as the applications to field theory, this axiom is often not essential.

Thus, a reader with little taste for formal mathematics could skip this part and still be able to follow the rest of this book.

6.1. Factorization algebras

A factorization algebra is a prefactorization algebra that satisfies a *local-to-global* axiom. This axiom is the analog of the gluing axiom for sheaves; it expresses how the values on big open sets are determined by the values on small open sets. For sheaves, the gluing axiom says that for any open set U and any cover of that open set, we can determine the value of the sheaf on U from the values on the open cover. For factorization algebras, we require our covers to be fine enough that they capture all the "multiplicative structure" — the structure maps — of a factorization algebra.

We will describe the local-to-global axiom for factorization algebras taking values in vector spaces or chain complexes, but the generalization to an arbitrary symmetric monoidal category is straightforward. In fact, a factorization algebra will be a cosheaf with respect to a modified notion of cover.

6.1.0.5 Definition. *Let U be an open set. A collection of open sets $\mathfrak{U} = \{U_i \mid i \in I\}$ is a* Weiss cover *of U if for any finite collection of points $\{x_1, \ldots, x_k\}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \ldots, x_k\} \subset U_i$.*

The Weiss covers define a Grothendieck topology on $Opens(M)$, the poset category of open subsets of a space M. We call it the *Weiss topology* of M.

Remark: A Weiss cover is certainly a cover in the usual sense, but a Weiss cover typically contains an enormous number of opens. It is a kind of "exponentiation" of the usual notion of cover, because a Weiss cover is well-suited to studying all configuration spaces of finitely many points in U. For instance, given a Weiss cover $\mathfrak{U}$ of U, the collection

$$\{U_i^n \mid i \in I\}$$

provides a cover in the usual sense of $U^n \subset M^n$ for every positive integer n. ◊

Example: For a smooth n-manifold M, there is a simple way to construct a Weiss cover for M. Fix a Riemannian metric on M, and consider

$$\mathfrak{B} = \{B_r(x) \, : \, \forall x \in M, \text{ with } 0 < r < InjRad(x)\},$$

the collection of open balls, running over each point $x \in M$, whose radii are less than the injectivity radius at x. We obtain a Weiss cover by taking the collection of all finite tuples of disjoint balls in $\mathfrak{B}$. Another construction is simply to take the collection of open sets in M diffeomorphic to a disjoint union of finitely many copies of the open n-ball. ◊

The examples above suggest the following.

6.1.0.6 Definition. *We say that a cover $\mathfrak{U} = \{U_\alpha\}$ of M* generates *the Weiss cover $\mathfrak{V}$ if every open $V \in \mathfrak{V}$ is given by a finite disjoint union of opens U_α from $\mathfrak{U}$.*

6.1.1. Strict factorization algebras.

The value of a factorization algebra on U is determined by its behavior on a Weiss cover, just as the value of a cosheaf on an open set U is determined by its value on any cover of U.

In order to motivate our definition of factorization algebra, let us write briefly recall the cosheaf axiom. A precosheaf Φ on M is a cosheaf if, for every open cover $\{U_i \mid i \in I\}$ of an open set $U \subset M$, the sequence

$$\bigoplus_{i,j} \Phi(U_i \cap U_j) \to \bigoplus_k \Phi(U_k) \to \Phi(U)$$

is exact on the right. (Alternatively, one can say the map $\oplus_k \Phi(U_k) \to \Phi(U)$ coequalizes the pair of maps $\oplus_{i,j} \Phi(U_i \cap U_j) \rightrightarrows \oplus_k \Phi(U_k)$.)

6.1.1.1 Definition. *A prefactorization algebra is a* lax factorization algebra *if it has the property: For every open subset $U \subset M$ and every Weiss cover $\{U_i \mid i \in I\}$ of U, the sequence*

$$\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_k \mathcal{F}(U_k) \to \mathcal{F}(U)$$

is exact on the right. That is, $\mathcal{F}$ is a lax factorization algebra if it is a cosheaf with respect to the Weiss topology.

A lax factorization algebra is a strict factorization algebra *if, in addition, for every pair of disjoint open sets $U, V \in M$, the natural map*

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$$

is an isomorphism.

To summarize, a strict factorization algebra satisfies two conditions: a (co)descent axiom and a factorization axiom. The descent axiom says that it is a cosheaf with respect to the Weiss topology, and the factorization axiom says that its value on finite collections of disjoint opens factors into a tensor product of the values on each open.

6.1.2. The Čech complex and homotopy factorization algebras. Now suppose we have a prefactorization algebra $\mathcal{F}$ taking values in complexes. We will define what it means for $\mathcal{F}$ to be a *homotopy factorization algebra*. This will happen when $\mathcal{F}(U)$ is quasi-isomorphic to the Čech complex constructed from any Weiss cover.

To motivate the definition, let us first recall the definition of a homotopy cosheaf. Let Φ be a pre-cosheaf on M, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a cover of some open subset U of M. The *Čech complex of* $\mathfrak{U}$ *with coefficients in* Φ is is defined in the usual way as

$$\check{C}(\mathfrak{U}, \Phi) = \bigoplus_{k=1}^{\infty} \left(\bigoplus_{\substack{j_1,\dots,j_k \in I \\ j_i \text{ all distinct}}} \Phi(U_{j_1} \cap \cdots \cap U_{j_k})[k-1] \right)$$

where the differential is defined in the usual way. Note that this Čech complex is the normalized cochain complex arising from a simplicial cochain complex, where we evaluate Φ on the simplicial space $\mathfrak{U}_\bullet$ associated to the cover $\mathfrak{U}$.

We say that Φ is a *homotopy cosheaf* if the natural map from the Čech complex to $\Phi(U)$ is a quasi-isomorphism for every open $U \subset M$ and every open cover of U.

6.1.2.1 Definition. *A* lax homotopy factorization algebra *on X is a prefactorization algebra $\mathcal{F}$ valued in cochain complexes, with the property that for every open set $U \subset X$ and Weiss cover $\mathfrak{U}$ of U, the natural map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a quasi-isomorphism. That is, $\mathcal{F}$ is a homotopy cosheaf with respect to the Weiss topology.

A lax homotopy factorization algebra is a homotopy factorization algebra *if, in addition, for every pair U, V of disjoint open subsets of X, the natural map*

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$$

is a quasi-isomorphism.

Remark: The notion of strict factorization algebra is not appropriate for the world of cochain complexes. Whenever we refer to a factorization algebra in cochain complexes, we will mean a homotopy factorization algebra. ◊

6.1.3. Factorization algebras valued in a multicategory. The factorization algebras of ultimate interest to us take values not in the symmetric monoidal category of cochain complexes, but in the multicategory of differentiable cochain complexes (section B). In this setting, the descent axiom continues to make sense but the factorization axiom does not, as we cannot speak about tensoring the values on disjoint opens.

Thus, we need to define what it means to be a factorization algebra in a multicategory $\mathcal{C}$. We will assume that $\mathcal{C}$ is equipped with a realization functor from the category $\mathcal{C}^{\triangle}$ of simplicial objects of $\mathcal{C}$ to the original category $\mathcal{C}$. This will allow us to define the Čech complex of an object of $\mathcal{C}$. (In the category of differentiable cochain complexes, the geometric realization of a simplicial object is defined in the same way as it is in the category of ordinary cochain complexes.)

We will also assume that $\mathcal{C}$ is equipped with some notion of weak equivalence (weak equivalences in differentiable cochain complexes are defined in section B).

As before, let Disj_M be the multicategory whose objects are open sets in M and whose multi-morphisms are defined by

$$\mathrm{Hom}(U_1, \dots, U_n; V) = \begin{cases} * & \text{if } U_1 \sqcup \dots U_n \subset V \\ \emptyset & \text{otherwise.} \end{cases}$$

6.1.3.1 Definition. *Let $\mathcal{C}$ be a multicategory with the structures listed above. A* lax factorization algebra *$\mathcal{F}$ with values in $\mathcal{C}$ is a functor* $\mathrm{Disj}_M \to \mathcal{C}$ *with the property that, for all open subsets $U \subset M$ and all Weiss covers $\mathfrak{U}$ of U, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a weak equivalence.

Remark: Suppose that $\mathcal{C}$ is the multicategory underlying a symmetric monoidal category $\mathcal{C}^{\otimes}$. Then a factorization algebra valued in $\mathcal{C}$, considered as a multicategory, is the same as a lax factorization algebra valued in $\mathcal{C}^{\otimes}$, considered as a symmetric monoidal category. ◊

6.1.3.2 Definition. *A* differentiable factorization algebra *is a factorization algebra valued in the multicategory of differentiable cochain complexes.*

Remark: Although the factorization axiom does not directly extend to the setting of multicategories, we can examine the structure map

$$f \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(U), \mathcal{F}(V); \mathcal{F}(U \sqcup V))$$

for two disjoint opens U and V and ask about the image of f. For almost all the examples constructed in this book, one can see that the image is dense with respect to the natural topology on $\mathcal{F}(U \sqcup V)$, which is a cousin of the factorization axiom. In practice, though, it is the descent axiom that is important. ♢

6.1.4. Factorization algebras in quantum field theory. We have seen (section 1.4) how prefactorization algebras appear naturally when one thinks about the structure of observables of a quantum field theory. It is natural to ask whether the local-to-global axiom which distinguishes factorization algebras from prefactorization algebras also has a quantum-field theoretic interpretation.

The local-to-global axiom we posit states, roughly speaking, that all observables on an open set $U \subset M$ can be built up as sums of observables supported on arbitrarily small open subsets of M. To be concrete, let us consider a Weiss cover $\mathfrak{U}_\varepsilon$ of M, built out of all open balls in M of radius smaller than ε. Applied to this Weiss cover, our local-to-global axiom states that any observable $O \in \mathrm{Obs}(U)$ can be written as a sum of observables of the form $O_1 O_2 \cdots O_k$, where $O_i \in \mathrm{Obs}(B_{\delta_i}(x_i))$ and $x_1, \dots, x_k \in M$.

By taking ε to be very small, we see that our local-to-global axiom implies that all observables can be written as sums of products of observables which are supported as close as we like to points in U.

This is a physically reasonable assumption: most of the observables (or operators) that are considered in quantum field theory textbooks are supported at points, so it might make sense to restrict attention to observables built from these.

However, more global observables are also considered in the physics literature. For example, in a gauge theory, one might consider the observable which measures the monodromy of a connection around some loop in the space-time manifold. How would such observables fit into the factorization algebra picture?

The answer reveals a key limitation of our axioms: *the concept of factorization algebra is only appropriate for perturbative quantum field theories*. Indeed, in a perturbative gauge theory, the gauge field (i.e., the connection) is taken to be an infinitesimally small perturbation $A_0 + \delta A$ of a fixed connection A_0, which is a solution to the equations of motion. There is a well-known formula (the time-ordered exponential) expressing the holonomy of $A_0 + \delta A$ as a power series in δA, where the coefficients of the power series are given as integrals over L^k, where L is the loop which we are considering.

This expression shows that the holonomy of $A_0 + \delta A$ can be built up from observables supported at points (which happen to lie on the loop L). Thus, the holonomy observable will form part of our factorization algebra.

However, if we are not working in a perturbative setting, this formula does not apply, and we would not expect (in general) that the prefactorization algebra of observables satisfies the local-to-global axiom.

6.2. Locally constant factorization algebras

If M is an n-dimensional manifold, then prefactorization algebras locally bear a resemblance to E_n algebras. After all, a prefactorization algebra prescribes a way to combine the elements associated to k distinct balls into an element associated to a big ball containing all k balls. In fact, E_n algebras form a full subcategory of factorization algebras on $\mathbb{R}^n$.

6.2.0.1 Definition. *A factorization algebra $\mathcal{F}$ on an n-manifold M is* locally constant *if for each inclusion of open sets $U \subset U'$ where U is a deformation retraction of U', then the map $\mathcal{F}(U) \to \mathcal{F}(U')$ is a quasi-isomorphism of cochain complexes.*

A central example is the factorization algebra $\mathcal{F}_A$ on $\mathbb{R}$ given by an associative algebra A.

Lurie [**Lurb**] has shown the following vast extension of this example.

6.2.0.2 Theorem. *There is an equivalence of $(\infty, 1)$-categories between E_n algebras and locally constant factorization algebras on $\mathbb{R}^n$.*

We remark that Lurie (and others) uses a different gluing axiom than we do. A careful comparison of the different axioms and a proof of their equivalence (for locally constant factorization algebras) can be found in [**Mat**].

6.2.1. Many examples of E_n algebras arise naturally from topology, such as labelled configuration spaces (as discussed in the work of Segal, McDuff, Bodigheimer, Salvatore, and Lurie). We will discuss an important example, that of mapping spaces.

Recall that (in the appropriate category of spaces) there is an isomorphism $\text{Maps}(U \sqcup V, X) \cong \text{Maps}(U, X) \times \text{Maps}(V, X)$. This fact suggests that we might fix a target space X and define a prefactorization algebra by sending an open set U to $\text{Maps}(U, X)$. This construction almost works, but it is not clear how to "extend" a map $f : U \to X$ from U

to a larger open set $V \ni U$. By working with "compactly-supported" maps, we solve this issue.

Fix (X, p) a pointed space. Let F denote the prefactorization algebra on M sending an open set U to the space of compactly-supported maps from U to (X, p). (Here, "f is compactly-supported" means that the closure of $f^{-1}(X - p)$ is compact.) Then F is a prefactorization algebra in the category of pointed spaces. (Composing with the singular chains functor gives a prefactorization algebra in abelian groups, but we will work at the level of spaces.)

Note that this prefactorization algebra is locally constant: if $U \hookrightarrow U'$ is an inclusion of open subsets where U is a deformation retraction of U', then the map $F(U) \to F(U')$ is a weak homotopy equivalence.

Note also that there is a natural isomorphism

$$F(U_1 \sqcup U_2) = F(U_1) \times F(U_2)$$

if U_1, U_2 are disjoint.

Let's consider for a moment the case when $M = \mathbb{R}^n$. If $D \subset \mathbb{R}^n$ is a ball, there is a equivalence

$$F(D) \simeq \Omega_p^n X$$

between the space of compactly supported maps $D \to X$ and the n-fold based loop space of X, based at the point p. (We are using the topologist's notation $\Omega_p^n X$ for the n-fold loop space; hopefully, this use does not confuse due to the standard notation for the space of n-forms.)

To see this equivalence, note that a compactly supported map $f : D \to X$ extends uniquely to a map from the closed ball $\overline{D}$, sending the boundary $\partial\overline{D}$ to the base point p of X. Since $\Omega_p^n X$ is defined to be the space of maps of pairs $(\overline{D}, \partial\overline{D}) \to (X, p)$, we have constructed the desired map from $F(D)$ to $\Omega_p^n X$. It is easily verified that this map is a homotopy equivalence.

If D_1, D_2 are disjoint balls contained in a ball D_3, the prefactorization structure gives us a map

$$F(D_1) \times F(D_2) \to F(D_3).$$

These maps correspond to the standard E_n structure on the n-fold loop space.

For a particularly nice example, consider the case where the source manifold is $M = \mathbb{R}$. The structure maps of F then describe the standard concatentation product on the space $\Omega_p X$ of based loops in X. At the level of components, we recover the standard product on $\pi_0 \Omega X = \pi_1(X, x)$.

This prefactorization algebra F does not always satisfy the gluing axiom. However, Salvatore [Sal99] and Lurie [Lurb] have shown that if X is sufficiently connected, this prefactorization algebra is in fact a factorization algebra.

6.2.2. The direct relationship between E_n algebras and locally constant factorization algebras on $\mathbb{R}^n$ raises the question of what the local-to-global axiom means from an algebraic point of view. Evaluating a locally constant factorization algebra on a manifold M is known as *factorization homology* or *topological chiral homology*.

The following result, for $n = 1$, is striking and helpful.

6.2.2.1 Theorem. *For A an E_1 algebra (e.g., an associative algebra), there is a weak equivalence*

$$\mathcal{F}_A(S^1) \simeq HH_*(A),$$

where $\mathcal{F}_A$ denotes the locally constant factorization algebra on $\mathbb{R}$ associated to $A$and $HH_*(A)$ denotes the Hochschild homology of A.

Here $HH_*(A)$ means any cochain complex quasi-isomorphic to the usual bar complex (i.e., we are interested in more than the mere cohomology groups).

This result has several proofs in the literature, depending on choice of gluing axiom and level of generality (for instance, one can work with algebra objects in more general ∞-categories). It is one of the primary motivations for the higher dimensional generalizations.

There is a generalization of this result even in dimension 1. Note that there are more general ways to extend $\mathcal{F}_A$ from the real line to the circle, by allowing “monodromy.” Let σ denote an automorphism of A. Pick an orientation of S^1 and fix a point p in S^1. Let $\mathcal{F}_A^\sigma$ denote the prefactorization algebra on S^1 such that on $S^1 - \{p\}$ it agrees with $\mathcal{F}_A$ but where the structure maps across p use the automorphism σ. For instance, if L is a small interval to the left of p, R is a small interval to the right of p, and M is an interval containing both L and R, then the structure map is

$$\begin{array}{ccc} A \otimes A & \to & A \\ a \otimes b & \mapsto & a \otimes \sigma(b) \end{array}$$

where the leftmost copy of A corresponds to L and so on. It is natural to view the copy of A associated to an interval containing p as the $A - A$ bimodule A^σ where A acts as the left by multiplication and the right by σ-twisted multiplication.

6.2.2.2 Theorem. *There is a weak equivalence $\mathcal{F}_A^\sigma(S^1) \simeq HH_*(A, A^\sigma)$.*

Beyond the one-dimensional setting, some of the most useful insights into the meaning of factorization homology arise from its connection to the cobordism hypothesis and extended topological field theories. See section 4 of [**Lur09b**] for an overview of these ideas, and [**Sch**] for further developments and detailed proofs.

For a deeper discussion of factorization homology than we've given, we recommend [**Fra**] to start, as it combines a clear overview with a wealth of applications. A lovely expository account is [**Gin**]. Locally constant factorization algebras already possess a substantial literature, as they sit at the nexus of manifold topology and higher algebra. See, for instance, [**And**], [**AFT**], [**Fra13**], [**GTZb**], [**GTZa**], [**Hora**], [**Horb**], [**Lurb**], [**Mat**], and [**MW12**].

6.3. Factorization algebras from cosheaves

The goal of this section is to describe a natural class of factorization algebras. The factorization algebras that we construct from classical and quantum field theory will be closely related to the factorization algebras discussed here.

The main result of this section is that, given a nice cosheaf of vector spaces or cochain complexes F on a manifold M, the functor $\mathrm{Sym}^* F : U \mapsto \mathrm{Sym}(F(U))$ is a factorization algebra. It is clear how this functor is a prefactorization algebra (see example **??**); the hard part is verifying that it satisfies the local-to-global axiom. The examples in which we are ultimately interested arise from cosheaves F that are compactly supported sections of a vector bundle, so we focus on cosheaves of this form.

We begin by providing the definitions necessary to state the main result of this section. We then state the main result and explain its role for the rest of the book. Finally, we prove the lemmas that culminate in the proof of the main result.

6.3.1. Preliminary definitions.

6.3.1.1 Definition. *A* local cochain complex *on M is a graded vector bundle E on M (with finite rank), whose smooth sections will be denoted by $\mathscr{E}$, equipped with a differential operator* $\mathrm{d} : \mathscr{E} \to \mathscr{E}$ *of cohomological degree* 1 *satisfying* $\mathrm{d}^2 = 0$.

Recall the notation from section 3.5. For E be a local cochain complex on M and U an open subset of M, we use $\mathscr{E}(U)$ to denote the cochain complex of smooth sections of E on U, and we use $\mathscr{E}_c(U)$ to denote the cochain complex of compactly supported sections of E on U. Similarly, let $\overline{\mathscr{E}}(U)$ denote the distributional sections on U and let $\overline{\mathscr{E}}_c(U)$ denote the compactly supported distributional sections of E on U. In the appendix (B) it is shown that these four cochain complexes are differentiable cochain complexes in a natural way.

We use $E^! = E \otimes \mathrm{Dens}_M$ to denote the appropriate dual object. We give $E^!$ the differential that is the formal adjoint to d on E. Note that, ignoring the differential, $\overline{\mathscr{E}}_c(U)$ is the continuous dual to $\mathscr{E}^!(U)$ and that $\mathscr{E}_c(U)$ is the continuous dual to $\overline{\mathscr{E}}^!(U)$.

The factorization algebras we will discuss are constructed from the symmetric algebra on the vector spaces $\mathscr{E}_c(U)$ and $\overline{\mathscr{E}}^!_c(U)$. Note that, since $\overline{\mathscr{E}}^!_c(U)$ is dual to $\mathscr{E}(U)$, we can view $\widehat{\mathrm{Sym}}\,\overline{\mathscr{E}}^!_c(U)$ as the algebra of formal power series on $\mathscr{E}(U)$. Thus, we often write

$$\widehat{\mathrm{Sym}}\,\overline{\mathscr{E}}^!_c(U) = \mathscr{O}(\mathscr{E}(U)),$$

because we view this algebra as the space of "functions on $\mathscr{E}(U)$."

6.3.2. Note that if $U \to V$ is an inclusion of open sets in M, then there are natural maps of commutative dg algebras

$$\begin{aligned}
\mathrm{Sym}^*\,\mathscr{E}_c(U) &\to \mathrm{Sym}^*\,\mathscr{E}_c(V)\\
\widehat{\mathrm{Sym}}^*\mathscr{E}_c(U) &\to \widehat{\mathrm{Sym}}^*\mathscr{E}_c(V)\\
\mathrm{Sym}^*\,\overline{\mathscr{E}}_c(U) &\to \mathrm{Sym}^*\,\overline{\mathscr{E}}_c(V)\\
\widehat{\mathrm{Sym}}^*\overline{\mathscr{E}}_c(U) &\to \widehat{\mathrm{Sym}}^* br\mathscr{E}_c(V).
\end{aligned}$$

Thus, each of these symmetric algebras forms a precosheaf of commutative algebras, and thus a prefactorization algebra. We denote these prefactorization algebras by $\mathrm{Sym}^*\,\mathscr{E}_c$ and so on.

6.3.3. The main result of this section is the following.

6.3.3.1 Theorem. *We have the following parallel results for vector bundles and local cochain complexes.*

1. *Let E be a vector bundle on M. Then*
   (a) $\mathrm{Sym}^*\,\mathscr{E}_c$ *and* $\mathrm{Sym}^*\,\overline{\mathscr{E}}_c$ *are strict (non-homotopical) factorization algebras valued in the category of differentiable vector spaces, and*
   (b) $\widehat{\mathrm{Sym}}^*\mathscr{E}_c$ *and* $\widehat{\mathrm{Sym}}^*\overline{\mathscr{E}}_c$ *are strict (non-homotopical) factorization algebras valued in the category of differentiable pro-vector spaces.*
2. *Let E be a local cochain complex on M. Then*
   (a) $\mathrm{Sym}^*\,\mathscr{E}_c$ *and* $\mathrm{Sym}^*\,\mathscr{E}_c$ *are homotopy factorization algebras valued in the category of differentiable cochain complexes, and*
   (b) $\widehat{\mathrm{Sym}}^*\mathscr{E}_c$ *and* $\widehat{\mathrm{Sym}}^*\mathscr{E}_c$ *are homotopy factorization algebras valued in the category of differentiable pro-cochain complexes.*

PROOF. Let us first prove the strict (non-homotopy) version of the result. To start with, consider the case of $\operatorname{Sym}^* \mathscr{E}_c$. We need to verify the local-to-global axiom (section 6.1).

Let U be an open set in M and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a Weiss cover of U. We need to prove that $\operatorname{Sym}^* \mathscr{E}_c(U)$ is the cokernel of the map

$$\bigoplus_{i,j \in I} \operatorname{Sym}^*(\mathscr{E}_c(U_i \cap U_i)) \to \bigoplus_{i \in I} \operatorname{Sym}^*(\mathscr{E}_c(U_i)).$$

This map is compatible with the decomposition of $\operatorname{Sym}^* \mathscr{E}_c(U)$ into symmetric powers. Thus, it suffices to show that

$$\operatorname{Sym}^m \mathscr{E}_c(U) = \operatorname{coker}\left(\oplus_{i,j \in I} \operatorname{Sym}^m(\mathscr{E}_c(U_i \cap U_j)) \to \oplus_{i \in I} \operatorname{Sym}^m \mathscr{E}_c(U_i)\right),$$

for all m.

Now, observe that

$$\mathscr{E}_c(U)^{\otimes m} = \mathscr{E}_c^{\boxtimes m}(U^m)$$

where $\mathscr{E}_c^{\boxtimes m}$ is the cosheaf on U^m of compactly supported smooth sections of the vector bundle $E^{\boxtimes m}$, which denotes the external product of E with itself m times.

Thus it is enough to show that

$$\mathscr{E}_c^{\boxtimes m}(U^m) = \operatorname{coker}\left(\bigoplus_{i,j \in I} \mathscr{E}_c^{\boxtimes m}\left((U_i \cap U_j)^m\right) \to \bigoplus_{i \in I} \mathscr{E}_c^{\boxtimes m}\left(U_i^m\right)\right).$$

Our cover $\mathfrak{U}$ is a Weiss cover. This means that, for every finite set of points $x_1, \dots, x_k \in M$, we can find an open U_i in the cover $\mathfrak{U}$ containing every x_j. This implies that the subsets of U^m of the form $(U_i)^m$, where $i \in I$, cover U^m. Further,

$$(U_i)^m \cap (U_j)^m = (U_i \cap U_j)^m.$$

The desired isomorphism now follows from the fact that $\mathscr{E}_c^{\boxtimes m}$ is a cosheaf on M^m.

The same argument applies to show that $\operatorname{Sym}^* \mathscr{E}_c^!$ is a factorization algebra. In the completed case, essentially the same argument applies, with the subtlety (see B) that, when working with pro-cochain complexes, the direct sum is completed.

For the homotopy case, the argument is similar. Let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a Weiss cover of an open subset U of M. Let $\mathcal{F} = \operatorname{Sym}^* \mathscr{E}_c$ denote the prefactorization algebras we are considering (the argument will below will apply when we use the completed symmetric product or use $\overline{\mathscr{E}}_c$ instead of $\mathscr{E}_c$). We need to show that map

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is an equivalence, where the left hand side is equipped with the standard Čech differential.

Let $\mathcal{F}^m(U) = \mathrm{Sym}^m \mathscr{E}_c$. Both sides of the displayed equation above split as a direct sum over m, and the map is compatible with this splitting. (If we use the completed symmetric product, this decomposition is as a product rather than a sum.)

We thus need to show that the map

$$\bigoplus_{i_1,\dots,i_n} \mathrm{Sym}^m \left(U_{i_1} \cap \cdots \cap U_{i_n}\right)[n-1] \to \mathrm{Sym}^m \left(\mathscr{E}_c(U)\right)$$

is a weak equivalence.

For $i \in I$, we get an open subset $U_i^m \subset U^m$. Since $\mathfrak{U}$ is a Weiss cover of U, these open subsets form a cover of U^m. Note that

$$U_{i_1}^m \cap \cdots \cap U_{i_n}^m = \left(U_{i_1} \cap \cdots \cap U_{i_n}\right)^m .$$

Note that $\mathscr{E}_c(U)^{\otimes m}$ can be naturally identified with $\Gamma_c(U^m, E^{\boxtimes m})$ (where the tensor product is the completed projective tensor product).

Thus, to show that the Čech descent axiom holds, we need to verify that the map

$$\bigoplus_{i_1,\dots,i_n \in PI} \Gamma_c\left(U_{i_1}^m \cap \cdots \cap U_{i_n}^m, E^{\boxtimes m}\right)[n-1] \to \Gamma_c(V^m, E^{\boxtimes m})$$

is a quasi-isomorphism. The left hand side above is the Čech complex for the cosheaf of compactly supported sections of $E^{\boxtimes m}$ on V^m. Standard partition-of-unity arguments show that this map is a weak equivalence. □

6.4. Factorization algebras from local Lie algebras

We just showed that for a local cochain complex E, the prefactorization algebra $\widehat{\mathrm{Sym}}\mathscr{E}_c$ is, in fact, a factorization algebra. What this construction says is that the "functions on $\mathscr{E}$," viewed as a space, satisfy a locality condition on the manifold M over which $\mathscr{E}$ lives. We can reconstruct functions about $\mathscr{E}(M)$ from knowing about functions on $\mathscr{E}$ with very small support on M. But $\mathscr{E}$ is a simple kind of space, as it is linear in nature. (We should remark that $\mathscr{E}$ is a simple kind of *derived* space because it is a cochain complex.) We now extend to a certain type of nonlinear situation.

In section 3.6, we introduced the notion of a local dg Lie algebra. Since a local cochain complex is an Abelian local dg Lie algebra, we might hope that the Chevalley-Eilenberg cochain complex $C^*\mathcal{L}$ of a local Lie algebra $\mathcal{L}$ also forms a factorization algebra. As we explain in chapter 10, a local dg Lie algebra can be interpreted as a derived space that is nonlinear in nature. In this setting, the Chevalley-Eilenberg cochain complex are the "functions" on this space. Hence, if $C^*\mathcal{L}$ is a factorization algebra, we would know that functions on this nonlinear space $\mathcal{L}$ can also be reconstructed from functions localized on the manifold M.

In section 3.6, we also constructed an important class of prefactorization algebras: the factorization envelope of a local dg Lie algebra. Again, in the preceding section, we showed $\mathrm{Sym}\,\mathscr{E}_c$ is a factorization algebra, which we can view as the factorization envelope of the Abelian local Lie algebra $\mathscr{E}[-1]$. Thus, we might expect that the factorization envelope of a local Lie algebra satisfies the local-to-global axiom.

We will now demonstrate that both these Lie-theoretic constructions are factorization algebras.

6.4.0.2 Theorem. *Let L be a local dg Lie algebra on a manifold M. Then the prefactorization algebras*

$$\begin{aligned} \mathbb{U}\mathcal{L} :& U \mapsto C_*(\mathcal{L}_c(U)) \\ \mathbb{O}\mathcal{L} :& U \mapsto C^*(\mathcal{L}(U)) \end{aligned}$$

are factorization algebras.

Remark: The argument below applies, with very minor changes, to a local L_∞ algebra, a modest generalization we introduce later (see definition **??**). ◊

PROOF. The proof is a spectral sequence argument, and we will reuse this idea throughout the book (notably in proving the quantum observables form a factorization algebra).

We start with the factorization envelope. Note that for any dg Lie algebra $\mathfrak{g}$, the Chevalley-Eilenberg chains $C_*\mathfrak{g}$ have a natural filtration $F^n = \mathrm{Sym}^{\leq n}(\mathfrak{g}[1])$ compatible with the differential. The first page of the associated spectral sequence is simply the cohomology of $\mathrm{Sym}(\mathfrak{g}[1])$, where $\mathfrak{g}$ is viewed as a cochain complex rather than a Lie algebra (i.e., we extend the differential on $\mathfrak{g}[1]$ as a coderivation to the cocommutative coalgebra $\mathrm{Sym}(\mathfrak{g}[1])$).

Consider the Čech complex of $\mathbb{U}\mathcal{L}$ with respect to some Weiss cover $\mathfrak{U}$ for an open U in M. Applying the filtration above to each side of the map

$$\check{C}(\mathfrak{U}, \mathbb{U}\mathcal{L}) \to \mathbb{U}\mathcal{L}(U),$$

we get a map of spectral sequences. On the first page, we have a quasi-isomorphism by theorem 6.3.3.1. Hence the original map of filtered complexes is a quasi-isomorphism.

Now we provide the analogous argument for $\mathbb{O}\mathcal{L}$. For any dg Lie algebra $\mathfrak{g}$, the Chevalley-Eilenberg cochains $C^*\mathfrak{g}$ have a natural filtration $F^n = \widehat{\mathrm{Sym}}^{\geq n}(\mathfrak{g}^\vee[-1])$ compatible with the differential. The first page of the associated spectral sequence is simply the cohomology of $\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])$, where we view $\mathfrak{g}^\vee[-1]$ as a cochain complex and extend its differential as a derivation to the completed symmetric algebra.

Consider the Čech complex of $\mathcal{O}\mathcal{L}$ with respect to some Weiss cover $\mathfrak{U}$ for an open U in M. Applying the filtration above to each side of the map

$$\check{C}(\mathfrak{U}, \mathcal{O}\mathcal{L}) \to \mathcal{O}\mathcal{L}(U),$$

we get a map of spectral sequences. On the first page, we have a quasi-isomorphism by theorem 6.3.3.1. Hence the original map of filtered complexes is a quasi-isomorphism. ☐

6.5. Some examples of computations

The examples at the heart of this book have an appealing aspect: it is straightforward to compute the global sections of the factorization algebra – its factorization homology – because we do not need to use the gluing axiom directly. The theorems in the preceding sections allow us to use analysis to compute the colimit of the complicated diagram.

In this section, we compute the global sections of several examples we've already studied in the preceding chapters.

6.5.1. Enveloping algebras. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^{\mathbb{R}}$ denote the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ on the real line $\mathbb{R}$. Recall proposition 3.4.0.1 which showed that the factorization envelope $\mathbb{U}\mathfrak{g}^{\mathbb{R}}$ recovers the universal enveloping algebra $U\mathfrak{g}$.

This factorization algebra $\mathbb{U}\mathfrak{g}^{\mathbb{R}}$ is also defined on the circle.

6.5.1.1 Proposition. *There is a weak equivalence*

$$\mathbb{U}\mathfrak{g}^{\mathbb{R}}(S^1) \simeq C_*(\mathfrak{g}, U\mathfrak{g}^{ad}) \simeq HH_*(U\mathfrak{g}),$$

where in the middle we mean the Lie algebra homology complex for $U\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint action

$$v \cdot a = va - av,$$

where $v \in \mathfrak{g}$ and $a \in U\mathfrak{g}$ and va denotes multiplication in $U\mathfrak{g}$.

The second equivalence is a standard fact about Hochschild homology (see, e.g., [Lod98]). It arises by constructing the natural map from the Chevalley-Eilenberg complex to the Hochschild complex, which is a quasi-isomorphism because the filtration (inherited from the tensor algebra) induces a map of spectral sequences that is an isomorphism on the E_1 page.

Note one interesting consequence of this theorem: the structure map for an interval mapping into the whole circle corresponds to the trace map $U\mathfrak{g} \to U\mathfrak{g}/[U\mathfrak{g}, U\mathfrak{g}]$.

PROOF. The wonderful fact here is that we do not need to pick a Weiss cover and work with the Čech complex. Instead, we simply need to examine $C_*(\Omega^*(S^1) \otimes \mathfrak{g})$.

One approach is to use the natural spectral sequence arising from the filtration $F^k = \mathrm{Sym}^{\leq k}$. The E_1 page is $C_*(\mathfrak{g}, U\mathfrak{g})$, and one must verify that there are no further pages.

Alternatively, recall that the circle is formal, so $\Omega^*(S^1)$ is quasi-isomorphic to its cohomology $H^*(S^1)$ as a dg algebra. Thus, we get a homotopy equivalence of dg Lie algebras

$$\Omega^*(S^1) \otimes \mathfrak{g} \simeq \mathfrak{g} \oplus \mathfrak{g}[-1],$$

where, on the right, $\mathfrak{g}$ acts by the (shifted) adjoint action on $\mathfrak{g}[-1]$. Now,

$$C_*(\mathfrak{g} \oplus \mathfrak{g}[-1]) \cong C_*(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}),$$

where $\mathrm{Sym}\, \mathfrak{g}$ obtains a $\mathfrak{g}$-module structure from the Chevalley-Eilenberg homology complex. Direct computation verifies this action is precisely the adjoint action of $\mathfrak{g}$ on $U\mathfrak{g}$ (we use, of course, that $U\mathfrak{g}$ and $\mathrm{Sym}\, \mathfrak{g}$ are isomorphic as vector spaces). $\square$

6.5.2. The free scalar field in dimension 1.

Recall from section 4.3 that the Weyl algebra is recovered from the factorization algebra of quantum observables Obs^q for the free scalar field on $\mathbb{R}$. We know that the global sections of Obs^q on a circle should thus have some relationship with the Hochschild homology of the Weyl algebra, but we will see that the relationship depends on the ratio of the mass to the radius of the circle.

For simplicity, we restrict attention to the case where S^1 has a rotation-invariant metric, radius r, and total length $L = 2\pi r$. Let S^1_L denote this circle $\mathbb{R}/\mathbb{Z}L$. We also have our observables live over $\mathbb{C}$ rather than $\mathbb{R}$.

6.5.2.1 Proposition. *If the mass $m = 0$, then $H^k \mathrm{Obs}^q(S^1_L)$ is $\mathbb{C}[\hbar]$ for $k = 0, -1$ and vanishes for all other k.*

If the mass m satisfies $mL = in$ for some integer n, then $H^k \mathrm{Obs}^q(S^1_L)$ is $\mathbb{C}[\hbar]$ for $k = 0, -2$ and vanishes for all other k.

For all other values of mass m, $H^0 \mathrm{Obs}^q(S^1_L) \cong \mathbb{C}[\hbar]$ and all other cohomology groups are zero.

Hochschild homology with monodromy (recall theorem 6.2.2.2) provides an explanation for this result. The equations of motion for the free scalar field locally have a two-dimensional space of solutions, but on a circle the space of solutions depends on the relationship between the mass and the length of the circle. In the massless case, a constant function is always a solution, no matter the length. In the massive case, there is either a two-dimensional space of solutions (for certain imaginary masses, because our conventions) or a zero-dimensional space. Viewing the space of local solutions as a local system, we have monodromy around the circle, determined by the Hamiltonian.

When the monodromy is trivial (so $mL \in i\mathbb{Z}$), we are simply computing the Hochschild homology of the Weyl algebra. Otherwise, we have a nontrivial automorphism of the Weyl algebra.

PROOF. We directly compute the global sections, in terms of the analysis of the local Lie algebra

$$\mathcal{L} = \left(C^\infty(S^1_L) \xrightarrow{\triangle + m^2} C^\infty(S^1_L)[-1] \right) \oplus \mathbb{C}\hbar$$

where $\hbar$ has cohomological degree 1 and the bracket is

$$[\phi^0, \phi^1] = \hbar \int_{S^1_L} \phi^0 \phi^1$$

with ϕ^k a smooth function with cohomological degree $k = 0$ or 1. We need to compute $C_*(\mathcal{L})$.

Fourier analysis allows us to make this computation easily. We know that the exponentials $e^{ikx/L}$, with $k \in \mathbb{Z}$, form a topological basis for smooth functions on S^1 and that

$$(\triangle + m^2) e^{ikx/L} = \left(\frac{k^2}{L^2} + m^2 \right) e^{ikx/L}.$$

Hence, for instance, when $m^2 L^2$ is not a square integer, it is easy to verify that the complex

$$C^\infty(S^1_L) \xrightarrow{\triangle + m^2} C^\infty(S^1_L)[-1]$$

has trivial cohomology. Thus, $H^*\mathcal{L} = \mathbb{C}\hbar$ and so $H^*(C_*\mathcal{L}) = \mathbb{C}[\hbar]$, where in the Chevalley-Eilenberg complex $\hbar$ is shifted into degree 0.

When $m = 0$, we have

$$H^*\mathcal{L} = \mathbb{R}x \oplus \mathbb{R}\xi \oplus \mathbb{R}\hbar,$$

where x represents the constant function in degree 0 and ξ represents the constant function in degree 1. The bracket on this Lie algebra is

$$[x, \xi] = \hbar L.$$

Hence $H^0(C_*\mathcal{L}) \cong \mathbb{C}[\hbar]$ and $H^{-1}(C_*\mathcal{L}) \cong \mathbb{C}[\hbar]$, and the remaining cohomology groups are zero.

When $m = in/L$ for some integer n, we see that the operator $\triangle + m^2$ has two-dimensional kernel and cokernel, both spanned by $e^{\pm inx/L}$. By a parallel argument to the case with $m = 0$, we see $H^0(C_*\mathcal{L}) \cong \mathbb{C}[\hbar] \cong H^{-2}(C_*\mathcal{L})$ and all other groups are zero. □

6.5.3. The Kac-Moody factorization algebras. Recall from example **??** and section **??** that there is a factorization algebra on a Riemann surface Σ associated to every affine Kac-Moody Lie algebra. For $\mathfrak{g}$ a simple Lie algebra with symmetric invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$, we have a shifted central extension of the local Lie algebra $\Omega^{0,*}_{\Sigma} \otimes \mathfrak{g}$ with shift

$$\omega(\alpha, \beta) = \int_{\Sigma} \langle \alpha, \partial\beta \rangle_{\mathfrak{g}}.$$

Let $\mathbb{U}_{\omega}\mathfrak{g}^{\Sigma}$ denote the twisted factorization envelope for this extended local Lie algebra.

6.5.3.1 Proposition. *Let g denote the genus of a closed Riemann surface Σ. Then*

$$H^*(\mathbb{U}^{\sigma}_{\omega}\mathfrak{g}(\Sigma)) \cong H^*(\mathfrak{g}, U\mathfrak{g}^{\otimes g})[c]$$

where c denotes a parameter of cohomological degree zero.

PROOF. We need to understand the Lie algebra homology

$$C_*(\Omega^{0,*}(\Sigma) \otimes \mathfrak{g} \oplus \mathbb{C}c),$$

whose differential has the form $\overline{\partial} + \mathrm{d}_{\mathrm{Lie}}$. (Note that in the Lie algebra, c has cohomological degree 1.)

Consider the filtration $F^k = \mathrm{Sym}^{\leq k}$ by polynomial degree. The purely analytic piece $\overline{\partial}$ preserves polynomial degree while the Lie part $\mathrm{d}_{\mathrm{Lie}}$ lowers polynomial degree by 1. Thus the E_1 page of the associated spectral sequence has $\mathrm{Sym}(H^*(\Sigma, \mathscr{O}) \otimes \mathfrak{g} \oplus \mathbb{C}c)$ as its underlying graded vector space. The differential now depends purely on the Lie-theoretic aspects. Moreover, the differential on further pages of the spectral sequence are zero.

In the untwisted case (where there is no extension), we find that the E_1 page is precisely

$$H^*(\mathfrak{g}, U\mathfrak{g}^{\otimes g}),$$

by computations directly analogous to those for proposition 6.5.1.1.

In the twisted case, the only subtlety is to understand what happens to the extension. Note that $H^0(\Omega^{0,*}(\Sigma)$ is spanned by the constant functions. The pairing ω vanishes if a constant function is an input, so we know that the central extension does not contribute to the differential on the E_1 page.

An alternative proof is to use the fact the $\Omega^{0,*}(\Sigma)$ is homotopy equivalent to its cohomology as a dg algebra. Hence we can compute $C_*(H^*(\Sigma, \mathscr{O}) \otimes \mathfrak{g} \oplus \mathbb{C}c)$ instead. □

We can use the ideas of sections 4.5 and **??** to understand the "correlation functions" of this factorization algebra $\mathbb{U}_{\omega}\mathfrak{g}^{\Sigma}$. More precisely, given a structure map

$$\mathbb{U}_{\omega}\mathfrak{g}^{\Sigma}(V_1) \otimes \cdots \otimes \mathbb{U}_{\omega}\mathfrak{g}^{\Sigma}(V_n) \to \mathbb{U}_{\omega}\mathfrak{g}^{\Sigma}(\Sigma)$$

for some collection of disjoint opens $V_1, \ldots, V_n \subset \Sigma$, we will provide a method for describing the image of this structure map. It has the flavor of a Wick's formula.

Let $[\mathcal{O}]$ denote the cohomology class of an element $\mathcal{O}$ of $\mathbb{U}_\omega \mathfrak{g}^\Sigma$. We want to encode relations between cohomology classes.

As Σ is closed, Hodge theory lets us construct an operator $\overline{\partial}^{-1}$, which vanishes on the harmonic functions and (0,1)-forms but provides an inverse to $\overline{\partial}$ on the complementary spaces. (We must make a choice of Riemannian metric, of course, to do this.) Now consider an element

$$a_1 \cdots a_k$$

of cohomological degree 0 in $\mathbb{U}_\omega \mathfrak{g}^\Sigma(\Sigma)$, where each a_j lives in

$$\Omega^{0,1}(\Sigma) \otimes \mathfrak{g} \oplus \mathbb{C}c.$$

Then we have

$$\begin{aligned}(\overline{\partial} + \mathrm{d}_{\mathrm{Lie}}) \left((\overline{\partial}^{-1} a_1) a_2 \cdots a_k \right) &= \overline{\partial}(\overline{\partial}^{-1} a_1) a_2 \cdots a_k + \mathrm{d}_{\mathrm{Lie}} \left((\overline{\partial}^{-1} a_1) a_2 \cdots a_k \right) \\ &= \overline{\partial}(\overline{\partial}^{-1} a_1) a_2 \cdots a_k + \sum_{j=2}^{k} [\overline{\partial}^{-1} a_1, a_j] a_2 \cdots \widehat{a_j} \cdots a_k \\ &\quad + \sum_{j=2}^{k} \int_\Sigma \left\langle \overline{\partial}^{-1} a_1, \partial a_j \right\rangle_{\mathfrak{g}} c\, a_2 \cdots \widehat{a_j} \cdots a_k.\end{aligned}$$

When a_1 is in the complementary space to the harmonic forms, we know $\overline{\partial}\overline{\partial}^{-1} a_1 = a_1$ so we have the relation

$$0 = [a_1 \cdots a_k] + \sum_{j=2}^{k} [[\overline{\partial}^{-1} a_1, a_j] a_2 \cdots \widehat{a_j} \cdots a_k] + \sum_{j=2}^{k} \int_\Sigma \left\langle \overline{\partial}^{-1} a_1, \partial a_j \right\rangle_{\mathfrak{g}} [c\, a_2 \cdots \widehat{a_j} \cdots a_k],$$

which allows us to iteratively reduce the "polynomial" degree of the original term $[a_1 \cdots a_k]$ (from k to $k-1$ or less, in this case).

This relation looks somewhat complicated but it is easy to understand for k small. For instance, in $k = 1$, we see that

$$0 = [a_1]$$

whenever a_1 is in the space orthogonal to the harmonic forms. For $k = 2$, we get

$$0 = [a_1 a_2] + [[\overline{\partial}^{-1} a_1,\, a_2]] + \int_\Sigma \left\langle \overline{\partial}^{-1} a_1, \partial a_2 \right\rangle_{\mathfrak{g}} [c].$$

6.5.4. The free $\beta\gamma$ system. In section 5.3 we studied the local structure of the free $\beta\gamma$ theory; in other words, we carefully examined the simplest structure maps for the factorization algebra Obs^q of quantum observables on the plane $\mathbb{C}$. It is natural to ask about the global sections on a closed Riemann surface.

There are many ways, however, to extend this theory to a Riemann surface Σ. Let $\mathcal{L}$ be a holomorphic line bundle on Σ. Then the $\mathcal{L}$-twisted free $\beta\gamma$ system has fields

$$\mathscr{E} = \Omega^{0,*}(\Sigma, \mathcal{L}) \oplus \Omega^{1,*}(\Sigma, \mathcal{L}^\vee),$$

where $\mathcal{L}^\vee$ denotes the dual line bundle. The -1-symplectic pairing on fields is to apply pointwise the evaluation pairing between $\mathcal{L}$ and $\mathcal{L}^\vee$ and then to integrate the resulting density. The differential is the $\overline{\partial}$ operator for these holomorphic line bundles.

Let $\mathrm{Obs}^q_{\mathcal{L}}$ denote the factorization algebra of quantum observables for the $\mathcal{L}$-twisted $\beta\gamma$ system. Locally on Σ, we know how to understand the structure maps: pick a trivialization of $\mathcal{L}$ and employ our work from section 5.3.

6.5.4.1 Proposition. *Let $b_k = \dim H^k(\Sigma, \mathcal{L})$. Then $H^* \mathrm{Obs}^q_{\mathcal{L}}(\Sigma)$ is a rank-one free module over $\mathbb{C}[\hbar]$ and concentrated in degree $-b_0 - b_1$.*

PROOF. As usual, we use the spectral sequence arising from the filtration $F^k = \mathrm{Sym}^{\leq k}$ on Obs^q. The first page just depends on the cohomology with respect to $\overline{\partial}$, so the underlying graded vector space is

$$\mathrm{Sym}\left(H^*(\Sigma, \mathcal{L})[1] \oplus H^*(\Sigma, \mathcal{L}^\vee \otimes \Omega^1_{hol})[1]\right)[\hbar],$$

where Ω^1_{hol} denotes the holomorphic cotangent bundle on Σ. By Serre duality, we know this is the symmetric algebra on a $+1$-symplectic vector space, concentrated in degrees 0 and -1 and with dimension $b_0 + b_1$ in each degree. The remaining differential in the spectral sequence is the BV Laplacian for the pairing, and so the cohomology is spanned by the maximal purely odd element, which has degree $-b_0 - b_1$. ☐

CHAPTER 7

Formal aspects of factorization algebras

7.1. Pushing forward factorization algebras

A crucial feature of factorization algebras is that they push forward nicely. Let M and N be topological spaces admitting Weiss covers and let $f : M \to N$ be a continuous map. Given a Weiss cover $\mathfrak{U} = \{U_\alpha\}$ of an open $U \subset N$, let $f^{-1}\mathfrak{U} = \{f^{-1}U_\alpha\}$ denote the preimage cover of $f^{-1}U \subset M$. Observe that $f^{-1}\mathfrak{U}$ is Weiss: given a finite collection of points $\{x_1, \ldots, x_n\}$ in $f^{-1}U$, the image points $\{f(x_1), \ldots, f(x_n)\}$ are contained in some U_α in $\mathfrak{U}$ and hence $f^{-1}U_\alpha$ contains the x_j.

7.1.0.2 Definition. *Given a factorization algebra $\mathcal{F}$ on a space M and a continuous map $f : M \to N$, the* pushforward factorization algebra *$f_*\mathcal{F}$ on N is defined by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Note that for the map to a point $f : M \to pt$, the pushforward factorization algebra $f_*\mathcal{F}$ is simply the global sections of $\mathcal{F}$. We also call this the *factorization homology* of $\mathcal{F}$ on M.

7.2. The category of factorization algebras

In this section, we explain how prefactorization algebras and factorization algebras form categories. In fact, they naturally form multicategories (or colored operads). We also explain how these multicategories are enriched in simplicial sets when the (pre)factorization algebras take values in cochain complexes.

7.2.1. Morphisms and the category structure.

7.2.1.1 Definition. *A* morphism of prefactorization algebras *$\phi : F \to G$ consists of a map $\phi_U : F(U) \to G(U)$ for each open $U \subset M$, compatible with the structure maps. That is, for any open V and any finite collection $U_1, \ldots, U_k$ of pairwise disjoint open sets, each contained in V, the* *following diagram commutes:*

$$\begin{tikzcd} F(U_1) \otimes \cdots \otimes F(U_k) \arrow[r, "\phi_{U_1} \otimes \cdots \otimes \phi_{U_k}"] \arrow[d] & G(U_1) \otimes \cdots \otimes G(U_k) \arrow[d] \\ F(V) \arrow[r, "\phi_V"] & G(V) \end{tikzcd} .$$

Likewise, all the obvious associativity relations are respected.

Remark: When our prefactorization algebras take values in cochain complexes, we require the ϕ_U to be cochain maps, i.e., they each have degree 0 and commute with the differentials. When our prefactorization algebras take values in differentiable cochain complexes, we require in addition that the maps ϕ_U are smooth. ◊

7.2.1.2 Definition. *On a space X, we denote the category of prefactorization algebras on X taking values in the multicategory $\mathcal{C}$ by $PreFA(X, \mathcal{C})$. The category of factorization algebras, $FA(X, \mathcal{C})$, is the full subcategory whose objects are the factorization algebras.*

In practice, $\mathcal{C}$ will normally be the multicategory of differentiable cochain complexes.

7.2.2. The multicategory structure. Let $S\mathcal{C}$ denote the universal symmetric monoidal category containing the multicategory $\mathcal{C}$. Any prefactorization algebra valued in $\mathcal{C}$ gives rise to one valued in $S\mathcal{C}$.

There is a natural tensor product on $PreFA(X, S\mathcal{C})$, as follows. Let F, G be prefactorization algebras. We define $F \otimes G$ by

$$F \otimes G(U) = F(U) \otimes G(U),$$

and we simply define the structure maps as the tensor product of the structure maps. For instance, if $U \subset V$, then the structure map is

$$F(U \subset V) \otimes G(U \subset V) : F \otimes G(U) = F(U) \otimes G(U) \to F(V) \otimes G(V) = F \otimes G(V).$$

7.2.2.1 Definition. *Let $PreFA_{mc}(X, \mathcal{C})$ denote the multicategory arising from the symmetric monoidal product on $PreFA(X, S\mathcal{C})$. That is, if F_i, G are prefactorization algebras valued in $\mathcal{C}$, we define the set of multi-morphisms by*

$$PreFA_{mc}(F_1, \cdots, F_n \mid G)$$

to be the set of maps of $S\mathcal{C}$-valued prefactorization algebras

$$F_1 \otimes \cdots \otimes F_n \to G.$$

Factorization algebras inherit this multicategory structure.

7.2.3. Enrichment over simplicial sets. In this subsection we will explain how the multicategory of factorization algebras with values in an appropriate multicategory is simplicially enriched, in a natural way. In order to define this simplicial enrichment, we need to introduce some notation.

The first thing to define is the algebra $\Omega^*(X)$ of smooth forms on a simplicial set X. An element $\omega \in \Omega^i(X)$ consists an i-form $f^*\Omega^i(\triangle^n)$ for every n-simplex $f : \triangle^n \to X$ satisfying the condition that for $\sigma : \triangle^m \to \triangle^n$ a face or degeneracy map, we have the equality $\sigma^* f^* \omega = (f \circ \sigma)^* \omega$.

If V is a differentiable cochain complex, we can define a complex

$$\Omega^*(X, V) = \Omega^*(X) \otimes_{C^\infty(X)} C^\infty(X, V)$$

where $C^\infty(X, V)$ refers to the cochain complex of smooth maps from X to V. In this way, we see that the multicategory of differentiable cochain complexes is tensored over the opposite category SSet^{op} to the category of simplicial sets.

This allows us to lift the multicategory DVS of differentiable cochain complexes to a simplicially enriched multicategory, where we define the n-simplices in the simplicial set of multimorphisms by

$$\mathrm{Hom}(V_1, \dots, V_n \mid W)[n] = \mathrm{Hom}(V_1, \dots, V_n \mid \Omega^*(\triangle^n, W)),$$

where on the right hand side Hom denotes smooth multilinear cochain maps

$$V_1 \times \cdots \times V_n \to \Omega^*(\triangle^n, W)$$

which are compatible with differentials.

Now, in general, suppose we have a multicategory $\mathcal{C}$ which is tensored over SSet^{op}. Then the multicategory $\mathrm{PreFA}(X, \mathcal{C})$ is also tensored over SSet^{op}: the tensor product $F \boxtimes X$ of a prefactorization algebra F with a simplicial set X is defined by

$$(F \boxtimes X)(U) = F(U) \boxtimes X.$$

Then, we can lift our multicategory of $\mathcal{C}$-valued prefactorization algebras to a simplicially enriched multicategory, by defining

$$PreFA^{\triangle}_{mc}(F_1, \dots, F_n \mid G) = PreFA^{\triangle}_{mc}(F_1, \dots, F_n \mid G \boxtimes X).$$

In particular, we see that the multicategory of prefactorization algebras valued in differentiable cochain complexes is simplicially enriched.

7.2.4. Equivalences. In the appendix B, we define a notion of *weak equivalence* or *quasi-isomorphism* of differentiable cochain complexes.

7.2.4.1 Definition. *Let F, G be factorization algebras valued in cochain complexes. Let $\phi : F \to G$ be a map. We say that ϕ is a* weak equivalence *if, for all open subsets $U \subset M$, the map $F(U) \to G(U)$ is a quasi-isomorphism of cochain complexes.*

Similarly, let F, G be factorization algebras valued in differentiable cochain complexes. We say a map $\phi : F \to G$ is a weak equivalence *if, for all U, the map $F(U) \to G(U)$ is a weak equivalence of differentiable cochain complexes.*

We now provide an explicit criterion for checking weak equivalences, using the notion of a factorizing basis (see definition 7.4.0.2).

7.2.4.2 Lemma. *A map $F \to G$ between differentiable factorization algebras is a weak equivalence if and only if, for every factorizing basis $\mathfrak{U}$ of X and every U in $\mathfrak{U}$, the map*

$$F(U) \to G(U)$$

is a weak equivalence.

PROOF. For any open subset $V \subset X$, let $\mathfrak{U}_V$ denote the Weiss cover of V generated by all open subsets in $\mathfrak{U}$ that lie in V. By the descent axiom, the map

$$\check{C}(\mathfrak{U}_V, F) \to F(V)$$

is a weak equivalence, and similarly for G. Thus, it suffices to check that the map

$$\check{C}(\mathfrak{U}_V, F) \to \check{C}(\mathfrak{U}_V, G)$$

is a weak equivalence, for the following reason. Every Čech complex has a natural filtration by number of intersections. We thus obtain of spectral sequences from the map of Čech complexes. The fact that the maps

$$F(U_1 \cap \cdots \cap U_k) \to G(U_1 \cap \cdots \cap U_k)$$

are all weak equivalences implies that we have a quasi-isomorphism on the first page of the map of spectral sequences, so our original map is a quasi-isomorphism. ☐

7.3. Descent

Strict factorization algebras – those $\mathcal{F}$ for which $\mathcal{F}(U \sqcup V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V)$ for any pair of disjoint opens – satisfy an *a priori* different gluing axiom, which is operadic in nature. This axiom illuminates the origins and meaning of the Weiss topology, and it also insures that strict factorization algebras satisfy a version of descent with respect to the usual topology. For the remainder of this section, a factorization algebra will always mean a strict factorization algebra.

We want a factorization algebra to be a prefactorization algebra whose behavior on big opens is determined by its behavior on small opens. There is a natural operadic way to phrase this condition, as follows.

Let X be a topological space. Recall from section **??** that Disj_X denotes the colored operad whose colors are the opens of X and whose operations are

$$\mathrm{Disj}_X(V_1, \ldots, V_n \mid W) = \begin{cases} *, & \text{if } V_j \text{ are pairwise disjoint and contained in } W \\ \emptyset, & \text{else} \end{cases}$$

A prefactorization algebra in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is an algebra over this colored operad.

It is straightforward to talk about a prefactorization algebra on "some set of small opens." Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a cover with respect to the usual topology (not the Weiss topology). Let $\mathrm{Disj}_{\mathfrak{U}}$ be the colored operad obtained from Disj_X by simply restricting to the opens V such that V is contained in some element U_i of $\mathfrak{U}$. (In other words, $\mathrm{Disj}_{\mathfrak{U}}$ is the full sub-colored operad with objects from the sieve of the cover $\mathfrak{U}$.) Then a prefactorization algebra *subordinate to* $\mathfrak{U}$ is an algebra over the colored operad $\mathrm{Disj}_{\mathfrak{U}}$.

Let $\mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_X)$ denote the category of algebras in $\mathcal{C}$ for the colored operad Disj_X, and let $\mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_{\mathfrak{U}})$ denote the category of algebras over $\mathrm{Disj}_{\mathfrak{U}}$. The inclusion of colored operads $i : \mathrm{Disj}_{\mathfrak{U}} \to \mathrm{Disj}_X$ induces a pullback (or forgetful) functor

$$i^* : \mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_X) \to \mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_{\mathfrak{U}})$$

and, provided $\mathcal{C}$ contains the appropriate colimits (we discuss this condition below), a left adjoint functor

$$i_! : \mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_{\mathfrak{U}}) \to \mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_X),$$

which extends a prefactorization algebra subordinate to $\mathfrak{U}$ to a prefactorization algebra on X. It constructs all the values and structure maps for bigger opens out of the data from the opens "smaller than the cover." Thus, it is an operadic version of gluing.

A natural question is then whether, for a factorization algebra $\mathcal{F}$, the extension $i_! i^* \mathcal{F}$ is equivalent to $\mathcal{F}$. In other words, we are asking whether we can recover $\mathcal{F}$ just by knowing the values and structure maps of $\mathcal{F}$ on opens subordinate to the cover $\mathfrak{U}$. Note that this operadic extension *a priori* has nothing to do with a cosheaf gluing axiom, which only depends on the unary operations (i.e., inclusions of opens).

As we are typically interested in the homotopical version of these constructions, we want to work with derived versions of i^* and $i_!$. How to provide the necessary derived replacements depends on $\mathcal{C}$ and one's taste in homotopical algebra. We momentarily delay making specific choices and simply denote the derived replacement for $i_!$ by $\mathbb{L}i_!$. Our goal is thus to verify that

$$\mathbb{L}i_! i^* \mathcal{F} \simeq \mathcal{F}$$

for $\mathcal{F}$ a homotopy factorization algebra with values in a good symmetric monoidal category $\mathcal{C}$. In the remainder of this section, we always mean homotopical factorization algebra, unless we say otherwise, although we simply write factorization algebra.

This result states, in particular, that the gluing axiom for the Weiss topology agrees with the derived operadic left Kan extension. In the course of the proof, it will become clear how the Weiss topology is forced to appear by the nature of the structure maps in a prefactorization algebra.

In the next subsection, we will give a precise statement of this result, pinning down conditions on $\mathcal{C}$ in particular. In the following subsection we will discuss applications of this result, notably how it relates to descent for the category of factorization algebras. In the final subsection, we prove the result.

7.3.1. A precise version of the main result. The target symmetric monoidal category $(\mathcal{C}, \otimes)$ for our prefactorization algebras must have several properties for our proof to work. We want a well-behaved, homotopical version of colimit and clean interaction with the tensor product, so that it is sensible to talk about a derived or homotopical operadic left Kan extension $\mathbb{L}i_!\mathcal{F}$ of $\mathcal{F} \in \mathrm{Alg}_{\mathcal{C}}(\mathrm{Disj}_{\mathfrak{U}})$. What we need for the result, in particular, is that this derived operadic extension $\mathbb{L}i_!\mathcal{F}$ is weakly equivalent to the derived left Kan extension $\mathbb{L}\,\mathrm{Lan}_i\,\mathcal{F}$ merely as a functor.

In order to make a precise statement, we must choose some formalism for higher categories, and here we will use model categories. (When we give the proof, we will first explain its structure, so that the reader can formulate a version in an alternative formalism.) We depend on the machinery developed by Berger and Moerdijk [BM07] for working with algebras over colored operads in a homotopical fashion. We also follow closely the discussion of these issues in [Hora], which pins down the relevant details.

Following [Hora], we require $\mathcal{C}$ to be a cofibrantly generated symmetric monoidal simplicial model category. We want it to have the following additional properties.

- For any colored operad $\mathcal{O}$, the algebras $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$ obtain a model category structure in which weak equivalences and fibrations are objectwise. That is, for example, a map of $\mathcal{O}$-algebras $F : A \to B$ is a weak equivalence if $F(c) : A(c) \to B(c)$ is a weak equivalence for every color c in $\mathcal{O}$.
- For any map of colored operads $j : \mathcal{O} \to \mathcal{P}$, the adjunction
$$j_! : \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}) \leftrightarrows \mathrm{Alg}_{\mathcal{C}}(\mathcal{P}) : j^*$$
is a Quillen adjunction.
- Let $Col(\mathcal{O})$ denote the discrete category associated to $\mathcal{O}$, where the objects are simply the colors of $\mathcal{O}$ and the only morphisms are identities. The forgetful functor from $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$ to $\mathrm{Alg}_{\mathcal{C}}(Col(\mathcal{O}))$ preserves cofibrations.

Following Horel, we say such a category $\mathcal{C}$ has a *good theory of algebras*. As Horel shows, $\mathcal{C}$ has a good theory of algebras if $\mathcal{C}$ is also a left proper model category with a monoidal fibrant replacement functor and a cofibrant unit.

Remark: These conditions insure a good theory of algebras for all colored operads, whereas we are only working here with a rather restricted class of colored operads and maps. But these conditions certainly suffice to obtain our result, and they are fairly natural. ◊

We now state the main result, using the notations from above.

7.3.1.1 Theorem. *Let $\mathcal{C}$ have a good theory of algebras. Let X be a topological space and $\mathfrak{U}$ an ordinary cover. For $\mathcal{F}$ a homotopy factorization algebra on a space X that is not lax, the extension $\mathbb{L}i_! i^* \mathcal{F}$ is weakly equivalent to $\mathcal{F}$.*

In particular, we have $\mathbb{L}i_! i^* \mathcal{F}(V) \simeq \mathcal{F}(V)$ for every open V in X.

In other words, a factorization algebra also satisfies an operadic gluing axiom, in addition to the cosheaf gluing axiom.

7.3.2. Consequences of the theorem. The main application of this theorem is to the question of descent. Given an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X in the usual sense (i.e., every point in X is contained in some U_i), suppose we have a factorization algebra $\mathcal{F}_i$ on each U_i, with gluing data on double intersections, coherence data on triple intersections, and so on. In this subsection we will show that we can construct a unique factorization algebra $\mathcal{F}$ on X from this descent data.

We will need the following simple construction. If $U \subset X$ is an open subset and $\mathcal{F}$ is a factorization algebra on X, then the restriction $\mathcal{F}|_U$ is a factorization algebra on U. Of course, an open embedding $i_U : U \hookrightarrow X$ of one manifold into another identifies U with an open subset of X. We will also use $i_U^* \mathcal{F}$ to denote the "pullback" factorization algebra on U.

The input to our gluing construction is the following. We have, for each finite subset $J \subset I$, a factorization algebra $\mathcal{F}_J$ on

$$U_J = \bigcap_{j \in J} U_j.$$

Further, we have weak equivalences

$$\mathcal{F}_J \to r_j^* \mathcal{F}_{J \setminus \{j\}}$$

of factorization algebras on U_J, for each $j \in J$, where

$$r_j : U_J \to U_{J \setminus \{j\}}$$

is the natural inclusion. Finally, we require that, for every J and every $j, j' \in J$, the diagram of factorization algebras on U_J

$$\begin{tikzcd}
\mathcal{F}_J \arrow[r] \arrow[d] & r_j^* \mathcal{F}_{J \setminus \{j\}} \arrow[d] \\
r_{j'}^* \mathcal{F}_{J \setminus \{j'\}} \arrow[r] & r_j^* r_{j'}^* \mathcal{F}_{I \setminus \{j,j'\}}
\end{tikzcd}$$

commutes.

We can think of this data as defining a factorization algebra $\mathcal{F}_i$ for each $i \in I$, together with weak equivalences on double intersections, provided by $\mathcal{F}_{\{i,j\}}$, and with coherences provided by the factorization algebras $\mathcal{F}_J$ for $J \subset I$ of higher cardinality. We call this a *descent datum*.

For simplicity, we restrict our attention to covers that are locally finite (i.e., each point in X is contained in only finitely many elements of the cover). Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a locally finite cover. For any open W subordinate to the cover, there is then a maximal finite set $J \subset I$ such that

$$W \subset U_J = \bigcap_{j \in J} U_j,$$

by running over all the elements of the cover containing W. We denote that maximal set for W by $[W]$. Given a descent datum $\{\mathcal{F}_J \mid \text{finite } J \subset I\}$ on a locally finite cover $\mathfrak{U} = \{U_i\}_{i \in I}$, we obtain a factorization algebra $\widetilde{\mathcal{F}}$ subordinate to $\mathfrak{U}$ by setting

$$\widetilde{\mathcal{F}}(W) = \mathcal{F}_{[W]}(W).$$

The structure maps of $\widetilde{\mathcal{F}}$ use the weak equivalences in the descent datum.

7.3.2.1 Proposition. *Given a descent datum $\{F_J \mid$ finite $J \subset I\}$ for a factorization algebra on a locally finite cover $\mathfrak{U}$, the operadic extension $\mathbb{L}i_! \widetilde{\mathcal{F}}$ to a prefactorization algebra on X is a factorization algebra.*

PROOF. Our proof relies on the proofs of theorem 7.3.1.1 and proposition 7.4.0.4, although we will try to clearly indicate where we are using ideas from those proofs.

Recall that the sieve of the cover $\mathfrak{U}$ is the collection of all opens subordinate to $\mathfrak{U}$. We use $\widehat{\mathfrak{U}}$ to denote the Weiss cover generated by this sieve: it is the collection of all finite disjoint unions of opens subordinate to the cover $\mathfrak{U}$. Note that it is a factorizing basis (see definition 7.4.0.2).

First, observe that

$$\mathbb{L}i_! \widetilde{\mathcal{F}}(U \sqcup V) \simeq \widetilde{\mathcal{F}}(U) \otimes \widetilde{\mathcal{F}}(V)$$

for U and V subordinate to $\mathfrak{U}$. This holds because the element $[U, V]$ in $\mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}}$ is homotopy terminal in the diagram computing the operadic Kan extension. Moreover, following the argument in the proof of theorem 7.3.1.1, this diagram is equivalent to the Čech gluing diagram using the Weiss cover for $U \sqcup V$ obtained from $\widehat{U}$.

Thus, $\mathbb{L}i_! \widetilde{\mathcal{F}}$ defines a $\widehat{\mathfrak{U}}$-factorization algebra, in the sense that it is a factorization algebra when restricted to this factorizing basis.

Again, following the proof of theorem 7.3.1.1, we see that the extension of the operadic extension *as a* $\widehat{\mathfrak{U}}$*-factorization algebra* agrees with the operadic extension $\mathbb{L}i_! \widetilde{\mathcal{F}}$. Hence it is a factorization algebra. ☐

7.3.3. Proof of the theorem. We start by explaining the idea of the proof and then give a detailed version, using the model category language.

7.3.3.1. As $\mathcal{F}$ is a strict factorization algebra, we know that we can recover its value on any open V from a Weiss cover. Although we have employed Čech diagrams to describe the cosheaf property, it will be convenient here to use the language of sieves instead. Recall that the *sieve of a cover* $\mathfrak{V}$ is the partially ordered set of all opens of X subordinate to the cover. In other words, it is the full subcategory of $Opens_X$ over the cover $\mathfrak{V}$. As $\mathcal{F}$ is a cosheaf in the Weiss topology, we know

$$\text{hocolim}_{Sieve(\mathfrak{V})} \mathcal{F} \to \mathcal{F}(V)$$

is a weak equivalence for every Weiss cover $\mathfrak{V}$ of V. (The homotopy colimit can be computed by the Čech diagram associated to the cover because the Čech diagram is homotopy terminal in the diagram over the sieve.)

When $\mathcal{C}$ is a well-behaved symmetric monoidal category, the derived operadic left Kan extension $\mathbb{L}i_! G(V)$ can be computed as a homotopy colimit over a diagram built from the inclusion $i : \text{Disj}_{\mathfrak{U}} \to \text{Disj})X$. The key observation is that this diagram is just a fattened version of a diagram over a sieve.

To be more precise, let $Sieve(\mathfrak{U})$ denote the sieve for the ordinary cover $\mathfrak{U}$. The opens in this sieve do not form a Weiss cover, but we can generate a Weiss cover from it. Let $\widehat{\mathfrak{U}}$ denote the Weiss cover

$$\{V_1 \sqcup \cdots \sqcup V_k \mid \text{ for each } 1 \leq j \leq k, V_j \subset U_i \text{ for some } U_i \in \mathfrak{U}\}.$$

Let $D_{\mathfrak{U}}$ denote the diagram such that

$$\mathbb{L}i_! i^* \mathcal{F}(X) = \text{hocolim}_{D_{\mathfrak{U}}} \mathcal{F}.$$

We will show that the map $D_{\mathfrak{U}} \to Sieve(\widehat{\mathfrak{U}})$ is homotopy terminal. Thus, we know the two homotopy colimits agree. (A parallel argument works for an arbitrary open V rather than the whole X. One simply has to work with the natural Weiss cover of V concocted from $\mathfrak{U}$ and with the diagram computing $\mathbb{L}i_! i^* \mathcal{F}$.)

7.3.3.2. We will now explain how to construct the diagram $D_{\mathfrak{U}}$. To start, we work at the 1-categorical level; the derived version comes after.

The first step is to replace colored operads with their symmetric monoidal envelopes. Thus, we can view $\mathcal{F}$ as a symmetric monoidal functor $\mathcal{F} : \mathbf{S}\,\mathrm{Disj}_X \to \mathcal{C}$, rather than an algebra over a colored operad. This shift of emphasis is just to work with categories rather than multicategories.

Note that we will use $\mathcal{F}$ for this symmetric monoidal functor, not $\mathbf{S}\mathcal{F}$. Likewise, we will use i for the functor $\mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}} \to \mathbf{S}\,\mathrm{Disj}_X$, rather than $\mathbf{S}i$.

The functor $i_!$ is now the symmetric monoidal left Kan extension. By this we mean that $i_!$ takes as input a symmetric monoidal functor $G : \mathrm{Disj}_{\mathfrak{U}} \to \mathcal{C}$ and outputs a *symmetric monoidal* functor $\tilde{G} : \mathrm{Disj}_X \to \mathcal{C}$ together with a monoidal natural transformation

$$G \Rightarrow \tilde{G} \circ i,$$

initial among all such pairs of a symmetric monoidal functor and monoidal natural transformation.

Just as the left Kan extension for functors has an objectwise formula if $\mathcal{C}$ possesses adequate colimits, the symmetric monoidal left Kan extension also has a nice formula, although we now need the tensor product to play nicely with colimits as well. In fact, Getzler [Get09b] gives conditions under which the left Kan extension as a functor *is* the symmetric monoidal left Kan extension:

- $\mathcal{C}$ possesses all colimits,
- the tensor product preserves colimits on each side (e.g., the functor $x \otimes -$ preserves colimits for every object x in $\mathcal{C}$), and
- the functor i^* is symmetric monoidal (not just lax symmetric monoidal).

Thus, in many cases, we can simply work with the usual left Kan extension.

For us, the functor i^* is always symmetric monoidal, since we're simply restricting to a full subcategory. Hence, we just need $\mathcal{C}$ to be a nice symmetric monoidal category, like cochain complexes.

Let us assume that $i_!G$ agrees with the usual left Kan extension. Then we know that

$$\begin{aligned} i_!G(X) &= \mathbf{S}\,\mathrm{Disj}_X(i(-), X) \otimes_{\mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}}} G(-) \\ &= \operatorname*{colim}_{D_{\mathfrak{U}}} G \end{aligned}$$

where $D_{\mathfrak{U}}$ denotes the diagram of all arrows $[V_1, \ldots, V_k] \to X$ in $\mathbf{S}\,\mathrm{Disj}_X$ where the source $[V_1, \ldots, V_k]$ lives in the subcategory $\mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}}$. In other words, we compute the colimit of G on the overcategory $i \downarrow \mathbf{S}\,\mathrm{Disj}_X$.

Remark: This diagram $D_{\mathfrak{U}}$ looks very similar to the Weiss cover $\widehat{\mathfrak{U}}$: in both cases, we are interested in finite tuples of opens subordinate to the cover $\mathfrak{U}$. One could reverse the direction of our logic in this book and start with the operadic approach and then introduce the Weiss topology, motivated by this colimit diagram. ◊

We want to show that when $G = i^*\mathcal{F}$ for $\mathcal{F}$ a factorization algebra (in the non-homotopical sense), then there is an equivalence

$$\operatorname*{colim}_{D_{\mathfrak{U}}} i^*\mathcal{F} \to \operatorname*{colim}_{Sieve(\widehat{\mathfrak{U}})} \mathcal{F}.$$

This equivalence is the underived version of the theorem, because the colimit on the right is equivalent to $\mathcal{F}(X)$, as $\mathcal{F}$ is a cosheaf for the Weiss topology.

First, observe that there is a natural map between the two colimits. An arrow $[V_1, \ldots, V_k] \to X$ in $D_{\mathfrak{U}}$ consists of a finite tuple of disjoint opens V_j where each is subordinate to the cover $\mathfrak{U}$. Hence, we see that the union $V_1 \sqcup \cdots \sqcup V_k$ is in the Weiss cover $\widehat{U}$. Denote this map of diagrams by $\sqcup : D_{\mathfrak{U}} \to Sieve(\widehat{\mathfrak{U}}$. Moreover, by hypothesis,

$$i^*\mathcal{F}([V_1, \ldots, V_k]) = \mathcal{F}(V_1) \otimes \cdots \otimes \mathcal{F}(V_k) \cong \mathcal{F}(V_1 \sqcup \cdots \sqcup V_k),$$

so that we get a map between the colimits.

Second, this map is terminal, so that the colimits agree. To verify this, we need to show that for any open W in $Sieve(\widehat{\mathfrak{U}})$, the undercategory $W \downarrow \sqcup$ is nonempty and connected. By construction, W is the finite union of opens $W_1 \sqcup \cdots \sqcup W_j$ that are subordinate to the cover $\mathfrak{U}$, so we know that the undercategory is nonempty. Connectedness is also simple: if we have two maps

$$V_1 \sqcup \cdots \sqcup V_k \leftarrow W \to V_1' \sqcup \cdots \sqcup V_l',$$

then these both factor through

$$\bigsqcup_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} (V_i \cap V_j'),$$

which is also in the image of the map $\sqcup$.

7.3.3.3. The structure of the underived proof carries over easily to the derived setting. Essentially, we replace colimits with homotopy colimits. The technical issues are

(1) to find an analog of Getzler's result – that the derived left Kan extension is the derived operadic left Kan extension – and
(2) to verify the map of diagrams $\sqcup$ is homotopy terminal.

We use model categories to do this, following [**Hora**].

We use the following result, proposition 2.15, of [**Hora**].

7.3.3.1 Proposition. *Let $\mathcal{C}$ have a good theory of algebras and a cofibrant unit. Let G be an algebra over* $\mathrm{Disj}_{\mathfrak{U}}$*. Then the derived operadic left Kan extension $\mathbb{L}i_!G$ is weakly equivalent to the homotopy left Kan extension of G along i (i.e., viewing G as just a functor, not a symmetric monoidal functor).*

Let QG denote a cofibrant replacement of G as an algebra over $\mathrm{Disj}_{\mathfrak{U}}$. The value of the homotopy left Kan extension at X can be computed as

$$\mathbf{S}\,\mathrm{Disj}_X(i(-),X) \otimes^{\mathbb{L}}_{\mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}}} G(-) = |\,\mathrm{B}_\bullet(\mathbf{S}\,\mathrm{Disj}_X(i(-),X), \mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}}, G)| \quad = |\,\mathrm{B}_\bullet(*, i\downarrow X, G)|,$$

the realization of the usual bar construction. For $G = i^*\mathcal{F}$, this formula is the derived replacement for $i_!i^*\mathcal{F}(X)$.

Similarly, we know that $\mathrm{hocolim}_{Sieve(\widehat{\mathfrak{U}})}\,\mathcal{F}$, the derived replacement of the other diagram, can be computed using the bar construction $|\,\mathrm{B}_\bullet(*, Sieve(\widehat{\mathfrak{U}}), Q\mathcal{F})|$.

Recall that there is a natural map $\sqcup : \mathbf{S}\,\mathrm{Disj}_{\mathfrak{U}} \to Sieve(\widehat{\mathfrak{U}})$ sending $[V_1,\dots,V_k]$ to $V_1 \sqcup \cdots \sqcup V_k$. Once we show it is homotopy terminal, we know that the two homotopy colimits agree. Given $W \in Sieve(\widehat{\mathfrak{U}})$, we need to show the undercategory $W \downarrow \sqcup$ is nonempty and contractible. We've already seen that it is nonempty. Contractibility follows because it is cofiltered: given any finite set of elements in the undercategory, namely

$$W \to V_1^{(i)} \sqcup \cdots \sqcup V_{k_i}^{(i)}$$

with $1 \leq i \leq n$, all the maps factor through

$$\bigsqcup_{f \in \prod_{i=1}^n \{1,\dots,k_i\}} \cap_{i=1}^n V_{f(i)}^{(i)}.$$

Thus the homotopy colimits are equivalent.

7.4. Extension from a basis

Let X be a topological space, and let $\mathfrak{U}$ be a basis for X, which is closed under taking finite intersections. It is well-known that there is an equivalence of categories between sheaves on X and sheaves that are only defined for open sets in the basis $\mathfrak{U}$. In this section we will prove a similar statement for *lax* factorization algebras. (When $\mathcal{F}$ is a *lax* factorization algebra, the procedure above for the operadic extension does not necessarily apply. The arguments often relied in some way on the factorizing property.)

We begin with a paired set of definitions.

7.4.0.2 Definition. *A* factorizing basis $\mathfrak{U} = \{U_i\}_{i\in I}$ *for a space X is a basis for the topology of X with the following properties:*

(1) *for every finite set* $\{x_1, \dots, x_n\} \subset X$*, there exists* $i \in I$ *such that* $\{x_1, \dots, x_n\} \subset U_i$*;*
(2) *if* U_i *and* U_j *are disjoint, then* $U_i \cup U_j$ *is in* $\mathfrak{U}$*;*
(3) $U_i \cap U_j \in \mathfrak{U}$ *for every* U_i *and* U_j *in* $\mathfrak{U}$*.*

In particular, a factorizing basis is a Weiss cover of X.

7.4.0.3 Definition. *Given a factorizing basis* $\mathfrak{U}$*, a* $\mathfrak{U}$-prefactorization algebra *consists of the following:*

(1) *for every* $U_i \in \mathfrak{U}$*, an object* $\mathcal{F}(U_i)$*;*
(2) *for every finite tuple of pairwise disjoint opens* $U_{i_0}, \dots, U_{i_n}$ *all contained in* U_j *— with all these opens from* $\mathfrak{U}$ *— a structure map*

$$\mathcal{F}(U_{i_0}) \otimes \cdots \otimes \mathcal{F}(U_{i_n}) \to \mathcal{F}(U_j);$$

(3) $\mathcal{F}(\emptyset) \simeq \mathbb{1}$*;*
(4) *the structure maps are covariant and associative.*

In other words, a $\mathfrak{U}$*-prefactorization algebra* $\mathcal{F}$ *is like a factorization algebra, except that* $\mathcal{F}(U)$ *is only defined for sets* U *in* $\mathfrak{U}$*.*

A $\mathfrak{U}$-factorization algebra *is a* $\mathfrak{U}$*-prefactorization algebra with the property that, for every* U *in* $\mathfrak{U}$ *and every Weiss cover* $\mathfrak{V}$ *of* U *consisting of open sets in* $\mathfrak{U}$*,*

$$\check{C}(\mathfrak{V}, \mathcal{F}) \simeq \mathcal{F}(U),$$

where $\check{C}(\mathfrak{V}, \mathcal{F})$ *denotes the Čech complex described earlier (section 6.1).*

Note that we have *not* required the factorizing property, so that we are focused here on *lax* factorization algebras. Throughout this section, factorization algebra means lax factorization algebra.

In this section we will show that any $\mathfrak{U}$-factorization algebra on X extends to a factorization algebra on X. This extension is unique up to quasi-isomorphism.

Let $\mathcal{F}$ be a $\mathfrak{U}$-factorization algebra. Let us define a prefactorization algebra $ext(\mathcal{F})$ on X by

$$ext(\mathcal{F})(V) = \check{C}(\mathfrak{U}_V, \mathcal{F}),$$

for each open $V \subset X$.

With these definitions in hand, it should be clear why we can recover a factorization algebra on X from just a factorization algebra on a factorizing basis $\mathfrak{U}$. The first property of $\mathfrak{U}$ insures that the "atomic" structure maps (multiplication out of a finite set of points) factor through the basis. The second property insures that we know how to multiply within the factorizing basis. In particular, we know

$$\mathcal{F}(U_{i_0}) \otimes \cdots \otimes \mathcal{F}(U_{i_n}) \to \mathcal{F}(U_{i_0} \cup \cdots \cup U_{i_n}),$$

and associativity insures that this map plus the unary structure maps determine all the other multiplication maps. Finally, the third property insures that we know $\mathcal{F}$ on the intersections that appear in the gluing condition.

7.4.0.4 Proposition. *With this definition, $ext(\mathcal{F})$ is a factorization algebra whose restriction to open sets in the cover $\mathfrak{U}$ is quasi-isomorphic to $\mathcal{F}$.*

7.4.1. The proof.

Our goal here is to prove that there is an equivalence of categories

$$\begin{tikzcd} FactAlg_X \arrow[r, bend left, "res"] & FactAlg_{\mathfrak{U}} \arrow[l, bend left, "ext"] \end{tikzcd}$$

where *res* is the functor of restricting $\mathcal{F}$ to the factorizing basis $\mathfrak{U}$ and *ext* is a functor extending from that basis. Before we can prove equivalence, we need to explicitly construct ext.

Let $\mathfrak{U}$ be a factorizing basis and $\mathcal{F}$ a $\mathfrak{U}$-factorization algebra. We will construct a factorization algebra $ext(\mathcal{F})$ on X in several stages:

- we give the value of $ext(\mathcal{F})$ on every open V in X,
- we construct the structure maps and verify associativity and covariance,
- we verify that $ext(\mathcal{F})$ satisfies the gluing axiom.

Finally, it will be manifest from our construction how to extend maps of $\mathfrak{U}$-factorization algebras to maps of their extensions.

We use the following notations. Given a simplicial cochain complex $A_\bullet$ (so each A_n is a cochain complex), let $C(A_\bullet)$ denote the totalization of the double complex obtained by taking the *un*normalized cochains. Let $C_N(A_\bullet)$ denote the totalization of the double complex by taking the *normalized* cochains. Finally, let

$$sh_{AB} : C_N(A_\bullet) \otimes C_N(B_\bullet) \to C_N(A_\bullet \otimes B_\bullet)$$

denote the Eilenberg-Zilber shuffle map, which is a lax symmetric monoidal functor from simplicial cochain complexes to cochain complexes.

7.4.2. Extending values.

For an open $V \subset X$, recall $\mathfrak{U}_V$ denotes the Weiss cover of V given by all the opens in $\mathfrak{U}$ contained inside V. We then define

$$ext(\mathcal{F})(V) := \check{C}(\mathfrak{U}_V, \mathcal{F}).$$

This construction provides a precosheaf on X.

7.4.3. Extending structure maps. The Čech complex for the factorization gluing axiom arises as the normalized cochain complex of a simplicial cochain complex, as should be clear from its construction. We use $\check{C}(\mathfrak{V},\mathcal{F})_\bullet$ to denote this simplicial cochain complex, so that

$$\check{C}(\mathfrak{V},\mathcal{F}) = C_N(\check{C}(\mathfrak{V},\mathcal{F})_\bullet),$$

with $\mathcal{F}$ a factorization algebra and $\mathfrak{V}$ a Weiss cover.

We construct the structure maps by using the simplicial cochain complexes $\check{C}(\mathfrak{V},\mathcal{F})_\bullet$, as many properties are manifest at that level. For instance, the unary maps $ext(\mathcal{F})(V) \to ext(\mathcal{F})(W)$ arising from inclusions $V \hookrightarrow W$ are easy to understand: $\mathfrak{U}_V$ is a subset of $\mathfrak{U}_W$ and thus we get a map between every piece of the simplicial cochain complex.

We now explain in detail the map

$$m_{VV'} : ext(\mathcal{F})(V) \otimes ext(\mathcal{F})(V') \to ext(\mathcal{F})(V \cup V'),$$

where $V \cap V' = \emptyset$. Note that knowing this map, we recover every other multiplication map

$$\begin{tikzcd}
ext(\mathcal{F})(V) \otimes ext(\mathcal{F})(V') \arrow[rr] \arrow[dr, "m_{VV'}"'] & & ext(\mathcal{F})(W) \\
& ext(\mathcal{F})(V \cup V') \arrow[ur] &
\end{tikzcd}$$

by postcomposing with a unary map.

The n-simplices of $\check{C}(\mathfrak{U}_V,\mathcal{F})_n$ are the direct sum of terms

$$\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n})$$

with the U_{i_k}'s in V. The n-simplices of

$$\check{C}(\mathfrak{U}_V,\mathcal{F})_\bullet \otimes \check{C}(\mathfrak{U}_{V'},\mathcal{F})_\bullet$$

are precisely the levelwise tensor product

$$\check{C}(\mathfrak{U}_V,\mathcal{F})_n \otimes \check{C}(\mathfrak{U}_{V'},\mathcal{F})_n,$$

which breaks down into a direct sum of terms

$$\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n}) \otimes \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_n})$$

with the U_{i_k}'s in V and the U_{j_k}'s in V'. We need to define a map

$$m_{VV',n} : \check{C}(\mathfrak{U}_V,\mathcal{F})_n \otimes \check{C}(\mathfrak{U}_{V'},\mathcal{F})_n \to \check{C}(\mathfrak{U}_{V\cup V'},\mathcal{F})_n$$

for every n, and we will express it in terms of the direct summands.

Now

$$(U_{i_0} \cap \cdots \cap U_{i_n}) \cup (U_{j_0} \cap \cdots \cap U_{j_n}) = (U_{i_0} \cup U_{j_0}) \cap \cdots \cap (U_{i_n} \cup U_{j_n}).$$

Thus we have a given map

$$\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n}) \otimes \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_n}) \to \mathcal{F}\Big((U_{i_0} \cup U_{j_0}) \cap \cdots \cap (U_{i_n} \cup U_{j_n}) \Big),$$

because $\mathcal{F}$ is defined on a factorizing basis. The right hand term is one of the direct summands for $\check{C}(\mathfrak{U}_{V \cup V'}, \mathcal{F})_n$. Summing over the direct summands on the left, we obtain the desired levelwise map.

Finally, the composition

$$\begin{tikzcd}
C_N\check{C}(\mathfrak{U} \cap V, \mathcal{F})_\bullet) \otimes C_N\check{C}(\mathfrak{U}_{V'}, \mathcal{F})_\bullet) \arrow[d, "sh"] \\
C_N\big(\check{C}(\mathfrak{U} \cap V, \mathcal{F})_\bullet \otimes \check{C}(\mathfrak{U}_{V'}, \mathcal{F})_\bullet\big) \arrow[d, "C_N m_{VV',\bullet})"] \\
C_N\check{C}(\mathfrak{U}_{V \cup V'}, \mathcal{F})_\bullet)
\end{tikzcd}$$

gives us $m_{VV'}$.

A parallel argument works to construct the multiplication maps from n disjoint opens to a bigger open.

The desired associativity and covariance are clear at the level of the simplicial cochain complexes $\check{C}(\mathfrak{U}_V, \mathcal{F})$, since they are inherited from $\mathcal{F}$ itself.

7.4.4. Verifying gluing.

We have constructed $ext(\mathcal{F})$ as a prefactorization algebra, but it remains to verify that it is a factorization algebra. Thus, our goal is the following.

7.4.4.1 Proposition. *The extension $ext(\mathcal{F})$ is a cosheaf with respect to the Weiss topology. In particular, for every open subset W and every Weiss cover $\mathfrak{W}$, the complex $\check{C}(\mathfrak{W}, ext(\mathcal{F}))$ is quasi-isomorphic to $ext(\mathcal{F})(W)$.*

As our gluing axiom is simply the axiom for cosheaf — but using a funny class of covers — the standard refinement arguments about Čech homology apply. We now spell this out.

For W an open subset of X and $\mathfrak{W} = \{W_j\}_{j \in J}$ a Weiss cover of W, there are two associated covers that we will use:

(a) $\mathfrak{U}_W = \{U_i \subset W \mid i \in I\}$ and
(b) $\mathfrak{U}_{\mathfrak{W}} = \{U_i \mid \exists j \text{ such that } U_i \subset W_j\}$.

The first is just the factorizing basis of W induced by $\mathfrak{U}$, but the second consists of the opens in $\mathfrak{U}$ subordinate to the cover $\mathfrak{W}$. Both are Weiss covers of W.

To prove the proposition, we break the argument into two steps and exploit the intermediary Weiss cover $\mathfrak{U}_{\mathfrak{W}}$.

7.4.4.2 Lemma. *There is a natural quasi-isomorphism*

$$f : \check{C}(\mathfrak{W}, ext(\mathcal{F})) \xrightarrow{\simeq} \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}),$$

for every Weiss cover $\mathfrak{W}$ of an open W.

PROOF OF LEMMA. This argument boils down to combinatorics with the covers. Some extra notation will clarify what's going on. In the Čech complex for the cover $\mathfrak{W}$, for instance, we run over $n+1$-fold intersections $W_{j_0} \cap \cdots \cap W_{j_n}$. We will denote this open by $W_{\vec{j}}$, where $\vec{j} = (j_0, \ldots, j_n) \in J^{n+1}$, with J the index set for $\mathfrak{W}$. Since we are only interested in intersections for which all the indices are pairwise distinct, we let $\widehat{J^{n+1}}$ denote this subset of J^{n+1}.

First, we must exhibit the desired map f of cochain complexes. The source complex $\check{C}(\mathfrak{W}, ext(\mathcal{F}))$ is constructed out of $\mathcal{F}$'s behavior on the opens U_i. Explicitly, we have

$$\check{C}(\mathfrak{W}, ext(\mathcal{F})) = \bigoplus_{n \geq 0} \bigoplus_{\vec{j} \in \widehat{J^{n+1}}} \check{C}(\mathfrak{U}_{W_{\vec{j}}}, \mathcal{F})[n].$$

Note that each term $\mathcal{F}(U_{\vec{i}})$ appearing in this source complex appears *only once* in the target complex $\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$. Let f send each such term $\mathcal{F}(U_{\vec{i}})$ to its unique image in the target complex via the identity. This map f is a cochain map: it clearly respects the internal differential of each term $\mathcal{F}(U_{\vec{i}})$, and it is compatible with the Čech differential by construction.

Second, we need to show f is a quasi-isomorphism. We will show this by imposing a filtration on the map and showing the induced spectral sequence is a quasi-isomorphism on the first page.

We filter the target complex $\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$ by

$$F^n \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}) := \bigoplus_{k \leq n} \bigoplus_{\vec{i} \in \widehat{I^{k+1}}} \mathcal{F}(U_{\vec{i}})[k].$$

Equip the source complex $\check{C}(\mathfrak{W}, ext(\mathcal{F}))$ with a filtration by pulling this filtration back along f. In particular, for any $U_{\vec{i}} = U_{i_0} \cap \cdots \cap U_{i_n}$, the preimage under f consists of a direct sum over all tuples $W_{\vec{j}} = W_{j_0} \cap \cdots \cap W_{j_m}$ such that every U_{i_k} is a subset of $W_{\vec{j}}$. Here, m can be any nonnegative integer (in particular, it can be bigger than n).

Consider the associated graded complexes with respect to these filtrations. The source complex has

$$\operatorname{Gr}\check{C}(\mathfrak{U}_{\mathfrak{W}},\mathcal{F})=\bigoplus_{n\geq 0}\bigoplus_{\vec{i}\in\widehat{I^{n+1}}}\mathcal{F}(U_{\vec{i}})[n]\otimes\left(\bigoplus_{\vec{j}\in\widehat{J^{m+1}}\text{ such that }U_{i_k}\subset W_{\vec{j}}\,\forall k}\mathbb{C}[m]\right).$$

The rightmost term (after the tensor product) corresponds to the chain complex for a simplex — here, the simplex is infinite-dimensional — and hence is contractible. In consequence, the map of spectral sequences is a quasi-isomorphism. □

We now wish to relate the Čech complex on the intermediary $\mathfrak{U}_{\mathfrak{W}}$ to that on $\mathfrak{U}_W$.

7.4.4.3 Lemma. *The complexes $\check{C}(\mathfrak{U}_W,\mathcal{F})$ and $\check{C}(\mathfrak{U}_{\mathfrak{W}},\mathcal{F})$ are quasi-isomorphic.*

PROOF OF LEMMA. We will produce a roof

$$\check{C}(\mathfrak{U}_W,\mathcal{F})\overset{\simeq}{\leftarrow}\check{C}(\mathfrak{U}_W,ext^{\mathfrak{W}}(\mathcal{F}))\overset{\simeq}{\to}\check{C}(\mathfrak{U}_{\mathfrak{W}},\mathcal{F}).$$

Recall that $\mathfrak{U}_W$ is a factorizing basis for W. Then $\mathcal{F}$, restricted to W, is a $\mathfrak{U}_W$-factorizing basis. Hence, for any $V\in\mathfrak{U}_W$ and any Weiss cover $\mathfrak{V}\subset\mathfrak{U}_W$ of V, we have

$$\check{C}(\mathfrak{V},\mathcal{F})\overset{\simeq}{\to}\mathcal{F}(V).$$

For any $V\in\mathfrak{U}_W$, let $\mathfrak{U}_{\mathfrak{W}}|_V=\{U_i\in\mathfrak{U}\mid U_i\subset V\cap W_j\text{ for some }j\in J\}$. Note that this is a Weiss cover for V. We define

$$ext^{\mathfrak{W}}(\mathcal{F})(V):=\check{C}(\mathfrak{U}_{\mathfrak{W}}|_V,\mathcal{F}).$$

By construction, the natural map

$$ext^{\mathfrak{W}}(\mathcal{F})(V)\to\mathcal{F}(V)\tag{7.4.4.1}$$

is a quasi-isomorphism.

Thus we have a quasi-isomorphism

$$\check{C}(\mathfrak{U}_W,ext^{\mathfrak{W}}(\mathcal{F}))\overset{\simeq}{\to}\check{C}(\mathfrak{U}_W,\mathcal{F}),$$

by using the natural map (7.4.4.1) on each open in the Čech complex for $\mathfrak{U}_W$.

The map

$$\check{C}(\mathfrak{U}_W,ext^{\mathfrak{W}}(\mathcal{F}))\overset{\simeq}{\to}\check{C}(\mathfrak{U}_{\mathfrak{W}},\mathcal{F})$$

arises by mimicking the construction in the preceding lemma. □

7.5. Pulling back along an open immersion

Factorization algebras do not pull back along an arbitrary continuous map, at least not in a simple way. Nonetheless, they do pull back along open embeddings (i.e., they restrict to open subsets in an obvious way). In this section we will discuss a generalization of this situation, namely pulling back along open immersions.

Let $f : N \to M$ be an open immersion. Let $\mathfrak{U}_f$ be the cover of N consisting of those open subsets $U \subset N$ with the property that

$$f\mid_U : U \to f(U)$$

is a homeomorphism. (To say that f is an open immersion means that sets of this form cover N.)

For any $U \in \mathfrak{U}_f$, we obtain a factorization algebra $f_U^*\mathcal{F}$ on U by defining

$$f_U^*\mathcal{F}(V) = \mathcal{F}(f(V)).$$

It is simply the pullback of $\mathcal{F}|_{f(U)}$ along the embedding $f : U \to f(U)$. It is immediate that these $f_U^*\mathcal{F}$ satisfy the coherence conditions to glue over the cover $\mathfrak{U}_f$, following proposition 7.3.2.1. (One should first take a locally finite refinement of this cover.)

7.5.0.4 Definition. *For an open immersion $f : N \to M$ and a factorization algebra $\mathcal{F}$ on M, the* pullback factorization algebra *$f^*\mathcal{F}$ is the factorization algebra on N constructed by gluing the factorization algebras $f_U^*\mathcal{F}$ over the cover $\mathfrak{U}_f$.*

7.6. Equivariant factorization algebras and descent along a torsor

Let G be a discrete group acting on a space X.

7.6.0.5 Definition. *A G-equivariant factorization algebra on X is a factorization algebra $\mathcal{F}$ on X together with isomorphisms*

$$\rho_g : g^*\mathcal{F} \cong \mathcal{F},$$

for each $g \in G$, such that

$$\rho_{\mathrm{Id}} = \mathrm{Id} \quad \text{and} \quad \rho_{gh} = \rho_h \circ h^*(\rho_g) : h^*g^*\mathcal{F} \to \mathcal{F}.$$

7.6.0.6 Proposition. *Let G be a discrete group acting properly discontinuously on X, so that $X \to X/G$ is a principal G-bundle. Then there is an equivalence of categories between G-equivariant factorization algebras on X and factorization algebras on X/G.*

PROOF. If $\mathcal{F}$ is a factorization algebra on X/G, then $f^*\mathcal{F}$ is a G-equivariant factorization algebra on X.

Conversely, let $\mathcal{F}$ be a G-equivariant factorization algebra on $\mathcal{F}$. Let $\mathfrak{U}_{con}$ be the open cover of X/G consisting of those connected sets where the G-bundle $X \to X/G$ admits a section. Let $\mathfrak{U}$ denote the factorizing basis for X/G generated by $\mathfrak{U}_{con}$. We will define a $\mathfrak{U}$-factorization algebra $\mathcal{F}^G$ by defining

$$\mathcal{F}^G(U) = \mathcal{F}(\sigma(U)),$$

where σ is any section of the G-bundle $\pi^{-1}(U) \to U$.

Because $\mathcal{F}$ is G-equivariant, $\mathcal{F}(\sigma(U))$ is independent of the section σ chosen. Since $\mathfrak{U}$ is a factorizing basis, $\mathcal{F}^G$ extends canonically to a factorization algebra on X/G. ☐

CHAPTER 8

Structured factorization algebras and quantization

In this chapter we will define what it means to have a factorization algebra endowed with the structure of an algebra over an operad. Not all operads work for this construction: only operads endowed with an extra structure – that of a *Hopf operad* – can be used. Add pointer to appendix! The issue is that we need to mix the structure maps of the factorization algebra with those of an algebra over an operad P, so we need to know how to tensor together P-algebras.

After explaining the relevant machinery, we focus on the cases of interest for us: the P_0 and BD operads that appear in the classical and quantum BV formalisms, respectively. These operads play a central role in our quantization theorem, the main result of this book, and thus we will have formulated the goal toward which the next two parts of the book are devoted.

Since, in this book, we are principally concerned with factorization algebras taking values in the category of differentiable cochain complexes we will restrict attention to this case in the present section.

8.1. Structured factorization algebras

8.1.0.7 Definition. *A* Hopf operad *is an operad in the category of differential graded cocommutative coalgebras.*

Any Hopf operad P is, in particular, a differential graded operad. In addition, the cochain complexes $P(n)$ are endowed with the structure of differential graded commutative coalgebra. The operadic composition maps

$$\circ_i : P(n) \otimes P(m) \to P(n+m-1)$$

are maps of coalgebras, as are the maps arising from the symmetric group action on $P(n)$.

If P is a Hopf operad, then the category of dg P-algebras becomes a symmetric monoidal category. If A, B are P-algebras, the tensor product $A \otimes_{\mathbb{C}} B$ is also a P-algebra. The structure map

$$P_{A\otimes B} : P(n) \otimes (A \otimes B)^{\otimes n} \to A \otimes B$$

is defined to be the composition

$$P(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{c(n)} P(n) \otimes P(n) \otimes A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{P_A \otimes P_B} A \otimes B.$$

In this diagram, $c(n) : P(n) \to P(n)^{\otimes 2}$ is the comultiplication on $c(n)$.

Any dg operad that is the homology operad of an operad in topological spaces is a Hopf operad (because topological spaces are automatically cocommutative coalgebras, with comultiplication defined by the diagonal map). For example, the commutative operad Com is a Hopf operad, with coproduct defined on the generator $\star \in \mathrm{Com}(2)$ by

$$c(\star) = \star \otimes \star.$$

With the comultiplication defined in this way, the tensor product of commutative algebras is the usual one. If A and B are commutative algebras, the product on $A \otimes B$ is defined by

$$(a \otimes b) \star (a' \otimes b') = (-1)^{|a'||b|} (a \star a') \otimes (b \star b').$$

The Poisson operad is also a Hopf operad, with coproduct defined (on the generators $\star, \{-,-\}$ by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

If A, B are Poisson algebras, then the tensor product $A \otimes B$ is a Poisson algebra with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|} (a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

8.1.0.8 Definition. *Let P be a differential graded Hopf operad. A* prefactorization P-algebra *is a prefactorization algebra with values in the multicategory of P-algebras. A* factorization P-algebra *is a prefactorization P-algebra, such that the underlying prefactorization algebra with values in cochain complexes is a factorization algebra.*

We can unpack this definition as follows. Suppose that $\mathcal{F}$ is a factorization P-algebra. Then $\mathcal{F}$ is a factorization algebra; and, in addition, for all $U \subset M$, $\mathcal{F}(U)$ is a P-algebra. The structure maps

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

(defined when $U_1, \ldots, U_n$ are disjoint open subsets of V) are required to be P-algebra maps in the sense defined above.

The following was floating free elsewhere in the book:
If P is a dg Hopf operad, then we can talk about P-algebras in the multicategory of differentiable cochain complexes. If F is a differentiable cochain complex, then we can define

the endomorphism dg operad End(F) whose n^{th} component is

$$\mathrm{End}(F)(n) = \mathrm{Hom}(F, \dots, F \mid F).$$

Then, a P-structure on F is a map of dg operads

$$P \to \mathrm{End}(F).$$

We call such an object a differentiable P-algebra.
Such P-algebras themselves form a multicategory, in a natural way. To see this, let S dgDiff denote the universal symmetric monoidal dg category containing the dg multicategory dgDiff of differentiable cochain complexes. If $F_1, \dots, F_k$ are differentiable P-algebras, then they are P-algebras in S dgDiff. Since P is a Hopf operad $F_1 \otimes \dots \otimes F_k$ is then a P-algebra in S dgDiff. A morphism in the multicategory of P-algebras is then a map of P algebras

$$F_1 \otimes \dots \otimes F_k \to G.$$

Note that forgetting the the forgetful functor from the multicategory of differentiable P-algebras to that of differentiable cochain complexes is faithful. That is, the map of sets

$$\mathrm{Hom}_P(F_1, \dots, F_k \mid G) \to \mathrm{Hom}_{\mathrm{dgDiff}}(F_1, \dots, F_k \mid G)$$

is injective. Further, the image of this map lies in the space of closed degree 0 multimorphisms; these are the same as multilinear cochain maps.
Thus, if F_i, G are differentiable P-algebras, and if

$$\phi : F_1 \times \dots \times F_k \to G$$

is a smooth multilinear cochain map, we can ask whether ϕ is a map of P-algebras.

8.1.0.9 Definition. *Let P be a differential graded Hopf operad. A prefactorization differentiable P-algebra is a prefactorization algebra with values in the multicategory of differentiable P-algebras. A factorization P-algebra is a prefactorization P-algebra, such that the underlying prefactorization algebra with values in differentiable cochain complexes is a factorization algebra.*

8.2. Commutative factorization algebras

One of the most important examples is when P is the operad Com of commutative algebras. Then, we find that $\mathcal{F}(U)$ is a commutative algebra for each U. Further, if $U_1, \dots, U_k \subset V$ are as above, the product map

$$m : \mathcal{F}(U_1) \times \dots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$

is compatible with the commutative algebra structures, in the following sense.

(1) If $1 \in \mathcal{F}(U_i)$ is the unit for the commutative product on each $F(U_i)$, then

$$m(1, \dots, 1) = 1.$$

(2) If $\alpha_i, \beta_i \in \mathcal{F}(U_i)$, then
$$m(\alpha_1\beta_1, \dots, \alpha_k\beta_k) = \pm m(\alpha_1, \dots, \alpha_k) m(\beta_1, \dots, \beta_k)$$
where ± indicates the usual Koszul rule of signs.

Note that the axioms of a factorization algebra imply that $\mathcal{F}(\emptyset)$ is the ground ring k (which we normally take to be $\mathbb{R}$ or $\mathbb{C}$ for classical theories and $\mathbb{R}[[\hbar]]$ or $\mathbb{C}[[\hbar]]$ for quantum field theories). The axioms above, in the case that $k = 1$ and $U_1 = \emptyset$, imply that the map
$$\mathcal{F}(\emptyset) \to \mathcal{F}(U)$$
is a map of unital commutative algebras.

If $\mathcal{F}$ is a commutative prefactorization algebra, then we can recover $\mathcal{F}$ uniquely from the underlying cosheaf of commutative algebras. Indeed, the maps
$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$
can be described in terms of the commutative product on $\mathcal{F}(V)$ and the maps $\mathcal{F}(U_i) \to \mathcal{F}(V)$.

8.3. The P_0 operad

Recall that the collection of observables in quantum mechanics form an associative algebra. The observables of a classical mechanical system form a Poisson algebra. In the deformation quantization approach to quantum mechanics, one starts with a Poisson algebra A^{cl}, and attempts to construct an associative algebra A^q, which is an algebra flat over the ring $\mathbb{C}[[\hbar]]$, together with an isomorphism of associative algebras $A^q/\hbar \cong A^{cl}$. In addition, if $a, b \in A^{cl}$, and $\widetilde{a}, \widetilde{b}$ are any lifts of a, b to A^q, then
$$\lim_{\hbar \to 0} \frac{1}{\hbar}[\widetilde{a}, \widetilde{b}] = \{a, b\} \in A^{cl}.$$

This book concerns the analog, in quantum field theory, of the deformation quantization picture in quantum mechanics. We have seen that the sheaf of solutions to the Euler-Lagrange equation of a classical field theory can be encoded by a commutative factorization algebra. A commutative factorization algebra is the analog, in our setting, of the commutative algebra appearing in deformation quantization. We have argued (section 1.6) that the observables of a quantum field theory should form a factorization algebra. This factorization algebra is the analog of the associative algebra appearing in deformation quantization.

In deformation quantization, the commutative algebra of classical observables has an extra structure – a Poisson bracket – which makes it “want” to deform into an associative algebra. In this section we will explain the analogous structure on a commutative factorization algebra which makes it want to deform into a factorization algebra. Later (section 12.2) we will see that the commutative factorization algebra associated to a classical field theory has this extra structure.

8.3.1. The E_0 operad.

8.3.1.1 Definition. *Let E_0 be the operad defined by*

$$E_0(n) = \begin{cases} 0 & \text{if } n > 0 \\ \mathbb{R} & \text{if } n = 0 \end{cases}$$

Thus, an E_0 algebra in the category of real vector spaces is a real vector space with a distinguished element in it. More generally, an E_0 algebra in a symmetric monoidal category $\mathcal{C}$ is the same thing as an object A of $\mathcal{C}$ together with a map $1_{\mathcal{C}} \to A$

The reason for the terminology E_0 is that this operad can be interpreted as the operad of little 0-discs.

The inclusion of the empty set into every open set implies that, for any factorization algebra $\mathcal{F}$, there is a unique map from the unit factorization algebra $\mathbb{R} \to \mathcal{F}$.

8.3.2. The P_0 operad. The Poisson operad is an object interpolating between the commutative operad and the associative (or E_1) operad. We would like to find an analog of the Poisson operad which interpolates between the commutative operad and the E_0 operad.

Let us define the P_k operad to be the operad whose algebras are commutative algebras equipped with a Poisson bracket of degree $1 - k$. With this notation, the usual Poisson operad is the P_1 operad.

Recall that the homology of the E_n operad is the P_n operad, for $n > 1$. Thus, just as the semi-classical version of an algebra over the E_1 operad is a Poisson algebra in the usual sense (that is, a P_1 algebra), the semi-classical version of an E_n algebra is a P_n algebra.

Thus, we have the following table:

Classical	Quantum
?	E_0 operad
P_1 operad	E_1 operad
P_2 operad	E_2 operad
$\vdots$	$\vdots$

This immediately suggests that the P_0 operad is the semi-classical version of the E_0 operad.

Note that the P_0 operad is a Hopf operad: the coproduct is defined by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

In concrete terms, this means that if A and B are P_0 algebras, their tensor product $A \otimes B$ is again a P_0 algebra, with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|} (a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

8.3.3. P_0 **factorization algebras.** Since the P_0 operad is a Hopf operad, it makes sense to talk about P_0 factorization algebras. We can give an explicit description of this structure. A P_0 factorization algebra is a commutative factorization algebra $\mathcal{F}$, together with a Poisson bracket of cohomological degree 1 on each commutative algebra $\mathcal{F}(U)$, with the following additional properties. Firstly, if $U \subset V$, the map

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

must be a homomorphism of P_0 algebras.

The second condition is that observables coming from disjoint sets must Poisson commute. More precisely, let U_1, U_2 be disjoint subsets f V. . Let $j_i : \mathcal{F}(U_i) \to \mathcal{F}(V)$ be the natural maps. Let $\alpha_i \in \mathcal{F}(U_i)$, and $j_i(\alpha_i) \in \mathcal{F}(V)$. Then, we require that

$$\{j_1(\alpha_1), j_2(\alpha_2)\} = 0 \in \mathcal{F}(V)$$

where $\{-,-\}$ is the Poisson bracket on $\mathcal{F}(V)$.

8.3.4. Quantization of P_0 **algebras.** We know what it means to quantize an Poisson algebra in the ordinary sense (that is, a P_1 algebra) into an E_1 algebra.

There is a similar notion of quantization for P_0 algebras. A quantization is simply an E_0 algebra over $\mathbb{R}[[\hbar]]$ which, modulo $\hbar$, is the original P_0 algebra, and for which there is a certain compatibility between the Poisson bracket on the P_0 algebra and the quantized E_0 algebra.

Let A be a commutative algebra in the category of cochain complexes. Let A_1 be an E_0 algebra flat over $\mathbb{R}[[\hbar]]/\hbar^2$, and suppose that we have an isomorphism of chain complexes

$$A_1 \otimes_{\mathbb{R}[[\hbar]]/\hbar^2} \mathbb{R} \cong A.$$

In this situation, we can define a bracket on A of degree 1, as follows.

We have an exact sequence

$$0 \to \hbar A \to A_1 \to A \to 0.$$

The boundary map of this exact sequence is a cochain map

$$D : A \to A$$

(well-defined up to homotopy).

Let us define a bracket on A by the formula

$$\{a, b\} = D(ab) - (-1)^{|a|} aDb - (Da)b.$$

Because D is well-defined up to homotopy, so is this bracket. However, unless D is an order two differential operator, this bracket is simply a cochain map $A \otimes A \to A$, and not a Poisson bracket of degree 1.

In particular, this bracket induces one on the cohomology $H^*(A)$ of A. The cohomological bracket is independent of any choices.

8.3.4.1 Definition. *Let A be a P_0 algebra in the category of cochain complexes. Then a* quantization of A *is an E_0 algebra $\widetilde{A}$ over $\mathbb{R}[[\hbar]]$, together with a quasi-isomorphism of E_0 algebras*

$$\widetilde{A} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A,$$

which satisfies the following correspondence principle*: the bracket on $H^*(A)$ induced by $\widetilde{A}$ must coincide with that given by the P_0 structure on A.*

In the next section we will consider a more sophisticated, operadic notion of quantization, which is strictly stronger than this one. To distinguish between the two notions, one could call the definition of quantization presented here a *weak quantization*, while the definition introduced later will be called a *strong quantization*.

8.4. The Beilinson-Drinfeld operad

Beilinson and Drinfeld [BD04] constructed an operad over the formal disc which generically is equivalent to the E_0 operad, but which at 0 is equivalent to the P_0 operad. We call this operad the Beilinson-Drinfeld operad.

The operad P_0 is generated by a commutative associative product $- \star -$, of degree 0; and a Poisson bracket $\{-, -\}$ of degree $+1$.

8.4.0.2 Definition. *The* Beilinson-Drinfeld (or *BD*) operad *is the differential graded operad over the ring $\mathbb{R}[[\hbar]]$ which, as a graded operad, is simply*

$$BD = P_0 \otimes \mathbb{R}[[\hbar]];$$

but with differential defined by

$$\mathrm{d}(- \star -) = \hbar\{-, -\}.$$

If M is a flat differential graded $\mathbb{R}[[\hbar]]$ module, then giving M the structure of a BD algebra amounts to giving M a commutative associative product, of degree 0, and a Poisson bracket of degree 1, such that the differential on M is a derivation of the Poisson bracket, and the following identity is satisfied:

$$\mathrm{d}(m \star n) = (\mathrm{d}m) \star n + (-1)^{|m|} m \star (\mathrm{d}n) + (-1)^{|m|} \hbar \{m, n\}$$

8.4.0.3 Lemma. *There is an isomorphism of operads,*

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong P_0,$$

and a quasi-isomorphism of operads over $\mathbb{R}((\hbar))$*,*

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)) \simeq E_0 \otimes \mathbb{R}((\hbar)).$$

Thus, the operad BD interpolates between the P_0 operad and the E_0 operad.

BD is an operad in the category of differential graded $\mathbb{R}[[\hbar]]$ modules. Thus, we can talk about BD algebras in this category, or in any symmetric monoidal category enriched over the category of differential graded $\mathbb{R}[[\hbar]]$ modules.

The BD algebra is, in addition, a Hopf operad, with coproduct defined in the same way as in the P_0 operad. Thus, one can talk about BD factorization algebras.

8.4.1. BD quantization of P_0 algebras.

8.4.1.1 Definition. *Let A be a P_0 algebra (in the category of cochain complexes). A* BD quantization of *A is a flat $\mathbb{R}[[\hbar]]$ module A^q, flat over $\mathbb{R}[[\hbar]]$, which is equipped with the structure of a BD algebra, and with an isomorphism of P_0 algebras*

$$A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A.$$

Similarly, an order k BD quantization of A is a differential graded $\mathbb{R}[[\hbar]]/\hbar^{k+1}$ module A^q, flat over $\mathbb{R}[[\hbar]]/\hbar^{k+1}$, which is equipped with the structure of an algebra over the operad

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/\hbar^{k+1},$$

and with an isomorphism of P_0 algebras

$$A^q \otimes_{\mathbb{R}[[\hbar]]/\hbar^{k+1}} \mathbb{R} \cong A.$$

This definition applies without any change in the world of factorization algebras.

8.4.1.2 Definition. *Let $\mathcal{F}$ be a P_0 factorization algebra on M. Then a* BD quantization of *$\mathcal{F}$ is a BD factorization algebra $\widetilde{\mathcal{F}}$ equipped with a quasi-isomorphism*

$$\widetilde{\mathcal{F}} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \simeq \mathcal{F}$$

of P_0 factorization algebras on M.

8.4.2. Operadic description of ordinary deformation quantization. We will finish this section by explaining how the ordinary deformation quantization picture can be phrased in similar operadic terms.

Consider the following operad BD_1 over $\mathbb{R}[[\hbar]]$. BD_1 is generated by two binary operations, a product $*$ and a bracket $[-,-]$. The relations are that the product is associative; the bracket is antisymmetric and satisfies the Jacobi identity; the bracket and the product satisfy a certain Leibniz relation, expressed in the identity

$$[ab, c] = a[b, c] \pm [b, c]a$$

(where ± indicates the Koszul sign rule); and finally the relation

$$a * b \mp b * a = \hbar[a, b]$$

holds. This operad was introduced by Ed Segal [Seg10].

Note that, modulo $\hbar$, BD_1 is the ordinary Poisson operad P_1. If we set $\hbar = 1$, we find that BD_1 is the operad E_1 of associative algebras. Thus, BD_1 interpolates between P_1 and E_1 in the same way that BD_0 interpolates between P_0 and E_0.

Let A be a P_1 algebra. Let us consider possible lifts of A to a BD_1 algebra.

8.4.2.1 Lemma. *A lift of A to a BD_1 algebra, flat over $\mathbb{R}[[\hbar]]$, is the same as a deformation quantization of A in the usual sense.*

PROOF. We need to describe BD_1 structures on $A[[\hbar]]$ compatible with the given Poisson structure. To give such a BD_1 structure is the same as to give an associative product on $A[[\hbar]]$, linear over $\mathbb{R}[[\hbar]]$, and which modulo $\hbar$ is the given commutative product on A. Further, the relations in the BD_1 operad imply that the Poisson bracket on A is related to the associative product on $A[[\hbar]]$ by the formula

$$\hbar^{-1}(a * b \mp b * a) = \{a, b\} \mod \hbar.$$

□

Part 3

Classical field theory

CHAPTER 9

Introduction to classical field theory

Our goal here is to describe how the observables of a classical field theory naturally form a factorization algebra (section 6.1). More accurately, we are interested in what might be called classical perturbative field theory. “Classical” means that the main object of interest is the sheaf of solutions to the Euler-Lagrange equations for some local action functional. “Perturbative” means that we will only consider those solutions which are infinitesimally close to a given solution. Much of this part of the book is devoted to providing a precise mathematical definition of these ideas, with inspiration taken from deformation theory and derived geometry. In this chapter, then, we will simply sketch the essential ideas.

9.1. The Euler-Lagrange equations

The fundamental objects of a physical theory are the observables of a theory, that is, the measurements one can make in that theory. In a classical field theory, the fields that appear “in nature” are constrained to be solutions to the Euler-Lagrange equations (also called the equations of motion). Thus, the measurements one can make are the functions on the space of solutions to the Euler-Lagrange equations.

However, it is essential that we do not take the naive moduli space of solutions. Instead, we consider the *derived* moduli space of solutions. Since we are working perturbatively — that is, infinitesimally close to a given solution — this derived moduli space will be a “formal moduli problem” [**?**, **Lur11**]. In the physics literature, the procedure of taking the derived critical locus of the action functional is implemented by the BV formalism. Thus, the first step (chapter 10.1.3) in our treatment of classical field theory is to develop a language to treat formal moduli problems cut out by systems of partial differential equations on a manifold M. Since it is essential that the differential equations we consider are elliptic, we call such an object a *formal elliptic moduli problem*.

Since one can consider the solutions to a differential equation on any open subset $U \subset M$, a formal elliptic moduli problem $\mathcal{F}$ yields, in particular, a sheaf of formal moduli problems on M. This sheaf sends U to the formal moduli space $\mathcal{F}(U)$ of solutions on U.

We will use the notation $\mathcal{E}\mathcal{L}$ to denote the formal elliptic moduli problem of solutions to the Euler-Lagrange equation on M; thus, $\mathcal{E}\mathcal{L}(U)$ will denote the space of solutions on an open subset $U \subset M$.

9.2. Observables

In a field theory, we tend to focus on measurements that are localized in spacetime. Hence, we want a method that associates a set of observables to each region in M. If $U \subset M$ is an open subset, the observables on U are

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathcal{E}\mathcal{L}(U)),$$

our notation for the algebra of functions on the formal moduli space $\mathcal{E}\mathcal{L}(U)$ of solutions to the Euler-Lagrange equations on U. (We will be more precise about which class of functions we are using later.) As we are working in the derived world, $\mathrm{Obs}^{cl}(U)$ is a differential-graded commutative algebra. Using these functions, we can answer any question we might ask about the behavior of our system in the region U.

The factorization algebra structure arises naturally on the observables in a classical field theory. Let U be an open set in M, and $V_1, \ldots, V_k$ a disjoint collection of open subsets of U. Then restriction of solutions from U to each V_i induces a natural map

$$\mathcal{E}\mathcal{L}(U) \to \mathcal{E}\mathcal{L}(V_1) \times \cdots \times \mathcal{E}\mathcal{L}(V_k).$$

Since functions pullback under maps of spaces, we get a natural map

$$\mathrm{Obs}^{cl}(V_1) \otimes \cdots \otimes \mathrm{Obs}^{cl}(V_k) \to \mathrm{Obs}^{cl}(U)$$

so that Obs^{cl} forms a *pre*factorization algebra. To see that Obs^{cl} is indeed a factorization algebra, it suffices to observe that the functor $\mathcal{E}\mathcal{L}$ is a sheaf.

Since the space $\mathrm{Obs}^{cl}(U)$ of observables on a subset $U \subset M$ is a commutative algebra, and not just a vector space, we see that the observables of a classical field theory form a commutative factorization algebra (section 8).

9.3. The symplectic structure

Above, we outlined a way to construct, from the elliptic moduli problem associated to the Euler-Lagrange equations, a commutative factorization algebra. This construction, however, would apply equally well to any system of differential equations. The Euler-Lagrange equations, of course, have the special property that they arise as the critical points of a functional.

In finite dimensions, a formal moduli problem which arises as the derived critical locus (section 11.1) of a function is equipped with an extra structure: a symplectic form of cohomological degree -1. For us, this symplectic form is an intrinsic way of indicating that a formal moduli problem arises as the critical locus of a functional. Indeed, any formal moduli problem with such a symplectic form can be expressed (non-uniquely) in this way.

We give (section 11.2) a definition of symplectic form on an elliptic moduli problem. We then simply *define* a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

Given a local action functional satisfying certain non-degeneracy properties, we construct (section 11.3.1) an elliptic moduli problem describing the corresponding Euler-Lagrange equations, and show that this elliptic moduli problem has a symplectic form of degree -1.

In ordinary symplectic geometry, the simplest construction of a symplectic manifold is as a cotangent bundle. In our setting, there is a similar construction: given any elliptic moduli problem $\mathcal{F}$, we construct (section 11.6) a new elliptic moduli problem $T^*[-1]\mathcal{F}$ which has a symplectic form of degree -1. It turns out that many examples of field theories of interest in mathematics and physics arise in this way.

9.4. The P_0 structure

In finite dimensions, if X is a formal moduli problem with a symplectic form of degree -1, then the dg algebra $\mathscr{O}(X)$ of functions on X is equipped with a Poisson bracket of degree 1. In other words, $\mathscr{O}(X)$ is a P_0 algebra (section 8.3).

In infinite dimensions, we show that something similar happens. If $\mathcal{F}$ is a classical field theory, then we show that on every open U, the commutative algebra $\mathscr{O}(\mathcal{F}(U)) = \mathrm{Obs}^{cl}(U)$ has a P_0 structure. We then show that the commutative factorization algebra Obs^{cl} forms a P_0 factorization algebra. This is not quite trivial; it is at this point that we need the assumption that our Euler-Lagrange equations are elliptic.

CHAPTER 10

Elliptic moduli problems

The essential data of a classical field theory is the moduli space of solutions to the equations of motion of the field theory. For us, it is essential that we take not the naive moduli space of solutions, but rather the *derived* moduli space of solutions. In the physics literature, the procedure of taking the derived moduli of solutions to the Euler-Lagrange equations is known as the classical Batalin-Vilkovisky formalism.

The derived moduli space of solutions to the equations of motion of a field theory on X is a sheaf on X. In this chapter we will introduce a general language for discussing sheaves of "derived spaces" on X that are cut out by differential equations.

Our focus in this book is on perturbative field theory, so we sketch the heuristic picture from physics before we introduce a mathematical language that formalizes the picture. Suppose we have a field theory and we have found a solution to the Euler-Lagrange equations ϕ_0. We want to find the nearby solutions, and a time-honored approach is to consider a formal series expansion around ϕ_0,

$$\phi_t = \phi_0 + t\phi_1 + t^2\phi_2 + \cdots,$$

and to solve iteratively the Euler-Lagrange equations for the higher terms ϕ_n. Of course, such an expansion is often not convergent in any reasonable sense, but this perturbative method has provided insights into many physical problems. In mathematics, particularly the deformation theory of algebraic geometry, this method has also flourished and acquired a systematic geometric interpretation. Here, though, we work in place of t with a parameter ε that is nilpotent, so that there is some integer n such that $\varepsilon^{n+1} = 0$. Let

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots + \varepsilon^n\phi_n.$$

Again, the Euler-Lagrange equation applied to ϕ becomes a system of simpler differential equations organized by each power of ε. As we let the order of ε go to infinity and find the nearby solutions, we describe the *formal neighborhood* of ϕ_0 in the space of all solutions to the Euler-Lagrange equations. (Although this procedure may seem narrow in scope, its range expands considerably by considering families of solutions, rather a single fixed solution. Our formalism is built to work in families.)

In this chapter we will introduce a mathematical formalism for this procedure, which includes derived perturbations (i.e., ε has nonzero cohomological degree). In mathematics, this formalism is part of derived deformation theory or formal derived geometry. Thus, before we discuss the concepts specific to classical field theory, we will explain some general techniques from deformation theory. A key role is played by a deep relationship between Lie algebras and formal moduli spaces.

10.1. Formal moduli problems and Lie algebras

In ordinary algebraic geometry, the fundamental objects are commutative algebras. In derived algebraic geometry, commutative algebras are replaced by commutative differential graded algebras concentrated in non-positive degrees (or, if one prefers, simplicial commutative algebras; over $\mathbb{Q}$, there is no difference).

We are interested in formal derived geometry, which is described by nilpotent commutative dg algebras.

10.1.0.2 Definition. *An* Artinian dg algebra *over a field K of characteristic zero is a differential graded commutative K-algebra R, concentrated in degrees ≤ 0, such that*

(1) each graded component R^i is finite dimensional, and $R^i = 0$ for $i \ll 0$;
(2) R has a unique maximal differential ideal m such that $R/m = K$, and such that $m^N = 0$ for $N \gg 0$.

Given the first condition, the second condition is equivalent to the statement that $H^0(R)$ is Artinian in the classical sense.

The category of Artinian dg algebras is simplicially enriched in a natural way. A map $R \to S$ is simply a map of dg algebras taking the maximal ideal m_R to that of m_S. Equivalently, such a map is a map of non-unital dg algebras $m_R \to m_S$. An n-simplex in the space $\mathrm{Maps}(R, S)$ of maps from R to S is defined to be a map of non-unital dg algebras

$$m_R \to m_S \otimes \Omega^*(\triangle^n)$$

where $\Omega^*(\triangle^n)$ is some commutative algebra model for the cochains on the n-simplex. (Normally, we will work over $\mathbb{R}$, and $\Omega^*(\triangle^n)$ will be the usual de Rham complex.)

We will (temporarily) let Art_k denote the simplicially enriched category of Artinian dg algebras over k.

10.1.0.3 Definition. *A formal moduli problem over a field k is a functor (of simplicially enriched categories)*

$$F : \mathrm{Art}_k \to \mathrm{sSets}$$

from Art_k *to the category* sSets *of simplicial sets, with the following additional properties.*

(1) *$F(k)$ is contractible.*
(2) *F takes surjective maps of dg Artinian algebras to fibrations of simplicial sets.*
(3) *Suppose that A, B, C are dg Artinian algebras, and that $B \to A$, $C \to A$ are surjective maps. Then we can form the fiber product $B \times_A C$. We require that the natural map*

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

We remark that such a moduli problem F is *pointed*: F assigns to k a point, up to homotopy, since $F(k)$ is contractible. Since we work mostly with pointed moduli problems in this book, we will not emphasize this issue. Whenever we work with more general moduli problems, we will indicate it explicitly.

Note that, in light of the second property, the fiber product $F(B) \times_{F(A)} F(C)$ coincides with the homotopy fiber product.

The category of formal moduli problems is itself simplicially enriched, in an evident way. If F, G are formal moduli problems, and $\phi : F \to G$ is a map, we say that ϕ is a weak equivalence if for all dg Artinian algebras R, the map

$$\phi(R) : F(R) \to G(R)$$

is a weak homotopy equivalence of simplicial sets.

10.1.1. Formal moduli problems and L_∞ algebras. One very important way in which formal moduli problems arise is as the solutions to the Maurer-Cartan equation in an L_∞ algebra. As we will see later, all formal moduli problems are equivalent to formal moduli problems of this form.

If $\mathfrak{g}$ is an L_∞ algebra, and (R, m) is a dg Artinian algebra, we will let

$$\mathrm{MC}(\mathfrak{g} \otimes m)$$

denote the simplicial set of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes m$. Thus, an n-simplex in this simplicial set is an element

$$\alpha \in \mathfrak{g} \otimes m \otimes \Omega^*(\triangle^n)$$

of cohomological degree 1, which satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \sum_{n \geq 2} \tfrac{1}{n!} l_n(\alpha, \dots, \alpha) = 0.$$

It is a well-known result in derived deformation theory that sending R to $\mathrm{MC}(\mathfrak{g} \otimes m)$ defines a formal moduli problem (see [Get09a], [Hin01]). We will often use the notation $B\mathfrak{g}$ to denote this formal moduli problem.

If $\mathfrak{g}$ is finite dimensional, then a Maurer-Cartan element of $\mathfrak{g} \otimes m$ is the same thing as a map of commutative dg algebras

$$C^*(\mathfrak{g}) \to R$$

which takes the maximal ideal of $C^*(\mathfrak{g})$ to that of R.

Thus, we can think of the Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ as the algebra of functions on $B\mathfrak{g}$.

Under the dictionary between formal moduli problems and L_∞ algebras, a dg vector bundle on $B\mathfrak{g}$ is the same thing as a dg module over $\mathfrak{g}$. The cotangent complex to $B\mathfrak{g}$ corresponds to the $\mathfrak{g}$-module $\mathfrak{g}^\vee[-1]$, with the shifted coadjoint action. The tangent complex corresponds to the $\mathfrak{g}$-module $\mathfrak{g}[1]$, with the shifted adjoint action.

If M is a $\mathfrak{g}$-module, then sections of the corresponding vector bundle on $B\mathfrak{g}$ is the Chevalley-Eilenberg cochains with coefficients in M. Thus, we can define $\Omega^1(B\mathfrak{g})$ to be

$$\Omega^1(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]).$$

Similarly, the complex of vector fields on $B\mathfrak{g}$ is

$$\mathrm{Vect}(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1]).$$

Note that, if $\mathfrak{g}$ is finite dimensional, this is the same as the cochain complex of derivations of $C^*(\mathfrak{g})$. Even if $\mathfrak{g}$ is not finite dimensional, the complex $\mathrm{Vect}(B\mathfrak{g})$ is, up to a shift of one, the Lie algebra controlling deformations of the L_∞ structure on $\mathfrak{g}$.

10.1.2. The fundamental theorem of deformation theory. The following statement is at the heart of the philosophy of deformation theory:

> There is an equivalence of $(\infty, 1)$ categories between the category of differential graded Lie algebras and the category of formal pointed moduli problems.

In a different guise, this statement goes back to Quillen's work [Qui69] on rational homotopy theory. A precise formulation of this theorem has been proved by Hinich [Hin01]; more general theorems of this nature are considered in [Lur11], [?] and in [?], which is also an excellent survey of these ideas.

It would take us too far afield to describe the language in which this statement can be made precise. We will simply use this statement as motivation: we will only consider formal moduli problems described by L_∞ algebras, and this statement asserts that we lose no information in doing so.

10.1.3. Elliptic moduli problems. We are interested in formal moduli problems which describe solutions to differential equations on a manifold M. Since we can discuss solutions to a differential equation on any open subset of M, such an object will give a sheaf of derived moduli problems on M, described by a sheaf of homotopy Lie algebras. Let us give a formal definition of such a sheaf.

10.1.3.1 Definition. *Let M be a manifold. A* local L_∞ algebra *on M consists of the following data.*

1. *A graded vector bundle L on M, whose space of smooth sections will be denoted $\mathcal{L}$.*
2. *A differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$, of cohomological degree 1 and square 0.*
3. *A collection of poly-differential operators*

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

for $n \geq 2$, which are alternating, are of cohomological degree $2 - n$, and endow $\mathcal{L}$ with the structure of L_∞ algebra.

10.1.3.2 Definition. *An* elliptic L_∞ algebra *is a local L_∞ algebra $\mathcal{L}$ as above with the property that $(\mathcal{L}, \mathrm{d})$ is an elliptic complex.*

Remark: The reader who is not comfortable with the language of L_∞ algebras will lose little by only considering elliptic dg Lie algebras. Most of our examples of classical field theories will be described using dg Lie algebra rather than L_∞ algebras.

If $\mathcal{L}$ is a local L_∞ algebra on a manifold M, then it yields a presheaf $B\mathcal{L}$ of formal moduli problems on M. This presheaf sends a dg Artinian algebra (R, m) and an open subset $U \subset M$ to the simplicial set

$$B\mathcal{L}(U)(R) = \mathrm{MC}(\mathcal{L}(U) \otimes m)$$

of Maurer-Cartan elements of the L_∞ algebra $\mathcal{L}(U) \otimes m$ (where $\mathcal{L}(U)$ refers to the sections of L on U). We will think of this as the R-points of the formal pointed moduli problem associated to $\mathcal{L}(U)$. One can show, using the fact that $\mathcal{L}$ is a fine sheaf, that this sheaf of formal moduli problems is actually a homotopy sheaf, i.e. it satisfies Čech descent. Since this point plays no role in our work, we will not elaborate further.

10.1.3.3 Definition. *A* formal pointed elliptic moduli problem *(or simply* elliptic moduli problem*) is a sheaf of formal moduli problems on M that is represented by an elliptic L_∞ algebra.*

The basepoint of the moduli problem corresponds, in the setting of field theory, to the distinguished solution we are expanding around.

10.2. Examples of elliptic moduli problems related to scalar field theories

10.2.1. The free scalar field theory. Let us start with the most basic example of an elliptic moduli problem, that of harmonic functions. Let M be a Riemannian manifold. We want to consider the formal moduli problem describing functions ϕ on M that are harmonic, namely, functions that satisfy $\mathrm{D}\phi = 0$ where D is the Laplacian. The base point of this formal moduli problem is the zero function.

The elliptic L_∞ algebra describing this formal moduli problem is defined by

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

This complex is thus situated in degrees 1 and 2. The products l_n in this L_∞ algebra are all zero for $n \geq 2$.

In order to justify this definition, let us analyze the Maurer-Cartan functor of this L_∞ algebra. Let R be an ordinary (not dg) Artinian algebra, and let m be the maximal ideal of R. The set of 0-simplices of the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m \mid \mathrm{D}\phi = 0.\}$$

Indeed, because the L_∞ algebra $\mathcal{L}$ is Abelian, the set of solutions to the Maurer-Cartan equation is simply the set of closed degree 1 elements of the cochain complex $\mathcal{L} \otimes m$. All higher simplices in the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ are constant. To see this, note that if $\phi \in \mathcal{L} \otimes m \otimes \Omega^*(\triangle^n)$ is a closed element in degree 1, then ϕ must be in $C^\infty(M) \otimes m \otimes \Omega^0(\triangle^n)$. The fact that ϕ is closed amounts to the statement that $\mathrm{D}\phi = 0$ and that $\mathrm{d}_{dR}\phi = 0$, where d_{dR} is the de Rham differential on $\Omega^*(\triangle^n)$.

Let us now consider the Maurer-Cartan simplicial set associated to a differential graded Artinian algebra (R, m) with differential d_R. The the set of 0-simplices of $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m^0,\ \ \psi \in C^\infty(M) \otimes m^{-1} \mid \mathrm{D}\phi = \mathrm{d}_R\psi.\}$$

(The superscripts on m indicate the cohomological degree.) Thus, the 0-simplices of our simplicial set can be identified with the set R-valued smooth functions ϕ on M that are harmonic up to a homotopy given by ψ and also vanish modulo the maximal ideal m.

Next, let us identify the set of 1-simplices of the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$. This is the set of closed degree 1 elements of $\mathcal{L} \otimes m \otimes \Omega^*([0,1])$. Such a closed degree 1 element has four terms:

$$\begin{aligned}
\phi_0(t) &\in C^\infty(M) \otimes m^0 \otimes \Omega^0([0,1]) \\
\phi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^1([0,1]) \\
\psi_0(t) &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^0([0,1]) \\
\psi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-2} \otimes \Omega^1([0,1]).
\end{aligned}$$

Being closed amounts to satisfying the three equations

$$\begin{aligned} \mathrm{D}\,\phi_0(t) &= \mathrm{d}_R\psi_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_0(t) &= \mathrm{d}_R\phi_1(t) \\ \mathrm{D}\,\phi_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\psi_0(t) &= \mathrm{d}_R\psi_1(t). \end{aligned}$$

These equations can be interpreted as follows. We think of $\phi_0(t)$ as providing a family of R-valued smooth functions on M, which are harmonic up to a homotopy specified by $\psi_0(t)$. Further, $\phi_0(t)$ is independent of t, up to a homotopy specified by $\phi_1(t)$. Finally, we have a coherence condition among our two homotopies.

The higher simplices of the simplicial set have a similar interpretation.

10.2.2. Interacting scalar field theories. Next, we will consider an elliptic moduli problem that arises as the Euler-Lagrange equation for an interacting scalar field theory. Let ϕ denote a smooth function on the Riemannian manifold M with metric g. The action functional is

$$S(\phi) = \int_M \tfrac{1}{2}\phi\,\mathrm{D}\,\phi + \tfrac{1}{4!}\phi^4\,\mathrm{dvol}_g\,.$$

The Euler-Lagrange equation for the action functional S is

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = 0,$$

a nonlinear PDE, whose space of solutions is hard to describe.

Instead of trying to describe the actual space of solutions to this nonlinear PDE, we will describe the formal moduli problem of solutions to this equation where ϕ is infinitesimally close to zero.

The formal moduli problem of solutions to this equation can be described as the solutions to the Maurer-Cartan equation in a certain elliptic L_∞ algebra which continue we call $\mathcal{L}$. As a cochain complex, $\mathcal{L}$ is

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

Thus, $C^\infty(M)$ is situated in degrees 1 and 2, and the differential is the Laplacian.

The L_∞ brackets l_n are all zero except for l_3. The cubic bracket l_3 is the map

$$\begin{aligned} l_3 : C^\infty(M)^{\otimes 3} &\to C^\infty(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3. \end{aligned}$$

Here, the copy of $C^\infty(M)$ appearing in the source of l_3 is the one situated in degree 1, whereas that appearing in the target is the one situated in degree 2.

If R is an ordinary (not dg) Artinian algebra, then the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ associated to R has for 0-simplices the set $\phi \in C^\infty(M) \otimes m$ such that $\mathrm{D}\phi + \frac{1}{3!}\phi^3 = 0$. This equation may look as complicated as the full nonlinear PDE, but it is substantially simpler than the original problem. For example, consider $R = \mathbb{R}[\varepsilon]/(\varepsilon^2)$, the "dual numbers." Then $\phi = \varepsilon\phi_1$ and the Maurer-Cartan equation becomes $\mathrm{D}\phi_1 = 0$. For $R = \mathbb{R}[\varepsilon]/(\varepsilon^4)$, we have $\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3$ and the Maurer-Cartan equation becomes a triple of simpler *linear* PDE:

$$\mathrm{D}\phi_1 = 0,\ \mathrm{D}\phi_2 = 0,\ \text{and } D\phi_3 + \tfrac{1}{2}\phi_1^3 = 0.$$

We are simply reading off the ε^k components of the Maurer-Cartan equation. The higher simplices of this simplicial set are constant.

If R is a dg Artinian algebra, then the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ has for 0-simplices the set of pairs $\phi \in C^\infty(M) \otimes m^0$ and $\psi \in C^\infty(M) \otimes m^{-1}$ such that

$$\mathrm{D}\phi + \tfrac{1}{3!}\phi^3 = \mathrm{d}_R\psi.$$

We should interpret this as saying that ϕ satisfies the Euler-Lagrange equations up to a homotopy given by ψ.

The higher simplices of this simplicial set have an interpretation similar to that described for the free theory.

10.3. Examples of elliptic moduli problems related to gauge theories

10.3.1. Flat bundles. Next, let us discuss a more geometric example of an elliptic moduli problem: the moduli problem describing flat bundles on a manifold M. In this case, because flat bundles have automorphisms, it is more difficult to give a direct definition of the formal moduli problem.

Thus, let G be a Lie group, and let $P \to M$ be a principal G-bundle equipped with a flat connection ∇_0. Let $\mathfrak{g}_P$ be the adjoint bundle (associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$). Then $\mathfrak{g}_P$ is a bundle of Lie algebras on M, equipped with a flat connection that we will also denote ∇_0.

For R be an Artinian dg algebra, we want to define the simplicial set $\mathrm{Def}_P(R)$ of R-families of flat G-bundles on M that deform P. The question is "what local L_∞ algebra yields this elliptic moduli problem?"

The *answer* is $\mathcal{L} = \Omega^*(M, \mathfrak{g}_P)$, where the differential is d_{∇_0}, the de Rham differential coupled to our connection ∇_0. But we need to explain how to find this answer so we will provide the reasoning behind our answer. This reasoning is a model for finding the local L_∞ algebras associated to field theories.

As the underlying topological bundle of P is rigid, we can only deform the flat connection on P. Let's consider deformations over a dg Artinian ring R with maximal ideal m. A deformation of the connection on P is given by an element

$$A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0.$$

We would like to ask that A is flat up to homotopy. The curvature $F(A)$ is

$$F(A) = \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g}_P) \otimes m.$$

Note that, by the Bianchi identity, $\mathrm{d}_{\nabla_0} F(A) + [A, F(A)] = 0$.

For A to be flat up to homotopy, we should ask that $F(A)$ is exact in the cochain complex $\Omega^2(M, \mathfrak{g}_P) \otimes m$ of m-valued 2-forms on M. However, we should also ask that $F(A)$ is exact in a way compatible with the Bianchi identity.

Thus, as a first approximation, we will define the 0-simplices of the deformation functor by

$$\begin{aligned}&\mathrm{Def}_P^{prelim}(R)[0] = \\ &\qquad \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m, B \in \Omega^2(M, \mathfrak{g}_P) \otimes m \mid F(A) = \mathrm{d}_R B,\ \mathrm{d}_{\nabla_0} B + [A, B] = 0\}.\end{aligned}$$

Here, A is of cohomological degree 1 and B is of cohomological degree 0.

Note that if m is of square zero, then the Bianchi constraint on B just says that $\mathrm{d}_\nabla B = 0$. This leads to a problem: the sheaf of closed 2-forms on M is *not* fine: it has higher cohomology groups. Thus, we cannot hope to construct a deformation functor with values in homotopy sheaves of simplicial sets on M in this way.

Instead, we will ask that B satisfy the Bianchi constraint up a sequence of higher homotopies. Thus, the 0-simplices of our simplicial set of deformations are defined by

$$\begin{aligned}\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m, B \in \Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m \\ \mid F(A) + \mathrm{d}_{\nabla_0} B + [A, B] + \tfrac{1}{2}[B, B] = 0.\}.\end{aligned}$$

Here, d refers to the total differential on the tensor product cochain complex $\Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m$. As before, A is of cohomological degree 1 and B is of cohomological degree 0.

If we let $B_i \in \Omega^i(M, \mathfrak{g}_P) \otimes m$, then the first few constraints on the B_i can be written as

$$\mathrm{d}_{\nabla_0} B_2 + [A, B_2] + \mathrm{d}_R B_3 = 0$$
$$\mathrm{d}_{\nabla_0} B_3 + [A, B_3] + \tfrac{1}{2}[B_2, B_2] + \mathrm{d}_R B_4 = 0.$$

Thus, B_2 satisfies the Bianchi constraint up to a homotopy defined by B_3, and so on.

The higher simplices of this simplicial set must relate gauge-equivalent flat connections. If the dg algebra R is concentrated in degree 0 (and so has zero differential), then we can define the simplicial set $\mathrm{Def}_P(R)$ to be the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$, which acts on $\mathrm{Def}_P(R)[0]$ in the standard way (see, for instance, [**KS**] or [**Man09**]).

If R is not concentrated in degree 0, however, then the higher simplices of $\mathrm{Def}_P(R)$ must also involve elements of R of negative cohomological degree. Indeed, degree 0 elements of R should be thought of as homotopies between degree 1 elements of R, and so should contribute 1-simplices to our simplicial set.

A slick way to define a simplicial set with both desiderata is to set

$$\mathrm{Def}_P(R)[n] = \{A \in \Omega^*(M, \mathfrak{g}_P) \otimes m \otimes \Omega^*(\triangle^n) \mid \mathrm{d}_{\nabla_0} A + \mathrm{d}_R A + \tfrac{1}{2}[A, A] = 0\}.$$

Suppose that R is concentrated in degree 0 (so that the differential on R is zero). Then, the higher forms on M don't play any role, and

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m \mid \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] = 0\}.$$

One can show (see [**Get09a**]) that in this case, the simplicial set $\mathrm{Def}_P(R)$ is weakly homotopy equivalent to the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$. Indeed, a 1-simplex in the simplicial set $\mathrm{Def}_P(R)$ is given by a family of the form $A_0(t) + A_1(t)\mathrm{d}t$, where $A_0(t)$ is a smooth family of elements of $\Omega^1(M, \mathfrak{g}_P) \otimes m$ depending on $t \in [0, 1]$, and $A_1(t)$ is a smooth family of elements of $\Omega^0(M, \mathfrak{g}_P) \otimes m$. The Maurer-Cartan equation in this context says that

$$\begin{aligned} \mathrm{d}_{\nabla_0} A_0(t) + \tfrac{1}{2}[A_0(t), A_0(t)] &= 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} A_0(t) + [A_1(t), A_0(t)] &= 0. \end{aligned}$$

The first equation says that $A_0(t)$ defines a family of flat connections. The second equation says that the gauge equivalence class of $A_0(t)$ is independent of t. In this way, gauge equivalences are represented by 1-simplices in $\mathrm{Def}_P(R)$.

It is immediate that the formal moduli problem $\mathrm{Def}_P(R)$ is represented by the elliptic dg Lie algebra

$$\mathcal{L} = \Omega^*(M, \mathfrak{g}).$$

The differential on $\mathcal{L}$ is the de Rham differential d_{∇_0} on M coupled to the flat connection on $\mathfrak{g}$. The only nontrivial bracket is l_2, which just arises by extending the bracket of $\mathfrak{g}$ over the commutative dg algebra $\Omega^*(M)$ in the appropriate way.

10.3.2. Self-dual bundles. Next, we will discuss the formal moduli problem associated to the self-duality equations on a 4-manifold. We won't go into as much detail as we did for flat connections; instead, we will simply write down the elliptic L_∞ algebra representing this formal moduli problem. (For a careful explanation, see the original article [**AHS78**].)

Let M be an oriented 4-manifold. Let G be a Lie group, and let $P \to M$ be a principal G-bundle, and let $\mathfrak{g}_P$ be the adjoint bundle of Lie algebras. Suppose we have a connection A on P with anti-self-dual curvature:

$$F(A)_+ = 0 \in \Omega^2_+(M, \mathfrak{g}_P)$$

(here $\Omega^2_+(M)$ denotes the space of self-dual two-forms).

Then, the elliptic Lie algebra controlling deformations of (P, A) is described by the diagram

$$\Omega^0(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}} \Omega^1(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_+} \Omega^2_+(M, \mathfrak{g}_P).$$

Here d_+ is the composition of the de Rham differential (coupled to the connection on $\mathfrak{g}_P$) with the projection onto $\Omega^2_+(M, \mathfrak{g}_P)$.

Note that this elliptic Lie algebra is a quotient of that describing the moduli of flat G-bundles on M.

10.3.3. Holomorphic bundles. In a similar way, if M is a complex manifold and if $P \to M$ is a holomorphic principal G-bundle, then the elliptic dg Lie algebra $\Omega^{0,*}(M, \mathfrak{g}_P)$, with differential $\overline{\partial}$, describes the formal moduli space of holomorphic G-bundles on M.

10.4. Cochains of a local L_∞ algebra

Let L be a local L_∞ algebra on M. If $U \subset M$ is an open subset, then $\mathcal{L}(U)$ denotes the L_∞ algebra of smooth sections of L on U. Let $\mathcal{L}_c(U) \subset \mathcal{L}(U)$ denote the sub-L_∞ algebra of compactly supported sections.

In the appendix (section B.8) we defined the algebra of functions on the space of sections on a vector bundle on a manifold. We are interested in the algebra

$$\mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq 0} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R}\right)_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps.

This space is naturally a graded differentiable vector space (that is, we can view it as a sheaf of graded vector spaces on the site of smooth manifolds). However, it is important that we treat this object as a differentiable pro-vector space. Basic facts about differentiable pro-vector spaces are developed in the Appendix B. The pro-structure comes from the filtration

$$F^i \mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq i} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R}\right)_{S_n},$$

which is the usual filtration on “power series.”

The L_∞ algebra structure on $\mathcal{L}(U)$ gives, as usual, a differential on $\mathscr{O}(\mathcal{L}(U)[1])$, making $\mathscr{O}(\mathcal{L}(U)[1])$ into a differentiable pro-cochain complex.

10.4.0.1 Definition. *Define the* Lie algebra cochain complex $C^*(\mathcal{L}(U))$ *to be*

$$C^*(\mathcal{L}(U)) = \mathscr{O}(\mathcal{L}(U)[1])$$

equipped with the usual Chevalley-Eilenberg differential. Similarly, define

$$C^*_{red}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

to be the reduced Chevalley-Eilenberg complex*, that is, the kernel of the natural augmentation map* $C^*(\mathcal{L}(U)) \to \mathbb{R}$*. These are both differentiable pro-cochain complexes.*

One defines $C^*(\mathcal{L}_c(U))$ in the same way, everywhere substituting $\mathcal{L}_c$ for $\mathcal{L}$.

We will think of $C^*(\mathcal{L}(U))$ as the algebra of functions on the formal moduli problem $B\mathcal{L}(U)$ associated to the L_∞ algebra $\mathcal{L}(U)$.

10.4.1. Cochains with coefficients in a module.

Let L be a local L_∞ algebra on M, and let $\mathcal{L}$ denote the smooth sections. Let E be a graded vector bundle on M and equip the global smooth sections $\mathscr{E}$ with a differential that is a differential operator.

10.4.1.1 Definition. *A* local action of $\mathcal{L}$ on $\mathscr{E}$ *is an action of* $\mathcal{L}$ *on* $\mathscr{E}$ *with the property that the structure maps*

$$\mathcal{L}^{\otimes n} \otimes \mathscr{E} \to \mathscr{E}$$

(defined for $n \geq 1$*) are all polydifferential operators.*

Note that $\mathcal{L}$ has an action on itself, called the adjoint action, where the differential on $\mathcal{L}$ is the one coming from the L_∞ structure, and the action map

$$\mu_n : \mathcal{L}^{\otimes n} \otimes \mathcal{L} \to \mathcal{L}$$

is the L_∞ structure map l_{n+1}.

Let $L^! = L^\vee \otimes_{C^\infty_M} \mathrm{Dens}_M$. Then, $\mathcal{L}^!$ has a natural local $\mathcal{L}$-action, which we should think of as the coadjoint action. This action is defined by saying that if $\alpha_1, \dots, \alpha_n \in \mathcal{L}$, the differential operator

$$\mu_n(\alpha_1, \dots, \alpha_n, -) : \mathcal{L}^! \to \mathcal{L}^!$$

is the formal adjoint to the corresponding differential operator arising from the action of $\mathcal{L}$ on itself.

has the structure of a local module over $\mathcal{L}$.

If E is a local module over L, then, for each $U \subset M$, we can define the Chevalley-Eilenberg cochains

$$C^*(\mathcal{L}(U), \mathscr{E}(U))$$

of $\mathcal{L}(U)$ with coefficients in $\mathscr{E}(U)$. As above, one needs to take account of the topologies on the vector spaces $\mathcal{L}(U)$ and $\mathscr{E}(U)$ when defining this Chevalley-Eilenberg cochain complex. Thus, as a graded vector space,

$$C^*(\mathcal{L}(U), \mathscr{E}(U)) = \prod_{n \geq 0} \operatorname{Hom}((\mathcal{L}(U)[1])^{\otimes n}, \mathscr{E}(U))_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps. Again, we treat this object as a differentiable pro-cochain complex.

As explained in the section on formal moduli problems (section 10.1), we should think of a local module E over L as providing, on each open subset $U \subset M$, a vector bundle on the formal moduli problem $B\mathcal{L}(U)$ associated to $\mathcal{L}(U)$. Then the Chevalley-Eilenberg cochain complex $C^*(\mathcal{L}(U), \mathscr{E}(U))$ should be thought of as the space of sections of this vector bundle.

10.5. D-modules and local L_∞ algebras

Our definition of a local L_∞ algebra is designed to encode the derived moduli space of solutions to a system of non-linear differential equations. An alternative language for describing differential equations is the theory of D-modules. In this section we will show how our local L_∞ algebras can also be viewed as L_∞ algebras in the symmetric monoidal category of D-modules.

The main motivation for this extra layer of formalism is that local action functionals — which play a central role in classical field theory — are elegantly described using the language of D-modules.

Let C^∞_M denote the sheaf of smooth functions on the manifold M, let Dens_M denote the sheaf of smooth densities, and let D_M the sheaf of differential operators with smooth coefficients. The ∞-jet bundle $Jet(E)$ of a vector bundle E is the vector bundle whose fiber at a point $x \in M$ is the space of jets (or formal germs) at x of sections of E. The sheaf of sections of $Jet(E)$, denoted $J(E)$, is equipped with a canonical D_M-module structure, i.e., the natural flat connection sometimes known as the Cartan distribution. This flat connection is characterized by the property that flat sections of $J(E)$ are those sections which arise by taking the jet at every point of a section of the vector bundle E. (For motivation, observe that a field ϕ (a section of E) gives a section of $Jet(E)$ that encodes all the *local* information about ϕ.)

The category of D_M modules has a symmetric monoidal structure, given by tensoring over C^∞_M. The following lemma allows us to translate our definition of local L_∞ algebra into the world of D-modules.

10.5.0.2 Lemma. *Let $E_1, \dots, E_n, F$ be vector bundles on M, and let $\mathscr{E}_i, \mathscr{F}$ denote their spaces of global sections. Then, there is a natural bijection*

$$\mathrm{PolyDiff}(\mathscr{E}_1 \times \cdots \times \mathscr{E}_n, \mathscr{F}) \cong \mathrm{Hom}_{D_M}(J(E_1) \otimes \cdots \otimes J(E_n), J(F))$$

where PolyDiff *refers to the space of polydifferential operators. On the right hand side, we need to consider maps which are continuous with respect to the natural adic topology on the bundle of jets.*

Further, this bijection is compatible with composition.

A more formal statement of this lemma is that the multi-category of vector bundles on M, with morphisms given by polydifferential operators, is a full subcategory of the symmetric monoidal category of D_M modules. The embedding is given by taking jets. The proof of this lemma (which is straightforward) is presented in [Cos11c], Chapter 5.

This lemma immediately tells us how to interpret a local L_∞ algebra in the language of D-modules.

10.5.0.3 Corollary. *Let L be a local L_∞ algebra on M. Then $J(L)$ has the structure of L_∞ algebra in the category of D_M modules.*

Indeed, the lemma implies that to give a local L_∞ algebra on M is the same as to give a graded vector bundle L on M together with an L_∞ structure on the D_M module $J(L)$.

We are interested in the Chevalley-Eilenberg cochains of $J(L)$, but taken now in the category of D_M modules. Because $J(L)$ is an inverse limit of the sheaves of finite-order jets, some care needs to be taken when defining this Chevalley-Eilenberg cochain complex.

In general, if E is a vector bundle, let $J(E)^\vee$ denote the sheaf $\mathrm{Hom}_{C^\infty_M}(J(E), C^\infty_M)$, where $\mathrm{Hom}_{C^\infty_M}$ denotes continuous linear maps of C^∞_M-modules. This sheaf is naturally a D_M-module. We can form the completed symmetric algebra

$$\begin{aligned}\mathscr{O}_{red}(J(E)) &= \prod_{n>0} \mathrm{Sym}^n_{C^\infty_M}(J(E)^\vee) \\ &= \prod_{n>0} \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)_{S_n}.\end{aligned}$$

Note that $\mathscr{O}_{red}(J(E)$ is a D_M-algebra, as it is defined by taking the completed symmetric algebra of $J(E)^\vee$ in the symmetric monoidal category of D_M-modules where the tensor product is taken over C^∞_M.

We can equivalently view $J(E)^\vee$ as an infinite-rank vector bundle with a flat connection. The symmetric power sheaf $\mathrm{Sym}^n_{C^\infty_M}(J(E)^\vee)$ is the sheaf of sections of the infinite-rank bundle whose fibre at x is the symmetric power of the fibre of $J(E)^\vee$ at x.

In the case that E is the trivial bundle $\underline{\mathbb{R}}$, the sheaf $J(\underline{\mathbb{R}})^\vee$ is naturally isomorphic to D_M as a left D_M-module. In this case, sections of the sheaf $\mathrm{Sym}^n_{C^\infty_M}(D_M)$ are objects which in local coordinates are finite sums of expressions like

$$f(x_i)\partial_{I_1}\dots\partial_{I_n}.$$

where ∂_{I_j} is the partial differentiation operator corresponding to a multi-index.

We should think of an element of $\mathscr{O}_{red}(J(E))$ as a Lagrangian on the space $\mathscr{E}$ of sections of E (a Lagrangian in the sense that an action functional is given by a Lagrangian density). Indeed, every element of $\mathscr{O}_{red}(J(E))$ has a Taylor expansion $F = \sum F_n$ where each F_n is a section

$$F_n \in \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)^{S_n}.$$

Each such F_n is a multilinear map which takes sections $\phi_1, \dots, \phi_n \in \mathscr{E}$ and yields a smooth function $F_n(\phi_1, \dots, \phi_n) \in C^\infty(M)$, with the property that $F_n(\phi_1, \dots, \phi_n)(x)$ only depends on the ∞-jet of ϕ_i at x.

In the same way, we can interpret an element $F \in \mathscr{O}_{red}(J(E))$ as something that takes a section $\phi \in \mathscr{E}$ and yields a smooth function

$$\sum F_n(\phi, \dots, \phi) \in C^\infty(M),$$

with the property that $F(\phi)(x)$ only depends on the jet of ϕ at x.

Of course, the functional F is a formal power series in the variable ϕ. One cannot evaluate most formal power series, since the putative infinite sum makes no sense. Instead, it only makes sense to evaluate a formal power series on infinitesimal elements. In particular, one can always evaluate a formal power series on nilpotent elements of a ring.

Indeed, a formal way to characterize a formal power series is to use the functor of points perspective on Artinian algebras: if R is an auxiliary graded Artinian algebra with maximal ideal m and if $\phi \in \mathscr{E} \otimes m$, then $F(\phi)$ is an element of $C^\infty(M) \otimes m$. This assignment is functorial with respect to maps of graded Artin algebras.

10.5.1. Local functionals. We have seen that we can interpret $\mathscr{O}_{red}(J(E))$ as the sheaf of Lagrangians on a graded vector bundle E on M. Thus, the sheaf

$$\mathrm{Dens}_M \otimes_{C^\infty_M} \mathscr{O}_{red}(J(E))$$

is the sheaf of Lagrangian densities on M. A section F of this sheaf is something which takes as input a section $\phi \in \mathscr{E}$ of $\mathscr{E}$ and produces a density $F(\phi)$ on M, in such a way that

$F(\phi)(x)$ only depends on the jet of ϕ at x. (As before, F is a formal power series in the variable ϕ.)

The sheaf of local action functionals is the sheaf of Lagrangian densities modulo total derivatives. Two Lagrangian densities that differ by a total derivative define the same local functional on (compactly supported) sections because the integral of total derivative vanishes. Thus, we do not want to distinguish them, as they lead to the same physics. The formal definition is as follows.

10.5.1.1 Definition. *Let E be a graded vector bundle on M, whose space of global sections is $\mathscr{E}$. Then the space of* local action functionals *on $\mathscr{E}$ is*

$$\mathscr{O}_{loc}(\mathscr{E}) = \mathrm{Dens}_M \otimes_{D_M} \mathscr{O}_{red}(J(E)).$$

Here, Dens_M is the right D_M-module of densities on M.

Let $\mathscr{O}_{red}(\mathscr{E}_c)$ denote the algebra of functionals modulo constants on the space $\mathscr{E}_c$ of compactly supported sections of E. Integration induces a natural inclusion

$$\iota : \mathscr{O}_{loc}(\mathscr{E}) \to \mathscr{O}_{red}(\mathscr{E}_c),$$

where the Lagrangian density $S \in \mathscr{O}_{loc}(\mathscr{E})$ becomes the functional $\iota(S) : \phi \mapsto \int_M S(\phi)$. (Again, ϕ must be nilpotent and compactly supported.) From here on, we will use this inclusion without explicitly mentioning it.

10.5.2. Local Chevalley-Eilenberg complex of a local L_∞ algebra.

Let L be a local L_∞ algebra. Then we can form, as above, the reduced Chevalley-Eilenberg cochain complex $C^*_{red}(J(L))$ of L. This is the D_M-algebra $\mathscr{O}_{red}(J(L)[1])$ equipped with a differential encoding the L_∞ structure on L.

10.5.2.1 Definition. *If $\mathcal{L}$ is a local L_∞-algebra, define the local Chevalley-Eilenberg complex to be*

$$C^*_{red,loc}(\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

This is the space of local action functionals on $\mathcal{L}[1]$, equipped with the Chevalley-Eilenberg differential. In general, if $\mathfrak{g}$ is an L_∞ algebra, we think of the Lie algebra cochain complex $C^*(\mathfrak{g})$ as being the algebra of functions on $B\mathfrak{g}$. In this spirit, we sometimes use the notation $\mathscr{O}_{loc}(B\mathcal{L})$ for the complex $C^*_{red,loc}(\mathcal{L})$.

Note that $C^*_{red,loc}(\mathcal{L})$ is *not* a commutative algebra. Although the D_M-module $C^*_{red}(J(L))$ is a commutative D_M-module, the functor $\mathrm{Dens}_M \otimes_{D_M} -$ is not a symmetric monoidal functor from D_M-modules to cochain complexes, so it does not take commutative algebras to commutative algebras.

Note that there's a natural inclusion of cochain complexes

$$C^*_{red,loc}(\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(M)),$$

where $\mathcal{L}_c(M)$ denotes the L_∞ algebra of compactly supported sections of L. The complex on the right hand side was defined earlier (see definition 10.4.0.1) and includes *non*local functionals.

10.5.3. Central extensions and local cochains.

In this section we will explain how local cochains are in bijection with certain central extensions of a local L_∞ algebra. To avoid some minor analytical difficulties, we will only consider central extensions that are split as precosheaves of graded vector spaces.

10.5.3.1 Definition. *Let $\mathcal{L}$ be a local L_∞ algebra on M. A k-shifted* local central extension *of $\mathcal{L}$ is an L_∞ structure on the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$, where $\underline{\mathbb{C}}$ is the constant precosheaf which takes value $\mathbb{C}$ on any open subset. We use the notation $\widetilde{\mathcal{L}}_c$ for the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$. We require that this L_∞ structure has the following properties.*

1. *The sequence*
$$0 \to \underline{\mathbb{C}}[k] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$
is an exact sequence of precosheaves of L_∞ algebras, where $\underline{\mathbb{C}}[k]$ is given the abelian structure and $\mathcal{L}_c$ is given its original structure.
2. *This implies that the L_∞ structure on $\widetilde{\mathcal{L}}_c$ is determined from that on $\mathcal{L}_c$ by L_∞ structure maps*
$$\widetilde{l}_n : \mathcal{L}_c \to \underline{\mathbb{C}}[k]$$
for $n \geq 1$. We require that these structure maps are given by local *action functionals.*

Two such central extensions, say $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$, are isomorphic *if there is an L_∞-isomorphism*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathcal{L}}'_c$$

that is the identity on $\underline{\mathbb{C}[k]}$ and on the quotient $\mathcal{L}_c$. This L_∞ isomorphism must satisfy an additional property: the terms in this L_∞ -isomorphism, which are given (using the decomposition of $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$ as $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$) by functionals

$$\mathcal{L}_c^{\otimes n} \to \underline{\mathbb{C}}[k],$$

must be local.

This definition refines the definition of central extension given in section 3.6 to include an extra locality property.

Example: Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a Lie algebra with an invariant pairing. Let $\mathcal{L} = \Omega^{0,*}_\Sigma \otimes \mathfrak{g}$. Consider the Kac-Moody central extension, as defined in section 3.6 of 3 We let

$$\widetilde{\mathcal{L}}_c = \underline{\mathbb{C}} \cdot c \oplus \mathcal{L}_c,$$

where the central parameter c is of degree 1 and the Lie bracket is defined by

$$[\alpha, \beta]_{\widetilde{\mathcal{L}}_c} = [\alpha, \beta]_{\mathcal{L}_c} + c \int \alpha \partial \beta.$$

This is a local central extension. As shown in section 5.4 of chapter 5, the factorization envelope of this extension recovers the vertex algebra of an associated affine Kac-Moody algebra. ♢

10.5.3.2 Lemma. *Let $\mathcal{L}$ be a local L_∞ algebra on a manifold M. There is a bijection between isomorphism classes of k-shifted local central extensions of $\mathcal{L}$ and classes in $H^{k+2}(\mathscr{O}_{loc}(B\mathcal{L}))$.*

PROOF. This result is almost immediate. Indeed, any closed degree $k+2$ element of $\mathscr{O}_{loc}(B\mathcal{L})$ give a local L_∞ structure on $\underline{\mathbb{C}}[k] \oplus \mathcal{L}_c$, where the L_∞ structure maps

$$\widetilde{l}_n : \mathcal{L}_c(U) \to \mathbb{C}[k]$$

arise from the natural cochain map $\mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(U))$. The fact that we start with a closed element of $\mathscr{O}_{loc}(B\mathcal{L})$ corresponds to the fact that the L_∞ axioms hold. Isomorphisms of local central extensions correspond to adding an exact cocycle to a closed degree $k+2$ element in $\mathscr{O}_{loc}(B\mathcal{L})$. □

Particularly important is the case when we have a -1-shifted central extension. As explained in subsection 3.6.3 in Chapter 3, in this situation we can form the twisted factorization envelope, which is a factorization algebra over $\mathbb{C}[t]$ (where t is of degree 0) defined by sending an open subset U to the Chevalley-Eilenberg chain complex

$$U \mapsto C_*(\widetilde{\mathcal{L}}_c(U)).$$

We think of $\mathbb{C}[t]$ as the Chevalley-Eilenberg chains of the Abelian Lie algebra $\mathbb{C}[-1]$. In this situation, we can set t to be a particular value, leading to a *twisted* factorization envelope of $\mathcal{L}$. Twisted factorization envelopes will play a central role in our formulation of Noether's theorem at the quantum level in chapter 18.

10.5.4. Calculations of local L_∞ algebra cohomology play an important role in quantum field theory. Indeed, the obstruction-deformation complex describing quantizations of a classical field theory are local L_∞ algebra cohomology groups. Thus, it will be helpful to be able to compute some examples.

Before we start, let us describe a general result which will facilitate computation.

10.5.4.1 Lemma. *Let M be an oriented manifold and let $\mathcal{L}$ be a local L_∞-algebra on M. Then, there is a natural quasi-isomorphism*

$$\Omega^*(M, C^*_{red}(J(L)))[\mathrm{dimM}] \cong C^*_{red,loc}(\mathcal{L}).$$

PROOF. By definition,

$$\mathscr{O}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red} J(\mathcal{L})$$

where D_M is the sheaf of C^∞ differential operators. The D_M-module $C^*_{red}(J(\mathcal{L}))$ is flat (this was checked in [Cos11c]), so we can replace the tensor product over D_M with the left-derived tensor product.

Since M is oriented, we can replace Dens_M by Ω^d_M where $d = \dim M$. The right D_M-module Ω^d_M has a free resolution of the form

$$\cdots \to \Omega^{d-1}_M \otimes_{C^\infty_M} D_M \to \Omega^d M \otimes_{C^\infty_M} D_M$$

where $\Omega^i_M \otimes_{C^\infty_M} D_M$ is in cohomological degree $-i$, and the differential in this complex is the de Rham differential coupled to the left D_M-module structure on D_M. (This is sometimes called the Spenser resolution).

It follows that we the derived tensor product can be represented as

$$\Omega^d_M \otimes^{mbbL}_{D_M} C^*_{red}(J(\mathcal{L})) = \Omega^*(M, C^*_{red}(J(L)))[d]$$

as desired.

□

10.5.4.2 Lemma. *Let Σ be a Riemann surface. Let $\mathcal{L}$ be the local L_∞ algebra on Σ defined by $\mathcal{L}(U) = \Omega^{0,*}(U, TU)$. In other words, $\mathcal{L}$ is the Dolbeault resolution of the sheaf of holomorphic vector fields on Σ.*

Then,

$$H^i(\mathscr{O}(B\mathcal{L})) = H^*(\Sigma)[-1].$$

Remark: The class in $H^1(\mathscr{O}(B\mathcal{L}))$ corresponding to the class $1 \in H^0(\Sigma)$ leads to a local central extension of $\mathcal{L}$. One can check that the corresponding twisted factorization envelope corresponds to the Virasoro vertex algebra, in the same way that we showed in section 5.4 that the Kac-Moody extension above leads to the Kac-Moody vertex algebra. ◊

PROOF. The previous lemma tells us that we need to compute the de Rham cohomology with coefficients in the D_Σ-module $C^*_{red}(J(L))[2]$. Suppose we want to compute the de Rham cohomology with coefficients in any complex M of D_Σ-modules. There is a spectral sequence converging to this cohomology, associated to the filtration on $\Omega^*(\Sigma, M)$ by form degree. The E_2 page of this spectral sequence is the de Rham complex $\Omega^*(\Sigma, \mathcal{H}^*(M))$ with coefficients in the cohomology D_Σ-module $\mathcal{H}^*(M)$.

We will use this spectral sequence in our example. The first step is to compute the cohomology of the D_Σ-module $C^*_{red}(J(\mathcal{L}))$. We will compute the cohomology of the fibres of this sheaf at an arbitrary point $x \in \Sigma$. Let us choose a holomorphic coordinate z at

x. The fibre $J_x(\mathcal{L})$ at x is the dg Lie algebra $\mathbb{C}[[z,\overline{z},d\overline{z}]]\partial_z$ with differential $\overline{\partial}$. This dg Lie algebra is quasi-isomorphic to the Lie algebra of formal vector fields $\mathbb{C}[[z]]\partial_z$.

A calculation performed by Gelfand-Fuchs [] shows that the reduced Lie algebra cohomology of $\mathbb{C}[[z]]\partial_z$ is concentrated in degree 3, where it is one-dimensional. A cochain representative for the unique non-zero cohomology class is $\partial_z^\vee(z\partial_z)^\vee(z^2\partial_z)^\vee$ where $(z^k\partial_z)^\vee$ refers to the element in $(\mathbb{C}[[z]]\partial_z)^\vee$ in the dual basis.

Thus, we find that the cohomology of $C^*_{red}(J(L))$ is a rank one local system situated in cohomological degree 3. Choosing a formal coordinate at a point in a Riemann surface trivializes the fibre of this line bundle. The trivialization is independent of the coordinate choice, and compatible with the flat connection. From this we deduce that

$$\mathcal{H}^*(C^*_{red}(J(\mathcal{L}))) = C^\infty_\Sigma[-3]$$

is the trivial rank one local system, situated in cohomological degree 3.

Therefore, the cohomology of $\mathscr{O}_{loc}(B\mathcal{L})$ is a shift by -1 of the de Rham cohomology of this trivial flat line bundle, completing the result. □

10.5.5. Cochains with coefficients in a local module for a local L_∞ algebras. Let L be a local L_∞ algebra on M, and let E be a local module for L. Then $J(E)$ has an action of the L_∞ algebra $J(L)$, in a way compatible with the D_M-module on both $J(E)$ and $J(L)$.

10.5.5.1 Definition. *Suppose that E has a local action of L. Then the* local cochains $C^*_{loc}(\mathcal{L},\mathscr{E})$ *of $\mathcal{L}$ with coefficients in $\mathscr{E}$ is defined to be the flat sections of the D_M-module of cochains of $J(L)$ with coefficients in $J(E)$.*

More explicitly, the D_M-module $C^*(J(L),J(E)$ is

$$\prod_{n\geq 0} \mathrm{Hom}_{C^\infty_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

equipped with the usual Chevalley-Eilenberg differential. The sheaf of flat sections of this D_M module is the subsheaf

$$\prod_{n\geq 0} \mathrm{Hom}_{D_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

where the maps must be D_M-linear. In light of the fact that

$$\mathrm{Hom}_{D_M}\left(J(L)^{\otimes n}, J(E)\right) = \mathrm{PolyDiff}(\mathcal{L}^{\otimes n},\mathscr{E}),$$

we see that $C^*_{loc}(\mathcal{L},\mathscr{E})$ is precisely the subcomplex of the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L},\mathscr{E}) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}((\mathcal{L}[1])^{\otimes n},\mathscr{E})_{S_n}$$

consisting of those cochains built up from polydifferential operators.

CHAPTER 11

The classical Batalin-Vilkovisky formalism

In the preceding chapter we explained how to encode the formal neighborhood of a solution to the Euler-Lagrange equations — a formal elliptic moduli problem — by an elliptic L_∞ algebra. As we explain in this chapter, the elliptic moduli problems arising from action functionals possess even more structure: a shifted symplectic form, so that the formal moduli problem is a derived symplectic space.

Our starting point is the finite-dimensional model that motivates the Batalin-Vilkovisky formalism for classical field theory. With this model in mind, we then develop the relevant definitions in the language of elliptic L_∞ algebras. The end of the chapter is devoted to several examples of classical BV theories, notably *cotangent* field theories, which are the analogs of cotangent bundles in ordinary symplectic geometry.

11.1. The classical BV formalism in finite dimensions

Before we discuss the Batalin-Vilkovisky formalism for classical field theory, we will discuss a finite-dimensional toy model (which we can think of as a 0-dimensional classical field theory). Our model for the space of fields is a finite-dimensional smooth manifold manifold M. The "action functional" is given by a smooth function $S \in C^\infty(M)$. Classical field theory is concerned with solutions to the equations of motion. In our setting, the equations of motion are given by the subspace $\mathrm{Crit}(S) \subset M$. Our toy model will not change if M is a smooth algebraic variety or a complex manifold, or indeed a smooth formal scheme. Thus we will write $\mathscr{O}(M)$ to indicate whatever class of functions (smooth, polynomial, holomorphic, power series) we are considering on M.

If S is not a nice function, then this critical set can by highly singular. The classical Batalin-Vilkovisky formalism tells us to take, instead the *derived* critical locus of S. (Of course, this is exactly what a derived algebraic geometer — see [**Lur09b**], [**Toë06**] — would tell us to do as well.) We will explain the essential idea without formulating it precisely inside any particular formalism for derived geometry. For such a treatment, see [**Vez11**].

The critical locus of S is the intersection of the graph

$$\Gamma(\mathrm{d}S) \subset T^*M$$

with the zero-section of the cotangent bundle of M. Algebraically, this means that we can write the algebra $\mathscr{O}(\mathrm{Crit}(S))$ of functions on $\mathrm{Crit}(S)$ as a tensor product

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

Derived algebraic geometry tells us that the derived critical locus is obtained by replacing this tensor product with a derived tensor product. Thus, the derived critical locus of S, which we denote $\mathrm{Crit}^h(S)$, is an object whose ring of functions is the commutative dg algebra

$$\mathscr{O}(\mathrm{Crit}^h(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

In derived algebraic geometry, as in ordinary algebraic geometry, spaces are determined by their algebras of functions. In derived geometry, however, one allows differential-graded algebras as algebras of functions (normally one restricts attention to differential-graded algebras concentrated in non-positive cohomological degrees).

We will take this derived tensor product as a definition of $\mathscr{O}(\mathrm{Crit}^h(S))$.

11.1.1. An explicit model. It is convenient to consider an explicit model for the derived tensor product. By taking a standard Koszul resolution of $\mathscr{O}(M)$ as a module over $\mathscr{O}(T^*M)$, one sees that $\mathscr{O}(\mathrm{Crit}^h(S))$ can be realized as the complex

$$\mathscr{O}(\mathrm{Crit}^h(S)) \simeq \ldots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} \mathscr{O}(M).$$

In other words, we can identify $\mathscr{O}(\mathrm{Crit}^h(S))$ with functions on the "graded manifold" $T^*[-1]M$, equipped with the differential given by contracting with the 1-form $\mathrm{d}S$. This notation $T^*[-1]M$ denotes the ordinary smooth manifold M equipped with the graded-commutative algebra $\mathrm{Sym}_{C^\infty_M}(\Gamma(M, TM)[1])$ as its ring of functions.

Note that

$$\mathscr{O}(T^*[-1]M) = \Gamma(M, \wedge^* TM)$$

has a Poisson bracket of cohomological degree 1, called the Schouten-Nijenhuis bracket. This Poisson bracket is characterized by the fact that if $f, g \in \mathscr{O}(M)$ and $X, Y \in \Gamma(M, TM)$, then

$$\begin{aligned} \{X, Y\} &= [X, Y] \\ \{X, f\} &= Xf \\ \{f, g\} &= 0 \end{aligned}$$

and the Poisson bracket between other elements of $\mathscr{O}(T^*[-1]M)$ is inferred from the Leibniz rule.

The differential on $\mathscr{O}(T^*[-1]M)$ corresponding to that on $\mathscr{O}(\mathrm{Crit}^h(S))$ is given by

$$\mathrm{d}\phi = \{S, \phi\}$$

for $\phi \in \mathscr{O}(T^*[-1]M)$.

The derived critical locus of any function thus has a symplectic form of cohomological degree -1. It is manifest in this model and hence can be found in others. In the Batalin-Vilkovisky formalism, the space of fields always has such a symplectic structure. However, one does not require that the space of fields arises as the derived critical locus of a function.

11.2. The classical BV formalism in infinite dimensions

We would like to consider classical field theories in the BV formalism. We have already explained how the language of elliptic moduli problems captures the formal geometry of solutions to a system of PDE. Now we need to discuss the shifted symplectic structures possessed by a derived critical locus. For us, a classical field theory will be specified by an elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

We defined the notion of formal elliptic moduli problem on a manifold M using the language of L_∞ algebras. Thus, in order to give the definition of a classical field theory, we need to understand the following question: what extra structure on an L_∞ algebra $\mathfrak{g}$ endows the corresponding formal moduli problem with a symplectic form?

In order to answer this question, we first need to understand a little about what it means to put a shifted symplectic form on a (formal) derived stack.

In the seminal work of Schwarz [**Sch93**, **AKSZ97**], a definition of a shifted symplectic form on a dg manifold is given. Dg manifolds where an early attempt to develop a theory of derived geometry. It turns out that dg manifolds are sufficient to capture some aspects of the modern theory of derived geometry, including formal derived geometry.

In the world of dg manifolds, as in any model of derived geometry, all spaces of tensors are cochain complexes. In particular, the space of i-forms $\Omega^i(\mathcal{M})$ on a dg manifold is a cochain complex. The differential on this cochain complex is called the internal differential on i-forms. In addition to the internal differential, there is also a de Rham differential $d_{dR} : \Omega^i(\mathcal{M}) \to \Omega^{i+1}(\mathcal{M})$ which is a cochain map. Schwarz defined a symplectic form on a dg manifold $\mathcal{M}$ to be a two-form ω which is both closed in the differential on the complex of two-forms, and which is also closed under the de Rham differential mapping two-forms to three-forms. A symplectic form is also required to be non-degenerate. The symplectic two-form ω will have some cohomological degree, which for the case relevant to the BV formalsim is -1.

Following these ideas, Pantev et al. [**PTVV11**] give a definition of (shifted) symplectic structure in the more modern language of derived stacks. In this approach, instead of asking that the two-form defining the symplectic structure be closed both in the internal differential on two-forms and closed under the de Rham differential, one constructs a double complex

$$\Omega^{\geq 2} = \Omega^2 \to \Omega^3[-1] \to \dots$$

as the subcomplex of the de Rham complex consisting of 2 and higher forms. One then looks for an element of this double complex which is closed under the total differential (the sum of the de Rham differential and the internal differential on each space of k-forms) and whose 2-form component is non-degenerate in a suitable sense.

However, it turns out that, in the case of formal derived stacks, the definition given by Schwarz and that given by Pantev et al. coincides. One can also show that in this situation there is a Darboux lemma, showing that we can take the symplectic form to have constant coefficients. In order to explain what we mean by this, let us explain how to understand forms on a formal derived stack in terms of the associated L_∞-algebra.

Given a pointed formal moduli problem $\mathcal{M}$, the associated L_∞ algebra $\mathfrak{g}_{\mathcal{M}}$ has the property that

$$\mathfrak{g}_{\mathcal{M}} = T_p\mathcal{M}[-1].$$

Further, we can identify geometric objects on $\mathcal{M}$ in terms of $\mathfrak{g}_{\mathcal{M}}$ as follows.

$C^*(\mathfrak{g}_{\mathcal{M}})$	the algebra $\mathscr{O}(\mathcal{M})$ of functions on $\mathcal{M}$
$\mathfrak{g}_{\mathcal{M}}$-modules	$\mathscr{O}_{\mathcal{M}}$-modules
$C^*(\mathfrak{g}_{\mathcal{M}}, V)$	the $\mathscr{O}_{\mathcal{M}}$-module corresponding to the $\mathfrak{g}_{\mathcal{M}}$-module V
the $\mathfrak{g}_{\mathcal{M}}$-module $g_{\mathcal{M}}[1]$	$T\mathcal{M}$

Following this logic, we see that the complex of 2-forms on $\mathcal{M}$ is identified with $C^*(\mathfrak{g}_{\mathcal{M}}, \wedge^2(\mathfrak{g}_{\mathcal{M}}^\vee[-1]))$.

As we have seen, according to Schwarz, a symplectic form on $\mathcal{M}$ is a two-form on $\mathcal{M}$ which is closed for both the internal and de Rham differentials. Any constant-coefficient two-form is automatically closed under the de Rham differential. A constant-coefficient two-form of degree k is an element of $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}})^\vee$ of cohomological degree $k-2$, i.e. a symmetric pairing on $\mathfrak{g}_{\mathcal{M}}$ of this degree. Such a two-form is closed for the internal differential if and only if it is invariant.

To give a formal pointed moduli problem with a symplectic form of cohomological degree k is the same as to give an L_∞ algebra with an invariant and non-degenerate pairing of cohomological degree $k-2$.

Thus, we find that constant coefficient symplectic two-forms of degree k on $\mathcal{M}$ are precisely the same as non-degenerate symmetric invariant pairings on $\mathfrak{g}_{\mathcal{M}}$. The relation between derived symplectic geometry and invariant pairings on Lie algebras was first developed by Kontsevich [Kon93].

The following formal Darboux lemma makes this relationship into an equivalence.

11.2.0.1 Lemma. *Let $\mathfrak{g}$ be a finite-dimensional L_∞ algebra. Then, k-shifted symplectic structures on the formal derived stack $B\mathfrak{g}$ (in the sense of Pantev et al.) are the same as symmetric invariant non-degenerate pairings on $\mathfrak{g}$ of cohomological degree $k-2$.*

The proof is a little technical, and appears in an appendix **??**. The proof of a closely related statement in a non-commutative setting was given by Kontsevich and Soibelman [**KS06**]. In the statement of the lemma, "the same" means that simplicial sets parametrizing the two objects are canonically equivalent.

Following this idea, we will define a classical field theory to be an elliptic L_∞ algebra equipped with a non-degenerate invariant pairing of cohomological degree -3. Let us first define what it means to have an invariant pairing on an elliptic L_∞ algebra.

11.2.0.2 Definition. *Let M be a manifold, and let E be an elliptic L_∞ algebra on M. An* invariant pairing on E of cohomological degree k *is a symmetric vector bundle map*

$$\langle -,-\rangle_E : E \otimes E \to \mathrm{Dens}(M)[k]$$

satisfying some additional conditions:

1. Non-degeneracy*: we require that this pairing induces a vector bundle isomorphism*
$$E \to E^\vee \otimes \mathrm{Dens}(M)[-3].$$
2. Invariance*: let $\mathscr{E}_c$ denotes the space of compactly supported sections of E. The pairing on E induces an inner product on $\mathscr{E}_c$, defined by*
$$\begin{aligned} \langle -,-\rangle : \mathscr{E}_c \otimes \mathscr{E}_c &\to \mathbb{R} \\ \alpha \otimes \beta &\to \int_M \langle \alpha, \beta \rangle . \end{aligned}$$
We require it to be an invariant pairing on the L_∞ algebra $\mathscr{E}_c$.

Recall that a symmetric pairing on an L_∞ algebra $\mathfrak{g}$ is called invariant if, for all n, the linear map

$$\begin{aligned} \mathfrak{g}^{\otimes n+1} &\to \mathbb{R} \\ \alpha_1 \otimes \cdots \otimes \alpha_{n+1} &\mapsto \langle l_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1}\rangle \end{aligned}$$

is graded anti-symmetric in the α_i.

11.2.0.3 Definition. *A* formal pointed elliptic moduli problem with a symplectic form of cohomological degree k *on a manifold M is an elliptic L_∞ algebra on M with an invariant pairing of cohomological degree $k-2$.*

11.2.0.4 Definition. *In the BV formalism, a* (perturbative) classical field theory *on M is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.*

11.3. The derived critical locus of an action functional

The critical locus of a function f is, of course, the zero locus of the 1-form $\mathrm{d}f$. We are interested in constructing the derived critical locus of a local functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$ on the formal moduli problem associated to a local L_∞ algebra $\mathcal{L}$ on a manifold M. Thus, we need to understand what kind of object the exterior derivative $\mathrm{d}S$ of such an action functional S is.

If $\mathfrak{g}$ is an L_∞ algebra, then we should think of $C^*_{red}(\mathfrak{g})$ as the algebra of functions on the formal moduli problem $B\mathfrak{g}$ that vanish at the base point. Similarly, $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$ should be the thought of as the space of 1-forms on $B\mathfrak{g}$. The exterior derivative is thus a map

$$\mathrm{d} : C^*_{red}(\mathfrak{g}) \to C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]),$$

namely the universal derivation.

We will define a similar exterior derivative for a local Lie algebra $\mathcal{L}$ on M. The analog of $\mathfrak{g}^\vee$ is the $\mathcal{L}$-module $\mathcal{L}^!$, whose sections are (up to completion) the Verdier dual of the sheaf $\mathcal{L}$. Thus, our exterior derivative will be a map

$$\mathrm{d} : \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Recall that $\mathscr{O}_{loc}(B\mathcal{L})$ denotes the subcomplex of $C^*_{red}(\mathcal{L}_c(M))$ consisting of local functionals. The exterior derivative for the L_∞ algebra $\mathcal{L}_c(M)$ is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \mathcal{L}_c(M)^\vee[-1]).$$

Note that the dual $\mathcal{L}_c(M)^\vee$ of $\mathcal{L}_c(M)$ is the space $\overline{\mathcal{L}}^!(M)$ of distributional sections of the bundle $L^!$ on M. Thus, the exterior derivative is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)[-1]).$$

Note that

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \subset C^*(\mathcal{L}_c(M), \mathcal{L}^!(M)) \subset C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)).$$

We will now show that d preserves locality and more.

11.3.0.5 Lemma. *The exterior derivative takes the subcomplex $\mathscr{O}_{loc}(B\mathcal{L})$ of $C^*_{red}(\mathcal{L}_c(M))$ to the subcomplex $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ of $C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M))$.*

PROOF. The content of this lemma is the familiar statement that the Euler-Lagrange equations associated to a local action functional are differential equations. We will give a formal proof, but the reader will see that we only use integration by parts.

Any functional

$$F \in \mathscr{O}_{loc}(B\mathcal{L})$$

can be written as a sum $F = \sum F_n$ where

$$F_n \in \mathrm{Dens}_M \otimes_{D_M} \mathrm{Hom}_{C^\infty_M} \left(J(L)^{\otimes n}, C^\infty_M \right)_{S_n}.$$

Any such F_n can be written as a finite sum

$$F_n = \sum_i \omega D_1^i \dots D_n^i$$

where ω is a section of Dens_M and D_j^i are differential operators from $\mathcal{L}$ to C^∞_M. (The notation $\omega D_1^i \dots D_n^i$ means simply to multiply the density ω by the outputs of the differential operators, which are smooth functions.)

If we view $F \in \mathscr{O}(\mathcal{L}_c(M))$, then the nth Taylor component of F is the linear map

$$\mathcal{L}_c(M)^{\otimes n} \to \mathbb{R}$$

defined by

$$\phi_1 \otimes \dots \otimes \phi_n \to \sum_i \int_M \omega (D_1^i \phi_1) \dots (D_n^i \phi_n).$$

Thus, the $(n-1)$th Taylor component of $\mathrm{d}F$ is given by the linear map

$$\begin{aligned} \mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} &\to \overline{L}^!(M) = \mathcal{L}_c(M)^\vee \\ \phi_1 \otimes \dots \otimes \phi_{n-1} \sum_i &\mapsto \omega (D_1^i \phi_1) \dots (D_{n-1}^i \phi_{n-1}) D_n^i(-) + \text{symmetric terms} \end{aligned}$$

where the right hand side is viewed as a linear map from $\mathcal{L}_c(M)$ to $\mathbb{R}$. Now, by integration by parts, we see that

$$(\mathrm{d}F_n)(\phi_1, \dots, \phi_{n-1})$$

is in the subspace $\mathcal{L}^!(M) \subset \overline{L}^!(M)$ of smooth sections of the bundle $L^!(M)$, inside the space of distributional sections.

It is clear from the explicit expressions that the map

$$\mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} \to \mathcal{L}^!(M)$$

is a polydifferential operator, and so defines an element of $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ as desired. $\square$

11.3.1. Field theories from action functionals. Physicists normally think of a classical field theory as being associated to an action functional. In this section we will show how to construct a classical field theory in our sense from an action functional.

We will work in a very general setting. Recall (section 10.1.3) that we defined a local L_∞ algebra on a manifold M to be a sheaf of L_∞ algebras where the structure maps are given by differential operators. We will think of a local L_∞ algebra $\mathcal{L}$ on M as defining a formal moduli problem cut out by some differential equations. We will use the notation $B\mathcal{L}$ to denote this formal moduli problem.

We want to take the derived critical locus of a local action functional

$$S \in \mathscr{O}_{loc}(B\mathcal{L})$$

of cohomological degree 0. (We also need to assume that S is at least quadratic: this condition insures that the base-point of our formal moduli problem $B\mathcal{L}$ is a critical point of S). We have seen (section 11.3) how to apply the exterior derivative to a local action functional S yields an element

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]),$$

which we think of as being a local 1-form on $B\mathcal{L}$.

The critical locus of S is the zero locus of $\mathrm{d}S$. We thus need to explain how to construct a new local L_∞ algebra that we interpret as being the derived zero locus of $\mathrm{d}S$.

11.3.2. Finite dimensional model. We will first describe the analogous construction in finite dimensions. Let $\mathfrak{g}$ be an L_∞ algebra, M be a $\mathfrak{g}$-module of finite total dimension, and α be a closed, degree zero element of $C^*_{red}(\mathfrak{g}, M)$. The subscript *red* indicates that we are taking the reduced cochain complex, so that α is in the kernel of the augmentation map $C^*(\mathfrak{g}, M) \to M$.

We think of M as a dg vector bundle on the formal moduli problem $B\mathfrak{g}$, and so α is a section of this vector bundle. The condition that α is in the reduced cochain complex translates into the statement that α vanishes at the basepoint of $B\mathfrak{g}$. We are interested in constructing the L_∞ algebra representing the zero locus of α.

We start by writing down the usual Koszul complex associated to a section of a vector bundle. In our context, the commutative dg algebra representing this zero locus of α is given by the total complex of the double complex

$$\cdots \to C^*(\mathfrak{g}, \wedge^2 M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}, M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}).$$

In words, we have written down the symmetric algebra on the dual of $\mathfrak{g}[1] \oplus M[-1]$. It follows that this commutative dg algebra is the Chevalley-Eilenberg cochain complex of $\mathfrak{g} \oplus M[-2]$, equipped with an L_∞ structure arising from the differential on this complex.

Note that the direct sum $\mathfrak{g} \oplus M[-2]$ (without a differential depending on α) has a natural semi-direct product L_∞ structure, arising from the L_∞ structure on $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $M[-2]$. This L_∞ structure corresponds to the case $\alpha = 0$.

11.3.2.1 Lemma. *The L_∞ structure on $\mathfrak{g} \oplus M[-2]$ describing the zero locus of α is a deformation of the semidirect product L_∞ structure, obtained by adding to the structure maps l_n the maps*

$$\begin{aligned} D_n\alpha : \mathfrak{g}^{\otimes n} &\to M \\ X_1 \otimes \cdots \otimes X_n &\mapsto \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}\alpha. \end{aligned}$$

This is a curved L_∞ *algebra unless the section* α *vanishes at* $0 \in \mathfrak{g}$.

PROOF. The proof is a straightforward computation. □

Note that the maps $D_n\alpha$ in the statement of the lemma are simply the homogeneous components of the cochain α.

We will let $Z(\alpha)$ denote $\mathfrak{g} \oplus M[-2]$, equipped with this L_∞ structure arising from α.

Recall that the formal moduli problem $B\mathfrak{g}$ is the functor from dg Artin rings (R, m) to simplicial sets, sending (R, m) to the simplicial set of Maurer-Cartan elements of $\mathfrak{g} \otimes m$. In order to check that we have constructed the correct derived zero locus for α, we should describe the formal moduli problem associated $Z(\alpha)$.

Thus, let (R, m) be a dg Artin ring, and $x \in \mathfrak{g} \otimes m$ be an element of degree 1, and $y \in M \otimes m$ be an element of degree -1. Then (x, y) satisfies the Maurer-Cartan equation in $Z(\alpha)$ if and only if

(1) x satisfies the Maurer-Cartan equation in $\mathfrak{g} \otimes m$ and
(2) $\alpha(x) = \mathrm{d}_x y \in M$, where

$$\mathrm{d}_x = \mathrm{d}y + \mu_1(x, y) + \tfrac{1}{2!}\mu_2(x, x, y) + \cdots : M \to M$$

is the differential obtained by deforming the original differential by that arising from the Maurer-Cartan element x. (Here $\mu_n : \mathfrak{g}^{\otimes n} \otimes M \to M$ are the action maps.)

In other words, we see that an R-point of $BZ(\alpha)$ is both an R-point x of $B\mathfrak{g}$ and a homotopy between $\alpha(x)$ and 0 in the fiber M_x of the bundle M at $x \in B\mathfrak{g}$. The fibre M_x is the cochain complex M with differential d_x arising from the solution x to the Maurer-Cartan equation. Thus, we are described the homotopy fiber product between the section α and the zero section in the bundle M, as desired.

Let us make thigs

11.3.3. The derived critical locus of a local functional. Let us now return to the situation where $\mathcal{L}$ is a local L_∞ algebra on a manifold M and $S \in \mathscr{O}(B\mathcal{L})$ is a local functional that is at least quadratic. Let

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$

denote the exterior derivative of S. Note that $\mathrm{d}S$ is in the reduced cochain complex, i.e. the kernel of the augmentation map $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \to \mathcal{L}^![-1]$.

Let

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

be the nth Taylor component of $\mathrm{d}S$. The fact that $\mathrm{d}S$ is a local cochain means that $\mathrm{d}_n S$ is a polydifferential operator.

11.3.3.1 Definition. *The* derived critical locus of S *is the local* L_∞ *algebra obtained by adding the maps*

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

to the structure maps l_n *of the semi-direct product* L_∞ *algebra* $\mathcal{L} \oplus \mathcal{L}^![-3]$*. We denote this local* L_∞ *algebra by* $\mathrm{Crit}(S)$*.*

If (R, m) is an auxiliary Artinian dg ring, then a solution to the Maurer-Cartan equation in $\mathrm{Crit}(S) \otimes m$ consists of the following data:

1. a Maurer-Cartan element $x \in \mathcal{L} \otimes m$ and
2. an element $y \in \mathcal{L}^! \otimes m$ such that

$$(\mathrm{d}S)(x) = \mathrm{d}_x y.$$

Here $\mathrm{d}_x y$ is the differential on $\mathcal{L}^! \otimes m$ induced by the Maurer-Cartan element x. These two equations say that x is an R-point of $B\mathcal{L}$ that satisfies the Euler-Lagrange equations up to a homotopy specified by y.

11.3.4. Symplectic structure on the derived critical locus.

Recall that a classical field theory is given by a local L_∞ algebra that is elliptic and has an invariant pairing of degree -3. The pairing on the local L_∞ algebra $\mathrm{Crit}(S)$ constructed above is evident: it is given by the natural bundle isomorphism

$$(L \oplus L^![-3])^![-3] \cong L^![-3] \oplus L.$$

In other words, the pairing arises, by a shift, from the natural bundle map $L \otimes L^! \to \mathrm{Dens}_M$.

11.3.4.1 Lemma. *This pairing on* $\mathrm{Crit}(S)$ *is invariant.*

PROOF. The original L_∞ structure on $\mathcal{L} \oplus \mathcal{L}^![-3]$ (that is, the L_∞ structure not involving S) is easily seen to be invariant. We will verify that the deformation of this structure coming from S is also invariant.

We need to show that if

$$\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{L}_c \oplus \mathcal{L}_c^![-3]$$

are compactly supported sections of $L \oplus L^1[-3]$, then

$$\langle l_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle$$

is totally antisymmetric in the variables α_i. Now, the part of this expression that comes from S is just

$$\left(\frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_{n+1}} \right) S(0).$$

The fact that partial derivatives commute — combined with the shift in grading due to $C^*(\mathcal{L}_c) = \mathscr{O}(\mathcal{L}_c[1])$ — immediately implies that this term is totally antisymmetric. ☐

Note that, although the local L_∞ algebra $\mathrm{Crit}(S)$ always has a symplectic form, it does not always define a classical field theory, in our sense. To be a classical field theory, we also require that the local L_∞ algebra $\mathrm{Crit}(S)$ is elliptic.

11.4. A succinct definition of a classical field theory

We defined a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In this section we will rewrite this definition in a more concise (but less conceptual) way. This version is included largely for consistency with [Cos11c] — where the language of elliptic moduli problems is not used — and for ease of reference when we discuss the quantum theory.

11.4.0.2 Definition. *Let E be a graded vector bundle on a manifold M. A* degree -1 symplectic structure on E *is an isomorphism of graded vector bundles*

$$\phi : E \cong E^![-1]$$

that is anti-symmetric, in the sense that $\phi^* = -\phi$ where ϕ^* is the formal adjoint of ϕ.

Note that if L is an elliptic L_∞ algebra on M with an invariant pairing of degree -3, then the graded vector bundle $L[1]$ on M has a -1 symplectic form. Indeed, by definition, L is equipped with a symmetric isomorphism $L \cong L^![-3]$, which becomes an antisymmetric isomorphism $L[1] \cong (L[1])^![-1]$.

Note also that the tangent space at the basepoint to the formal moduli problem $B\mathcal{L}$ associated to $\mathcal{L}$ is $\mathcal{L}[1]$ (equipped with the differential induced from that on $\mathcal{L}$). Thus, the algebra $C^*(\mathcal{L})$ of cochains of $\mathcal{L}$ is isomorphic, as a graded algebra without the differential, to the algebra $\mathscr{O}(\mathcal{L}[1])$ of functionals on $\mathcal{L}[1]$.

Now suppose that E is a graded vector bundle equipped with a -1 symplectic form. Let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$, as defined in section 10.5.1.

11.4.0.3 Proposition. *For E a graded vector bundle equipped with a -1 symplectic form, let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Then we have the following.*

(1) *The symplectic form on $\mathscr{E}$ induces a Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$, of degree* $+1$.

(2) *Equipping* $E[-1]$ *with a local* L_∞ *algebra structure compatible with the given pairing on* $E[-1]$ *is equivalent to picking an element* $S \in \mathscr{O}_{loc}(\mathscr{E})$ *that has cohomological degree* 0*, is at least quadratic, and satisfies the classical master equation*

$$\{S, S\} = 0.$$

PROOF. Let $L = E[-1]$. Note that L is a local L_∞ algebra, with the zero differential and zero higher brackets (i.e., a totally abelian L_∞ algebra). We write $\mathscr{O}_{loc}(B\mathcal{L})$ or $C^*_{red,loc}(\mathcal{L})$ for the reduced local cochains of $\mathcal{L}$. This is a complex with zero differential which coincides with $\mathscr{O}_{loc}(\mathscr{E})$.

We have seen that the exterior derivative (section 11.3) gives a map

$$\mathrm{d} : \mathscr{O}_{loc}(\mathscr{E}) = \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Note that the isomorphism

$$\mathcal{L} \cong \mathcal{L}^![-3]$$

gives an isomorphism

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \cong C^*_{loc}(\mathcal{L}, \mathcal{L}[2]).$$

Finally, $C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is the L_∞ algebra controlling deformations of $\mathcal{L}$ as a local L_∞ algebra. It thus remains to verify that $\mathscr{O}_{loc}(B\mathcal{L}) \subset C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is a sub L_∞ algebra, which is straightforward. ☐

Note that the finite-dimensional analog of this statement is simply the fact that on a formal symplectic manifold, all symplectic derivations (which correspond, after a shift, to deformations of the formal symplectic manifold) are given by Hamiltonian functions, defined up to the addition of an additive constant. The additive constant is not mentioned in our formulation because $\mathscr{O}_{loc}(\mathscr{E})$, by definition, consists of functionals without a constant term.

Thus, we can make a concise definition of a field theory.

11.4.0.4 Definition. *A* pre-classical field theory *on a manifold* M *consists of a graded vector bundle* E *on* M*, equipped with a symplectic pairing of degree* -1*, and a local functional*

$$S \in \mathscr{O}_{loc}(\mathscr{E}_c(M))$$

of cohomological degree 0*, satisfying the following properties.*

(1) *S satisfies the classical master equation* $\{S, S\} = 0$.
(2) *S is at least quadratic (so that* $0 \in \mathscr{E}_c(M)$ *is a critical point of* S*).*

In this situation, we can write S as a sum (in a unique way)

$$S(e) = \langle e, Qe \rangle + I(e)$$

where $Q : \mathscr{E} \to \mathscr{E}$ is a skew self-adjoint differential operator of cohomological degree 1 and square zero.

11.4.0.5 Definition. *A pre-classical field is a* classical field theory *if the complex* $(\mathscr{E}, Q)$ *is elliptic.*

There is one more property we need of a classical field theories in order to be apply the quantization machinery of [**Cos11c**].

11.4.0.6 Definition. *A* gauge fixing operator *is a map*

$$Q^{GF} : \mathscr{E}(M) \to \mathscr{E}(M)$$

that is a differential operator of cohomological degree -1 *such that* $(Q^{GF})^2 = 0$ *and*

$$[Q, Q^{GF}] : \mathscr{E}(M) \to \mathscr{E}(M)$$

is a generalized Laplacian in the sense of [**BGV92**].

The only classical field theories we will try to quantize are those that admit a gauge fixing operator. Thus, we will only consider classical field theories which have a gauge fixing operator. An important point which will be discussed at length in the chapter on quantum field theory is the fact that the observables of the quantum field theory are independent (up to homotopy) of the choice of gauge fixing condition.

11.5. Examples of field theories from action functionals

Let us now give some basic examples of field theories arising as the derived critical locus of an action functional. We will only discuss scalar field theories in this section.

Let (M, g) be a Riemannian manifold. Let $\underline{\mathbb{R}}$ be the trivial line bundle on M and $Dens_M$ the density line bundle. Note that the volume form $\mathrm{d}Vol_g$ provides an isomorphism between these line bundles. Let

$$S(\phi) = \tfrac{1}{2} \int_M \phi \, \mathrm{D} \phi$$

denote the action functional for the free massless field theory on M. Here D is the Laplacian on M, viewed as a differential operator from $C^\infty(M)$ to $\mathrm{Dens}(M)$, so $D\phi = (\Delta_g \phi) \mathrm{d}Vol_g$.

The derived critical locus of S is described by the elliptic L_∞ algebra

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2]$$

where $\mathrm{Dens}(M)$ is the global sections of the bundle of densities on M. Thus, $C^\infty(M)$ is situated in degree 1, and the space $\mathrm{Dens}(M)$ is situated in degree 2. The pairing between $\mathrm{Dens}(M)$ and $C^\infty(M)$ gives the invariant pairing on $\mathcal{L}$, which is symmetric of degree -3 as desired.

11.5.1. Interacting scalar field theories. Next, let us write down the derived critical locus for a basic interacting scalar field theory, given by the action functional

$$S(\phi) = \tfrac{1}{2}\int_M \phi\,\mathrm{D}\,\phi + \tfrac{1}{4!}\int_M \phi^4.$$

The cochain complex underlying our elliptic L_∞ algebra is, as before,

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2].$$

The interacting term $\frac{1}{4!}\int_M \phi^4$ gives rise to a higher bracket l_3 on $\mathcal{L}$, defined by the map

$$\begin{aligned} C^\infty(M)^{\otimes 3} &\to \mathrm{Dens}(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3 \mathrm{d}Vol_g. \end{aligned}$$

Let (R, m) be a nilpotent Artinian ring, concentrated in degree 0. Then a section of $\phi \in C^\infty(M) \otimes m$ satisfies the Maurer-Cartan equation in this L_∞ algebra if and only if

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 \mathrm{d}Vol = 0.$$

Note that this is precisely the Euler-Lagrange equation for S. Thus, the formal moduli problem associated to $\mathcal{L}$ is, as desired, the derived version of the moduli of solutions to the Euler-Lagrange equations for S.

11.6. Cotangent field theories

We have defined a field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In geometry, cotangent bundles are the basic examples of symplectic manifolds. We can apply this construction in our setting: given any elliptic moduli problem, we will produce a new elliptic moduli problem – its shifted cotangent bundle – that has a symplectic form of degree -1. We call the field theories that arise by this construction *cotangent field theories*. It turns out that a surprising number of field theories of interest in mathematics and physics arise as cotangent theories, including, for example, both the A- and the B-models of mirror symmetry and their half-twisted versions.

We should regard cotangent field theories as the simplest and most basic class of non-linear field theories, just as cotangent bundles are the simplest class of symplectic manifolds. One can show, for example, that the phase space of a cotangent field theory is always an (infinite-dimensional) cotangent bundle, whose classical Hamiltonian function is linear on the cotangent fibers.

11.6.1. The cotangent bundle to an elliptic moduli problem. Let $\mathcal{L}$ be an elliptic L_∞ algebra on a manifold X, and let $\mathcal{M}_{\mathcal{L}}$ be the associated elliptic moduli problem.

Let $L^!$ be the bundle $L^\vee \otimes \mathrm{Dens}(X)$. Note that there is a natural pairing between compactly supported sections of L and compactly supported sections of $L^!$.

Recall that we use the notation $\mathcal{L}$ to denote the space of sections of L. Likewise, we will let $\mathcal{L}^!$ denote the space of sections of $L^!$.

11.6.1.1 Definition. *Let $T^*[k]B\mathcal{L}$ denote the elliptic moduli problem associated to the elliptic L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![k-2]$.*

This elliptic L_∞ algebra has a pairing of cohomological degree $k-2$.

The L_∞ structure on the space $\mathcal{L} \oplus \mathcal{L}^![k-2]$ of sections of the direct sum bundle $L \oplus L^![k-2]$ arises from the natural $\mathcal{L}$-module structure on $\mathcal{L}^!$.

11.6.1.2 Definition. *Let $\mathcal{M} = B\mathcal{L}$ be an elliptic moduli problem corresponding to an elliptic L_∞ algebra $\mathcal{L}$. Then the* cotangent field theory *associated to $\mathcal{M}$ is the -1-symplectic elliptic moduli problem $T^*[-1]\mathcal{M}$, whose elliptic L_∞ algebra is $\mathcal{L} \oplus \mathcal{L}^![-3]$.*

11.6.2. Examples. In this section we will list some basic examples of cotangent theories, both gauge theories and nonlinear sigma models.

In order to make the discussion more transparent, we will not explicitly describe the elliptic L_∞ algebra related to every elliptic moduli problem we describe. Instead, we may simply define the elliptic moduli problem in terms of the geometric objects it classifies. In all examples, it is straightforward using the techniques we have discussed so far to write down the elliptic L_∞ algebra describing the formal neighborhood of a point in the elliptic moduli problems we will consider.

11.6.3. Self-dual Yang-Mills theory. Let X be an oriented 4-manifold equipped with a conformal class of a metric. Let G be a compact Lie group. Let $\mathcal{M}(X,G)$ denote the elliptic moduli problem parametrizing principal G-bundles on X with a connection whose curvature is self-dual.

Then we can consider the cotangent theory $T^*[-1]\mathcal{M}(X,G)$. This theory is known in the physics literature as *self-dual Yang-Mills theory*.

Let us describe the L_∞ algebra of this theory explicitly. Observe that the elliptic L_∞ algebra describing the completion of $\mathcal{M}(X,G)$ near a point (P,∇) is

$$\Omega^0(X,\mathfrak{g}_P) \xrightarrow{\mathrm{d}_\nabla} \Omega^1(X,\mathfrak{g}_P) \xrightarrow{\mathrm{d}_-} \Omega^2_-(X,\mathfrak{g}_P)$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to the principal G-bundle P. Here d_- denotes the connection followed by projection onto the anti-self-dual 2-forms.

Thus, the elliptic L_∞ algebra describing $T^*[-1]\mathcal{M}$ is given by the diagram

$$\begin{tikzcd}
\Omega^0(X,\mathfrak{g}_P) \arrow[r, "\mathrm{d}_\nabla"] & \Omega^1(X,\mathfrak{g}_P) \arrow[r, "\mathrm{d}_-"] & \Omega^2_-(X,\mathfrak{g}_P) & \\
 & \oplus & \oplus & \\
 & \Omega^2_-(X,\mathfrak{g}_P) \arrow[r, "\mathrm{d}_\nabla"] & \Omega^3(X,\mathfrak{g}_P) \arrow[r, "\mathrm{d}_\nabla"] & \Omega^4(X,\mathfrak{g}_P)
\end{tikzcd}$$

This is a standard presentation of the fields of self-dual Yang-Mills theory in the BV formalism (see [CCRF$^+$98] and [Cos11c]). Note that it is, in fact, a dg Lie algebra, so there are no nontrivial higher brackets.

Ordinary Yang-Mills theory arises as a deformation of the self-dual theory. One simply deforms the differential in the diagram above by including a term that is the identity from $\Omega^2_-(X,\mathfrak{g}_P)$ in degree 1 to the copy of $\Omega^2_-(X,\mathfrak{g}_P)$ situated in degree 2.

11.6.4. The holomorphic σ-model.

Let E be an elliptic curve and let X be a complex manifold. Let $\mathcal{M}(E,X)$ denote the elliptic moduli problem parametrizing holomorphic maps from $E \to X$. As before, there is an associated cotangent field theory $T^*[-1]\mathcal{M}(E,X)$. (In [Cos11a] it is explained how to describe the formal neighborhood of any point in this mapping space in terms of an elliptic L_∞ algebra on E.)

In [Cos10], this field theory was called a holomorphic Chern-Simons theory, because of the formal similarities between the action functional of this theory and that of the holomorphic Chern-Simons gauge theory. In the physics literature ([Wit05], [Kap05]) this theory is known as the twisted $(0,2)$ supersymmetric sigma model, or as the curved $\beta-\gamma$ system.

This theory has an interesting role in both mathematics and physics. For instance, it was shown in [Cos10, Cos11a] that the partition function of this theory (at least, the part which discards the contributions of non-constant maps to X) is the Witten genus of X.

11.6.5. Twisted supersymmetric gauge theories.

Of course, there are many more examples of cotangent theories, as there are very many elliptic moduli problems. In [Cos11b], it is shown how twisted versions of supersymmetric gauge theories can be written as cotangent theories. We will focus on holomorphic (or minimal) twists. Holomorphic twists are richer than the more well-studied topological twists, but contain less information than the full untwisted supersymmetric theory. As explained in [Cos11b], one can obtain topological twists from holomorphic twists by applying a further twist.

The most basic example is the twisted $\mathcal{N}=1$ field theory. If X is a complex surface and G is a complex Lie group, then the $\mathcal{N}=1$ twisted theory is simply the cotangent theory to the elliptic moduli problem of holomorphic principal G-bundles on X. If we fix a principal G-bundle $P \to X$, then the elliptic L_∞ algebra describing this formal moduli problem near P is

$$\Omega^{0,*}(X, \mathfrak{g}_P),$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to P. It is a classic result of Kodaira and Spencer that this dg Lie algebra describes deformations of the holomorphic principal bundle P.

The cotangent theory to this elliptic moduli problem is thus described by the elliptic L_∞ algebra

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus \mathfrak{g}_P^\vee \otimes K_X[-1].).$$

Note that K_X denotes the canonical line bundle, which is the appropriate holomorphic substitute for the smooth density line bundle.

11.6.6. The twisted $\mathscr{N} = 2$ theory. Twisted versions of gauge theories with more supersymmetry have similar descriptions, as is explained in [Cos11b]. The $\mathscr{N} = 2$ theory is the cotangent theory to the elliptic moduli problem for holomorphic G-bundles $P \to X$ together with a holomorphic section of the adjoint bundle $\mathfrak{g}_P$. The underlying elliptic L_∞ algebra describing this moduli problem is

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1]).$$

Thus, the cotangent theory has

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1] \oplus \mathfrak{g}_P^\vee \otimes K_X \oplus \mathfrak{g}_P^\vee \otimes K_X[-1])$$

for its elliptic L_∞ algebra.

11.6.7. The twisted $\mathscr{N} = 4$ theory. Finally, we will describe the twisted $\mathscr{N} = 4$ theory. There are two versions of this twisted theory: one used in the work of Vafa-Witten [VW94] on S-duality, and another by Kapustin-Witten [KW06] in their work on geometric Langlands. Here we will describe only the latter.

Let X again be a complex surface and G a complex Lie group. Then the twisted $\mathscr{N} = 4$ theory is the cotangent theory to the elliptic moduli problem describing principal G-bundles $P \to X$, together with a holomorphic section $\phi \in H^0(X, T^*X \otimes \mathfrak{g}_P)$ satisfying

$$[\phi, \phi] = 0 \in H^0(X, K_X \otimes g_P).$$

Here T^*X is the holomorphic cotangent bundle of X.

The elliptic L_∞ algebra describing this is

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus T^*X \otimes \mathfrak{g}_P[-1] \oplus K_X \otimes \mathfrak{g}_P[-2]).$$

Of course, this elliptic L_∞ algebra can be rewritten as

$$(\Omega^{*,*}(X, \mathfrak{g}_P), \overline{\partial}),$$

where the differential is just $\overline{\partial}$ and does not involve ∂. The Lie bracket arises from extending the Lie bracket on $\mathfrak{g}_P$ by tensoring with the commutative algebra structure on the algebra $\Omega^{*,*}(X)$ of forms on X.

Thus, the corresponding cotangent theory has

$$\Omega^{*,*}(X, \mathfrak{g}_P) \oplus \Omega^{*,*}(X, \mathfrak{g}_P)[1]$$

for its elliptic Lie algebra.

CHAPTER 12

The observables of a classical field theory

So far we have given a definition of a classical field theory, combining the ideas of derived deformation theory and the classical BV formalism. Our goal in this chapter is to show that the observables for such a theory do indeed form a commutative factorization algebra, denoted Obs^{cl}, and to explain how to equip it with a shifted Poisson bracket. The first part is straightforward — implicitly, we have already done it! — but the Poisson bracket is somewhat subtle, due to complications that arise when working with infinite-dimensional vector spaces. We will exhibit a sub-factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$ of Obs^{cl} which is equipped with a commutative product and Poisson bracket, and such that the inclusion map $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ is a quasi-isomorphism.

12.1. The factorization algebra of classical observables

We have given two descriptions of a classical field theory, and so we provide the two descriptions of the associated observables.

Let $\mathcal{L}$ be the elliptic L_∞ algebra of a classical field theory on a manifold M. Thus, the associated elliptic moduli problem is equipped with a symplectic form of cohomological degree -1.

12.1.0.1 Definition. *The* observables with support in the open subset U *is the commutative dg algebra*

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}*, assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open* U*.*

The interpretation of this definition should be clear from the preceding chapters. The elliptic L_∞ algebra $\mathcal{L}$ encodes the space of solutions to the Euler-Lagrange equations for the theory (more accurately, the formal neighborhood of the solution given by the base-point of the formal moduli problem). Its Chevalley-Eilenberg cochains $C^*(\mathcal{L}(U))$ on the open U are interpreted as the algebra of functions on the space of solutions over the open U.

By the results of section 6.4, we know that this construction is in fact a factorization algebra.

We often call Obs^{cl} simply the *classical observables*, in contrast to the factorization algebras of some quantization, which we will call the quantum observables.

Alternatively, let E be a graded vector bundle on M, equipped with a symplectic pairing of degree -1 and a local action functional S which satisfies the classical master equation. As we explained in section 11.4 this data is an alternative way of describing a classicla field theory. The bundle L whose sections are the local L_∞ algebra $\mathcal{L}$ is $E[-1]$.

12.1.0.2 Definition. *The* observables with support in the open subset U *is the commutative dg algebra*

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

equipped with the differential $\{S, -\}$.

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}, *assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open* U.

Recall that the operator $\{S, -\}$ is well-defined because the bracket with the local functional is always well-defined.

The underlying graded-commutative algebra of $\mathrm{Obs}^{cl}(U)$ is manifestly the functions on the fields $\mathscr{E}(U)$ over the open set U. The differential imposes the relations between observables arising from the Euler-Lagrange equations for S. In physical language, we are giving a cochain complex whose cohomology is the "functions on the fields that are on-shell."

It is easy to check that this definition of classical observables coincides with the one in terms of cochains of the sheaf of L_∞-algebras $\mathcal{L}(U)$.

12.2. The graded Poisson structure on classical observables

Recall the following definition.

12.2.0.3 Definition. *A P_0 algebra (in the category of cochain complexes) is a commutative differential graded algebra together with a Poisson bracket $\{-,-\}$ of cohomological degree* 1*, which satisfies the Jacobi identity and the Leibniz rule.*

The main result of this chapter is the following.

12.2.0.4 Theorem. *For any classical field theory (section 11.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a weak equivalence of commutative factorization algebras.*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

Concretely, $\widetilde{\mathrm{Obs}}^{cl}(U)$ is built from functionals on the space of solutions to the Euler-Lagrange equations that have more regularity than the functionals in $\mathrm{Obs}^{cl}(U)$.

The idea of the definition of the P_0 structure is very simple. Let us start with a finite-dimensional model. Let $\mathfrak{g}$ be an L_∞ algebra equipped with an invariant antisymmetric element $P \in \mathfrak{g} \otimes \mathfrak{g}$ of cohomological degree 3. This element can be viewed (according to the correspondence between formal moduli problems and Lie algebras given in section 10.1) as a bivector on $B\mathfrak{g}$, and so it defines a Poisson bracket on $\mathscr{O}(B\mathfrak{g}) = C^*(\mathfrak{g})$. Concretely, this Poisson bracket is defined, on the generators $\mathfrak{g}^\vee[-1]$ of $C^*(\mathfrak{g})$, as the map

$$\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{R}$$

determined by the tensor P.

Now let $\mathcal{L}$ be an elliptic L_∞ algebra describing a classical field theory. Then the kernel for the isomorphism $\mathcal{L}(U) \cong \mathcal{L}^!(U)[-3]$ is an element $P \in \overline{\mathcal{L}}(U) \otimes \overline{\mathcal{L}}(U)$, which is symmetric, invariant, and of degree 3.

We would like to use this idea to define the Poisson bracket on

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

As in the finite dimensional case, in order to define such a Poisson bracket, we would need an invariant tensor in $\mathcal{L}(U)^{\otimes 2}$. The tensor representing our pairing is instead in $\overline{\mathcal{L}}(U)^{\otimes 2}$, which contains $\mathcal{L}(U)^{\otimes 2}$ as a dense subspace. In other words, we run into a standard problem in analysis: our construction in finite-dimensional vector spaces does not port immediately to infinite-dimensional vector spaces.

We solve this problem by finding a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

such that the Poisson bracket *is* well-defined on the subcomplex and the inclusion is a weak equivalence. Up to quasi-isomorphism, then, we have the desired Poisson structure.

12.3. The Poisson structure for free field theories

In this section, we will construct a P_0 structure on the factorization algebra of observables of a free field theory. More precisely, we will construct for every open subset U, a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

of the complex of classical observables such that

(1) $\widetilde{\mathrm{Obs}}^{cl}$ forms a sub-commutative factorization algebra of Obs^{cl};
(2) the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$ is a weak equivalence of differentiable pro-cochain complexes for every open set U; and
(3) $\widetilde{\mathrm{Obs}}^{cl}$ has the structure of P_0 factorization algebra.

The complex $\mathrm{Obs}^{cl}(U)$ consists of a product over all n of certain distributional sections of a vector bundle on U^n. The complex $\widetilde{\mathrm{Obs}}^{cl}$ is defined by considering instead smooth sections on U^n of the same vector bundle.

Let us now make this definition more precise. Recall that a free field theory is a classical field theory associated to an elliptic L_∞ algebra $\mathcal{L}$ that is abelian, i.e., where all the brackets $\{l_n \mid n \geq 2\}$ vanish.

Thus, let L be the graded vector bundle associated to an abelian elliptic L_∞ algebra, and let $\mathcal{L}(U)$ be the elliptic complex of sections of L on U. To say that L defines a field theory means we have a symmetric isomorphism $\mathcal{L} \cong \mathcal{L}^![-3]$.

Recall (section B.2) that we use the notation $\overline{\mathcal{L}}(U)$ to denote the space of distributional sections of L on U. A lemma of Atiyah-Bott (section B.10) shows that the inclusion

$$\mathcal{L}(U) \hookrightarrow \overline{\mathcal{L}}(U)$$

is a continuous homotopy equivalence of topological cochain complexes.

It follows that the natural map

$$C^*(\overline{\mathcal{L}}(U)) \hookrightarrow C^*(\mathcal{L}(U))$$

is a cochain homotopy equivalence. Indeed, because we are dealing with an abelian L_∞ algebra, the Chevalley-Eilenberg cochains become quite simple:

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}(U)^\vee[-1]),$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}(U)^\vee[-1]),$$

where, as always, the symmetric algebra is defined using the completed tensor product. The differential is simply the differential on, for instance, $\mathcal{L}(U)^\vee$ extended as a derivation, so that we are simply taking the completed symmetric algebra of a complex. The complex $C^*(\mathcal{L}(U))$ is built from distributional sections of the bundle $(L^!)^{\boxtimes n}[-n]$ on U^n, and the complex $C^*(\overline{\mathcal{L}}(U))$ is built from smooth sections of the same bundle.

Note that

$$\mathcal{L}(U)^\vee = \overline{\mathcal{L}}_c^!(U) = \overline{\mathcal{L}}_c(U)[3].$$

Thus,

$$\begin{aligned} C^*(\mathcal{L}(U)) &= \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}_c(U)[2]), \\ C^*(\overline{\mathcal{L}}(U)) &= \widehat{\mathrm{Sym}}(\mathcal{L}_c(U)[2]). \end{aligned}$$

We can define a Poisson bracket of degree 1 on $C^*(\overline{\mathcal{L}}(U))$ as follows. On the generators $\mathcal{L}_c(U)[2]$, it is defined to be the given pairing

$$\langle -, - \rangle : \mathcal{L}_c(U) \times \mathcal{L}_c(U) \to \mathbb{R},$$

since we *can* pair smooth sections. This pairing extends uniquely, by the Leibniz rule, to continuous bilinear map

$$C^*(\overline{\mathcal{L}}(U)) \times C^*(\overline{\mathcal{L}}(U)) \to C^*(\overline{\mathcal{L}}(U)).$$

In particular, we see that $C^*(\overline{\mathcal{L}}(U))$ has the structure of a P_0 algebra in the multicategory of differentiable cochain complexes.

Let us define the modified observables in this theory by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*(\overline{\mathcal{L}}(U)).$$

We have seen that $\widetilde{\mathrm{Obs}}^{cl}(U)$ is homotopy equivalent to $\mathrm{Obs}^{cl}(U)$ and that $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a P_0 structure.

12.3.0.5 Lemma. $\mathrm{Obs}^{cl}(U)$ *has the structure of a* P_0 *factorization algebra.*

PROOF. It remains to verify that if $U_1, \dots, U_n$ are disjoint open subsets of M, each contained in an open subset W, then the map

$$\widetilde{\mathrm{Obs}}^{cl}(U_1) \times \cdots \times \widetilde{\mathrm{Obs}}^{cl}(U_n) \to \widetilde{\mathrm{Obs}}^{cl}(W)$$

is compatible with the P_0 structures. This map automatically respects the commutative structure, so it suffices to verify that for $\alpha \in \widetilde{\mathrm{Obs}}^{cl}(U_i)$ and $\beta \in \widetilde{\mathrm{Obs}}^{cl}(U_j)$, where $i \neq j$, then

$$\{\alpha, \beta\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

That this bracket vanishes follows from the fact that if two "linear observables" $\phi, \psi \in \mathcal{L}_c(W)$ have disjoint support, then

$$\langle \phi, \psi \rangle = 0.$$

Every Poisson bracket reduces to a sum of brackets between linear terms by applying the Leibniz rule repeatedly. $\square$

12.4. The Poisson structure for a general classical field theory

In this section we will prove the following.

12.4.0.6 Theorem. *For any classical field theory (section 11.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a quasi-isomorphism of commutative factorization algebras*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

12.4.1. Functionals with smooth first derivative.

For a free field theory, we defined a subcomplex $\widetilde{\mathrm{Obs}}^{cl}$ of observables which are built from smooth sections of a vector bundle on U^n, instead of distributional sections as in the definition of Obs^{cl}. It turns out that, for an interacting field theory, this subcomplex of Obs^{cl} is not preserved by the differential. Instead, we have to find a subcomplex built from distributions on U^n which are not smooth but which satisfy a mild regularity condition. We will call also this complex $\widetilde{\mathrm{Obs}}^{cl}$ (thus introducing a conflict with the terminology introduced in the case of free field theories).

Let $\mathcal{L}$ be an elliptic L_∞ algebra on M that defines a classical field theory. Recall that the cochain complex of observables is

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)),$$

where $\mathcal{L}(U)$ is the L_∞ algebra of sections of L on U.

Recall that as a graded vector space, $C^*(\mathcal{L}(U))$ is the algebra of functionals $\mathscr{O}(\mathcal{L}(U)[1])$ on the graded vector space $\mathcal{L}(U)[1]$. In the appendix (section B.8), given any graded vector bundle E on M, we define a subspace

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}(\mathscr{E}(U))$$

of functionals that have "smooth first derivative". A function $\Phi \in \mathscr{O}(\mathscr{E}(U))$ is in $\mathscr{O}^{sm}(\mathscr{E}(U))$ precisely if

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U).$$

(The exterior derivative of a general function in $\mathscr{O}(\mathscr{E}(U))$ will lie *a priori* in the larger space $\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}_c^!(U)$.) The space $\mathscr{O}^{sm}(\mathscr{E}(U))$ is a differentiable pro-vector space.

Recall that if $\mathfrak{g}$ is an L_∞ algebra, the exterior derivative maps $C^*(\mathfrak{g})$ to $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$. The complex $C^*_{sm}(\mathcal{L}(U))$ of cochains with smooth first derivative is thus defined to be the subcomplex of $C^*(\mathcal{L}(U))$ consisting of those cochains whose first derivative lies in $C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1])$, which is a subcomplex of $C^*(\mathcal{L}(U), \mathcal{L}(U)^\vee[-1])$.

In other words, $C^*_{sm}(\mathcal{L}(U))$ is defined by the fiber diagram

$$\begin{tikzcd}
C^*_{sm}(\mathcal{L}(U)) \arrow[r, "\mathrm{d}"] \arrow[d] & C^*(\mathcal{L}(U), \mathcal{L}^!_c(U)[-1]) \arrow[d] \\
C^*(\mathcal{L}(U)) \arrow[r, "\mathrm{d}"] & C^*(\mathcal{L}(U), \overline{\mathcal{L}_c}^!(U)[-1]).
\end{tikzcd}$$

(Note that differentiable pro-cochain complexes are closed under taking limits, so that this fiber product is again a differentiable pro-cochain complex; more details are provided in the appendix B.8).

Note that

$$C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

is a sub-commutative dg algebra for every open U. Furthermore, as U varies, $C^*_{sm}(\mathcal{L}(U))$ defines a sub-commutative prefactorization algebra of the prefactorization algebra defined by $C^*(\mathcal{L}(U))$.

We define

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U)) = \mathrm{Obs}^{cl}(U).$$

The next step is to construct the Poisson bracket.

12.4.2. The Poisson bracket.

Because the elliptic L_∞ algebra L defines a classical field theory, it is equipped with an isomorphism $L \cong L^![-3]$. Thus, we have an isomorphism

$$\Phi : C^*(\mathcal{L}(U), \mathcal{L}^!_c(U)[-1]) \cong C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]).$$

In the appendix (section B.9), we show that $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ — which we think of as vector fields on the formal manifold $B\mathcal{L}(U)$ — has a natural structure of a dg Lie algebra in the multicategory of differentiable pro-cochain complexes. The bracket is, of course, a version of the bracket of vector fields. Further, $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ acts on $C^*(\mathcal{L}(U))$ by derivations. This action is in the multicategory of differentiable pro-cochain complexes: the map

$$C^*(\mathcal{L}(U), \mathcal{L}(U)[1]) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is a smooth bilinear cochain map. We will write $\mathrm{Der}(C^*(\mathcal{L}(U)))$ for this dg Lie algebra $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$.

Thus, composing the map Φ above with the exterior derivative d and with the inclusion $\mathcal{L}_c(U) \hookrightarrow \mathcal{L}(U)$, we find a cochain map

$$C^*_{sm}(\mathcal{L}(U)) \to C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1].$$

If $f \in C^*_{sm}(\mathcal{L}(U))$, we will let $X_f \in \mathrm{Der}(C^*(\mathcal{L}(U)))$ denote the corresponding derivation. If f has cohomological degree k, then X_f has cohomological degree $k+1$.

If $f, g \in C^*_{sm}(\mathcal{L}(U)) = \widetilde{\mathrm{Obs}}^{cl}(U)$, we define

$$\{f, g\} = X_f g \in \widetilde{\mathrm{Obs}}^{cl}(U).$$

This bracket defines a bilinear map

$$\widetilde{\mathrm{Obs}}^{cl}(U) \times \widetilde{\mathrm{Obs}}^{cl}(U) \to \widetilde{\mathrm{Obs}}^{cl}(U).$$

Note that we are simply adopting the usual formulas to our setting.

12.4.2.1 Lemma. *This map is smooth, i.e., a bilinear map in the multicategory of differentiable pro-cochain complexes.*

PROOF. This follows from the fact that the map

$$\mathrm{d} : \widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1]$$

is smooth, which is immediate from the definitions, and from the fact that the map

$$\mathrm{Der}(C^*(\mathcal{L}(U)) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is smooth (which is proved in the appendix B.9). ☐

12.4.2.2 Lemma. *This bracket satisfies the Jacobi rule and the Leibniz rule. Further, for U, V disjoint subsets of M, both contained in W, and for any $f \in \widetilde{\mathrm{Obs}}^{cl}(U)$, $g \in \widetilde{\mathrm{Obs}}^{cl}(V)$, we have*

$$\{f, g\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

PROOF. The proof is straightforward. ☐

Following the argument for lemma 12.3.0.5, we obtain a P_0 factorization algebra.

12.4.2.3 Corollary. $\widetilde{\mathrm{Obs}}^{cl}$ *defines a P_0 factorization algebra in the valued in the multicategory of differentiable pro-cochain complexes.*

The final thing we need to verify is the following.

12.4.2.4 Proposition. *For all open subset $U \subset M$, the map*

$$\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

is a weak equivalence.

PROOF. It suffices to show that it is a weak equivalence on the associated graded for the natural filtration on both sides. Now, $\mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U)$ fits into a fiber diagram

$$\begin{tikzcd}
\mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U) \arrow[r] \arrow[d] & \mathrm{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1]) \otimes \mathcal{L}^!_c(U) \arrow[d] \\
\mathrm{Gr}^n \mathrm{Obs}^{cl}(U) \arrow[r] & \mathrm{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1]) \otimes \overline{\mathcal{L}}^!_c(U).
\end{tikzcd}$$

Note also that

$$\mathrm{Gr}^n \mathrm{Obs}^{cl}(U) = \mathrm{Sym}^n \overline{\mathcal{L}}^!_c(U).$$

The Atiyah-Bott lemma B.10 shows that the inclusion

$$\mathcal{L}^!_c(U) \hookrightarrow \overline{\mathcal{L}}^!_c(U)$$

is a continuous cochain homotopy equivalence. We can thus choose a homotopy inverse

$$P : \overline{\mathcal{L}}^!_c(U) \to \mathcal{L}^!_c(U)$$

and a homotopy

$$H : \overline{\mathcal{L}}^!_c(U) \to \overline{\mathcal{L}}^!_c(U)$$

such that $[\mathrm{d}, H] = P - \mathrm{Id}$ and such that H preserves the subspace $\mathcal{L}^!_c(U)$.

Now,

$$\mathrm{Sym}^n \mathcal{L}^!_c(U) \subset \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Sym}^n \overline{\mathcal{L}}^!_c(U).$$

Using the projector P and the homotopy H, one can construct a projector

$$P_n = P^{\otimes n} : \overline{\mathcal{L}}^!_c(U)^{\otimes n} \to \mathcal{L}^!_c(U)^{\otimes n}.$$

We can also construct a homotopy

$$H_n : \overline{\mathcal{L}}^!_c(U)^{\otimes n} \to \mathcal{L}^!_c(U)^{\otimes n}.$$

The homotopy H_n is defined inductively by the formula

$$H_n = H \otimes P_{n-1} + 1 \otimes H_{n-1}.$$

This formula defines a homotopy because

$$[\mathrm{d}, H_n] = P \otimes P_{n-1} - 1 \otimes P_{n-1} + 1 \otimes P_{n-1} - 1 \otimes 1.$$

Notice that the homotopy H_n preserves all the subspaces of the form

$$\overline{\mathcal{L}}^!_c(U)^{\otimes k} \otimes \mathcal{L}^!_c(U) \otimes \overline{\mathcal{L}}^!_c(U)^{\otimes n-k-1}.$$

This will be important momentarily.

Next, let

$$\pi : \overline{\mathcal{L}}^!_c(U)^{\otimes n}[-n] \to \mathrm{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1])$$

be the projection, and let

$$\Gamma_n = \pi^{-1} \operatorname{Gr}^n \widetilde{\operatorname{Obs}}^{cl}(U).$$

Then Γ_n is acted on by the symmetric group S_n, and the S_n invariants are $\widetilde{\operatorname{Obs}^{cl}}(U)$.

Thus, it suffices to show that the inclusion

$$\Gamma_n \hookrightarrow \overline{\mathcal{L}}_c(U)^{\otimes n}$$

is a weak equivalence of differentiable spaces. We will show that it is continuous homotopy equivalence.

The definition of $\widetilde{\operatorname{Obs}}^{cl}(U)$ allows one to identify

$$\Gamma_n = \cap_{k=0}^{n-1} \overline{\mathcal{L}}_c^!(U)^{\otimes k} \otimes \mathcal{L}_c^!(U) \otimes \overline{\mathcal{L}}_c^!(U)^{\otimes n-k-1}.$$

The homotopy H_n preserves Γ_n, and the projector P_n maps

$$\overline{\mathcal{L}}_c^!(U)^{\otimes n} \to \mathcal{L}_c(U)^{\otimes n} \subset \Gamma_n.$$

Thus, P_n and H_n provide a continuous homotopy equivalence between $\overline{\mathcal{L}}_c^!(U)^{\otimes n}$ and Γ_n, as desired. $\square$

Part 4

Quantum field theory

CHAPTER 13

Introduction to quantum field theory

As explained in the introduction, this book develops a version of deformation quantization for field theories, rather than mechanics. In the chapters on classical field theory, we showed that the observables of a classical BV theory naturally form a commutative factorization algebra, with a homotopical P_0 structure. In the following chapters, we will show that every quantization of a classical BV theory produces a factorization algebra (in Beilinson-Drinfeld algebras) that we call the quantum observables of the quantum field theory. To be precise, the main theorem of this part is the following.

13.0.2.5 Theorem. *Any quantum field theory on a manifold M, in the sense of* [**Cos11c**], *gives rise to a factorization algebra* Obs^q *on M of* quantum observables. *This is a factorization algebra over $\mathbb{C}[[\hbar]]$, valued in differentiable pro-cochain complexes, and it quantizes (in the weak sense of 1.7) the P_0 factorization algebra of classical observables of the corresponding classical field theory.*

For free field theories, this factorization algebra of quantum observables is essentially the same as the one discussed in Chapter 4. (The only difference is that, when discussing free field theories, we normally set $\hbar = 1$ and took our observables to be polynomial functions of the fields. When we discuss interacting theories, we take our observables to be power series on the space of fields, and we take $\hbar$ to be a formal parameter).

Chapter 14 is thus devoted to reviewing the formalism of [**Cos11c**], stated in a form most suitable to our purposes here. It's important to note that, in contrast to the deformation quantization of Poisson manifolds, a classical BV theory may not possess any quantizations (i.e., quantization may be *obstructed*) or it may have many quantizations. A central result of [**Cos11c**], stated in section 14.5, is that there is a space of BV quantizations. Moreover, this space can be constructed as a tower of fibrations, where the fiber between any pair of successive layers is described by certain cohomology groups of local functionals. These cohomology groups can be computed just from the classical theory.

The machinery of [**Cos11c**] allows one to construct many examples of quantum field theories, by calculating the appropriate cohomology groups. For example, in [**Cos11c**], the quantum Yang-Mills gauge theory is constructed. Theorem 13.0.2.5, together with the results of [**Cos11c**], thus produces many interesting examples of factorization algebras.

Remark: We forewarn the reader that our definitions and constructions involve a heavy use of functional analysis and (perhaps more surprisingly) simplicial sets, which is our preferred way of describing a space of field theories. Making a quantum field theory typically requires many choices, and as mathematicians, we wish to pin down precisely how the quantum field theory depends on these choices. The machinery we use gives us very precise statements, but statements that can be forbidding at first sight. We encourage the reader, on a first pass through this material, to simply make all necessary choices (such as a parametrix) and focus on the output of our machine, namely the factorization algebra of quantum observables. Keeping track of the dependence on choices requires careful bookkeeping (aided by the machinery of simplicial sets) but is straightforward once the primary construction is understood. ◊

The remainder of this chapter consists of an introduction to the quantum BV formalism, building on our motivation for the classical BV formalism in section 11.1.

13.1. The quantum BV formalism in finite dimensions

In section 11.1, we motivated the classical BV formalism with a finite-dimensional toy model. To summarize, we described the *derived* critical locus of a function S on a smooth manifold M of dimension n. The functions on this derived space $\mathscr{O}(\mathrm{Crit}^h(S))$ form a commutative dg algebra,

$$\Gamma(M, \wedge^n TM) \xrightarrow{\vee \mathrm{d}S} \ldots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} C^\infty(M),$$

the polyvector fields $PV(M)$ on M with the differential given by contraction with $\mathrm{d}S$. This complex remembers how $\mathrm{d}S$ vanishes and not just where it vanishes.

The quantum BV formalism uses a deformation of this *classical BV complex* to encode, in a homological way, oscillating integrals.

In finite dimensions, there already exists a homological approach to integration: the de Rham complex. For instance, on a compact, oriented n-manifold without boundary, M, we have the commuting diagram

$$\begin{tikzcd}
\Omega^n(M) \arrow[rr, "\int_M"] \arrow[dr, "{[-]}"'] & & \mathbb{R} \\
& H^n(M) \arrow[ur, "{\langle [M], - \rangle}"'] &
\end{tikzcd}$$

where $[\mu]$ denotes the cohomology class of the top form μ and $\langle [M], - \rangle$ denotes pairing the class with the fundamental class of M. Thus, integration factors through the de Rham cohomology.

Suppose μ is a smooth probability measure, so that $\int_M \mu = 1$ and μ is everywhere nonnegative (which depends on the choice of orientation). Then we can interpret the expected value of a function f on M — an "observable on the space of fields M" — as the cohomology class $[f\mu] \in H^n(M)$.

The BV formalism in finite dimensions secretly exploits this use of the de Rham complex, as we explain momentarily. For an infinite-dimensional manifold, though, the de Rham complex ceases to encode integration over the whole manifold because there are no top forms. In contrast, the BV version scales to the infinite-dimensional setting. Infinite dimensions, of course, introduces extra difficulties to do with the fact that integration in infinite dimensions is not well-defined. These difficulties manifest themselves as ultraviolte divergences of quantum field theory, and we deal with them using the techniques developed in [**Cos11c**].

In the classical BV formalism, we work with the polyvector fields rather than de Rham forms. A choice of probability measure μ, however, produces a map between these graded vector spaces

$$\begin{tikzcd}
\Gamma(M, \wedge^n TM) \arrow[d, "\vee\mu"'] & \cdots & \Gamma(M, \wedge^2 TM) \arrow[d, "\vee\mu"'] & \Gamma(M, TM) \arrow[d, "\vee\mu"'] & C^\infty(M) \arrow[d, "\vee\mu"'] \\
C^\infty(M) & \cdots & \Omega^{n-2}(M) & \Omega^{n-1}(M) & \Omega^n(M)
\end{tikzcd}$$

where $\vee\mu$ simply contracts a k-polyvector field with μ to get a $n-k$-form. When μ is nowhere-vanishing (i.e., when μ is a *volume form*), this map is an isomorphism and so we can "pull back" the exterior derivative to equip the polyvector fields with a differential. This differential is usually called the *divergence operator for* μ, so we denote it div_μ.

By the *divergence complex for* μ, we mean the polyvector fields (concentrated in nonpositive degrees) with differential div_μ. Its cohomology is isomorphic, by construction, to $H^*_{dR}(M)[n]$. In particular, given a function f on M, viewed as living in degree zero and providing an "observable," we see that its cohomology class $[f]$ in the divergence complex corresponds to the expected value of f against μ. More precisely, we can define the ratio $[f]/[1]$ as the expected value of f. Under the map $\vee\mu$, it goes to the usual expected value.

What we've done above is provide an alternative homological approach to integration. More accurately, we've shown how "integration against a volume form" can be encoded by an appropriate choice of differential on the polyvector fields. Cohomology classes in this divergence complex encode the expected values of functions against this measure. Of course, this is what we want from the path integral! The divergence complex is the motivating example for the quantum BV formalism, and so it is also called a *quantum BV complex*.

We can now explain why this approach to homological integration is more suitable to extension to infinite dimensions than the usual de Rham picture. Even for an infinite-dimensional manifold M, the polyvector fields are well-defined (although one must make choices in how to define them, depending on one's preferences with functional analysis). One can still try to construct a "divergence-type operator" and view it as the effective replacement for the probability measure. By taking cohomology classes, we compute the expected values of observables. The difficult part is making sense of the divergence operator; this is achieved through renormalization.

This vein of thought leads to a question: how to characterize, in an abstract fashion, the nature of a divergence operator? An answer leads, as we've shown, to a process for defining a homological path integral. Below, we'll describe one approach, but first we examine a simple case.

Remark: The cohomology of the complex (both in the finite and infinite dimensional settings) always makes sense, but H^0 is not always one-dimensional. For example, on a manifold X that is not closed, the de Rham cohomology often vanishes at the top. If the manifold is disconnected but closed, the top de Rham cohomology has dimension equal to the number of components of the manifold. In general, one must choose what class of functions to integrate against the volume form, and the cohomology depends on this choice (e.g., consider compactly supported de Rham cohomology).

Instead of computing expected values, the cohomology provides relations between expected values of observables. We will see how the cohomology encodes relations in the example below. In the setting of conformal field theory, for instance, one often uses such relations to obtain formulas for the operator product expansion. ◊

13.2. The "free scalar field" in finite dimensions

A concrete example is in order. We will work with a simple manifold, the real line, equipped with the Gaussian measure and recover the baby case of Wick's lemma. The generalization to a finite-dimensional vector space will be clear.

Remark: This example is especially pertinent to us because in this book we are working with perturbative quantum field theories. Hence, for us, there is always a free field theory — whose space of fields is a vector space equipped with some kind of Gaussian measure — that we've modified by adding an interaction to the action functional. The underlying vector space is equipped with a linear pairing that yields the BV Laplacian, as we work with it. As we will see in this example, the usual BV formalism relies upon the underlying "manifold" being linear in nature. To extend to a global nonlinear situation, on e needs to develop new techniques (see, for instance, [**Cos11a**]). ◊

Before we undertake the Gaussian measure, let’s begin with the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. This is not a probability measure, but it is nowhere-vanishing, which is the only property necessary to construct a divergence operator. In this case, we compute

$$\mathrm{div}_{Leb} : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x}.$$

In one popular notion, we use ξ to denote the vector field $\partial/\partial x$, and the polyvector fields are then $C^\infty(\mathbb{R})[\xi]$, where ξ has cohomological degree -1. The divergence operator becomes

$$\mathrm{div}_{Leb} = \frac{\partial}{\partial x}\frac{\partial}{\partial \xi},$$

which is also the standard example of the BV Laplacian $\triangle$. (In short, the usual BV Laplacian on $\mathbb{R}^n$ is simply the divergence operator for the Lebesgue measure.) We will use $\triangle$ for it, as this notation will continue throughout the book.

It is easy to see, by direct computation or the Poincaré lemma, that the cohomology of the divergence complex for the Lebesgue measure is simply $H^{-1} \cong \mathbb{R}$ and $H^0 \cong \mathbb{R}$.

Let μ_b be the usual Gaussian probability measure on $\mathbb{R}$ with variance b:

$$\mu_b = \sqrt{\frac{1}{2\pi b}}e^{-x^2/2b}\mathrm{d}x.$$

As μ is a nowhere-vanishing probability measure, we obtain a divergence operator

$$\mathrm{div}_b : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x} - \frac{x}{b}f.$$

We have

$$\mathrm{div}_b = \triangle + \vee \mathrm{d}S$$

where $S = -x^2/2b$. Note that this complex is a deformation of the classical BV complex for S by adding the BV Laplacian $\triangle$.

This divergence operator preserves the subcomplex of polynomial polyvector fields. That is, a vector field with polynomial coefficient goes to a polynomial function.

Explicitly, we see

$$\mathrm{div}_b\left(x^n\frac{\partial}{\partial x}\right) = nx^{n-1} - \frac{1}{b}x^{n+1}.$$

Hence, at the level of cohomology, we see $[x^{n+1}] = bn[x^{n-1}]$. We have just obtained the following, by a purely cohomological process.

13.2.0.6 Lemma (Baby case of Wick’s lemma). *The expected value of x^n with respect to the Gaussian measure is zero if n odd and $b^k(2k-1)(2k-3)\cdots 5\cdot 3$ if $n = 2k$.*

Since Wick's lemma appears by this method, it should be clear that one can recover the usual Feynman diagrammatic expansion. Indeed, the usual arguments with integration by parts are encoded here by the relations between cohomology classes.

Note that for any function $S : \mathbb{R} \to \mathbb{R}$, the volume form $e^S \mathrm{d}x$ has divergence operator

$$\mathrm{div}_S = \triangle + \frac{\partial S}{\partial x}\frac{\partial}{\partial x},$$

and using the Schouten bracket $\{-,-\}$ on polyvector fields, we can write it as

$$\mathrm{div}_S = \triangle + \{S, -\}.$$

The *quantum master equation* (QME) is the equation $\mathrm{div}_S^2 = 0$. The *classical master equation* (CME) is the equation $\{S, S\} = 0$, which just encodes the fact that the differential of the classical BV complex is square-zero. (In the examples we've discussed so far, this property is immediate, but in many contexts, such as gauge theories, finding such a function S can be a nontrivial process.)

13.3. An operadic description

Before we provide abstract properties that characterize a divergence operator, we should recall properties that characterize the classical BV complex. Of course, functions on the derived critical locus are a commutative dg algebra. Polyvector fields, however, also have the Schouten bracket — the natural extension of the Lie bracket of vector fields and functions — which is a Poisson bracket of cohomological degree 1 and which is compatible with the differential $\vee S = \{S, -\}$. Thus, we introduced the notion of a P_0 algebra, where P_0 stands for "Poisson-zero," in section 8.3. In chapter 12, we showed that the factorization algebra of observables for a classical BV theory have a lax P_0 structure.

Examining the divergence complex for a measure of the form $e^S \mathrm{d}x$ in the preceding section, we saw that the divergence operator was a deformation of $\{S, -\}$, the differential for the classical BV complex. Moreover, a simple computation shows that a divergence operator satisfies

$$\mathrm{div}(ab) = (\mathrm{div}\, a)b + (-1)^{|a|}a(\mathrm{div}\, b) + (-1)^{|a|}\{a, b\}$$

for any polyvector fields a and b. (This relation follows, under the polyvector-de Rham isomorphism given by the measure, from the fact that the exterior derivative is a derivation for the wedge product.) Axiomatizing these two properties, we obtain the notion of a Beilinson-Drinfeld algebra, discussed in section 8.4. The differential of a BD algebra possesses many of the essential properties of a divergence operator, and so we view a BD algebra as a homological way to encode integration on (a certain class of) derived spaces.

In short, the quantum BV formalism aims to find, for a P_0 algebra A^{cl}, a BD algebra A^q such that $A^{cl} = A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/(\hbar)$. We view it as moving from studying functions on the derived critical locus of some action functional S to the divergence complex for $e^S \mathscr{D}\phi$.

This motivation for the definition of a BD algebra is complementary to our earlier motivation, which emphasizes the idea that we simply want to deform from a commutative factorization algebra to a "plain," or E_0, factorization algebra. It grows out of the path integral approach to quantum field theory, rather than extending to field theory the deformation quantization approach to mechanics.

For us, the basic situation is a formal moduli space $\mathcal{M}$ with -1-symplectic pairing. Its algebra of functions is a P_0 algebra. By a version of the Darboux lemma for formal moduli spaces, we can identify $\mathcal{M}$ with an L_∞ algebra $\mathfrak{g}$ equipped with an invariant symmetric pairing. Geometrically, this means the symplectic pairing is translation-invariant and all the nonlinearity is pushed into the brackets. As the differential d on $\mathscr{O}(\mathcal{M})$ respects the Poisson bracket, we view it as a symplectic vector field of cohomological degree 1, and in this formal situation, we can find a Hamiltonian function S such that $d = \{S, -\}$.

Comparing to our finite-dimensional example above, we are seeing the analog of the fact that any nowhere-vanishing volume form on $\mathbb{R}^n$ can be written as $e^S \mathrm{d}x_1 \cdots \mathrm{d}x_n$. The associated divergence operator looks like $\triangle + \{S, -\}$, where the BV Laplacian $\triangle$ is the divergence operator for Lebesgue measure.

The translation-invariant Poisson bracket on $\mathscr{O}(\mathcal{M})$ also produces a translation-invariant BV Laplacian $\triangle$. Quantizing then amounts to finding a function $I \in \hbar\mathscr{O}(\mathcal{M})[[\hbar]]$ such that

$$\{S, -\} + \{I, -\} + \hbar\triangle$$

is square-zero. In the BV formalism, we call I a "solution to the quantum master equation for the action S." As shown in chapter 6 of [**Cos11c**], we have the following relationship.

13.3.0.7 Proposition. *Let $\mathcal{M}$ be a formal moduli space with -1-symplectic structure. There is an equivalence of spaces*

$$\{\textit{solutions of the QME}\} \simeq \{\textit{BD quantizations}\}.$$

13.4. Equivariant BD quantization and volume forms

We now return to our discussion of volume forms and formulate a precise relationship with BD quantization. This relationship, first noted by Koszul [**Kos85**], generalizes naturally to the setting of cotangent field theories. In section 16.4, we explain how cotangent quantizations provide volume forms on elliptic moduli problems.

For a smooth manifold M, there is a special feature of a divergence complex that we have not yet discussed. Polyvector fields have a natural action of the multiplicative group

$\mathbb{G}_m$, where functions have weight zero, vector fields have weight -1, and k-vector fields have weight $-k$. This action arises because polyvector fields are functions on the shifted cotangent bundle $T^*[-1]M$, and there is always a scaling action on the cotangent fibers.

We can make the classical BV complex into a $\mathbb{G}_m$-equivariant P_0 algebra, as follows. Simply equip the Schouten bracket with weight 1 and the commutative product with weight zero. We now ask for a $\mathbb{G}_m$-equivariant BD quantization.

To make this question precise, we rephrase our observations operadically. Equip the operad P_0 with the $\mathbb{G}_m$ action where the commutative product is weight zero and the Poisson bracket is weight 1. An equivariant P_0 algebra is then a P_0 algebra with a $\mathbb{G}_m$ action such that the bracket has weight 1 and the product has weight zero. Similarly, equip the operad BD with the $\mathbb{G}_m$ action where $\hbar$ has weight -1, the product has weight zero, and the bracket has weight 1. A filtered BD algebra is a BD algebra with a $\mathbb{G}_m$ action with the same weights.

Given a volume form μ on M, the $\hbar$-weighted divergence complex

$$(PV(M)[[\hbar]], \hbar \operatorname{div}_\mu)$$

is a filtered BD algebra.

On an smooth manifold, we saw that each volume form μ produced a divergence operator div_μ, via "conjugating" the exterior derivative d by the isomorphism $\vee\mu$. In fact, any rescaling $c\mu$, with $c \in \mathbb{R}^\times$, produces the same divergence operator. Since we want to work with probability measures, this fact meshes well with our objectives: we would always divide by the integral $\int_X \mu$ anyway. In fact, one can show that every filtered BD quantization of the P_0 algebra $PV(M)$ arises in this way.

13.4.0.8 Proposition. *There is a bijection between projective volume forms on M, and filtered BV quantizations of $PV(M)$.*

See [Cos11a] for more details on this.

13.5. How renormalization group flow interlocks with the BV formalism

So far, we have introduced the quantum BV formalism in the finite dimensional setting and extracted the essential algebraic structures. Applying these ideas in the setting of field theories requires nontrivial work. Much of this work is similar in flavor to our construction of a lax P_0 structure on Obs^{cl}: issues with functional analysis block the most naive approach, but there are alternative approaches, often well-known in physics, that accomplish our goal, once suitably reinterpreted.

Here, we build on the approach of [**Cos11c**]. The book uses exact renormalization group flow to define the notion of effective field theory and develops an effective version of the BV formalism. In chapter 14, we review these ideas in detail. We will sketch how to apply the BV formalism to formal elliptic moduli problems $\mathcal{M}$ with -1-symplectic pairing.

The main problem here is the same as in defining a shifted Poisson structure on the classical observables: the putative Poisson bracket $\{-,-\}$, arising from the symplectic structure, is well-defined only on a subspace of all observables. As a result, the associated BV Laplacian $\triangle$ is also only partially-defined.

To work around this problem, we use the fact that every parametrix Φ for the elliptic complex underlying $\mathcal{M}$ yields a mollified version $\triangle_\Phi$ of the BV Laplacian, and hence a mollified bracket $\{-,-\}_\Phi$. An *effective field theory* consists of a BD algebra Obs_Φ for every parametrix and a homotopy equivalence for any two parametrices, $\mathrm{Obs}_\Phi \simeq \mathrm{Obs}_\Psi$, satisfying coherence relations. In other words, we get a family of BD algebras over the space of parametrices. The renormalization group (RG) flow provides the homotopy equivalences for any pair of parametrices. Modulo $\hbar$, we also get a family Obs^{cl}_Φ of P_0 algebras over the space of parametrices. The tree-level RG flow produces the homotopy equivalences modulo $\hbar$.

An effective field theory is a quantization of $\mathcal{M}$ if, in the limit as $\triangle_\Phi$ goes to $\triangle$, the P_0 algebra goes to the functions $\mathscr{O}(\mathcal{M})$ on the formal moduli problem.

The space of parametrices is contractible, so an effective field theory describes just one BD algebra, up to homotopy equivalence. From the perspective developed thus far, we interpret this BD algebra as encoding integration over $\mathcal{M}$.

There is another way to interpret this definition, though, that may be attractive. The RG flow amounts to a Feynman diagram expansion, and hence we can see it as a definition of functional integration (in particular, flowing from energy scale Λ to Λ' integrates over the space of functions with energies between those scales). In [**Cos11c**], the RG flow is extended to the setting where the underlying free theory is an elliptic complex, not just given by an elliptic operator.

13.6. Overview of the rest of this Part

Here is a detailed summary of the chapters on quantum field theory.

(1) In section 15.1 we recall the definition of a free theory in the BV formalism and construct the factorization algebra of quantum observables of a general free theory, using the factorization envelope construction of section 3.6 of Chapter 3. This generalizes the discussion in chapter 4.

(2) In sections **??** to 14.5 we give an overview of the definition of QFT developed in [**Cos11c**].
(3) In section 15.2 we show how the definition of a QFT leads immediately to a construction of a BD algebra of "global observables" on the manifold M, which we denote $\mathrm{Obs}^q_{\mathscr{P}}(M)$.
(4) In section 15.3 we start the construction of the factorization algebra associated to a QFT. We construct a cochain complex $\mathrm{Obs}^q(M)$ of global observables, which is quasi-isomorphic to (but much smaller than) the BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$.
(5) In section 15.5 we construct, for every open subset $U \subset M$, the subspace $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U.
(6) Section 15.6 accomplishes the primary aim of the chapter. In it, we prove that the cochain complexes $\mathrm{Obs}^q(U)$ form a factorization algebra. The proof of this result is the most technical part of the chapter.
(7) In section 16.1 we show that translation-invariant theories have translation-invariant factorization algebras of observables, and we treat the holomorphic situation as well.
(8) In section 16.4 we explain how to interpret our definition of a QFT in the special case of a cotangent theory: roughly speaking, a quantization of the cotangent theory to an elliptic moduli problem yields a locally-defined volume form on the moduli problem we start with.

CHAPTER 14

Effective field theories and Batalin-Vilkovisky quantization

In this chapter, we will give a summary of the definition of a QFT as developed in [**Cos11c**]. We will emphasize the aspects used in our construction of the factorization algebra associated to a QFT. This means that important aspects of the story there — such as the concept of renormalizability — will not be mentioned. The introductory chapter of [**Cos11c**] is a leisurely exposition of the main physical and mathematical ideas, and we encourage the reader to examine it before delving into what follows. The approach there is perturbative and hence has the flavor of formal geometry (that is, geometry with formal manifolds).

A perturbative field theory is defined to be a family of effective field theories parametrized by some notion of "scale." The notion of scale can be quite flexible; the simplest version is where the scale is a positive real number, the length. In this case, the effective theory at a length scale L is obtained from the effective theory at scale ε by integrating out over fields with length scale between ε and L. In order to construct factorization algebras, we need a more refined notion of "scale," where there is a scale for every parametrix Φ of a certain elliptic operator. We denote such a family of effective field theories by $\{I[\Phi]\}$, where $I[\Phi]$ is the "interaction term" in the action functional $S[\Phi]$ at "scale" Φ. We always study families with respect to a fixed free theory.

A local action functional (see section 14.1) S is a real-valued function on the space of fields such that $S(\phi)$ is given by integrating some function of the field and its derivatives over the base manifold (the "spacetime"). The main result of [**Cos11c**] states that the space of perturbative QFTs is the "same size" as the space of local action functionals. More precisely, the space of perturbative QFTs defined modulo $\hbar^{n+1}$ is a torsor over the space of QFTs defined modulo $\hbar^n$ for the abelian group of local action functionals. In consequence, the space of perturbative QFTs is non-canonically isomorphic to local action functionals with values in $\mathbb{R}[[\hbar]]$ (where the choice of isomorphism amounts to choosing a way to construct counterterms).

The starting point for many physical constructions — such as the path integral — is a local action functional. However, a naive application of these constructions to such an action functional yields a nonsensical answer. Many of these constructions *do* work if, instead of applying them to a local action functional, they are applied to a family $\{I[\Phi]\}$ of effective action functionals. Thus, one can view the family of effective action functionals

$\{I[\Phi]\}$ as a quantum version of the local action functional defining classical field theory. The results of [**Cos11c**] allow one to construct such families of action functionals. Many formal manipulations with path integrals in the physics literature apply rigorously to families $\{I[\Phi]\}$ of effective actions. Our strategy for constructing the factorization algebra of observables is to mimic path-integral definitions of observables one can find in the physics literature, but replacing local functionals by families of effective actions.

14.1. Local action functionals

In studying field theory, there is a special class of functions on the fields, known as local action functionals, that parametrize the possible classical physical systems. Let M be a smooth manifold. Let $\mathscr{E} = C^\infty(M, E)$ denote the smooth sections of a $\mathbb{Z}$-graded super vector bundle E on M, which has finite rank when all the graded components are included. We call $\mathscr{E}$ the *fields*.

Various spaces of functions on the space of fields are defined in the appendix B.8.

14.1.0.9 Definition. *A* functional F *is an element of*

$$\mathscr{O}(\mathscr{E}) = \prod_{n=0}^{\infty} \mathrm{Hom}_{DVS}(\mathscr{E}^{\times n}, \mathbb{R})^{S_n}.$$

This is also the completed symmetric algebra of $\mathscr{E}^\vee$*, where the tensor product is the completed projective one.*

Let $\mathscr{O}_{red}(\mathscr{E}) = \mathscr{O}(\mathscr{E})/\mathbb{C}$ *be the space of functionals on* $\mathscr{E}$ *modulo constants.*

Note that every element of $\mathscr{O}(\mathscr{E})$ has a Taylor expansion whose terms are smooth multilinear maps

$$\mathscr{E}^{\times n} \to \mathbb{C}.$$

Such smooth mulitilinear maps are the same as compactly-supported distributional sections of the bundle $(E^!)^{\boxtimes n}$ on M^n. Concretely, a functional is then an infinite sequence of vector-valued distributions on powers of M.

The local functionals depend only on the local behavior of a field, so that at each point of M, a local functional should only depend on the jet of the field at that point. In the Lagrangian formalism for field theory, their role is to describe the permitted actions, so we call them *local action functionals*. A local action functional is the essential datum of a *classical* field theory.

14.1.0.10 Definition. *A functional* F *is* local *if each homogeneous component* F_n *is a finite sum of terms of the form*

$$F_n(\phi) = \int_M (D_1\phi) \cdots (D_n\phi)\, d\mu,$$

where each D_i is a differential operator from $\mathscr{E}$ to $C^\infty(M)$ and $d\mu$ is a density on M.

We let

$$\mathscr{O}_{loc}(\mathscr{E}) \subset \mathscr{O}_{red}(\mathscr{E})$$

denote denote the space of local action functionals modulo constants.

As explained in section 11.4, a classical BV theory is a choice of local action functional S of cohomological degree 0 such that $\{S, S\} = 0$. That is, S must satisfy the classical master equation.

14.2. The definition of a quantum field theory

In this section, we will give the formal definition of a quantum field theory. The definition is a little long and somewhat technical. The reader should consult the first chapter of [**Cos11c**] for physical motivations for this definition. We will provide some justification for the definition from the point of view of homological algebra shortly (section 15.2).

14.2.1.

14.2.1.1 Definition. *A* free BV theory *on a manifold M consists of the following data:*

1. *a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that is of finite rank;*
2. *a graded antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of cohomological degree -1 that is fiberwise nondegenerate. It induces a graded antisymmetric pairing of degree -1 on compactly supported smooth sections $\mathscr{E}_c$ of E:*
$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$
3. *a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

In our constructions, we require the existence of a *gauge-fixing operator* $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the following properties:

1. it is a square-zero differential operator of cohomological degree -1 ;
2. it is self adjoint for the symplectic pairing;
3. $D = [Q, Q^{GF}]$ is a generalized Laplacian on M, in the sense of [**BGV92**]. This means that D is an order 2 differential operator whose symbol $\sigma(D)$, which is an endomorphism of the pullback bundle p^*E on the cotangent bundle $p : T^*M \to M$, is
$$\sigma(D) = g\,\mathrm{Id}_{p^*E}$$

where g is some Riemannian metric on M, viewed as a function on T^*M.

All our constructions vary homotopically with the choice of gauge fixing operator. In practice, there is a natural contractible space of gauge fixing operators, so that our constructions are independent (up to contractible choice) of the choice of gauge fixing operator. (As an example of contractibility, if the complex $\mathscr{E}$ is simply the de Rham complex, each metric gives a gauge fixing operator d^*. The space of metrics is contractible.)

14.2.2. Operators and kernels. Let us recall the relationship between kernels and operators on $\mathscr{E}$. Any continuous linear map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ can be represented by a kernel

$$K_F \in \mathcal{D}(M^2, E \boxtimes E^!).$$

Here $\mathcal{D}(M, -)$ denotes distributional sections. We can also identify this space as

$$\begin{aligned}\mathcal{D}(M^2, E \boxtimes E^!) &= \mathrm{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c, \mathbb{C}) \\ &= \mathrm{Hom}_{DVS}(\mathscr{E}_c, \overline{\mathscr{E}}) \\ &= \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}^!.\end{aligned}$$

Here $\widehat{\otimes}_\pi$ denotes the completed projective tensor product.

The symplectic pairing on $\mathscr{E}$ gives an isomorphism between $\overline{\mathscr{E}}$ and $\overline{\mathscr{E}}^![-1]$. This allows us to view the kernel for any continuous linear map F as an element

$$K_F \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}} = \mathrm{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c^!, \mathbb{C})$$

. If F is of cohomological degree k, then the kernel K_F is of cohomological degree $k+1$.

If the map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ has image in $\overline{\mathscr{E}}_c$ and extends to a continuous linear map $\mathscr{E} \to \overline{\mathscr{E}}_c$, then the kernel K_F has compact support. If F has image in $\mathscr{E}$ and extends to a continuous linear map $\overline{\mathscr{E}}_c \to \mathscr{E}$, then the kernel K_F is smooth.

Our conventions are such that the following hold.

1. $K_{[Q,F]} = QK_F$, where Q is the total differential on $\overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}$.
2. Suppose that $F : \mathscr{E}_c \to \mathscr{E}_c$ is skew-symmetric with respect to the degree -1 pairing on $\mathscr{E}_c$. Then K_F is symmetric. Similarly, if F is symmetric, then K_F is antisymmetric.

14.2.3. The heat kernel. In this section we will discuss heat kernels associated to the generalized Laplacian $D = [Q, Q^{GF}]$. These generalized heat kernels will not be essential to our story; most of our constructions will work with a general parametrix for the operator D, and the heat kernel simply provides a convenient example.

Suppose that we have a free BV theory with a gauge fixing operator Q^{GF}. As above, let $D = [Q, Q^{GF}]$. If our manifold M is compact, then this leads to a heat operator e^{-tD} acting on sections $\mathscr{E}$. The heat kernel K_t is the corresponding kernel, which is an element of $\overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}\widehat{\otimes}_\pi C^\infty(\mathbb{R}_{\geq 0})$. Further, if $t > 0$, the operator e^{-tD} is a smoothing operator, so that the kernel K_t is in $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$. Since the operator e^{-tD} is skew symmetric for the symplectic pairing on $\mathscr{E}$, the kernel K_t is symmetric.

The kernel K_t is uniquely characterized by the following properties:

(1) The heat equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}K_t + (D \otimes 1)K_t = 0.$$

(2) The initial condition that $K_0 \in \overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}$ is the kernel for the identity operator.

On a non-compact manifold M, there is more than one heat kernel satisfying these properties.

14.2.4. Parametrices.

In [Cos11c], two equivalent definitions of a field theory are given: one based on the heat kernel, and one based on a general parametrix. We will use exclusively the parametrix version in this book.

Before we define the notion of parametrix, we need a technical definition.

14.2.4.1 Definition. *If M is a manifold, a subset $V \subset M^n$ is* proper *if all of the projection maps $\pi_1, \ldots, \pi_n : V \to M$ are proper. We say that a function, distribution, etc. on M^n has proper support if its support is a proper subset of M^n.*

14.2.4.2 Definition. *A* parametrix Φ *is a distributional section*

$$\Phi \in \overline{\mathscr{E}}(M)\widehat{\otimes}_\pi\overline{\mathscr{E}}(M)$$

of the bundle $E \boxtimes E$ on $M \times M$ with the following properties.

1. *Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}}(M)\widehat{\otimes}_\pi\overline{\mathscr{E}}(M)$.*
2. *Φ is of cohomological degree 1.*
3. *Φ has proper support.*
4. *Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that*

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is a smooth section of $E \boxtimes E$ on $M \times M$. Thus,

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}} \in \mathscr{E}(M)\widehat{\otimes}_\pi\mathscr{E}(M).$$

(Here K_{Id} is the kernel corresponding to the identity operator).

Remark: For clarity's sake, note that our definition depends on a choice of Q^{GF}. Thus, we are defining here parametrices for the generalized Laplacian $[Q, Q^{GF}]$, *not* general parametrices for the elliptic complex $\mathscr{E}$. ♢

Note that the parametrix Φ can be viewed (using the correspondence between kernels and operators described above) as a linear map $A_\Phi : \mathscr{E} \to \mathscr{E}$. This operator is of cohomological degree 0, and has the property that

$$A_\Phi [Q, Q^{GF}] = \mathrm{Id} + \text{a smoothing operator}$$
$$[Q, Q^{GF}] A_\Phi = \mathrm{Id} + \text{a smoothing operator.}$$

This property – being both a left and right inverse to the operator $[Q, Q^{GF}]$, up to a smoothing operator – is the standard definition of a parametrix.

An example of a parametrix is the following. For M compact, let $K_t \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$ be the heat kernel. Then, the kernel $\int_0^L K_t \mathrm{d}t$ is a parametrix, for any $L > 0$.

It is a standard result in the theory of pseudodifferential operators (see e.g. [Tar87]) that every elliptic operator admits a parametrix. Normally a parametrix is not assumed to have proper support; however, if Φ is a parametrix satisfying all conditions except that of proper support, and if $f \in C^\infty(M \times M)$ is a smooth function with proper support that is 1 in a neighborhood of the diagonal, then $f\Phi$ is a parametrix with proper support. This shows that parametrices with proper support always exist.

Let us now list some key properties of parametrices, all of which are consequences of elliptic regularity.

14.2.4.3 Lemma. *(1) If Φ, Ψ are parametrices, then the section $\Phi - \Psi$ of the bundle $E \boxtimes E$ on $M \times M$ is smooth.*

(2) Any parametrix Φ is smooth away from the diagonal in $M \times M$.

(3) Any parametrix Φ is such that $(Q \otimes 1 + 1 \otimes Q)\Phi$ is smooth on all of $M \times M$. (Note that $Q \otimes 1 + 1 \otimes Q$ is the natural differential on the space $\overline{\mathscr{E}} \widehat{\otimes}_\beta \overline{\mathscr{E}}$).

PROOF. We will let Q denote $Q \otimes 1 + 1 \otimes Q$, and similarly $Q^{GF} = Q^{GF} \otimes 1 + 1 \otimes Q^{GF}$, acting on the space $\overline{\mathscr{E}} \widehat{\otimes}_\beta \overline{\mathscr{E}}$. Note that

$$[Q, Q^{GF}] = [Q, Q^{GF}] \otimes 1 + 1 \otimes [Q, Q^{GF}].$$

(1) Since $[Q, Q^{GF}](\Phi - \Psi)$ is smooth, and the operator $[Q, Q^{GF}]$ is elliptic, this follows from elliptic regularity.

(2) Away from the diagonal, Φ is annihilated by the elliptic operator $[Q, Q^{GF}]$, and so is smooth.

(3) Note that

$$[Q, Q^{GF}] Q\Phi = Q[Q, Q^{GF}]\Phi$$

and that $[Q, Q^{GF}]\Phi - 2K_{Id}$ is smooth, where K_{Id} is the kernel for the identity operator. Since $QK_{Id} = 0$, the statement follows.

□

If Φ, Ψ are parametrices, we say that $\Phi < \Psi$ if the support of Φ is contained in the support of Ψ. In this way, parametrices acquire a partial order.

14.2.5. The propagator for a parametrix. In what follows, we will use the notation $Q, Q^{GF}, [Q, Q^{GF}]$ for the operators $Q \otimes 1 + 1 \otimes Q$, etc.

If Φ is a parametrix, we let

$$P(\Phi) = \tfrac{1}{2} Q^{GF} \Phi \in \overline{\mathscr{E}} \widehat{\otimes}_{\pi} \overline{\mathscr{E}}.$$

This is the propagator associated to Φ. We let

$$K_{\Phi} = K_{\mathrm{Id}} - QP(\Phi)..$$

Note that

$$\begin{aligned} QP(\Phi)) &= \tfrac{1}{2}[Q, Q^{GF}]\Phi - Q\Phi \\ &= K_{id} + \text{ smooth kernels} . \end{aligned}$$

Thus, K_{Φ} is smooth.

An important identity we will often use is that

$$K_{\Phi} - K_{\Psi} = QP(\Psi) - QP(\Phi).$$

To relate to section 14.2.3 and [**Cos11c**], we note that if M is a compact manifold and if

$$\Phi = \int_0^L K_t \mathrm{d}t$$

is the parametrix associated to the heat kernel, then

$$P(\Phi) = P(0, L) = \int_0^L (Q^{GF} \otimes 1) K_t \mathrm{d}t$$

and

$$K_{\Phi} = K_L.$$

14.2.6. Classes of functionals. In the appendix B.8 we define various classes of functions on the space $\mathscr{E}_c$ of compactly-supported fields. Here we give an overview of those classes. Many of the conditions seem somewhat technical at first, but they arise naturally as one attempts both to discuss the support of an observable and to extend the algebraic ideas of the BV formalism in this infinite-dimensional setting.

We are interested, firstly, in functions modulo constants, which we call $\mathscr{O}_{red}(\mathscr{E}_c)$. Every functional $F \in \mathscr{O}_{red}(\mathscr{E}_c)$ has a Taylor expansion in terms of symmetric smooth linear maps

$$F_k : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

(for $k > 0$). Such linear maps are the same as distributional sections of the bundle $(E^!)^{\boxtimes k}$ on M^k. We say that F has *proper support* if the support of each F_k (as defined above) is a proper subset of M^k. The space of functionals with proper support is denoted $\mathscr{O}_P(\mathscr{E}_c)$ (as always in this section, we work with functionals modulo constants). This condition equivalently means that, when we think of F_k as an operator

$$\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!,$$

it extends to a smooth multilinear map

$$F_k : \mathscr{E}^{\times k-1} \to \overline{\mathscr{E}}^!.$$

At various points in this book, we will need to consider *functionals with smooth first derivative*, which are functionals satisfying a certain technical regularity constraint. Functionals with smooth first derivative are needed in two places in the text: when we define the Poisson bracket on classical observables, and when we give the definition of a quantum field theory. In terms of the Taylor components F_k, viewed as multilinear operators $\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!$, this condition means that the F_k has image in $\mathscr{E}^!$. (For more detail, see Appendix B, section B.8.)

We are interested in the functionals with smooth first derivative and with proper support. We denote this space by $\mathscr{O}_{P,sm}(\mathscr{E})$. These are the functionals with the property that the Taylor components F_k, when viewed as operators, give continuous linear maps

$$\mathscr{E}^{\times k-1} \to \mathscr{E}^!.$$

14.2.7. The renormalization group flow. Let Φ and Ψ be parametrices. Then $P(\Phi) - P(\Psi)$ is a smooth kernel with proper support.

Given any element

$$\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E} = C^\infty(M \times M, E \boxtimes E)$$

of cohomological degree 0, we define an operator

$$\partial_\alpha : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

This map is an order 2 differential operator, which, on components, is the map given by contraction with α:

$$\alpha \vee - : \operatorname{Sym}^n \mathscr{E}^\vee \to \operatorname{Sym}^{n-2} \mathscr{E}^\vee .$$

The operator ∂_α is the unique order 2 differential operator that is given by pairing with α on $\operatorname{Sym}^2 \mathscr{E}^\vee$ and that is zero on $\operatorname{Sym}^{\leq 1} \mathscr{E}^\vee$.

We define a map

$$\begin{aligned} W(\alpha, -) : \mathscr{O}^+(\mathscr{E})[[\hbar]] &\to \mathscr{O}^+(\mathscr{E})[[\hbar]] \\ F &\mapsto \hbar \log \left(e^{\hbar \partial_\alpha} e^{F/\hbar} \right), \end{aligned}$$

known as the *renormalization group flow* with respect to α. (When $\alpha = P(\Phi) - P(\Psi)$, we call it the RG flow from Ψ to Φ.) This formula is a succinct way of summarizing a Feynman diagram expansion. In particular, $W(\alpha, F)$ can be written as a sum over Feynman diagrams with the Taylor components F_k of F labelling vertices of valence k, and with α as propagator. (All of this, and indeed everything else in this section, is explained in far greater detail in chapter 2 of [**Cos11c**].) For this map to be well-defined, the functional F must have only cubic and higher terms modulo $\hbar$. The notation $\mathscr{O}^+(\mathscr{E})[[\hbar]]$ denotes this restricted class of functionals.

If $\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$ has proper support, then the operator $W(\alpha, -)$ extends (uniquely, of course) to a continuous (or equivalently, smooth) operator

$$W(\alpha, -) : \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]].$$

Our philosophy is that a parametrix Φ is like a choice of "scale" for our field theory. The renormalization group flow relating the scale given by Φ and that given by Ψ is $W(P(\Phi) - P(\Psi), -)$.

Because $P(\Phi)$ is not a smooth kernel, the operator $W(P(\Phi), -)$ is not well-defined. This is just because the definition of $W(P(\Phi), -)$ involves multiplying distributions. In physics terms, the singularities that appear when one tries to define $W(P(\Phi), -)$ are called ultraviolet divergences.

However, if $I \in \mathscr{O}^+_{P,sm}(\mathscr{E})$, the tree level part

$$W_0(P(\Phi), I) = W((P(\Phi), I) \bmod \hbar$$

is a well-defined element of $\mathscr{O}^+_{P,sm}(\mathscr{E})$. The $\hbar \to 0$ limit of $W(P(\Phi), I)$ is called the tree-level part because, whereas the whole object $W(P(\Phi), I)$ is defined as a sum over graphs, the $\hbar \to 0$ limit $W_0(P(\Phi), I)$ is defined as a sum over trees. It is straightforward to see that $W_0(P(\Phi), I)$ only involves multiplication of distributions with transverse singular support, and so is well defined.

14.2.8. The BD algebra structure associated to a parametrix. A parametrix also leads to a BV operator

$$\triangle_{\Phi} = \partial_{K_{\Phi}} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Again, this operator preserves the subspace $\mathscr{O}_{P,sm}(\mathscr{E})$ of functions with proper support and smooth first derivative. The operator $\triangle_{\Phi}$ commutes with Q, and it satisfies $(\triangle_{\Phi})^2 = 0$. In a standard way, we can use the BV operator $\triangle_{\Phi}$ to define a bracket on the space $\mathscr{O}(\mathscr{E})$, by

$$\{I, J\}_{\Phi} = \triangle_{\Phi}(IJ) - (\triangle_{\Phi} I)J - (-1)^{|I|} I \triangle_{\Phi} J.$$

This bracket is a Poisson bracket of cohomological degree 1. If we give the graded-commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$ the standard product, the Poisson bracket $\{-,-\}_{\Phi}$, and the differential $Q + \hbar\triangle_{\Phi}$, then it becomes a BD algebra.

The bracket $\{-,-\}_{\Phi}$ extends uniquely to a continuous linear map

$$\mathscr{O}_P(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Further, the space $\mathscr{O}_{P,sm}(\mathscr{E})$ is closed under this bracket. (Note, however, that $\mathscr{O}_{P,sm}(\mathscr{E})$ is *not* a commutative algebra if M is not compact: the product of two functionals with proper support no longer has proper support.)

A functional $F \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is said to satisfy the Φ-quantum master equation if

$$QF + \hbar\triangle_{\Phi} F + \tfrac{1}{2}\{F, F\}_{\Phi} = 0.$$

It is shown in [Cos11c] that if F satisfies the Φ-QME, and if Ψ is another parametrix, then $W(P(\Psi) - P(\Phi), F)$ satisfies the Ψ-QME. This follows from the identity

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = \triangle_{\Psi} - \triangle_{\Phi}$$

of order 2 differential operators on $\mathscr{O}(\mathscr{E})$. This relationship between the renormalization group flow and the quantum master equation is a key part of the approach to QFT of [Cos11c].

14.2.9. The definition of a field theory. Our definition of a field theory is as follows.

14.2.9.1 Definition. *Let $(\mathscr{E}, Q, \langle -,- \rangle)$ be a free BV theory. Fix a gauge fixing condition Q^{GF}. Then a* quantum field theory *(with this space of fields) consists of the following data.*

(1) *For all parametrices Φ, a functional*

$$I[\Phi] \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}_c)[[\hbar]]$$

that we call the scale Φ effective interaction. *As we explained above, the subscripts indicate that $I[\Phi]$ must have smooth first derivative and proper support. The superscript $+$ indicates that, modulo $\hbar$, $I[\Phi]$ must be at least cubic. Note that we work with functions modulo constants.*

(2) *For two parametrices* Φ, Ψ, $I[\Phi]$ *must be related by the renormalization group flow:*

$$I[\Phi] = W\left(P(\Phi) - P(\Psi), I[\Psi]\right).$$

(3) *Each* $I[\Phi]$ *must satisfy the* Φ-quantum master equation

$$(Q + \hbar \triangle_\Phi) e^{I[\Phi]/\hbar} = 0.$$

Equivalently,

$$QI[\Phi] + \hbar \triangle_\Phi I[\Phi] + \tfrac{1}{2}\{I[\Phi], I[\Phi]\}_\Phi.$$

(4) *Finally, we require that* $I[\Phi]$ *satisfies a locality axiom. Let*

$$I_{i,k}[\Phi] : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

be the kth Taylor component of the coefficient of $\hbar^i$ *in* $I[\Phi]$. *We can view this as a distributional section of the bundle* $(E^!)^{\boxtimes k}$ *on* M^k. *Our locality axiom says that, as* Φ *tends to zero, the support of*

$$I_{i,k}[\Phi]$$

becomes closer and closer to the small diagonal in M^k.

For the constructions in this book, it turns out to be useful to have precise bounds on the support of $I_{i,k}[\Phi]$. *To give these bounds, we need some notation. Let* $\mathrm{Supp}(\Phi) \subset M^2$ *be the support of the parametrix* Φ, *and let* $\mathrm{Supp}(\Phi)^n \subset M^2$ *be the subset obtained by convolving* $\mathrm{Supp}(\Phi)$ *with itself n times. (Thus,* $(x, y) \in \mathrm{Supp}(\Phi)^n$ *if there exists a sequence* $x = x_0, x_1, \ldots, x_n = y$ *such that* $(x_i, x_{i+1}) \in \mathrm{Supp}(\Phi)$.*)*

Our support condition is that, if $e_j \in \mathscr{E}_c$, *then*

$$I_{i,k}(e_1, \ldots, e_k) = 0$$

unless, for all $1 \leq r < s \leq k$,

$$\mathrm{Supp}(e_r) \times \mathrm{Supp}(e_s) \subset \mathrm{Supp}(\Phi)^{3i+k}.$$

Remark: (1) The locality axiom condition as presented here is a little unappealing. An equivalent axiom is that for all open subsets $U \subset M^k$ containing the small diagonal $M \subset M^k$, there exists a parametrix Φ_U such that

$$\mathrm{Supp}\, I_{i,k}[\Phi] \subset U \text{ for all } \Phi < \Phi_U.$$

In other words, by choosing a small parametrix Φ, we can make the support of $I_{i,k}[\Phi]$ as close as we like to the small diagonal on M^k.

We present the definition with a precise bound on the size of the support of $I_{i,k}[\Phi]$ because this bound will be important later in the construction of the factorization algebra. Note, however, that the precise exponent $3i + k$ which appears in the definition (in $\mathrm{Supp}(\Phi)^{3i+k}$) is not important. What is important is that we have some bound of this form.

(2) It is important to emphasize that the notion of quantum field theory is only defined once we have chosen a gauge fixing operator. Later, we will explain in detail how to understand the dependence on this choice. More precisely, we will construct a simplicial set of QFTs and show how this simplicial set only depends on the homotopy class of gauge fixing operator (in most examples, the space of natural gauge fixing operators is contractible).

◊

Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional (defined modulo constants) that satisfies the classical master equation

$$QI_0 + \tfrac{1}{2}\{I_0, I_0\} = 0.$$

Suppose that I_0 is at least cubic.

Then, as we have seen above, we can define a family of functionals

$$I_0[\Phi] = W_0\left(P(\Phi), I_0\right) \in \mathscr{O}_{P,sm}(\mathscr{E})$$

as the tree-level part of the renormalization group flow operator from scale 0 to the scale given by the parametrix Φ. The compatibility between this classical renormalization group flow and the classical master equation tells us that $I_0[\Phi]$ satisfies the Φ-classical master equation

$$QI_0[\Phi] + \tfrac{1}{2}\{I_0[\Phi], I_0[\Phi]\}_\Phi = 0.$$

14.2.9.2 Definition. *Let $I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$ be the collection of effective interactions defining a quantum field theory. Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional satisfying the classical master equation, and so defining a classical field theory. We say that the quantum field theory $\{I[\Phi]\}$ is a* quantization *of the classical field theory defined by I_0 if*

$$I[\Phi] = I_0[\Phi] \text{ mod } \hbar,$$

or, equivalently, if

$$\lim_{\Phi \to 0} I[\Phi] - I_0 \text{ mod } \hbar = 0.$$

14.3. Families of theories over nilpotent dg manifolds

Before discussing the interpretation of these axioms and also explaining the results of [Cos11c] that allow one to construct such quantum field theories, we will explain how to define families of quantum field theories over some base dg algebra. The fact that we can work in families in this way means that the moduli space of quantum field theories is something like a derived stack. For instance, by considering families over the base dg algebra of forms on the n-simplex, we see that the set of quantizations of a given classical field theory is a simplicial set.

One particularly important use of the families version of the theory is that it allows us to show that our constructions and results are independent, up to homotopy, of the choice of gauge fixing condition (provided one has a contractible — or at least connected — space of gauge fixing conditions, which happens in most examples).

In later sections, we will work implicitly over some base dg ring in the sense described here, although we will normally not mention this base ring explicitly.

14.3.0.3 Definition. *A* nilpotent dg manifold *is a manifold X (possibly with corners), equipped with a sheaf $\mathscr{A}$ of commutative differential graded algebras over the sheaf Ω^*_X, with the following properties.*

1. *$\mathscr{A}$ is concentrated in finitely many degrees.*
2. *Each $\mathscr{A}^i$ is a locally free sheaf of Ω^0_X-modules of finite rank. This means that $\mathscr{A}^i$ is the sheaf of sections of some finite rank vector bundle A^i on X.*
3. *We are given a map of dg Ω^*_X-algebras $\mathscr{A} \to C^\infty_X$.*

   We will let $\mathscr{I} \subset \mathscr{A}$ be the ideal which is the kernel of the map $\mathscr{A} \to C^\infty_X$: we require that $\mathscr{I}$, its powers $\mathscr{I}^k$, and each $\mathscr{A}/\mathscr{I}^k$ are locally free sheaves of C^∞_X-modules. Also, we require that $\mathscr{I}^k = 0$ for k sufficiently large.

Note that the differential d *on $\mathscr{A}$ is necessary a differential operator.*

We will use the notation $\mathscr{A}^\sharp$ to refer to the bundle of graded algebras on X whose smooth sections are $\mathscr{A}^\sharp$, the graded algebra underlying the dg algebra $\mathscr{A}$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are nilpotent dg manifolds, a map $(Y, \mathscr{B}) \to (X, \mathscr{A})$ is a smooth map $f : Y \to X$ together with a map of dg $\Omega^*(X)$-algebras $\mathscr{A} \to \mathscr{B}$.

Here are some basic examples.

1. $\mathscr{A} = C^\infty(X)$ and $\mathscr{I} = 0$. This describes the smooth manifold X.
2. $\mathscr{A} = \Omega^*(X)$ and $\mathscr{I} = \Omega^{>0}(X)$. This equips X with its de Rham complex as a structure sheaf. (Informally, we can say that "constant functions are the only functions on a small open" so that this dg manifold is sensitive to topological rather than smooth structure.)
3. If R is a dg Artinian $\mathbb{C}$-algebra with maximal ideal m, then R can be viewed as giving the structure of nilpotent graded manifold on a point.
4. If again R is a dg Artinian algebra, then for any manifold $(X, R \otimes \Omega^*(X))$ is a nilpotent dg manifold.
5. If X is a complex manifold, then $\mathscr{A} = (\Omega^{0,*}(X), \bar{\partial})$ is a nilpotent dg manifold.

Remark: We study field theories in families over nilpotent dg manifolds for both practical and structural reasons. First, we certainly wish to discuss familes of field theories over smooth manifolds. However, we would also like to access a "derived moduli space" of field theories.

In derived algebraic geometry, one says that a derived stack is a functor from the category of non-positively graded dg rings to that of simplicial sets. Thus, such non-positively graded dg rings are the "test objects" one uses to define derived algebraic geometry. Our use of nilpotent dg manifolds mimics this story: we could say that a C^∞ derived stack is a functor from nilpotent dg manifolds to simplicial sets. The nilpotence hypothesis is not a great restriction, as the test objects used in derived algebraic geometry are naturally pro-nilpotent, where the pro-nilpotent ideal consists of the elements in degrees < 0.

Second, from a practical point of view, our arguments are tractable when working over nilpotent dg manifolds. This is related to the fact that we choose to encode the analytic structure on the vector spaces we consider using the language of differentiable vector spaces. Differentiable vector spaces are, by definition, objects where one can talk about smooth families of maps depending on a smooth manifold. In fact, the definition of differentiable vector space is strong enough that one can talk about smooth families of maps depending on nilpotent dg manifolds. ♢

We can now give a precise notion of "family of field theories." We will start with the case of a family of field theories parameterized by the nilpotent dg manifold $X = (X, C^\infty_X)$, i.e. the sheaf of dg rings on X is just the sheaf of smooth functions.

14.3.0.4 Definition. *Let M be a manifold and let $(X, \mathscr{A})$ be a nilpotent dg manifold. A family over $(X, \mathscr{A})$ of free BV theories is the following data.*

1. *A graded bundle E on $M \times X$ of locally free $A^\sharp$-modules. We will refer to global sections of E as $\mathscr{E}$. The space of those sections $s \in \Gamma(M \times X, E)$ with the property that the map $\operatorname{Supp} s \to X$ is proper will be denoted $\mathscr{E}_c$. Similarly, we let $\overline{\mathscr{E}}$ denote the space of sections which are distributional on M and smooth on X, that is,*

   $$\overline{\mathscr{E}} = \mathscr{E} \otimes_{C^\infty(M\times X)} \left(\mathcal{D}(M)\widehat{\otimes}_\pi C^\infty(X)\right).$$

   (This is just the algebraic tensor product, which is reasonable as $\mathscr{E}$ is a finitely generated projective $C^\infty(M \times X)$-module).

   As above, we let

   $$E^! = \operatorname{Hom}_{A^\sharp}(E, A^\sharp) \otimes \operatorname{Dens}_M$$

   denote the "dual" bundle. There is a natural $\mathscr{A}^\sharp$-valued pairing between $\mathscr{E}$ and $\mathscr{E}^!_c$.
2. *A differential operator $Q : \mathscr{E} \to \mathscr{E}$, of cohomological degree 1 and square-zero, making $\mathscr{E}$ into a dg module over the dg algebra $\mathscr{A}$.*
3. *A map*

   $$E \otimes_{A^\sharp} E \to \operatorname{Dens}_M \otimes A^\sharp$$

which is of degree -1*, anti-symmetric, and leads to an isomorphism*

$$\mathrm{Hom}_{A^\sharp}(E, A^\sharp) \otimes \mathrm{Dens}_M \to E$$

of sheaves of $A^\sharp$*-modules on* $M \times X$*.*

This pairing leads to a degree -1 *anti-symmetric* $\mathscr{A}$*-linear pairing*

$$\langle -, - \rangle : \mathscr{E}_c \widehat{\otimes}_\pi \mathscr{E}_c \to \mathscr{A}.$$

We require it to be a cochain map. In other words, if $e, e' \in \mathscr{E}_c$*,*

$$\mathrm{d}_{\mathscr{A}} \left\langle e, e' \right\rangle = \left\langle Qe, e' \right\rangle + (-1)^{|e|} \left\langle e, Qe' \right\rangle.$$

14.3.0.5 Definition. *Let* $(E, Q, \langle -, - \rangle)$ *be a family of free BV theories on* M *parameterized by* $\mathscr{A}$*. A* gauge fixing condition *on* $\mathscr{E}$ *is an* $\mathscr{A}$*-linear differential operator*

$$Q^{GF} : \mathscr{E} \to \mathscr{E}$$

such that

$$\mathrm{D} = [Q, Q^{GF}] : \mathscr{E} \to \mathscr{E}$$

is a generalized Laplacian, in the following sense.

Note that D *is an* $\mathscr{A}$*-linear cochain map. Thus, we can form*

$$\mathrm{D}_0 : \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X) \to \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X)$$

by reducing modulo the maximal ideal $\mathscr{I}$ *of* $\mathscr{A}$*.*

Let $E_0 = E/I$ *be the bundle on* $M \times X$ *obtained by reducing modulo the ideal* I *in the bundle of algebras* A*. Let*

$$\sigma(\mathrm{D}_0) : \pi^* E_0 \to \pi^* E_0$$

be the symbol of the $C^\infty(X)$*-linear operator* D_0*. Thus,* $\sigma(\mathrm{D}_0)$ *is an endomorphism of the bundle of* $\pi^* E_0$ *on* $(T^*M) \times X$*.*

We require that $\sigma(\mathrm{D}_0)$ *is the product of the identity on* E_0 *with a smooth family of metrics on* M *parameterized by* X*.*

Throughout this section, we will fix a family of free theories on M, parameterized by $\mathscr{A}$. We will take $\mathscr{A}$ to be our base ring throughout, so that everything will be $\mathscr{A}$-linear. We would also like to take tensor products over $\mathscr{A}$. Since $\mathscr{A}$ is a topological dg ring and we are dealing with topological modules, the issue of tensor products is a little fraught. Instead of trying to define such things, we will use the following shorthand notations:

(1) $\mathscr{E} \otimes_{\mathscr{A}} \mathscr{E}$ is defined to be sections of the bundle

$$E \boxtimes_{A^\sharp} E = \pi_1^* E \otimes_{A^\sharp} \pi_2^* E$$

on $M \times M \times X$, with its natural differential which is a differential operator induced from the differentials on each copy of $\mathscr{E}$.

(2) $\overline{\mathscr{E}}$ is the space of sections of the bundle E on $M \times X$ which are smooth in the X-direction and distributional in the M-direction. Similarly for $\overline{\mathscr{E}}_c$, $\overline{\mathscr{E}}^!$, etc.

(3) $\overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$ is defined to be sections of the bundle $E \boxtimes_{A^\sharp} E$ on $M \times M \times X$, which are distributions in the M-directions and smooth as functions of X.

(4) If $x \in X$, let $\mathscr{E}_x$ denote the sections on M of the restriction of the bundle E on $M \times X$ to $M \times x$. Note that $\mathscr{E}_x$ is an $A_x^\sharp$-module. Then, we define $\mathscr{O}(\mathscr{E})$ to be the space of smooth sections of the bundle of topological (or differentiable) vector spaces on X whose fibre at x is

$$\mathscr{O}(\mathscr{E})_x = \prod_n \mathrm{Hom}_{DVS/A_x^\sharp}(\mathscr{E}_x^{\times n}, \mathscr{A}_x^\sharp)_{S_n}.$$

That is an element of $\mathscr{O}(\mathscr{E}_x)$ is something whose Taylor expansion is given by smooth $A_x^\sharp$-multilinear maps to $A_x^\sharp$.

If $F \in \mathscr{O}(\mathscr{E})$ is a smooth section of this bundle, then the Taylor terms of F are sections of the bundle $(E^!)^{\boxtimes_{A^\sharp} n}$ on $M^n \times X$ which are distributional in the M^n-directions, smooth in the X-directions, and whose support maps properly to X.

In other words: when we want to discuss spaces of functionals on $\mathscr{E}$, or tensor powers of $\mathscr{E}$ or its distributional completions, we just to everything we did before fibrewise on X and linear over the bundle of algebras $A^\sharp$. Then, we take sections of this bundle on X.

14.3.1. Now that we have defined free theories over a base ring $\mathscr{A}$, the definition of an interacting theory over $\mathscr{A}$ is very similar to the definition given when $\mathscr{A} = \mathbb{C}$. First, one defines a parametrix to be an element

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$$

with the same properties as before, but where now we take all tensor products (and so on) over $\mathscr{A}$. More precisely,

(1) Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(2) Φ is of cohomological degree 1.
(3) Φ is closed under the differential on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(4) Φ has proper support: this means that the map $\mathrm{Supp}\,\Phi \to M \times X$ is proper.
(5) Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is an element of $\mathscr{E} \otimes \mathscr{E}$ (where, as before, $K_{\mathrm{Id}} \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is the kernel for the identity map).

An interacting field theory is then defined to be a family of $\mathscr{A}$-linear functionals

$$I[\Phi] \in \mathscr{O}_{red}(\mathscr{E})[[\hbar]] = \prod_{n \geq 1} \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}^{\otimes_{\mathscr{A}} n}, \mathscr{A})_{S_n}[[\hbar]]$$

satisfying the renormalization group flow equation, quantum master equation, and locality condition, just as before. In order for the RG flow to make sense, we require that each $I[\Phi]$ has proper support and smooth first derivative. In this context, this means the following. Let $I_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathscr{A}$ be the kth Taylor component of the coefficient of $\hbar^i$ in $I_{i,k}[\Phi]$. Proper support means that any projection map

$$\operatorname{Supp} I_{i,k}[\Phi] \subset M^k \times X \to M \times X$$

is proper. Smooth first derivative means, as usual, that when we think of $I_{i,k}[\Phi]$ as an operator $\mathscr{E}^{\otimes k-1} \to \overline{\mathscr{E}}$, the image lies in $\mathscr{E}$.

If we have a family of theories over $(X, \mathscr{A})$, and a map

$$f : (Y, \mathscr{B}) \to (X, \mathscr{A})$$

of dg manifolds, then we can base change to get a family over $(Y, \mathscr{B})$. The bundle on Y of $B_x^\sharp$-modules of fields is defined, fibre by fibre, by

$$(f^*\mathscr{E})_y = \mathscr{E}_{f(y)} \otimes_{A^\sharp_{f(y)}} B^\sharp_y.$$

The gauge fixing operator

$$Q^{GF} : f^*\mathscr{E} \to f^*\mathscr{E}$$

is the $\mathscr{B}$-linear extension of the gauge fixing condition for the family of theories over $\mathscr{A}$.

If

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}} \subset f^*\overline{\mathscr{E}} \otimes_{\mathscr{B}} f^*\overline{\mathscr{E}}$$

is a parametrix for the family of free theories $\mathscr{E}$ over $\mathscr{A}$, then it defines a parametrix $f^*\Phi$ for the family of free theories $f^*\mathscr{E}$ over $\mathscr{B}$. For parametrices of this form, the effective action functionals

$$f^* I[f^*\Phi] \in \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]] = \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes_{\mathscr{A}} \mathscr{B}$$

is simply the image of the original effective action functional

$$I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \subset \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]].$$

For a general parametrix Ψ for $f^*\mathscr{E}$, the effective action functional is defined by the renormalization group equation

$$f^* I[\Psi] = W\left(P(\Psi) - P(f^*\Phi), f^* I[f^*\Phi]\right).$$

This is well-defined because

$$P(\Psi) - P(f^*\Phi) \in f^*\mathscr{E} \otimes_{\mathscr{B}} f^*\mathscr{E}$$

has no singularities.

The compatibility between the renormalization group equation and the quantum master equation guarantees that the effective action functionals $f^* I[\Psi]$ satisfy the QME for every parametrix Ψ. The locality axiom for the original family of effective action functionals

$I[\Phi]$ guarantees that the pulled-back family $f^* I[\Psi]$ satisfy the locality axiom necessary to define a family of theories over $\mathscr{B}$.

14.4. The simplicial set of theories

One of the main reasons for introducing theories over a nilpotent dg manifold $(X, \mathscr{A})$ is that this allows us to talk about the simplicial set of theories. This is essential, because the main result we will use from [Cos11c] is homotopical in nature: it relates the simplicial set of theories to the simplicial set of local functionals.

We introduce some useful notation. Let us fix a family of classical field theories on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$. As above, the fields of such a theory are a dg $\mathscr{A}$-module $\mathscr{E}$ equipped with an $\mathscr{A}$-linear local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation $QI + \frac{1}{2}\{I, I\} = 0$.

By pulling back along the projection map

$$(X \times \triangle^n, \mathscr{A} \otimes C^\infty(\triangle^n)) \to (X, \mathscr{A}),$$

we get a new family of classical theories over the dg base ring $\mathscr{A} \otimes C^\infty(\triangle^n)$, whose fields are $\mathscr{E} \otimes C^\infty(\triangle^n)$. We can then ask for a gauge fixing operator

$$Q^{GF} : \mathscr{E} \otimes C^\infty(\triangle^n) \to \mathscr{E} \otimes C^\infty(\triangle^n).$$

for this family of theories. This is the same thing as a smooth family of gauge fixing operators for the original theory depending on a point in the n-simplex.

14.4.0.1 Definition. *Let $(\mathscr{E}, I)$ denote the classical theory we start with over $\mathscr{A}$. Let $\mathscr{G}\mathscr{F}(\mathscr{E}, I)$ denote the simplicial set whose n-simplices are such families of gauge fixing operators over $\mathscr{A} \otimes C^\infty(\triangle^n)$. If there is no ambiguity as to what classical theory we are considering, we will denote this simplicial set by $\mathscr{G}\mathscr{F}$.*

Any such gauge fixing operator extends, by $\Omega^*(\triangle^n)$-linearity, to a linear map $\mathscr{E} \otimes \Omega^*(\triangle^n) \to \mathscr{E} \otimes \Omega^*(\triangle^n)$, which thus defines a gauge fixing operator for the family of theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ pulled back via the projection

$$(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A}).$$

(Note that $\Omega^*(\triangle^n)$ is equipped with the de Rham differential.)

Example: Suppose that $\mathscr{A} = \mathbb{C}$, and the classical theory we are considering is Chern-Simons theory on a 3-manifold M, where we perturb around the trivial bundle. Then, the space of fields is $\mathscr{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$ and $Q = \mathrm{d}_{dR}$. For every Riemannian metric on M, we find a gauge fixing operator $Q^{GF} = \mathrm{d}^*$. More generally, if we have a smooth family

$$\{g_\sigma \mid \sigma \in \triangle^n\}$$

of Riemannian metrics on M, depending on the point σ in the n-simplex, we get an n-simplex of the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators.

Thus, if $\operatorname{Met}(M)$ denotes the simplicial set whose n-simplices are the set of Riemannian metrics on the fibers of the submersion $M \times \triangle^n \to \triangle^n$, then we have a map of simplicial sets

$$\operatorname{Met}(M) \to \mathscr{G}\mathscr{F}.$$

Note that the simplicial set $\operatorname{Met}(M)$ is (weakly) contractible (which follows from the familiar fact that, as a topological space, the space of metrics on M is contractible).

A similar remark holds for almost all theories we consider. For example, suppose we have a theory where the space of fields

$$\mathscr{E} = \Omega^{0,*}(M, V)$$

is the Dolbeault complex on some complex manifold M with coefficients in some holomorphic vector bundle V. Suppose that the linear operator $Q : \mathscr{E} \to \mathscr{E}$ is the $\overline{\partial}$-operator. The natural gauge fixing operators are of the form $\overline{\partial}^*$. Thus, we get a gauge fixing operator for each choice of Hermitian metric on M together with a Hermitian metric on the fibers of V. This simplicial set is again contractible.

It is in this sense that we mean that, in most examples, there is a natural contractible space of gauge fixing operators. ♢

14.4.1. We will use the shorthand notation $(\mathscr{E}, I)$ to denote the classical field theory over $\mathscr{A}$ that we start with; and we will use the notation $(\mathscr{E}_{\triangle^n}, I_{\triangle^n})$ to refer to the family of classical field theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ obtained by base-change along the projection $(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A})$.

14.4.1.1 Definition. *We let $\mathscr{T}^{(n)}$ denote the simplicial set whose k-simplices consist of the following data.*

1. *A k-simplex $Q^{GF}_{\triangle^k} \in \mathscr{G}\mathscr{F}[k]$, defining a gauge-fixing operator for the family of theories $(\mathscr{E}_{\triangle^k}, I_{\triangle^k})$ over $\mathscr{A} \otimes \Omega^*(\triangle^k)$.*
2. *A quantization of the family of classical theories with gauge fixing operator $(\mathscr{E}_{\triangle^k}, I_{\triangle^k}, Q^{GF}_{\triangle^k})$, defined modulo $\hbar^{n+1}$.*

We let $\mathscr{T}^{(\infty)}$ denote the corresponding simplicial set where the quantizations are defined to all orders in $\hbar$.

Note that there are natural maps of simplicial sets $\mathscr{T}^{(n)} \to \mathscr{T}^{(m)}$, and that $\mathscr{T}^{(\infty)} = \varprojlim \mathscr{T}^{(n)}$. Further, there are natural maps $\mathscr{T}^{(n)} \to \mathscr{G}\mathscr{F}$.

Note further that $\mathscr{T}^{(0)} = \mathscr{GF}$.

This definition describes the most sophisticated version of the set of theories we will consider. Let us briefly explain how to interpret this simplicial set of theories.

Suppose for simplicity that our base ring $\mathscr{A}$ is just $\mathbb{C}$. Then, a 0-simplex of $\mathscr{T}^{(0)}$ is simply a gauge-fixing operator for our theory. A 0-simplex of $\mathscr{T}^{(n)}$ is a gauge fixing operator, together with a quantization (defined with respect to that gauge-fixing operator) to order n in $\hbar$.

A 1-simplex of $\mathscr{T}^{(0)}$ is a homotopy between two gauge fixing operators. Suppose that we fix a 0-simplex of $\mathscr{T}^{(0)}$, and consider a 1-simplex of $\mathscr{T}^{(\infty)}$ in the fiber over this 0-simplex. Such a 1-simplex is given by a collection of effective action functionals

$$I[\Phi] \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}) \otimes \Omega^{*}([0,1])[[\hbar]]$$

one for each parametrix Φ, which satisfy a version of the QME and the RG flow, as explained above.

We explain in some more detail how one should interpret such a 1-simplex in the space of theories. Let us fix a parametrix Φ on $\mathscr{E}$ and extend it to a parametrix for the family of theories over $\Omega^{*}([0,1])$. We can then expand our effective interaction $I[\Phi]$ as

$$I[\Phi] = J[\Phi](t) + J'[\Phi](t)\mathrm{d}t$$

where $J[\Phi](t), J'[\Phi](t)$ are elements

$$J[\Phi](t), J'[\Phi](t) \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}) \otimes C^{\infty}([0,1])[[\hbar]].$$

Here t is the coordinate on the interval $[0,1]$.

The quantum master equation implies that the following two equations hold, for each value of $t \in [0,1]$,

$$QJ[\Phi](t) + \tfrac{1}{2}\{J[\Phi](t), J[\Phi](t)\}_{\Phi} + \hbar\triangle_{\Phi} J[\Phi](t) = 0,$$

$$\frac{\partial}{\partial t} J[\Phi](t) + QJ'[\Phi](t) + \{J[\Phi](t), J'[\Phi](t)\}_{\Phi} + \hbar\triangle_{\Phi} J'[\Phi](t) = 0.$$

The first equation tells us that for each value of t, $J[\Phi](t)$ is a solution of the quantum master equation. The second equation tells us that the t-derivative of $J[\Phi](t)$ is homotopically trivial as a deformation of the solution to the QME $J[\Phi](t)$.

In general, if I is a solution to some quantum master equation, a transformation of the form

$$I \mapsto I + \varepsilon J = I + \varepsilon Q I' + \{I, I'\} + \hbar\triangle I'$$

is often called a "BV canonical transformation" in the physics literature. In the physics literature, solutions of the QME related by a canonical transformation are regarded as equivalent: the canonical transformation can be viewed as a change of coordinates on the space of fields.

For us, this interpretation is not so important. If we have a family of theories over $\Omega^*([0,1])$, given by a 1-simplex in $\mathscr{T}^{(\infty)}$, then the factorization algebra we will construct from this family of theories will be defined over the dg base ring $\Omega^*([0,1])$. This implies that the factorization algebras obtained by restricting to 0 and 1 are quasi-isomorphic.

14.4.2. Generalizations. We will shortly state the theorem which allows us to construct such quantum field theories. Let us first, however, briefly introduce a slightly more general notion of "theory."

We work over a nilpotent dg manifold $(X, \mathscr{A})$. Recall that part of the data of such a manifold is a differential ideal $I \subset \mathscr{A}$ whose quotient is $C^\infty(X)$. In the above discussion, we assumed that our classical action functional S was at least quadratic; we then split S as

$$S = \langle e, Qe \rangle + I(e)$$

into kinetic and interacting terms.

We can generalize this to the situation where S contains linear terms, as long as they are accompanied by elements of the ideal $\mathscr{I} \subset \mathscr{A}$. In this situation, we also have some freedom in the splitting of S into kinetic and interacting terms; we require only that linear and quadratic terms in the interaction I are weighted by elements of the nilpotent ideal $\mathscr{I}$.

In this more general situation, the classical master equation $\{S, S\} = 0$ does not imply that $Q^2 = 0$, only that $Q^2 = 0$ modulo the ideal $\mathscr{I}$. However, this does not lead to any problems; the definition of quantum theory given above can be easily modified to deal with this more general situation.

In the L_∞ language used in Chapter 10, this more general situation describes a family of curved L_∞ algebras over the base dg ring $\mathscr{A}$ with the property that the curving vanishes modulo the nilpotent ideal $\mathscr{I}$.

Recall that ordinary (not curved) L_∞ algebras correspond to formal pointed moduli problems. These curved L_∞ algebras correspond to families of formal moduli problems over $\mathscr{A}$ which are pointed modulo $\mathscr{I}$.

14.5. The theorem on quantization

Let M be a manifold, and suppose we have a family of classical BV theories on M over a nilpotent dg manifold $(X, \mathscr{A})$. Suppose that the space of fields on M is the $\mathscr{A}$-module $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})$ be the dg $\mathscr{A}$-module of local functionals with differential $Q + \{I, -\}$.

Given a cochain complex C, we denote the Dold-Kan simplicial set associated to C by $\mathrm{DK}(C)$. Its n-simplices are the closed, degree 0 elements of $C \otimes \Omega^*(\triangle^n)$.

14.5.0.1 Theorem. *All of the simplicial sets $\mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan complexes and $\mathscr{T}^{(\infty)}(\mathscr{E}, I)$. The maps $p : \mathscr{T}^{(n+1)}(\mathscr{E}, I) \to \mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan fibrations.*

Further, there is a homotopy fiber diagram of simplicial sets

$$\begin{tikzcd}
\mathscr{T}^{(n+1)}(\mathscr{E}, I) \arrow[r] \arrow[d, "p"'] & 0 \arrow[d] \\
\mathscr{T}^{(n)}(\mathscr{E}, I) \arrow[r, "O"] & \mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E})[1], Q + \{I, -\})
\end{tikzcd}$$

where O is the "obstruction map."

In more prosaic terms, the second part of the theorem says the following. If $\alpha \in \mathscr{T}^{(n)}(\mathscr{E}, I)[0]$ is a zero-simplex of $\mathscr{T}^{(n)}(\mathscr{E}, I)$, then there is an obstruction $O(\alpha) \in \mathscr{O}_{loc}(\mathscr{E})$. This obstruction is a closed degree 1 element. The simplicial set $p^{-1}(\alpha) \in \mathscr{T}^{(n+1)}(\mathscr{E}, I)$ of extensions of α to the next order in $\hbar$ is homotopy equivalent to the simplicial set of ways of making $O(\alpha)$ exact. In particular, if the cohomology class $[O(\alpha)] \in H^1(\mathscr{O}_{loc}(E), Q + \{I, -\})$ is non-zero, then α does not admit a lift to the next order in $\hbar$. If this cohomology class is zero, then the simplicial set of possible lifts is a torsor for the simplicial Abelian group $\mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E}))[1]$.

Note also that a first order deformation of the classical field theory $(\mathscr{E}, Q, I)$ is given by a closed degree 0 element of $\mathscr{O}_{loc}(\mathscr{E})$. Further, two such first order deformations are equivalent if they are cohomologous. Thus, this theorem tells us that the moduli space of QFTs is "the same size" as the moduli space of classical field theories: at each order in $\hbar$, the data needed to describe a QFT is a local action functional.

The first part of the theorem says can be interpreted as follows. A Kan simplicial set can be thought of as an "infinity-groupoid." Since we can consider families of theories over arbitrary nilpotent dg manifolds, we can consider $\mathscr{T}^{\infty}(\mathscr{E}, I)$ as a functor from the category of nilpotent dg manifolds to that of Kan complexes, or infinity-groupoids. Thus, the space of theories forms something like a "derived stack" [Toë06, Lur11].

This theorem also tells us in what sense the notion of "theory" is independent of the choice of gauge fixing operator. The simplicial set $\mathscr{T}^{(0)}(\mathscr{E}, I)$ is the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators. Since the map

$$\mathscr{T}^{(\infty)}(\mathscr{E}, I) \to \mathscr{T}^{(0)}(\mathscr{E}, I) = \mathscr{G}\mathscr{F}$$

is a fibration, a path between two gauge fixing conditions Q_0^{GF} and Q_1^{GF} leads to a homotopy between the corresponding fibers, and thus to an equivalence between the ∞-groupoids of theories defined using Q_0^{GF} and Q_1^{GF}.

As we mentioned several times, there is often a natural contractible simplicial set mapping to the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators. Thus, $\mathscr{G}\mathscr{F}$ often has a canonical "homotopy point". From the homotopical point of view, having a homotopy point is just as good as having an actual point: if $S \to \mathscr{G}\mathscr{F}$ is a map out of a contractible simplicial set, then the fibers in $\mathscr{T}^{(\infty)}$ above any point in S are canonically homotopy equivalent.

CHAPTER 15

The observables of a quantum field theory

15.1. Free fields

Before we give our general construction of the factorization algebra associated to a quantum field theory, we will give the much easier construction of the factorization algebra for a free field theory.

Let us recall the definition of a free BV theory.

15.1.0.2 Definition. *A free BV theory on a manifold M consists of the following data:*

(1) *a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that has finite rank;*

(2) *an antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of degree -1 that is fiberwise nondegenerate. It induces a symplectic pairing on compactly supported smooth sections $\mathscr{E}_c$ of E:*

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(3) *a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

Remark: When we consider deforming free theories into interacting theories, we will need to assume the existence of a "gauge fixing operator": this is a degree -1 operator Q^{GF} : $\mathscr{E} \to E$ such that $[Q, Q^{GF}]$ is a generalized Laplacian in the sense of [BGV92]. ◊

On any open set $U \subset M$, the commutative dg algebra of classical observables supported in U is

$$\mathrm{Obs}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^\vee(U)), Q),$$

where

$$\mathscr{E}^\vee(U) = \overline{\mathscr{E}}^!_c(U)$$

denotes the distributions dual to $\mathscr{E}$ with compact support in U and Q is the derivation given by extending the natural action of Q on the distributions.

In section 12.3 we constructed a sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}_c^!(U)), Q)$$

defined as the symmetric algebra on the compactly-supported smooth (rather than distributional) sections of the bundle $E^!$. We showed that the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$ is a weak equivalence of factorization algebras. Further, $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a Poisson bracket of cohomological degree 1, defined on the generators by the natural pairing

$$\mathscr{E}_c^!(U) \widehat{\otimes}_\pi \mathscr{E}_c^!(U) \to \mathbb{R},$$

which arises from the dual pairing on $\mathscr{E}_c(U)$. In this section we will show how to construct a quantization of the P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$.

15.1.1. The Heisenberg algebra construction. Our quantum observables on an open set U will be built from a certain Heisenberg Lie algebra.

Recall the usual construction of a Heisenberg algebra. If V is a symplectic vector space, viewed as an abelian Lie algebra, then the Heisenberg algebra $\mathrm{Heis}(V)$ is the central extension

$$0 \to \mathbb{C} \cdot \hbar \to \mathrm{Heis}(V) \to V$$

whose bracket is $[x, y] = \hbar \langle x, y \rangle$.

Since the element $\hbar \in \mathrm{Heis}(V)$ is central, the algebra $\widehat{U}(\mathrm{Heis}(V))$ is an algebra over $\mathbb{C}[[\hbar]]$, the completed universal enveloping algebra of the Abelian Lie algebra $\mathbb{C} \cdot \hbar$.

In quantum mechanics, this Heisenberg construction typically appears in the study of systems with quadratic Hamiltonians. In this context, the space V can be viewed in two ways. Either it is the space of solutions to the equations of motion, which is a linear space because we are dealing with a free field theory; or it is the space of linear observables dual to the space of solutions to the equations of motion. The natural symplectic pairing on V gives an isomorphism between these descriptions. The algebra $\widehat{U}(\mathrm{Heis}(V))$ is then the algebra of non-linear observables.

Our construction of the quantum observables of a free field theory will be formally very similar. We will start with a space of linear observables, which (after a shift) is a cochain complex with a symplectic pairing of cohomological degree 1. Then, instead of applying the usual universal enveloping algebra construction, we will take Chevalley-Eilenberg chain complex, whose cohomology is the Lie algebra homology.[1] This fits with our operadic philosophy: Chevalley-Eilenberg chains are the E_0 analog of the universal enveloping algebra.

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[1] As usual, we always use gradings such that the differential has degree +1.

15.1.2. The basic homological construction. Let us start with a 0-dimensional free field theory. Thus, let V be a cochain complex equipped with a symplectic pairing of cohomological degree -1. We will think of V as the space of fields of our theory. The space of linear observables of our theory is $V^\vee$; the Poisson bracket on $\mathscr{O}(V)$ induces a symmetric pairing of degree 1 on $V^\vee$. We will construct the space of all observables from a Heisenberg Lie algebra built on $V^\vee[-1]$, which has a symplectic pairing $\langle -,- \rangle$ of degree -1. Note that there is an isomorphism $V \cong V^\vee[-1]$ compatible with the pairings on both sides.

15.1.2.1 Definition. *The Heisenberg algebra* $\mathrm{Heis}(V)$ *is the Lie algebra central extension*

$$0 \to \mathbb{C} \cdot \hbar[-1] \to \mathrm{Heis}(V) \to V^\vee[-1] \to 0$$

whose bracket is

$$[v + \hbar a, w + \hbar b] = \hbar \langle v, w \rangle$$

The element $\hbar$ *labels the basis element of the center* $\mathbb{C}[-1]$.

Putting the center in degree 1 may look strange, but it is necessary to do this in order to get a Lie bracket of cohomological degree 0.

Let $\widehat{C}_*(\mathrm{Heis}(V))$ denote the completion[2] of the Lie algebra chain complex of $\mathrm{Heis}(V)$, defined by the product of the spaces $\mathrm{Sym}^n \mathrm{Heis}(V)$, instead of their sum.

In this zero-dimensional toy model, the classical observables are

$$\mathrm{Obs}^{cl} = \mathscr{O}(V) = \prod_n \mathrm{Sym}^n(V^\vee).$$

This is a commutative dg algebra equipped with the Poisson bracket of degree 1 arising from the pairing on V. Thus, $\mathscr{O}(V)$ is a P_0 algebra.

15.1.2.2 Lemma. *The completed Chevalley-Eilenberg chain complex* $\widehat{C}_*(\mathrm{Heis}(V))$ *is a BD algebra (section 8.4) which is a quantization of the* P_0 *algebra* $\mathscr{O}(V)$.

Proof. The completed Chevalley-Eilenberg complex for $\mathrm{Heis}(V)$ has the completed symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$ as its underlying graded vector space. Note that

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1]) = \mathrm{Sym}(V^\vee \oplus \mathbb{C} \cdot \hbar) = \widehat{\mathrm{Sym}}(V^\vee)[[\hbar]],$$

so that $\widehat{C}_*(\mathrm{Heis}(V))$ is a flat $\mathbb{C}[[\hbar]]$ module which reduces to $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$. The Chevalley-Eilenberg chain complex $\widehat{C}_*(\mathrm{Heis}(V))$ inherits a product, corresponding to the natural product on the symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$. Further, it has a natural Poisson bracket of cohomological degree 1 arising from the Lie bracket on $\mathrm{Heis}(V)$, extended

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[2] One doesn't need to take the completed Lie algebra chain complex. We do this to be consistent with our discussion of the observables of interacting field theories, where it is essential to complete.

to be a derivation of $\widehat{C}_*(\mathrm{Heis}(V))$. Note that, since $\mathbb{C}\cdot\hbar[-1]$ is central in $\mathrm{Heis}(V)$, this Poisson bracket reduces to the given Poisson bracket on $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$.

In order to prove that we have a BD quantization, it remains to verify that, although the commutative product on $\widehat{C}_*(\mathrm{Heis}(V))$ is not compatible with the product, it satisfies the BD axiom:

$$\mathrm{d}(a\cdot b) = (\mathrm{d}a)\cdot b + (-1)^{|a|}a\cdot(\mathrm{d}b) + \hbar\{a,b\}.$$

This follows by definition. ☐

15.1.3. Cosheaves of Heisenberg algebras. Next, let us give the analog of this construction for a general free BV theory E on a manifold M. As above, our classical observables are defined by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = \widehat{\mathrm{Sym}}\,\mathscr{E}_c^!(U)$$

which has a Poisson bracket arising from the pairing on $\mathscr{E}_c^!(U)$. Recall that this is a factorization algebra.

To construct the quantum theory, we define, as above, a Heisenberg algebra $\mathrm{Heis}(U)$ as a central extension

$$0 \to \mathbb{C}[-1]\cdot\hbar \to \mathrm{Heis}(U) \to \mathscr{E}_c^!(U)[-1] \to 0.$$

Note that $\mathrm{Heis}(U)$ is a pre-cosheaf of Lie algebras. The bracket in this Heisenberg algebra arises from the pairing on $\mathscr{E}_c^!(U)$.

We then define the quantum observables by

$$\mathrm{Obs}^q(U) = \widehat{C}_*(\mathrm{Heis}(U)).$$

The underlying cochain complex is, as before,

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1])$$

where the completed symmetric algebra is defined (as always) using the completed tensor product.

15.1.3.1 Proposition. *Sending U to $\mathrm{Obs}^q(U)$ defines a BD factorization algebra in the category of differentiable pro-cochain complexes over $\mathbb{R}[[\hbar]]$, which quantizes $\mathrm{Obs}^{cl}(U)$.*

PROOF. First, we need to define the filtration on $\mathrm{Obs}^q(U)$ making it into a differentiable pro-cochain complex. The filtration is defined, in the identification

$$\mathrm{Obs}^q(U) = \widehat{\mathrm{Sym}}\,\mathscr{E}_c^!(U)[[\hbar]]$$

by saying

$$F^n\,\mathrm{Obs}^q(U) = \prod_k \hbar^k\,\mathrm{Sym}^{\geq n-2k}\,\mathscr{E}_c^!(U).$$

This filtration is engineered so that the $F^n\,\mathrm{Obs}^q(U)$ is a subcomplex of $\mathrm{Obs}^q(U)$.

It is immediate that Obs^q is a BD pre-factorization algebra quantizing $\mathrm{Obs}^{cl}(U)$. The fact that it is a factorization algebra follows from the fact that $\mathrm{Obs}^{cl}(U)$ is a factorization algebra, and then a simple spectral sequence argument. (A more sophisticated version of this spectral sequence argument, for interacting theories, is given in section 15.6.) □

15.2. The BD algebra of global observables

In this section, we will try to motivate our definition of a quantum field theory from the point of view of homological algebra. All of the constructions we will explain will work over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, but to keep the notation simple we will not normally mention the base ring $\mathscr{A}$.

Thus, suppose that $(\mathscr{E}, I, Q, \langle -,-\rangle)$ is a classical field theory on a manifold M. We have seen (Chapter 12, section 12.2) how such a classical field theory gives immediately a commutative factorization algebra whose value on an open subset is

$$\mathrm{Obs}^{cl}(U) = (\mathscr{O}(\mathscr{E}(U)), Q + \{I, -\}).$$

Further, we saw that there is a P_0 sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\mathscr{O}_{sm}(\mathscr{E}(U)), Q + \{I, -\}).$$

In particular, we have a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ of global sections of this P_0 algebra. We can think of $\widetilde{\mathrm{Obs}}^{cl}(M)$ as the algebra of functions on the derived space of solutions to the Euler-Lagrange equations.

In this section we will explain how a quantization of this classical field theory will give a quantization (in a homotopical sense) of the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ into a BD algebra $\mathrm{Obs}^q(M)$ of global observables. This BD algebra has some locality properties, which we will exploit later to show that $\mathrm{Obs}^q(M)$ is indeed the global sections of a factorization algebra of quantum observables.

In the case when the classical theory is the cotangent theory to some formal elliptic moduli problem $B\mathcal{L}$ on M (encoded in an elliptic L_∞ algebra $\mathcal{L}$ on M), there is a particularly nice class of quantizations, which we call cotangent quantizations. Cotangent quantizations have a very clear geometric interpretation: they are locally-defined volume forms on the sheaf of formal moduli problems defined by $\mathcal{L}$.

15.2.1. The BD algebra associated to a parametrix.

Suppose we have a quantization of our classical field theory (defined with respect to some gauge fixing condition, or family of gauge fixing conditions). Then, for every parametrix Φ, we have seen how to construct a cohomological degree 1 operator

$$\triangle_\Phi : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

and a Poisson bracket

$$\{-,-\}_\Phi : \mathscr{O}(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

such that $\mathscr{O}(\mathscr{E})[[\hbar]]$, with the usual product, with bracket $\{-,-\}_\Phi$ and with differential $Q + \hbar\triangle_\Phi$, forms a BD algebra.

Further, since the effective interaction $I[\Phi]$ satisfies the quantum master equation, we can form a new BD algebra by adding $\{I[\Phi],-\}_\Phi$ to the differential of $\mathscr{O}(\mathscr{E})[[\hbar]]$.

15.2.1.1 Definition. *Let* $\mathrm{Obs}^q_\Phi(M)$ *denote the BD algebra*

$$\mathrm{Obs}^q_\Phi(M) = (\mathscr{O}(\mathscr{E})[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi),$$

with bracket $\{-,-\}_\Phi$ *and the usual product.*

Remark: Note that $I[\Phi]$ is not in $\mathscr{O}(\mathscr{E})[[\hbar]]$, but rather in $\mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$. However, as we remarked earlier in 14.2.8, the bracket

$$\{I[\Phi],-\}_\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

is well-defined. ◊

Remark: Note that we consider $\mathrm{Obs}^q_\Phi(M)$ as a BD algebra valued in the multicategory of differentiable pro-cochain complexes (see Appendix B). This structure includes a filtration on $\mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]]$. The filtration is defined by saying that

$$F^n \mathscr{O}(\mathscr{E})[[\hbar]] = \prod_i \hbar^i \,\mathrm{Sym}^{\geq (n-2i)}(\mathscr{E}^\vee);$$

it is easily seen that the differential $Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi$ preserves this filtration. ◊

We will show that for varying Φ, the BD algebras $\mathrm{Obs}^q_\Phi(M)$ are canonically weakly equivalent. Moreover, we will show that there is a canonical weak equivalence of P_0 algebras

$$\mathrm{Obs}^q_\Phi(M) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \simeq \widetilde{\mathrm{Obs}}^{cl}(M).$$

To show this, we will construct a family of BD algebras over the dg base ring of forms on a certain contractible simplicial set of parametrices that restricts to $\mathrm{Obs}^q_\Phi(M)$ at each vertex.

Before we get into the details of the construction, however, let us say something about how this result allows us to interpret the definition of a quantum field theory.

A quantum field theory gives a BD algebra for each parametrix. These BD algebras are all canonically equivalent. Thus, at first glance, one might think that the data of a QFT is entirely encoded in the BD algebra for a single parametrix. However, this does not take account of a key part of our definition of a field theory, that of *locality*.

The BD algebra associated to a parametrix Φ has underlying commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$, equipped with a differential which we temporarily denote

$$\mathrm{d}_\Phi = Q + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi.$$

If $K \subset M$ is a closed subset, we have a restriction map

$$\mathscr{E} = \mathscr{E}(M) \to \mathscr{E}(K),$$

where $\mathscr{E}(K)$ denotes germs of smooth sections of the bundle E on K. There is a dual map on functionals $\mathscr{O}(\mathscr{E}(K)) \to \mathscr{O}(\mathscr{E})$. We say a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is *supported on K* if it is in the image of this map.

As $\Phi \to 0$, the effective interaction $I[\Phi]$ and the BV Laplacian $\triangle_\Phi$ become more and more local (i.e., their support gets closer to the small diagonal). This tells us that, for very small Φ, the operator d_Φ only increases the support of a functional in $\mathscr{O}(\mathscr{E})[[\hbar]]$ by a small amount. Further, by choosing Φ to be small enough, we can increase the support by an arbitrarily small amount.

Thus, a quantum field theory is

(1) A family of BD algebra structures on $\mathscr{O}(\mathscr{E})[[\hbar]]$, one for each parametrix, which are all homotopic (and which all have the same underlying graded commutative algebra).
(2) The differential d_Φ defining the BD structure for a parametrix Φ increases support by a small amount if Φ is small.

This property of d_Φ for small Φ is what will allow us to construct a factorization algebra of quantum observables. If d_Φ did not increase the support of a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ at all, the factorization algebra would be easy to define: we would just set $\mathrm{Obs}^q(U) = \mathscr{O}(\mathscr{E}(U))[[\hbar]]$, with differential d_Φ. However, because d_Φ does increase support by some amount (which we can take to be arbitrarily small), it takes a little work to push this idea through.

Remark: The precise meaning of the statement that d_Φ increases support by an arbitrarily small amount is a little delicate. Let us explain what we mean. A functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ has an infinite Taylor expansion of the form $f = \sum \hbar^i f_{i,k}$, where $f_{i,k} : \mathscr{E}^{\widehat{\otimes}_\pi k} \to \mathbb{C}$ is a symmetric linear map. We let $\mathrm{Supp}_{\leq (i,k)} f$ be the unions of the supports of $f_{r,s}$ where $(r,s) \leq (i,k)$ in the lexicographical ordering. If $K \subset M$ is a subset, let $\Phi^n(K)$ denote the subset obtained by convolving n times with $\mathrm{Supp}\,\Phi \subset M^2$. The differential d_Φ has the following property: there are constants $c_{i,k} \in \mathbb{Z}_{>0}$ of a purely combinatorial nature (independent of the theory we are considering) such that, for all $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$,

$$\mathrm{Supp}_{\leq (i,k)}\, \mathrm{d}_\Phi f \subset \Phi^{c_{i,k}}(\mathrm{Supp}_{\leq (i,k)}\, f).$$

Thus, we could say that d_Φ increase support by an amount linear in $\mathrm{Supp}\,\Phi$. We will use this concept in the main theorem of this chapter. ◊

15.2.2. Let us now turn to the construction of the equivalences between $\mathrm{Obs}^q_\Phi(M)$ for varying parametrices Φ. The first step is to construct the simplicial set $\mathscr{P}$ of parametrices; we will then construct a BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$ over the base dg ring $\Omega^*(\mathscr{P})$, which we define below.

Let

$$V \subset C^\infty(M \times M, E \boxtimes E) = \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$$

denote the subspace of those elements which are cohomologically closed and of degree 1, symmetric, and have proper support.

Note that the set of parametrices has the structure of an affine space for V: if Φ, Ψ are parametrices, then

$$\Phi - \Psi \in V$$

and, conversely, if Φ is a parametrix and $A \in V$, then $\Phi + A$ is a new parametrix.

Let $\mathscr{P}$ denote the simplicial set whose n-simplices are affine-linear maps from $\triangle^n$ to the affine space of parametrices. It is clear that $\mathscr{P}$ is contractible.

For any vector space V, let $V_\triangle$ denote the simplicial set whose k-simplices are affine linear maps $\triangle^k \to V$. For any convex subset $U \subset V$, there is a sub-simplicial set $U_\triangle \subset V_\triangle$ whose k-simplices are affine linear maps $\triangle^k \to U$. Note that $\mathscr{P}$ is a sub-simplicial set of $\overline{\mathscr{E}}_\triangle^{\widehat{\otimes}_\pi 2}$, corresponding to the convex subset of parametrices inside $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$.

Let $\mathscr{C}\mathscr{P}[0] \subset \overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ denote the cone on the affine subspace of parametrices, with vertex the origin $\overline{0}$. An element of $\mathscr{C}\mathscr{P}[0]$ is an element of $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ of the form $t\Phi$, where Φ is a parametrix and $t \in [0,1]$. Let $\mathscr{C}\mathscr{P}$ denote the simplicial set whose k-simplices are affine linear maps to $\mathscr{C}\mathscr{P}[0]$.

Recall that the simplicial de Rham algebra $\Omega^*_\triangle(S)$ of a simplicial set S is defined as follows. Any element $\omega \in \Omega^i_\triangle(S)$ consists of an i-form

$$\omega(\phi) \in \Omega^i(\triangle^k)$$

for each k-simplex $\phi : \triangle^k \to S$. If $f : \triangle^k \to \triangle^l$ is a face or degeneracy map, then we require that

$$f^*\omega(\phi) = \omega(\phi \circ f).$$

The main results of this section are as follows.

15.2.2.1 Theorem. *There is a BD algebra* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *over* $\Omega^*(\mathscr{P})$ *which, at each* 0*-simplex* Φ*, is the BD algebra* $\mathrm{Obs}^q_\Phi(M)$ *discussed above.*

The underlying graded commutative algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *is* $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\mathscr{P})[[\hbar]]$*.*

For every open subset $U \subset M \times M$, let $\mathscr{P}_U$ denote the parametrices whose support is in U. Let $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ denote the restriction of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ to U. The differential on $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ increases support by an amount linear in U (in the sense explained precisely in the remark above).

The bracket $\{-,-\}_{\mathscr{P}_U}$ on $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ is also approximately local, in the following sense. If $O_1, O_2 \in \mathrm{Obs}^q_{\mathscr{P}_U}(M)$ have the property that

$$\mathrm{Supp}\, O_1 \times \mathrm{Supp}\, O_2 \cap U = \emptyset \in M \times M,$$

then $\{O_1, O_2\}_{\mathscr{P}_U} = 0$.

Further, there is a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ over $\Omega^*(\mathscr{C}\mathscr{P})$ equipped with a quasi-isomorphism of P_0 algebras over $\Omega^*(\mathscr{P})$,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\mathscr{P}} \simeq \mathrm{Obs}^q_{\mathscr{P}}(M) \text{ modulo } \hbar,$$

and with an isomorphism of P_0 algebras,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\bar{0}} \cong \widetilde{\mathrm{Obs}}^{cl}(M),$$

where $\widetilde{\mathrm{Obs}}^{cl}(M)$ is the P_0 algebra constructed in Chapter 12.

The underlying commutative algebra of $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is $\widetilde{\mathrm{Obs}}^{cl}(M) \otimes \Omega^*(\mathscr{C}\mathscr{P})$, the differential on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ increases support by an arbitrarily small amount, and the Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is approximately local in the same sense as above.

PROOF. We need to construct, for each k-simplex $\phi : \triangle^k \to \mathscr{P}$, a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$. We view the k-simplex as a subset of $\mathbb{R}^{k+1}$ by

$$\triangle^k := \left\{ (\lambda_0, \dots, \lambda_k) \subset [0,1]^{k+1} \;:\; \sum_i \lambda_i = 1 \right\}.$$

Since simplices in $\mathscr{P}$ are affine linear maps to the space of parametrices, the simplex ϕ is determined by $k+1$ parametrices $\Phi_0, \dots, \Phi_k$, with

$$\phi(\lambda_0, \dots, \lambda_k) = \sum_i \lambda_i \Phi_i$$

for $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$.

The graded vector space underlying our BD algebra is

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k).$$

The structure as a BD algebra will be encoded by an order two, $\Omega^*(\triangle^k)$-linear differential operator

$$\triangle_\phi : \mathrm{Obs}^q_\phi(M) \to \mathrm{Obs}^q_\phi(M).$$

We need to recall some notation in order to define this operator. Each parametrix Φ provides an order two differential operator $\triangle_\Phi$ on $\mathscr{O}(\mathscr{E})$, the BV Laplacian corresponding to Φ. Further, if Φ, Ψ are two parametrices, then the difference between the propagators $P(\Phi) - P(\Psi)$ is an element of $\mathscr{E} \otimes \mathscr{E}$, so that contracting with $P(\Phi) - P(\Psi)$ defines an order two differential operator $\partial_{P(\Phi)} - \partial_{P(\Psi)}$ on $\mathscr{O}(\mathscr{E})$. (This operator defines the infinitesimal version of the renormalization group flow from Ψ to Φ.) We have the equation

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = -\triangle_\Phi + \triangle_\Psi.$$

Note that although the operator $\partial_{P(\Phi)}$ is only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$, because $P(\Phi) \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$, the difference $\partial_{P(\Phi)}$ and $\partial_{P(\Psi)}$ is nonetheless well-defined on $\mathscr{O}(\mathscr{E})$ because $P(\Phi) - P(\Psi) \in \mathscr{E} \otimes \mathscr{E}$.

The BV Laplacian $\triangle_\phi$ associated to the k-simplex $\phi : \triangle^k \to \mathscr{P}$ is defined by the formula

$$\triangle_\phi = \sum_{i=0}^{k} \lambda_i \triangle_{\Phi_i} - \sum_{i=0}^{k} \mathrm{d}\lambda_i \partial_{P(\Phi_i)},$$

where the $\lambda_i \in [0,1]$ are the coordinates on the simplex $\triangle^k$ and, as above, the Φ_i are the parametrices associated to the vertices of the simplex ϕ.

It is not entirely obvious that this operator makes sense as a linear map $\mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$, because the operators $\partial_{P(\Phi)}$ are only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$. However, since $\sum \mathrm{d}\lambda_i = 0$, we have

$$\sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} = \sum \mathrm{d}\lambda_i (\partial_{P(\Phi_i)} - \partial_{P(\Phi_0)}),$$

and the right hand side is well defined.

It is immediate that $\triangle_\phi^2 = 0$. If we denote the differential on the classical observables $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^n)$ by $Q + \mathrm{d}_{dR}$, we have

$$[Q + \mathrm{d}_{dR}, \triangle_\phi] = 0.$$

To see this, note that

$$\begin{aligned}
[Q + \mathrm{d}_{dR}, \triangle_\phi] &= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} + \sum \mathrm{d}\lambda_i [Q, \partial_{\Phi_i} - \partial_{\Phi_0}] \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} - \sum \mathrm{d}\lambda_i (\triangle_{\Phi_i} - \triangle_{\Phi_0}) \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_0} \\
&= 0,
\end{aligned}$$

where we use various identities from earlier.

The operator $\triangle_\phi$ defines, in the usual way, an $\Omega^*(\triangle^k)$-linear Poisson bracket $\{-,-\}_\phi$ on $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$.

We have effective action functionals $I[\Psi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Ψ. Let

$$I[\phi] = I[\sum \lambda_i \Phi_i] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes C^\infty(\triangle^k).$$

The renormalization group equation tells us that $I[\sum \lambda_i \Phi_i]$ is smooth (actually polynomial) in the λ_i.

We define the structure of BD algebra on the graded vector space

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k)$$

as follows. The product is the usual one; the bracket is $\{-,-\}_\phi$, as above; and the differential is

$$Q + \mathrm{d}_{dR} + \hbar \triangle_\phi + \{I[\phi], -\}_\phi.$$

We need to check that this differential squares to zero. This is equivalent to the quantum master equation

$$(Q + \mathrm{d}_{dR} + \hbar \triangle_\phi) e^{I[\phi]/\hbar} = 0.$$

This holds as a consequence of the quantum master equation and renormalization group equation satisfied by $I[\phi]$. Indeed, the renormalization group equation tells us that

$$e^{I[\phi]/\hbar} = \exp\left(\hbar \sum \lambda_i \left(\partial_{P\Phi_i)} - \partial_{P(\Phi_0)}\right)\right) e^{I[\Phi_0]/\hbar}.$$

Thus,

$$\mathrm{d}_{dR} e^{I[\phi]/\hbar} = \hbar \sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} e^{I[\phi]/\hbar}$$

The QME for each $I[\sum \lambda_i \Phi_i]$ tells us that

$$(Q + \hbar \sum \lambda_i \triangle_{\Phi_i}) e^{I[\phi]/\hbar} = 0.$$

Putting these equations together with the definition of $\triangle_\phi$ shows that $I[\phi]$ satisfies the QME.

Thus, we have constructed a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$ for every simplex $\phi : \triangle^k \to \mathscr{P}$. It is evident that these BD algebras are compatible with face and degeneracy maps, and so glue together to define a BD algebra over the simplicial de Rham complex $\Omega^*_\triangle(\mathscr{P})$ of $\mathscr{P}$.

Let ϕ be a k-simplex of $\mathscr{P}$, and let

$$\mathrm{Supp}(\phi) = \cup_{\lambda \in \triangle^k} \mathrm{Supp}(\sum \lambda_i \Phi_i).$$

We need to check that the bracket $\{O_1, O_2\}_\phi$ vanishes for observables O_1, O_2 such that $(\mathrm{Supp}\, O_1 \times \mathrm{Supp}) O_2 \cap \mathrm{Supp}\, \phi = \emptyset$. This is immediate, because the bracket is defined by contracting with tensors in $\mathscr{E} \otimes \mathscr{E}$ whose supports sit inside $\mathrm{Supp}\, \phi$.

Next, we need to verify that, on a k-simplex ϕ of $\mathscr{P}$, the differential $Q + \{I[\phi], -\}_\phi$ increases support by an amount linear in $\mathrm{Supp}(\phi)$. This follows from the support properties satisfied by $I[\Phi]$ (which are detailed in the definition of a quantum field theory, definition 14.2.9.1).

It remains to construct the P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$. The construction is almost identical, so we will not give all details. A zero-simplex of $\mathscr{C}\mathscr{P}$ is an element of $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ of the form $\Psi = t\Phi$, where Φ is a parametrix. We can use the same formulae we used for parametrices to construct a propagator $P(\Psi)$ and Poisson bracket $\{-,-\}_\Psi$ for each $\Psi \in \mathscr{C}\mathscr{P}$. The kernel defining the Poisson bracket $\{-,-\}_\Psi$ need not be smooth. This means that the bracket $\{-,-\}_\Psi$ is only defined on the subspace $\mathscr{O}_{sm}(\mathscr{E})$ of functionals with smooth first derivative. In particular, if $\Psi = 0$ is the vertex of the cone $\mathscr{C}\mathscr{P}$, then $\{-,-\}_0$ is the Poisson bracket defined in Chapter 12 on $\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E})$.

For each $\Psi \in \mathscr{C}\mathscr{P}$, we can form a tree-level effective interaction

$$I_0[\Psi] = W_0\left(P(\Psi), I\right) \in \mathscr{O}_{sm,P}(\mathscr{E}),$$

where $I \in \mathscr{O}_{loc}(\mathscr{E})$ is the classical action functional we start with. There are no difficulties defining this expression because we are working at tree-level and using functionals with smooth first derivative. If $\Psi = 0$, then $I_0[0] = I$.

The P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$ is defined in almost exactly the same way as we defined the BD algebra over $\Omega^*_{\mathscr{P}}$. The underlying commutative algebra is $\mathscr{O}_{sm}(\mathscr{E}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. On a k-simplex ψ with vertices $\Psi_0, \dots, \Psi_k$, the Poisson bracket is

$$\{-,-\}_\psi = \sum \lambda_i \{-,-\}_{\Psi_i} + \sum \mathrm{d}\lambda_i \{-,-\}_{P(\Psi_i)},$$

where $\{-,-\}_{P(\Psi_i)}$ is the Poisson bracket of cohomological degree 0 defined using the propagator $P(\Psi_i) \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}$ as a kernel. If we let $I_0[\psi] = I_0[\sum \lambda_i \Psi_i]$, then the differential is

$$\mathrm{d}_\psi = Q + \{I_0[\psi], -\}_\psi.$$

The renormalization group equation and classical master equation satisfied by the $I_0[\Psi]$ imply that $\mathrm{d}_\psi^2 = 0$. If $\Psi = 0$, this P_0 algebra is clearly the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ constructed in Chapter 12. When restricted to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$, this P_0 algebra is the sub P_0 algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$ obtained by restricting to functionals with smooth first derivative; the inclusion

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M) \mid_{\mathscr{P}} \hookrightarrow \mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$$

is thus a quasi-isomorphism, using proposition 12.4.2.4 of Chapter 12. ☐

15.3. Global observables

In the next few sections, we will prove the first version (section 1.7) of our quantization theorem. Our proof is by construction, associating a factorization algebra on M to a quantum field theory on M, in the sense of [Cos11c]. This is a quantization (in the weak sense) of the P_0 factorization algebra associated to the corresponding classical field theory.

More precisely, we will show the following.

15.3.0.2 Theorem. *For any quantum field theory on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$, there is a factorization algebra Obs^q on M, valued in the multicategory of differentiable pro-cochain complexes flat over $\mathscr{A}[[\hbar]]$.*

There is an isomorphism of factorization algebras

$$\mathrm{Obs}^q \otimes_{\mathscr{A}[[\hbar]]} \mathscr{A} \cong \mathrm{Obs}^{cl}$$

between Obs^q modulo $\hbar$ and the commutative factorization algebra Obs^{cl}.

Further, Obs^q is a weak quantization (in the sense of Chapter 1, section 1.7) of the P_0 factorization algebra Obs^{cl} of classical observables.

15.3.1. So far we have constructed a BD algebra $\mathrm{Obs}^q_\Phi(M)$ for each parametrix Φ; these BD algebras are all weakly equivalent to each other. In this section we will define a cochain complex $\mathrm{Obs}^q(M)$ of global observables which is independent of the choice of parametrix. For every open subset $U \subset M$, we will construct a subcomplex $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U. The complexes $\mathrm{Obs}^q(U)$ will form our factorization algebra.

Thus, suppose we have a quantum field theory on M, with space of fields $\mathscr{E}$ and effective action functionals $\{I[\Phi]\}$, one for each parametrix (as explained in section 14.2).

An *observable* for a quantum field theory (that is, an element of the cochain complex $\mathrm{Obs}^q(M)$) is simply a first-order deformation $\{I[\Phi] + \delta O[\Phi]\}$ of the family of effective action functionals $I[\Phi]$, which satisfies a renormalization group equation but does not necessarily satisfy the locality axiom in the definition of a quantum field theory. Definition 15.3.1.3 makes this idea precise.

Remark: This definition is motivated by a formal argument with the path integral. Let $S(\phi)$ be the action functional for a field ϕ, and let $O(\phi)$ be another function of the field, describing a measurement that one could make. Heuristically, the expectation value of the observable is

$$\langle O \rangle = \frac{1}{Z_S} \int O(\phi) e^{-S(\phi)/\hbar} \mathscr{D}\phi,$$

where Z_S denotes the partition function, simply the integral without O. A formal manipulation shows that

$$\langle O \rangle = \frac{d}{d\delta} \frac{1}{Z_S} \int e^{(-S(\phi)+\hbar\delta O(\phi))/\hbar} \mathscr{D}\phi.$$

In other words, we can view O as a first-order deformation of the action functional S and compute the expectation value as the change in the partition function. Because the book [Cos11c] gives an approach to the path integral that incorporates the BV formalism, we can define and compute expectation values of observables by exploiting the second description of $\langle O \rangle$ given above. ◊

Earlier we defined cochain complexes $\mathrm{Obs}^q_\Phi(M)$ for each parametrix. The underlying graded vector space of $\mathrm{Obs}^q_\Phi(M)$ is $\mathscr{O}(\mathscr{E})[[\hbar]]$; the differential on $\mathrm{Obs}^q_\Phi(M)$ is

$$\widehat{Q}_\Phi = Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi.$$

15.3.1.1 Definition. *Define a linear map*

$$W^\Phi_\Psi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

by requiring that, for an element $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ of cohomological degree i,

$$I[\Phi] + \delta W^\Phi_\Psi(f) = W\left(P(\Phi) - P(\Psi), I[\Psi] + \delta f\right)$$

where δ is a square-zero parameter of cohomological degree $-i$.

15.3.1.2 Lemma. *The linear map*

$$W^\Phi_\Psi : \mathrm{Obs}^q_\Psi(M) \to \mathrm{Obs}^q_\Phi(M)$$

is an isomorphism of differentiable pro-cochain complexes.

PROOF. The fact that W^Φ_Ψ intertwines the differentials $\widehat{Q}_\Phi$ and $\widehat{Q}_\Psi$ follows from the compatibility between the quantum master equation and the renormalization group equation described in [Cos11c], Chapter 5 and summarized in section 14.2. It is not hard to verify that W^Φ_Ψ is a map of differentiable pro-cochain complexes. The inverse to W^Φ_Ψ is W^Ψ_Φ. □

15.3.1.3 Definition. *A global observable O of cohomological degree i is an assignment to every parametrix Φ of an element*

$$O[\Phi] \in \mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathcal{E})[[\hbar]]$$

of cohomological degree i such that

$$W^\Phi_\Psi O[\Psi] = O[\Phi].$$

If O is an observable of cohomological degree i, we let $\widehat{Q}O$ be defined by

$$\widehat{Q}(O)[\Phi] = \widehat{Q}_\Phi(O[\Phi]) = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar \triangle_\Phi O[\Phi].$$

This makes the space of observables into a differentiable pro-cochain complex, which we call $\mathrm{Obs}^q(M)$.

Thus, if $O \in \mathrm{Obs}^q(M)$ is an observable of cohomological degree i, and if δ is a square-zero parameter of cohomological degree $-i$, then the collection of effective interactions $\{I[\Phi] + \delta O[\Phi]\}$ satisfy most of the axioms needed to define a family of quantum field theories over the base ring $\mathbb{C}[\delta]/\delta^2$. The only axiom which is not satisfied is the locality axiom: we have not imposed any constraints on the behavior of the $O[\Phi]$ as $\Phi \to 0$.

15.4. Local observables

So far, we have defined a cochain complex $\mathrm{Obs}^q(M)$ of global observables on the whole manifold M. If $U \subset M$ is an open subset of M, we would like to isolate those observables which are "supported on U".

The idea is to say that an observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for sufficiently small parametrices, $O[\Phi]$ is supported on U. The precise definition is as follows.

15.4.0.4 Definition. *An observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for each $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, there exists a compact subset $K \subset U^k$ and a parametrix Φ, such that for all parametrices $\Psi \leq \Phi$*

$$\mathrm{Supp}\, O_{i,k}[\Psi] \subset K.$$

Remark: Recall that $O_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathbb{C}$ is the kth term in the Taylor expansion of the coefficient of $\hbar^i$ of the functional $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$. ◊

Remark: As always, the definition works over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, even though we suppress this from the notation. In this generality, instead of a compact subset $K \subset U^k$ we require $K \subset U^k \times X$ to be a set such that the map $K \to X$ is proper. ◊

We let $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ be the sub-graded vector space of observables supported on U.

15.4.0.5 Lemma. *$\mathrm{Obs}^q(U)$ is a sub-cochain complex of $\mathrm{Obs}^q(M)$. In other words, if $O \in \mathrm{Obs}^q(U)$, then so is $\widehat{Q}O$.*

PROOF. The only thing that needs to be checked is the support condition. We need to check that, for each (i,k), there exists a compact subset K of U^k such that, for all sufficiently small Φ, $\widehat{Q}O_{i,k}[\Phi]$ is supported on K.

Note that we can write

$$\widehat{Q}O_{i,k}[\Phi] = QO_{i,k}[\Phi] + \sum_{\substack{a+b=i \\ r+s=k+2}} \{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_{\Phi} + \Delta_{\Phi} O_{i-1,k+2}[\Phi].$$

We now find a compact subset K for $\widehat{Q}O_{i,k}[\Phi]$. We know that, for each (i,k) and for all sufficiently small Φ, $O_{i,k}[\Phi]$ is supported on $\tilde{K}$, where $\tilde{K}$ is some compact subset of U^k. It follows that $QO_{i,k}[\Phi]$ is supported on $\tilde{K}$.

By making $\tilde{K}$ bigger, we can assume that for sufficiently small Φ, $O_{i-1,k+2}[\Phi]$ is supported on L, where L is a compact subset of U^{k+2} whose image in U^k, under every projection map, is in $\tilde{K}$. This implies that $\Delta_\Phi O_{i-1,k+2}[\Phi]$ is supported on $\tilde{K}$.

The locality condition for the effective actions $I[\Phi]$ implies that, by choosing Φ to be sufficiently small, we can make $I_{i,k}[\Phi]$ supported as close as we like to the small diagonal in M^k. It follows that, by choosing Φ to be sufficiently small, the support of $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ can be taken to be a compact subset of U^k. Since there are only a finite number of terms like $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ in the expression for $(\widehat{Q}O)_{i,k}[\Phi]$, we see that for Φ sufficiently small, $(\widehat{Q}O)_{i,k}[\Phi]$ is supported on a compact subset K of U^k, as desired. □

15.4.0.6 Lemma. $\mathrm{Obs}^q(U)$ *has a natural structure of differentiable pro-cochain complex space.*

PROOF. Our general strategy for showing that something is a differentiable vector space is to ensure that everything works in families over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. Thus, suppose that the theory we are working with is defined over $(X, \mathscr{A})$. If Y is a smooth manifold, we say a smooth map $Y \to \mathrm{Obs}^q(U)$ is an observable for the family of theories over $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi C^\infty(Y))$ obtained by base-change along the map $X \times Y \to X$ (so this family of theories is constant over Y).

The filtration on $\mathrm{Obs}^q(U)$ (giving it the structure of pro-differentiable vector space) is inherited from that on $\mathrm{Obs}^q(M)$. Precisely, an observable $O \in \mathrm{Obs}^q(U)$ is in $F^k \mathrm{Obs}^q(U)$ if, for all parametrices Φ,

$$O[\Phi] \in \prod \hbar^i \mathrm{Sym}^{\geq (2k-i)} \mathscr{E}^\vee.$$

The renormalization group flow W_Φ^Ψ preserves this filtration.

So far we have verified that $\mathrm{Obs}^q(U)$ is a pro-object in the category of pre-differentiable cochain complexes. The remaining structure we need is a flat connection

$$\nabla : C^\infty(Y, \mathrm{Obs}^q(U)) \to \Omega^1(Y, \mathrm{Obs}^q(U))$$

for each manifold Y, where $C^\infty(Y, \mathrm{Obs}^q(U))$ is the space of smooth maps $Y \to \mathrm{Obs}^q(U)$.

This flat connection is equivalent to giving a differential on

$$\Omega^*(Y, \mathrm{Obs}^q(U)) = C^\infty(Y, \mathrm{Obs}^q(U)) \otimes_{C^\infty(Y)} \Omega^*(Y)$$

making it into a dg module for the dg algebra $\Omega^*(Y)$. Such a differential is provided by considering observables for the family of theories over the nilpotent dg manifold $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi \Omega^*(Y))$, pulled back via the projection map $X \times Y \to Y$. □

15.5. Local observables form a prefactorization algebra

At this point, we have constructed the cochain complex $\mathrm{Obs}^q(M)$ of global observables of our factorization algebra. We have also constructed, for every open subset $U \subset M$, a sub-cochain complex $\mathrm{Obs}^q(U)$ of observables supported on U.

In this section we will see that the local quantum observables $\mathrm{Obs}^q(U)$ of a quantum field on a manifold M form a prefactorization algebra.

The definition of local observables makes it clear that they form a pre-cosheaf: there are natural injective maps of cochain complexes

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$$

if $U \subset U'$ is an open subset.

Let U, V be disjoint open subsets of M. The structure of prefactorization algebra on the local observables is specified by the pre-cosheaf structure mentioned above, and a bilinear cochain map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

These product maps need to be maps of cochain complexes which are compatible with the pre-cosheaf structure and with reordering of the disjoint opens. Further, they need to satisfy a certain associativity condition which we will verify.

15.5.1. Defining the product map.

Suppose that $O \in \mathrm{Obs}^q(U)$ and $O' \in \mathrm{Obs}^q(V)$ are observables on U and V respectively. Note that $O[\Phi]$ and $O'[\Phi]$ are elements of the cochain complex

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], \widehat{Q}_\Phi\right)$$

which is a BD algebra and so a commutative algebra (ignoring the differential, of course). (The commutative product is simply the usual product of functions on $\mathscr{E}$.) In the definition of the prefactorization product, we will use the product of $O[\Phi]$ and $O'[\Phi]$ taken in the commutative algebra $\mathcal{O}(\mathcal{E})$. This product will be denoted $O[\Phi] * O'[\Phi] \in \mathcal{O}(\mathcal{E})$.

Recall (see definition 15.3.1.1) that we defined a linear renormalization group flow operator W_Φ^Ψ, which is an isomorphism between the cochain complexes $\mathrm{Obs}^q_\Phi(M)$ and $\mathrm{Obs}^q_\Psi(M)$.

The main result of this section is the following.

15.5.1.1 Theorem. *For all observables $O \in \mathrm{Obs}^q(U)$, $O' \in \mathrm{Obs}^q(V)$, where U and V are disjoint, the limit*

$$\lim_{\Psi \to 0} W_\Psi^\Phi \left(O[\Psi] * O'[\Psi]\right) \in \mathscr{O}(\mathscr{E})[[\hbar]]$$

exists. Further, this limit satisfies the renormalization group equation, so that we can define an observable $m(O, O')$ *by*

$$m(O, O')[\Phi] = \lim_{\Psi \to 0} W_{\Psi}^{\Phi}\left(O[\Psi] * O'[\Psi]\right).$$

The map

$$\begin{aligned} \mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) &\mapsto \mathrm{Obs}^q(U \amalg V) \\ O \times O' &\mapsto m(O, O') \end{aligned}$$

is a smooth bilinear cochain map, and it makes Obs^q *into a prefactorization algebra in the multicategory of differentiable pro-cochain complexes.*

PROOF. We will show that, for each i, k, the Taylor term

$$W_{\Phi}^{\Psi}(O[\Phi] * O'[\Phi])_{i,k} : \mathscr{E}^{\otimes k} \to \mathbb{C}$$

is independent of Φ for Φ sufficiently small. This will show that the limit exists.

Note that

$$W_{\Gamma}^{\Psi}\left(W_{\Phi}^{\Gamma}\left(O[\Phi] * O'[\Phi]\right)\right) = W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right).$$

Thus, to show that the limit $\lim_{\Phi \to 0} W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right)$ is eventually constant, it suffices to show that, for all sufficiently small Φ, Γ satisfying $\Phi < \Gamma$,

$$W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])_{i,k} = (O[\Gamma] * O'[\Gamma])_{i,k}.$$

This turns out to be an exercise in the manipulation of Feynman diagrams. In order to prove this, we need to recall a little about the Feynman diagram expansion of $W_{\Phi}^{\Gamma}(O[\Phi])$. (Feynman diagram expansions of the renormalization group flow are discussed extensively in [Cos11c].)

We have a sum of the form

$$W_{\Phi}^{\Gamma}(O[\Phi])_{i,k} = \sum_{G} \frac{1}{|\mathrm{Aut}(G)|} w_G\left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi)\right).$$

The sum is over all connected graphs G with the following decorations and properties.

(1) The vertices v of G are labelled by an integer $g(v) \in \mathbb{Z}_{\geq 0}$, which we call the genus of the vertex.
(2) The first Betti number of G, plus the sum of over all vertices of the genus $g(v)$, must be i (the "total genus").
(3) G has one special vertex.
(4) G has k tails (or external edges).

The weight $w_G\left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi)\right)$ is computed by the contraction of a collection of symmetric tensors. One places $O[\Phi]_{r,s}$ at the special vertex, when that vertex has genus r

and valency s; places $I[\Phi]_{g,v}$ at every other vertex of genus g and valency v; and puts the propagator $P(\Gamma) - P(\Phi)$ on each edge.

Let us now consider $W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])$. Here, we a sum over graphs with one special vertex, labelled by $O[\Phi] * O'[\Phi]$. This is the same as having two special vertices, one of which is labelled by $O[\Phi]$ and the other by $O'[\Phi]$. Diagrammatically, it looks like we have split the special vertex into two pieces. When we make this maneuver, we introduce possibly disconnected graphs; however, each connected component must contain at least one of the two special vertices.

Let us now compare this to the graphical expansion of

$$O[\Gamma] * O'[\Gamma] = W_{\Phi}^{\Gamma}(O[\Phi]) * W_{\Phi}^{\Gamma}(O'[\Phi]).$$

The Feynman diagram expansion of the right hand side of this expression consists of graphs with two special vertices, labelled by $O[\Phi]$ and $O'[\Phi]$ respectively (and, of course, any number of other vertices, labelled by $I[\Phi]$, and the propagator $P(\Gamma) - P(\Phi)$ labelling each edge). Further, the relevant graphs have precisely two connected components, each of which contains one of the special vertices.

Thus, we see that

$$W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi]) - W_{\Phi}^{\Gamma}(O[\Phi]) * W_{\Phi}^{\Gamma}(O'[\Phi]).$$

is a sum over *connected* graphs, with two special vertices, one labelled by $O[\Phi]$ and the other by $O'[\Phi]$. We need to show that the weight of such graphs vanish for Φ, Γ sufficiently small, with $\Phi < \Gamma$.

Graphs with one connected component must have a chain of edges connecting the two special vertices. (A chain is a path in the graph with no repeated vertices or edges.) For a graph G with "total genus" i and k tails, the length of any such chain is bounded by $2i + k$.

It is important to note here that we require a non-special vertex of genus zero to have valence at least three and a vertex of genus one to have valence at least one. See [**Cos11c**] for more discussion. If we are considering a family of theories over some dg ring, we do allow bivalent vertices to be accompanied by nilpotent parameters in the base ring; nilpotence of the parameter forces there to be a global upper bound on the number of bivalent vertices that can appear. The argument we are presenting works with minor modifications in this case too.

Each step along a chain of edges involves a tensor with some support that depends on the choice of parametrices *Phi* and Γ. As we move from the special vertex O toward the other O', we extend the support, and our aim is to show that we can choose Φ and Γ to be small enough so that the support of the chain, excluding $O'[\Phi]$, is disjoint from the support of $O'[\Phi]$. The contraction of a distribution and function with disjoint supports is zero, so that the weight will vanish. We now make this idea precise.

Let us choose arbitrarily a metric on M. By taking Φ and Γ to be sufficiently small, we can assume that the support of the propagator on each edge is within ε of the diagonal in this metric, and ε can be taken to be as small as we like. Similarly, the support of the $I_{r,s}[\Gamma]$ labelling a vertex of genus r and valency s can be taken to be within $c_{r,s}\varepsilon$ of the diagonal, where $c_{r,s}$ is a combinatorial constant depending only on r and s. In addition, by choosing Φ to be small enough we can ensure that the supports of $O[\Phi]$ and $O'[\Phi]$ are disjoint.

Now let G' denote the graph G with the special vertex for O' removed. This graph corresponds to a symmetric tensor whose support is within some distance $C_G\varepsilon$ of the small diagonal, where C_G is a combinatorial constant depending on the graph G'. As the supports K and K' (of O and O', respectively) have a finite distance d between them, we can choose ε small enough that $C_G\varepsilon < d$. It follows that, by choosing Φ and Γ to be sufficiently small, the weight of any connected graph is obtained by contracting a distribution and a function which have disjoint support. The graph hence has weight zero.

As there are finitely many such graphs with total genus i and k tails, we see that we can choose Γ small enough that for any $\Phi < \Gamma$, the weight of all such graphs vanishes.

Thus we have proved the first part of the theorem and have produced a bilinear map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

It is a straightforward to show that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization algebra. The fact that this is a smooth map of differentiable pro-vector spaces follows from the fact that this construction works for families of theories over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. ☐

15.6. Local observables form a factorization algebra

We have seen how to define a prefactorization algebra Obs^q of observables for our quantum field theory. In this section we will show that this prefactorization algebra is in fact a factorization algebra. In the course of the proof, we show that modulo $\hbar$, this factorization algebra is isomorphic to Obs^{cl}.

15.6.0.2 Theorem. *(1) The prefactorization algebra* Obs^q *of quantum observables is, in fact, a factorization algebra.*
(2) Further, there is an isomorphism

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$, and the factorization algebra of classical observables.

15.6.1. Proof of the theorem.

This theorem will be a corollary of a more technical proposition.

15.6.1.1 Proposition. *For any open subset $U \subset M$, filter $\operatorname{Obs}^q(U)$ by saying that the k-th filtered piece $G^k \operatorname{Obs}^q(U)$ is the sub $\mathbb{C}[[\hbar]]$-module consisting of those observables which are zero modulo $\hbar^k$. Note that this is a filtration by sub prefactorization algebras over the ring $\mathbb{C}[[\hbar]]$.*

Then, there is an isomorphism of prefactorization algebras (in differentiable pro-cochain complexes)

$$\operatorname{Gr} \operatorname{Obs}^q \simeq \operatorname{Obs}^{cl} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]].$$

This isomorphism makes $\operatorname{Gr} \operatorname{Obs}^q$ into a factorization algebra.

Remark: We can give $G^k \operatorname{Obs}^q(U)$ the structure of a pro-differentiable cochain complex, as follows. The filtration on $G^k \operatorname{Obs}^q(U)$ that defines the pro-structure is obtained by intersecting $G^k \operatorname{Obs}^q(U)$ with the filtration on $\operatorname{Obs}^q(U)$ defining the pro-structure. Then the inclusion $G^k \operatorname{Obs}^q(U) \hookrightarrow \operatorname{Obs}^q(U)$ is a cofibration of differentiable pro-vector spaces (see definition B.6.0.7). ♢

PROOF OF THE THEOREM, ASSUMING THE PROPOSITION. We need to show that for every open U and for every Weiss cover $\mathfrak{U}$, the natural map

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q) \to \operatorname{Obs}^q(U) \tag{†}$$

is a quasi-isomorphism of differentiable pro-cochain complexes.

The basic idea is that the filtration induces a spectral sequence for both $\check{C}(\mathfrak{U}, \operatorname{Obs}^q)$ and $\operatorname{Obs}^q(U)$, and we will show that the induced map of spectral sequences is an isomorphism on the first page. Because we are working with differentiable pro-cochain complexes, this is a little subtle. The relevant statements about spectral sequences in this context are developed in this context in Appendix B.

Note that $\check{C}(\mathfrak{U}, \operatorname{Obs}^q)$ is filtered by $\check{C}(\mathfrak{U}, G^k \operatorname{Obs}^q)$. The map (†) preserves the filtrations. Thus, we have a maps of inverse systems

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q / G^k \operatorname{Obs}^q) \to \operatorname{Obs}^q(U) / G^k \operatorname{Obs}^q(U).$$

These inverse systems satisfy the properties of Appedix B, lemma B.6.0.11. Further, it is clear that

$$\operatorname{Obs}^q(U) = \varprojlim \operatorname{Obs}^q(U) / G^k \operatorname{Obs}^q(U).$$

We also have

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q) = \varprojlim \check{C}(\mathfrak{U}, \operatorname{Obs}^q / G^k \operatorname{Obs}^q).$$

This equality is less obvious, and uses the fact that the Čech complex is defined using the completed direct sum as described in Appendix B, section B.6.

Using lemma B.6.0.11, we need to verify that the map

$$\check{C}(\mathfrak{U}, \operatorname{Gr} \operatorname{Obs}^q) \to \operatorname{Gr} \operatorname{Obs}^q(U)$$

is an equivalence. This follows from the proposition because $\operatorname{Gr}\operatorname{Obs}^q$ is a factorization algebra. ☐

PROOF OF THE PROPOSITION. The first step in the proof of the proposition is the following lemma.

15.6.1.2 Lemma. *Let* $\operatorname{Obs}^q_{(0)}$ *denote the prefactorization algebra of observables which are only defined modulo* $\hbar$*. Then there is an isomorphism*

$$\operatorname{Obs}^q_{(0)} \simeq \operatorname{Obs}^{cl}$$

of differential graded prefactorization algebras.

PROOF OF LEMMA. Let $O \in \operatorname{Obs}^{cl}(U)$ be a classical observable. Thus, O is an element of the cochain complex $\mathscr{O}(\mathscr{E}(U))$ of functionals on the space of fields on U. We need to produce an element of $\operatorname{Obs}^q_{(0)}$ from O. An element of $\operatorname{Obs}^q_{(0)}$ is a collection of functionals $O[\Phi] \in \mathscr{O}(\mathscr{E})$, one for every parametrix Φ, satisfying a classical version of the renormalization group equation and an axiom saying that $O[\Phi]$ is supported on U for sufficiently small Φ.

Given an element

$$O \in \operatorname{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

we define an element

$$\{O[\Phi]\} \in \operatorname{Obs}^q_{(0)}$$

by the formula

$$O[\Phi] = \lim_{\Gamma \to 0} W^{\Phi}_{\Gamma}(O) \text{ modulo } \hbar.$$

The Feynman diagram expansion of the right hand side only involves trees, since we are working modulo $\hbar$. As we are only using trees, the limit exists. The limit is defined by a sum over trees with one special vertex, where each edge is labelled by the propagator $P(\Phi)$, the special vertex is labelled by O, and every other vertex is labelled by the classical interaction $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ of our theory.

The map

$$\operatorname{Obs}^{cl}(U) \to \operatorname{Obs}^q_{(0)}(U)$$

we have constructed is easily seen to be a map of cochain complexes, compatible with the structure of prefactorization algebra present on both sides. (The proof is a variation on the argument in section 11, chapter 5 of [Cos11c], about the scale 0 limit of a deformation of the effective interaction I modulo $\hbar$.)

A simple inductive argument on the degree shows this map is an isomorphism.

Because the construction works over an arbitrary nilpotent dg manifold, it is clear that these maps are maps of differentiable cochain complexes. ☐

The next (and most difficult) step in the proof of the proposition is the following lemma. We use it to work inductively with the filtration of quantum observables.

Let $\mathrm{Obs}^q_{(k)}$ denote the prefactorization algebra of observables defined modulo $\hbar^{k+1}$.

15.6.1.3 Lemma. *For all open subsets $U \subset M$, the natural quotient map of differentiable pro-cochain complexes*

$$\mathrm{Obs}^q_{(k+1)}(U) \to \mathrm{Obs}^q_{(k)}(U)$$

is a fibration of differentiable pro-cochain complexes (see Appendix B, Definition B.6.0.7 for the definition of a fibration). The fiber is isomorphic to $\mathrm{Obs}^{cl}(U)$.

PROOF OF LEMMA. We give the set $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lexicographical ordering, so that $(i,k) > (r,s)$ if $i > r$ or if $i = r$ and $k > s$.

We will let $\mathrm{Obs}^q_{\leq(i,k)}(U)$ be the quotient of $\mathrm{Obs}^q_{(i)}$ consisting of functionals

$$O[\Phi] = \sum_{(r,s)\leq(i,k)} \hbar^r O_{(r,s)}[\Phi]$$

satisfying the renormalization group equation and locality axiom as before, but where $O_{(r,s)}[\Phi]$ is only defined for $(r,s) \leq (i,k)$. Similarly, we will let $\mathrm{Obs}^q_{<(i,k)}(U)$ be the quotient where the $O_{(r,s)}[\Phi]$ are only defined for $(r,s) < (i,k)$.

We will show that the quotient map

$$\mathsf{q} : \mathrm{Obs}^q_{\leq(i,k)}(U) \to \mathrm{Obs}^q_{<(i,k)}(U)$$

is a fibration. The result will follow.

Recall what it means for a map $f : V \to W$ of differentiable cochain complexes to be a fibration. For X a manifold, let $C^\infty_X(V)$ denote the sheaf of cochain complexes on X of smooth maps to V. We say f is a fibration if for every manifold X, the induced map of sheaves $C^\infty_X(V) \to C^\infty_X(W)$ is surjective in each degree. Equivalently, we require that for all smooth manifolds X, every smooth map $X \to W$ lifts locally on X to a map to V.

Now, by definition, a smooth map from X to $\mathrm{Obs}^q(U)$ is an observable for the constant family of theories over the nilpotent dg manifold $(X, C^\infty(X))$. Thus, in order to show q is a fibration, it suffices to show the following. For any family of theories over a nilpotent dg manifold $(X, \mathscr{A})$, any open subset $U \subset M$, and any observable α in the $\mathscr{A}$-module $\mathrm{Obs}^q_{<(i,k)}(U)$, we can lift α to an element of $\mathrm{Obs}^q_{\leq(i,k)}(U)$ locally on X.

To prove this, we will first define, for every parametrix Φ, a map

$$L_\Phi : \mathrm{Obs}^q_{<(i,k)}(U) \to \mathrm{Obs}^q_{\leq(i,k)}(M)$$

with the property that the composed map

$$\mathrm{Obs}^q_{<(i,k)}(U) \xrightarrow{L_\Phi} \mathrm{Obs}^q_{\leq(i,k)}(M) \to \mathrm{Obs}^q_{<(i,k)}(M)$$

is the natural inclusion map. Then, for every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we will show that $L_\Phi(O)$ is supported on U, for sufficiently small parametrices Φ, so that $L_\Phi(O)$ provides the desired lift.

For

$$O \in \mathrm{Obs}^q_{<(i,k)}(U),$$

we define

$$L_\Phi(O) \in \mathrm{Obs}^q_{\leq(i,k)}(M)$$

by

$$L_\Phi(O)_{r,s}[\Phi] = \begin{cases} O_{r,s}[\Phi] & \text{if } (r,s) < (i,k) \\ 0 & \text{if } (r,s) = (i,k) \end{cases}.$$

For $\Psi \neq \Phi$, we obtain $L_\Phi(O)_{r,s}[\Psi]$ by the renormalization group flow from $L_\Phi(O)_{r,s}[\Phi]$. The RG flow equation tells us that if $(r,s) < (i,k)$, then

$$L_\Phi(O)_{r,s}[\Psi] = O_{r,s}[\Psi].$$

However, the RG equation for $L_\Phi(O)_{r,s}$ is non-trivial and tells us that

$$I_{i,k}[\Psi] + \delta\left(L_\Phi(O)_{i,k}[\Psi]\right) = W_{i,k}\left(P(\Psi) - P(\Phi), I[\Phi] + \delta O[\Phi]\right)$$

for δ a square-zero parameter of cohomological degree opposite to that of O.

To complete the proof of this lemma, we prove the required local lifting property in the sublemma below. □

15.6.1.4 Sub-lemma. *For each $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we can find a parametrix Φ — locally over the parametrizing manifold X — so that $L_\Phi O$ lies in $\mathrm{Obs}^q_{\leq(i,k)}(U) \subset \mathrm{Obs}^q_{\leq(i,k)}(M)$.*

PROOF. Although the observables Obs^q form a factorization algebra on the manifold M, they also form a sheaf on the parametrizing base manifold X. That is, for every open subset $V \subset X$, let $\mathrm{Obs}^q(U) \mid_V$ denote the observables for our family of theories restricted to V. In other words, $\mathrm{Obs}^q(U) \mid_V$ denotes the sections of this sheaf $\mathrm{Obs}^q(U)$ on V.

The map L_Φ constructed above is then a map of sheaves on X.

For every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we need to find an open cover

$$X = \bigcup_\alpha Y_\alpha$$

of X, and on each Y_α a parametrix Φ_α (for the restriction of the family of theories to Y_α) such that

$$L_{\Phi_\alpha}(O \mid_{Y_\alpha}) \in \mathrm{Obs}^q_{\leq(i,k)}(U) \mid_{Y_\alpha} .$$

More informally, we need to show that locally in X, we can find a parametrix Φ such that for all sufficiently small Ψ, the support of $L_\Phi(O)_{(i,k)}[\Psi]$ is in a subset of $U^k \times X$ which maps properly to X.

This argument resembles previous support arguments (e.g., the product lemma from section 15.5). The proof involves an analysis of the Feynman diagrams appearing in the expression

$$(\star) \qquad L_\Phi(O)_{i,k}[\Psi] = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} w_\gamma \left(O[\Phi]; I[\Phi]; P(\Psi) - P(\Phi)\right).$$

The sum is over all connected Feynman diagrams of genus i with k tails. The edges are labelled by $P(\Psi) - P(\Phi)$. Each graph has one special vertex, where $O[\Phi]$ appears. More explicitly, if this vertex is of genus r and valency s, it is labelled by $O_{r,s}[\Phi]$. Each non-special vertex is labelled by $I_{a,b}[\Phi]$, where a is the genus and b the valency of the vertex. Note that only a finite number of graphs appear in this sum.

By assumption, O is supported on U. This means that there exists some parametrix Φ_0 and a subset $K \subset U \times X$ mapping properly to X such that for all $\Phi < \Phi_0$, $O_{r,s}[\Phi]$ is supported on K^s. (Here by $K^s \subset U^s \times X$ we mean the fibre product over X.)

Further, each $I_{a,b}[\Phi]$ is supported as close as we like to the small diagonal $M \times X$ in $M^k \times X$. We can find precise bounds on the support of $I_{a,b}[\Phi]$, as explained in section 14.2. To describe these bounds, let us choose metrics for X and M. For a parametrix Φ supported within ε of the diagonal $M \times X$ in $M \times M \times X$, the effective interaction $I_{a,b}[\Phi]$ is supported within $(2a + b)\varepsilon$ of the diagonal.

(In general, if $A \subset M^n \times X$, the ball of radius ε around A is defined to be the union of the balls of radius ε around each fibre A_x of $A \to X$. It is in this sense that we mean that $I_{a,b}[\Phi]$ is supported within $(2a + b)\varepsilon$ of the diagonal.)

Similarly, for every parametrix Ψ with $\Psi < \Phi$, the propagator $P(\Psi) - P(\Phi)$ is supported within ε of the diagonal.

In sum, there exists a set $K \subset U \times X$, mapping properly to X, such that for all $\varepsilon > 0$, there exists a parametrix Φ_ε, such that

(1) $O[\Phi_\varepsilon]_{r,s}$ is supported on K^s for all $(r,s) < (i,k)$.
(2) $I_{a,b}[\Phi_\varepsilon]$ is supported within $(2a+b)\varepsilon$ of the small diagonal.
(3) For all $\Psi < \Phi_\varepsilon$, $P(\Psi) - P(\Phi_\varepsilon)$ is supported within ε of the small diagonal.

The weight w_γ of a graph in the graphical expansion of the expression $(\star)$ above (using the parametrices Φ_ε and any $\Psi < \Phi_\varepsilon$) is thus supported in the ball of radius $c\varepsilon$ around K^k (where c is some combinatorial constant, depending on the number of edges and vertices in γ). There are a finite number of such graphs in the sum, so we can choose the combinatorial constant c uniformly over the graphs.

Since $K \subset U \times X$ maps properly to X, locally on X, we can find an ε so that the closed ball of radius $c\varepsilon$ is still inside $U^k \times X$. This completes the proof. □

□

15.7. The map from theories to factorization algebras is a map of presheaves

In [**Cos11c**], it is shown how to restrict a quantum field theory on a manifold M to any open subset U of M. Factorization algebras also form a presheaf in an obvious way. In this section, we will prove the following result.

15.7.0.5 Theorem. *The map from the simplicial set of theories on a manifold M to the ∞-groupoid of factorization algebras on M extends to a map of simplicial presheaves.*

The proof of this will rely on the results we have already proved, and in particular on thefact that observables form a factorization algebra.

As a corollary, we have the following very useful result.

15.7.0.6 Corollary. *For every open subset $U \subset M$, there is an isomorphism of graded differentiable vector spaces*

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^{cl}(U)[[\hbar]].$$

Note that what we have proved already is that there is a filtration on $\mathrm{Obs}^q(U)$ whose associated graded is $\mathrm{Obs}^{cl}(U)[[\hbar]]$. This result shows that this filtration is split as a filtration of differentiable vector spaces.

PROOF. By the theorem, $\mathrm{Obs}^q(U)$ can be viewed as global observables for the field theory obtained by restricting our field theory on M to one on U. Choosing a parametrix on U allows one to identify global observables with $\mathrm{Obs}^{cl}(U)[[\hbar]]$, with differential $\mathrm{d} + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi$. This is an isomorphism of differentiable vector spaces. □

The proof of this theorem is a little technical, and uses the same techniques we have discussed so far. Before we explain the proof of the theorem, we need to explain how to restrict theories to open subsets.

Let $\mathscr{E}(M)$ denote the space of fields for a field theory on M. In order to relate field theories on U and on M, we need to relate parametrices on U and on M. If

$$\Phi \in \overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\overline{\mathscr{E}}(M)$$

is a parametrix on M (with proper support as always), then the restriction

$$\Phi \mid_U \in \overline{\mathscr{E}}(U)\widehat{\otimes}_\beta\overline{\mathscr{E}}(U)$$

of Φ to U may no longer be a parametrix. It will satisfy all the conditions required to be a parametrix except that it will typically not have proper support.

We can modify $\Phi \mid_U$ so that it has proper support, as follows. Let $K \subset U$ be a compact set, and let f be a smooth function on $U \times U$ with the following properties:

1. f is 1 on $K \times K$.
2. f is 1 on a neighbourhood of the diagonal.
3. f has proper support.

Then, $f\Phi \mid_U$ does have proper support, and further, $f\Phi \mid_U$ is equal to Φ on $K \times K$.

Conversely, given any parametrix Φ on U, there exists a parametrix $\widetilde{\Phi}$ on M such that Φ and $\widetilde{\Phi}$ agree on K. One can construct $\widetilde{\Phi}$ by taking any parametrix Ψ on M, and observing that, when restricted to U, Ψ and Φ differ by a smooth section of the bundle $E \boxtimes E$ on $U \times U$.

We can then choose a smooth section σ of this bundle on $U \times U$ such that f has compact support and $\sigma = \Psi - \Phi$ on $K \times K$. Then, we let $\widetilde{\Phi} = \Psi - f$.

Let us now explain what it means to restrict a theory on M to one on U. Then we will state the theorem that there exists a unique such restriction.

Fix a parametrix Φ on U. Let $K \subset U$ be a compact set, and consider the compact set

$$L_n = (\operatorname{Supp} \Phi *)^n K \subset U.$$

Here we are using the convolution construction discussed earlier, whereby the collection of proper subsets of $U \times U$ acts on that of compact sets in U by convolution. Thus, L_n is the set of those $x \in U$ such that there exists a sequence $(y_0, \dots, y_n)$ where (y_i, y_{i+1}) is in $\operatorname{Supp} \Phi$, $y_n \in K$ and $y_0 = x$.

15.7.0.7 Definition. *Fix a theory on M, specified by a collection $\{I[\Psi]\}$ of effective interactions. Then a restriction of $\{I[\Psi]\}$ to U consists of a collection of effective interactions $\{I^U[\Phi]\}$ with the* *following propery. For every parametrix Φ on U, and for all compact sets $K \subset U$, let $L_n \subset U$ be as above.*

Let $\widetilde{\Phi}_n$ be a parametrix on M with the property that

$$\widetilde{\Phi}_n = \Phi \text{ on } L_n \times L_n.$$

Then we require that

$$I^U_{i,k}[\Phi](e_1, \dots, e_k) = I_{i,k}[\widetilde{\Phi}_n](e_1, \dots, e_k)$$

where $e_i \in \mathscr{E}_c(U)$ have support on K, and where $n \geq 2i + k$.

This definition makes sense in families with obvious modifications.

15.7.0.8 Theorem. *Any theory $\{I[\Psi]\}$ on M has a unique restriction on U.*

This restriction map works in families, and so defines a map of simplicial sets from the simplicial set of theories on M to that on U.

In this way, we have a simplicial presheaf $\mathcal{T}$ on M whose value on U is the simplicial set of theories on U (quantizing a given classical theory). This simplicial presheaf is a homotopy sheaf, meaning that it satisfies Čech descent.

PROOF. It is obvious that the restriction, it if exists, is unique. Indeed, we have specified each $I^U_{i,k}[\Phi]$ for every Φ and for every compact subset $K \subset U$. Since each $I^U_{i,k}[\Phi]$ must have compact support on U^k, it is determined by its behaviour on compact sets of the form K^k.

In [Cos11c], a different definition of restriction was given, defined not in terms of general parametrices but in terms of those defined by the heat kernel. One therefore needs to check that the notion of restriction defined in [Cos11c] coincides with the one discussed in this theorem. This is easy to see by a Feynman diagram argument similar to the ones we discussed earlier. The statement that the simplicial presheaf of theories satisfies Čech descent is proved in [Cos11c]. ☐

Now here is the main theorem in this section.

15.7.0.9 Theorem. *The map which assigns to a field theory the corresponding factorization algebra is a map of presheaves. Further, the map which assigns to an n-simplex in the simplicial set of theories, a factorization algebra over $\Omega^*(\triangle^n)$, is also a map of presheaves.*

Let us explain what this means concretely. Consider a theory on M and let Obs^q_M denote the corresponding factorization algebra. Let Obs^q_U denote the factorization algebra for the theory restricted to U, and let $Obs^q_M \mid_U$ denote the factorization algebra Obs^q_M restricted to U (that is, we only consider open subsets contained in M). Then there is a canonical isomorphism of factorization algebras on U,

$$\mathrm{Obs}^q_U \cong \mathrm{Obs}^q_M \mid_U .$$

In addition, this construction works in families, and in particular in families over $\Omega^*(\triangle^n)$.

PROOF. Let $V \subset U$ be an open set whose closure in U is compact. We will first construct an isomorphism of differentiable cochain complexes

$$\mathrm{Obs}^q_M(V) \cong \mathrm{Obs}^q_U(V).$$

Later we will check that this isomorphism is compatible with the product structures. Finally, we will use the codescent properties for factorization algebras to extend to an isomorphism of factorization algebras defined on all open subsets $V \subset U$, and not just those whose closure is compact.

Thus, let $V \subset U$ have compact closure, and let $O \in \mathrm{Obs}^q_M(V)$. Thus, O is something which assigns to every parametrix Φ on M a collection of functionals $O_{i,k}[\Phi]$ satisfying the renormalization group equation and a locality axiom stating that for each i, k, there exists a parametrix Φ_0 such that $O_{i,k}[\Phi]$ is supported on V for $\Phi \leq \Phi_0$.

We want to construct from such an observable a collection of functionals $\rho(O)_{i,k}[\Psi]$, one for each parametrix Ψ on U, satisfying the RG flow on U and the same locality axiom. It suffices to do this for a collection of parametrices which include parametrices which are arbitrarily small (that is, with support contained in an arbitrarily small neighbourhood of the diagonal in $U \times U$).

Let $L \subset U$ be a compact subset with the property that $\overline{V} \subset \mathrm{Int}\, L$. Choose a function f on $U \times U$ which is 1 on a neighbourhood of the diagonal, 1 on $L \times L$, and has proper support. If Ψ is a parametrix on M, we let Ψ^f be the parametrix on U obtained by multiplying the restriction of Ψ to $U \times U$ by f. Note that the support of Ψ^f is a subset of that of Ψ.

The construction is as follows. Choose (i, k). We define

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi]$$

for all $(r, s) \leq (i, k)$ and all Ψ sufficiently small. We will not spell out what we mean by sufficiently small, except that it in particular means it is small enough so that $O_{r,s}[\Psi]$ is supported on V for all $(r, s) \leq (i, k)$. The value of $\rho(O)_{r,s}$ for other parametrices is determined by the RG flow.

To check that this construction is well-defined, we need to check that if we take some parametrix $\widetilde{\Psi}$ on M which is also sufficiently small, then the $\rho(O)_{r,s}[\Psi^f]$ and $\rho(O)_{r,s}[\widetilde{\Psi}^f]$

are related by the RG flow for observables for the theory on U. This RG flow equation relating these two quantities is a sum over connected graphs, with one vertex labelled by $\rho(O)[\Psi^f]$, all other vertices labelled by $I^U[\Psi^f]$, and all internal edges labelled by $P(\widetilde{\Psi}^f) - P(\Psi^f)$. Since we are only considering $(r,s) \leq (i,k)$ only finitely many graphs can appear, and the number of internal edges of these graphs is bounded by $2i+k$. We are assuming that both Ψ and $\widetilde{\Psi}$ are sufficiently small so that $O_{r,s}[\Psi]$ and $O_{r,s}[\widetilde{\Psi}]$ have compact support on V. Also, by taking Ψ sufficiently small, we can assume that $I^U[\Psi]$ has support arbitrarily close to the diagonal. It follows that, if we choose both Ψ and $\widetilde{\Psi}$ to be sufficiently small, there is a compact set $L' \subset U$ containing V such that the weight of each graph appearing in the RG flow is zero if one of the inputs (attached to the tails) has support on the complement of L'. Further, by taking Ψ and $\widetilde{\Psi}$ sufficiently small, we can arrange so that L' is as small as we like, and in particular, we can assume that $L' \subset \operatorname{Int} L$ (where L is the compact set chosen above).

Recall that the weight of a Feynman diagram involves pairing quantities attached to edges with multilinear functionals attached to vertices. A similar combinatorial analysis tells us that, for each vertex in each graph appearing in this sum, the inputs to the multilinear functional attached to the vertex are all supported in L'.

Now, for Ψ sufficiently small, we have

$$I^U_{r,s}[\Psi^f](e_1, \dots, e_s) = I_{r,s}[\Psi](e_1, \dots, e_s)$$

if all of the e_i are supported in L'. (This follows from the definition of the restriction of a theory. Recall that I^U indicates the theory on U and I indicates the theory on M).

It follows that, in the sum over diagrams computing the RG flow, we get the same answer if we label the vertices by $I[\Psi]$ instead of $I^U[\Psi^f]$. The RG flow equation now follows from that for the original observable $O[\Psi]$ on M.

The same kind of argument tells us that if we change the choice of compact set $L \subset U$ with $\overline{V} \subset \operatorname{Int} L$, and if we change the bumb function f we chose, the map

$$\rho : \mathrm{Obs}^q_M(V) \to \mathrm{Obs}^q_U(V)$$

does not change.

A very similar argument also tells us that this map is a cochain map. It is immediate that ρ is an isomorphism, and that it commutes with the maps arising from inclusions $V \subset V'$.

We next need to verify that this map respects the product structure. Recall that the product of two observables O, O' in V, V' is defined by saying that $([\Psi]O'[\Psi])_{r,s}$ is simply the naive product in the symmetric algebra $\mathrm{Sym}^* \mathscr{E}_c^!(V \amalg V')$ for $(r,s) \leq (i,k)$ (some fixed (i,k)) and for Ψ sufficiently small.

Since, for $(r,s) \leq (i,k)$ and for Ψ sufficiently small, we defined

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi],$$

we see immediately that ρ respects products.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q_M \mid_U \cong \mathrm{Obs}^q_U$$

of prefactorization algebras on U, where we consider open subsets in U with compact closure. We need to extend this to an isomorphism of factorization algebras. To do this, we use the following property: for any open subset $W \subset U$,

$$\mathrm{Obs}^q_U(W) = \operatorname*{colim}_{V \subset W} \mathrm{Obs}^q_U(V)$$

where the colimit is over all open subsets with compact closure. (The colimit is taken, of course, in the category of filtered differentiable cochain complexes, and is simply the naive and not homotopy colimit). The same holds if we replace Obs^q_U by Obs^q_M. Thus we have constructed an isomorphism

$$\mathrm{Obs}^q_U(W) \cong \mathrm{Obs}^q_M(W)$$

for all open subsets W. The associativity axioms of prefactorization algebras, combined with the fact that $\mathrm{Obs}^q(W)$ is a colimit of $\mathrm{Obs}^q(V)$ for V with compact closure and the fact that the isomorphisms we have constructed respect the product structure for such open subsets V, implies that we have constructed an isomorphism of factorization algebras on U. $\square$

CHAPTER 16

Further aspects of quantum observables

16.1. Translation-invariant factorization algebras from translation-invariant quantum field theories

In this section, we will show that a translation-invariant quantum field theory on $\mathbb{R}^n$ gives rise to a smoothly translation-invariant factorization algebra on $\mathbb{R}^n$ (see section 4.7). We will also show that a holomorphically translation-invariant field theory on $\mathbb{C}^n$ gives rise to a holomorphically translation-invariant factorization algebra.

16.1.1. First, we need to define what it means for a field theory to be translation-invariant. Let us consider a classical field theory on $\mathbb{R}^n$. Recall that this is given by

1. A graded vector bundle E whose sections are $\mathscr{E}$;
2. An antisymmetric pairing $E \otimes E \to \mathrm{Dens}_{\mathbb{R}^n}$;
3. A differential operator $Q : \mathscr{E} \to \mathscr{E}$ making $\mathscr{E}$ into an elliptic complex, which is skew-self adjoint;
4. A local action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

A classical field theory is translation-invariant if

1. The graded bundle E is translation-invariant, so that we are given an isomorphism between E and the trivial bundle with fibre E_0.
2. The pairing, differential Q, and local functional I are all translation-invariant.

It takes a little more work to say what it means for a quantum field theory to be translation-invariant. Suppose we have a translation-invariant classical field theory, equipped with a translation-invariant gauge fixing operator Q^{GF}. As before, a quantization of such a field theory is given by a family of interactions $I[\Phi] \in \mathscr{O}_{sm,P}(\mathscr{E})$, one for each parametrix Φ.

16.1.1.1 Definition. *A translation-invariant quantization of a translation-invariant classical field theory is a quantization with the property that, for all translation-invariant parametrices Φ, $I[\Phi]$ is translation-invariant.*

Remark: In general, in order to give a quantum field theory on a manifold M, we do not need to give an effective interaction $I[\Phi]$ for all parametrices. We only need to specify $I[\Phi]$ for a collection of parametrices such that the intersection of the supports of Φ is the small diagonal $M \subset M^2$. The functional $I[\Psi]$ for all other parametrices Ψ is defined by the renormalization group flow. It is easy to construct a collection of translation-invariant parametrices satisfying this condition. $\Diamond$

16.1.1.2 Proposition. *The factorization algebra associated to a translation-invariant quantum field theory is smoothly translation-invariant (see section 4.7 in Chapter 4 for the definition).*

PROOF. Let Obs^q denote the factorization algebra of quantum observables for our translation-invariant theory. An observable supported on $U \subset \mathbb{R}^n$ is defined by a family $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$, one for each translation-invariant parametrix, which satisfiese the RG flow and (in the sense we explained in section 15.4) is supported on U for sufficiently small parametrices. The renormalization group flow

$$W_\Psi^\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

for translation-invariant parametrices Ψ, Φ commutes with the action of $\mathbb{R}^n$ on $\mathscr{O}(\mathscr{E})$ by translation, and therefore acts on $\mathrm{Obs}^q(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, let $T_x U$ denote the x-translate of U. It is immediate that the action of $x \in \mathbb{R}^n$ on $\mathrm{Obs}^q(\mathbb{R}^n)$ takes $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(\mathbb{R}^n)$ to $\mathrm{Obs}^q(T_x U)$. It is not difficult to verify that the resulting map

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(T_x U)$$

is an isomorphism of differentiable pro-cochain complexes and that it is compatible with the structure of a factorization algebra.

We need to verify the smoothness hypothesis of a smoothly translation-invariant factorization algebra. This is the following. Suppose that $U_1, \dots, U_k$ are disjoint open subsets of $\mathbb{R}^n$, all contained in an open subset V. Let $A' \subset \mathbb{R}^{nk}$ be the subset consisting of those $x_1, \dots, x_k$ such that the closures of $T_{x_i} U_i$ remain disjoint and in V. Let A be the connected component of 0 in A'. We need only examine the case where A is non-empty.

We need to show that the composed map

$$m_{x_1,\dots,x_k} : \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to \\ \mathrm{Obs}^q(T_{x_1} U_1) \times \cdots \times \mathrm{Obs}^q(T_{x_k} U_k) \to \mathrm{Obs}^q(V)$$

varies smoothly with $(x_1, \dots, x_k) \in A$. In this diagram, the first map is the product of the translation isomorphisms $\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i)$, and the second map is the product map of the factorization algebra.

The smoothness property we need to check says that the map $m_{x_1,\dots,x_k}$ lifts to a multilinear map of differentiable pro-cochain complexes

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to C^\infty(A, \mathrm{Obs}^q(V)),$$

where on the right hand side the notation $C^\infty(A, \mathrm{Obs}^q(V))$ refers to the smooth maps from A to $\mathrm{Obs}^q(V)$.

This property is local on A, so we can replace A by a smaller open subset if necessary.

Let us assume (replacing A by a smaller subset if necessary) that there exist open subsets U_i' containing U_i, which are disjoint and contained in V and which have the property that for each $(x_1, \dots, x_k) \in A$, $T_{x_i} U_i \subset U_i'$.

Then, we can factor the map $m_{x_1,\dots,x_k}$ as a composition

$$(\dagger) \quad \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \xrightarrow{i_{x_1} \times \cdots \times i_{x_k}} \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V).$$

Here, the map $i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$ is the composition

$$\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$$

of the translation isomorphism with the natural inclusion map $\mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$. The second map in equation (†) is the product map associated to the disjoint subsets $U_1', \dots, U_k' \subset V$.

By possibly replacing A by a smaller open subset, let us assume that $A = A_1 \times \cdots \times A_k$, where the A_i are open subsets of $\mathbb{R}^n$ containing the origin. It remains to show that the map

$$i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$$

is smooth in x_i, that is, extends to a smooth map

$$\mathrm{Obs}^q(U_i) \to C^\infty(A_i, \mathrm{Obs}^q(U_i')).$$

Indeed, the fact that the product map

$$m : \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V)$$

is a smooth multilinear map implies that, for every collection of smooth maps $\alpha_i : Y_i \to \mathrm{Obs}^q(U_i')$ from smooth manifolds Y_i, the resulting map

$$\begin{aligned} Y_1 \times \cdots \times Y_k &\to \mathrm{Obs}^q(V) \\ (y_1, \dots y_k) &\mapsto m(\alpha_1(y), \dots, \alpha_k(y)) \end{aligned}$$

is smooth.

Thus, we have reduced the result to the following statement: for all open subsets $A \subset \mathbb{R}^n$ and for all $U \subset U'$ such that $T_x U \subset U'$ for all $x \in A$, the map $i_x : \mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$ is smooth in $x \in A$.

But this statement is tractable. Let

$$O \in \mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(U') \subset \mathrm{Obs}^q(\mathbb{R}^n)$$

be an observable. It is obvious that the family of observables $T_x O$, when viewed as elements of $\mathrm{Obs}^q(\mathbb{R}^n)$, depends smoothly on x. We need to verify that it depends smoothly on x when viewed as an element of $\mathrm{Obs}^q(U')$.

This amounts to showing that the support conditions which ensure an observable is in $\mathrm{Obs}^q(U')$ hold uniformly for x in compact sets in A.

The fact that O is in $\mathrm{Obs}^q(U)$ means the following. For each (i,k), there exists a compact subset $K \subset U$ and $\varepsilon > 0$ such that for all translation-invariant parametrices Φ supported within ε of the diagonal and for all $(r,s) \leq (i,k)$ in the lexicographical ordering, the Taylor coefficient $O_{r,s}[\Phi]$ is supported on K^s.

We need to enlarge K to a subset $L \subset U' \times A$, mapping properly to A, such that $T_x O$ is supported on L in this sense (again, for $(r,s) \leq (i,k)$). Taking $L = K \times A$, embedded in $U' \times A$ by

$$(k,x) \mapsto (T_x k, x)$$

suffices. □

Remark: Essentially the same proof will give us the somewhat stronger result that for any manifold M with a smooth action of a Lie group G, the factorization algebra corresponding to a G-equivariant field theory on M is smoothly G-equivariant. ♢

16.2. Holomorphically translation-invariant theories and their factorization algebras

Similarly, we can talk about holomorphically translation-invariant classical and quantum field theories on $\mathbb{C}^n$. In this context, we will take our space of fields to be $\Omega^{0,*}(\mathbb{C}^n, V)$, where V is some translation-invariant holomorphic vector bundle on $\mathbb{C}^n$. The pairing must arise from a translation-invariant map of holomorphic vector bundles

$$V \otimes V \to K_{\mathbb{C}^n}$$

of cohomological degree $n-1$, where $K_{\mathbb{C}^n}$ denotes the canonical bundle. This means that the composed map

$$\Omega_c^{0,*}(\mathbb{C}^n, V)^{\otimes 2} \to \Omega_c^{0,*}(\mathbb{C}^n, K_{\mathbb{C}^n}) \xrightarrow{\int} \mathbb{C}$$

is of cohomological degree -1.

Let

$$\eta_i = \frac{\partial}{\partial \overline{z}_i} \vee - : \Omega^{0,k}(\mathbb{C}^n, V) \to \Omega^{0,k-1}(\mathbb{C}^n, V)$$

be the contraction operator. The cohomological differential operator Q on $\Omega^{0,*}(V)$ must be of the form

$$Q = \overline{\partial} + Q_0$$

where Q_0 is translation-invariant and satisfies the following conditions:

(1)
(2) Q_0 (and hence Q) must be skew self-adjoint with respect to the pairing on $\Omega_c^{0,*}(\mathbb{C}^n, V)$.
(3) We assume that Q_0 is a purely holomorphic differential operator, so that we can write Q_0 as a finite sum

$$Q_0 = \sum \frac{\partial}{\partial z^I} \mu_I$$

where $\mu_I : V \to V$ are linear maps of cohomological degree 1. (Here we are using multi-index notation). Note that this implies that

$$[Q_0, \eta_i] = 0,$$

for $i = 1, \dots, n$. In terms of the μ_I, the adjointness condition says that μ_I is skew-symmetric if $|I|$ is even and symmetric if $|I|$ is odd.

The other piece of data of a classical field theory is the local action functional $I \in \mathscr{O}_{loc}(\Omega^{0,*}(\mathbb{C}^n, V))$. We assume that I is translation-invariant, of course, but also that

$$\eta_i I = 0$$

for $i = 1 \dots n$, where the linear map η_i on $\Omega^{0,*}(\mathbb{C}^n, V)$ is extended in the natural way to a derivation of the algebra $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$ preserving the subspace of local functionals.

Any local functional I on $\Omega^{0,*}(\mathbb{C}^n, V)$ can be written as a sum of functionals of the form

$$\phi \mapsto \int_{\mathbb{C}^n} \mathrm{d}z_1 \dots \mathrm{d}z_n A(D_1\phi \dots D_k\phi)$$

where $A : V^{\otimes k} \to \mathbb{C}$ is a linear map, and each D_i is in the space

$$\mathbb{C}\left[\mathrm{d}\overline{z}_i, \eta_i, \frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial z_i}\right].$$

(Recall that η_i indicates $\frac{\partial}{\partial \mathrm{d}\overline{z}_i}$). The condition that $\eta_i I = 0$ for each i means that we only consider those D_i which are in the subspace

$$\mathbb{C}\left[\eta_i, \frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial z_i}\right].$$

In other words, as a differential operator on the graded algebra $\Omega^{0,*}(\mathbb{C}^n)$, each D_i has constant coefficients.

It turns out that, under some mild hypothesis, any such action functional I is equivalent (in the sense of the BV formalism) to one which has only z_i derivatives, and no $\overline{z}_i$ or $\mathrm{d}\overline{z}_i$ derivatives.

16.2.0.3 Lemma. *Suppose that $Q = \overline{\partial}$, so that $Q_0 = 0$. Then, any interaction I satisfying the classical master equation and the condition that $\eta_i I = 0$ for $i = 1, \dots, n$ is equivalent to one only involving derivatives in the z_i.*

PROOF. Let $\mathscr{E} = \Omega^{0,*}(\mathbb{C}^n, V)$ denote the space of fields of our theory, and let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})^{hol}$ denote those functions which are translation-invariant and are in the kernel of the operators η_i, and let $\mathscr{O}_{loc}(\mathscr{E})^{hol'}$ denote those which in addition have only z_i derivatives. We will show that the inclusion map

$$\mathscr{O}_{loc}(\mathscr{E})^{hol'} \to \mathscr{O}_{loc}(\mathscr{E})^{hol}$$

is a quasi-isomorphism, where both are equipped with just the $\overline{\partial}$ differential. Both sides are graded by polynomial degree of the local functional, so it suffices to show this for local functionals of a fixed degree.

Note that the space V is filtered, by saying that F^i consists of those elements of degrees $\geq i$. This induces a filtration on $\mathscr{E}$ by the subspaces $\Omega^{0,*}(\mathbb{C}^n, F^i V)$. After passing to the associated graded, the operator Q becomes $\overline{\partial}$. By considering a spectral sequence with respect to this filtration, we see that it suffices to show we have a quasi-isomorphism in the case $Q = \overline{\partial}$.

But this follows immediately from the fact that the inclusion

$$\mathbb{C}\left[\frac{\partial}{\partial z_i}\right] \hookrightarrow \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right]$$

is a quasi-isomorphism, where the right hand side is equipped with the differential $[\overline{\partial}, -]$. To see that this map is a quasi-isomorphism, note that the $\overline{\partial}$ operator sends η_i to $\frac{\partial}{\partial \overline{z}_i}$. □

Recall that the action functional I induces the structure of L_∞ algebra on $\Omega^{0,*}(\mathbb{C}^n, V)[-1]$ whose differential is Q, and whose L_∞ structure maps are encoded by the Taylor components of I. Under the hypothesis of the previous lemma, this L_∞ algebra is L_∞ equivalent to one which is the Dolbeault complex with coefficients in a translation-invariant local L_∞ algebra whose structure maps only involve z_i derivatives.

There are many natural examples of holomorphically translation-invariant classical field theories. Geometrically, they arise from holomorphic moduli problems. For instance, one could take the cotangent theory to the derived moduli of holomorphic G bundles on $\mathbb{C}^n$, or the cotangent theory to the derived moduli space of such bundles equipped with holomorphic sections of some associated bundles, or the cotangent theory to the moduli of holomorphic maps from C^n to some complex manifold.

As is explained in great detail in [], holomorphically translation-invariant field theories arise very naturally in physics as holomorphic (or minimal) twists of supersymmetric field theories in even dimensions.

16.2.1. A holomorphically translation invariant classical theory on $\mathbb{C}^n$ has a natural gauge fixing operator, namely

$$\overline{\partial}^* = -\sum \eta_i \frac{\partial}{\partial \overline{z}_i}.$$

Since $[\eta_i, Q_0] = 0$, we see that $[Q, \overline{\partial}^*] = [\overline{\partial}, \overline{\partial}^*]$ is the Laplacian. (More generally, we can consider a family of gauge fixing operators coming from the $\overline{\partial}^*$ operator for a family of flat Hermitian metrics on $\mathbb{C}^n$. Since the space of such metrics is $GL(n,\mathbb{C})/U(n)$ and thus contractible, we see that everything is independent up to homotopy of the choice of gauge fixing operator.)

We say a translation-invariant parametrix

$$\Phi \in \overline{\Omega}^{0,*}(\mathbb{C}^n, V)^{\otimes 2}$$

is *holomorphically translation-invariant* if

$$(\eta_i \otimes 1 + 1 \otimes \eta_i)\Phi = 0$$

for $i = 1, \dots, n$. For example, if Φ_0 is a parametrix for the scalar Laplacian

$$\triangle = -\sum \frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z}_i}$$

then

$$\Phi_0 \prod_{i=1}^{n} \mathrm{d}(\overline{z}_i - \overline{w}_i) c$$

defines such a parametrix. Here z_i and w_i indicate the coordinates on the two copies of $\mathbb{C}^n$, and $c \in V \otimes V$ is the inverse of the pairing on v. Clearly, we can find holomorphically translation-invariant parametrices which are supported arbitrarily close to the diagonal. This means that we can define a field theory by only considering $I[\Phi]$ for holomorphically translation-invariant parametrices Φ.

16.2.1.1 Definition. *A holomorphically translation-invariant quantization of a holomorphically translation-invariant classical field theory as above is a translation-invariant quantization such that for each holomorphically translation-invariant parametrix Φ, the effective interaction $I[\Phi]$ satisfies*

$$\eta_i I[\Phi] = 0$$

for $i = 1, \dots, n$. Here η_i abusively denotes the natural extension of the contraction η_i to a derivation on $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$.

The usual obstruction theory arguments hold for constructing holomorphically-translation invariant quantizations. At each order in $\hbar$, the obstruction-deformation complex is the subcomplex of the complex $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^n}$ of translation-invariant local functionals which are also in the kernel of the operators η_i.

16.2.1.2 Proposition. *A holomorphically translation-invariant quantum field theory on $\mathbb{C}^n$ leads to a holomorphically translation-invariant factorization algebra.*

PROOF. This follows immediately from proposition 16.1.1.2. Indeed, quantum observables form a smoothly translation-invariant factorization algebra. Such an observable O on U is specified by a family $O[\Phi] \in \mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ of functionals defined for each holomorphically translation-invariant parametrix Φ, which are supported on U for Φ sufficiently small. The operators $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i$ act in a natural way on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ by derivations, and each commutes with the renormalization group flow W_Ψ^Φ for holomorphically translation-invariant parametrices Ψ, Φ. Thus, $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}$ and η_i define derivations of the factorization algebra Obs^q. Explicitly, if $O \in \mathrm{Obs}^q(U)$ is an observable, then for each holomorphically translation-invariant parametrix Φ,

$$\left(\frac{\partial}{\partial z_i} O\right)[\Phi] = \frac{\partial}{\partial z_i}(O[\Phi]),$$

and similarly for $\frac{\partial}{\partial \overline{z}_i}$ and η_i.

By definition (Definition 5.1.1.1), a holomorphically translation-invariant factorization algebra is a translation-invariant factorization algebra where the derivation operator $\frac{\partial}{\partial \overline{z}_i}$ on observables is homotopically trivialized.

Note that, for a holomorphically translation-invariant parametrix Φ, $[\eta_i, \triangle_\Phi] = 0$ and η_i is a derivation for the Poisson bracket $\{-,-\}_\Phi$. It follows that

$$[Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi, \eta_i] = [Q, \eta_i]$$

as operators on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$. Since we wrote $Q = \overline{\partial} + Q_0$ and required that $[Q_0, \eta_i] = 0$, we have

$$[Q, \eta_i] = [\overline{\partial}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}.$$

Since the differential on $\mathrm{Obs}^q(U)$ is defined by

$$(\widehat{Q}O)[\Phi] = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi],$$

we see that $[\widehat{Q}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}$, as desired. □

As we showed in Chapter 5, a holomorphically translation invariant factorization algebra in one complex dimension, with some mild additional conditions, gives rise to a vertex algebra. Let us verify that these conditions hold in the examples of interest. We first need a definition.

16.2.1.3 Definition. *A holomorphically translation-invariant field theory on $\mathbb{C}$ is S^1-invariant if the following holds. First, we have an S^1 action on the vector space V, inducing an action of S^1 on the space $\mathscr{E} = \Omega^{0,*}(\mathbb{C}, V)$ of fields, by combining the S^1 action on V with the natural one* *on $\Omega^{0,*}(\mathbb{C})$ coming from rotation on $\mathbb{C}$. . We suppose that all the structures of the field theory are S^1-invariant. More precisely, the symplectic pairing on $\mathscr{E}$ and the differential Q on $\mathscr{E}$ must be S^1-invariant. Further, for every S^1-invariant parametrix Φ, the effective interaction $I[\Phi]$ is S^1-invariant.*

16.2.1.4 Lemma. *Suppose we have a holomorphically translation invariant field theory on $\mathbb{C}$ which is also S^1-invariant. Then, the corresponding factorization algebra satisfies the conditions stated in theorem 5.2.2.1 of Chapter 5 allowing us to construct a vertex algebra structure on the cohomology.*

PROOF. Let $\mathcal{F}$ denote the factorization algebra of observables of our theory. Note that if $U \subset \mathbb{C}$ is an S^1-invariant subset, then S^1 acts on $\mathcal{F}(U)$.

Recall that $\mathcal{F}$ is equipped with a complete decreasing filtration, and is viewed as a factorization algebra valued in pro-differentiable cochain complexes. Recall that we need to check the following properties.

(1) The S^1 action on $\mathcal{F}(D(0,r))$ extends to a smooth action of the algebra $\mathcal{D}(S^1)$ of distributions on S^1.

(2) Let $\operatorname{Gr}^m \mathcal{F}(D(0,r))$ denote the associated graded with respect to the filtration on $\mathcal{F}(D(0,r))$. Let $\operatorname{Gr}^m_k \mathcal{F}(D(0,r))$ refer to the kth S^1-eigenspace in $\operatorname{Gr}^m \mathcal{F}(D(0,r))$. Then, we require that the map

$$\operatorname{Gr}^m_k \mathcal{F}(D(0,r)) \to \operatorname{Gr}^m_k \mathcal{F}(D(0,r'))$$

is a quasi-isomorphism of differentiable vector spaces.

(3) The differentiable vector space $H^*(\operatorname{Gr}^m_k \mathcal{F}(D(0,r)))$ is finite-dimensional for all k and is zero for $k \gg 0$.

Let us first check that the S^1 action extends to a $\mathcal{D}(S^1)$-action. If $\lambda \in S^1$ let ρ^*_λ denote this action. We need to check that for any observable $\{O[\Phi]\}$ and for every distribution $D(\lambda)$ on $\mathcal{S}^1$ the expression

$$\int_{\lambda \in S^1} D(\lambda) \rho^*_\lambda O[\Phi]$$

makes sense and defines another observable. Further, this construction must be smooth in both $D(\lambda)$ and the observable $O[\Phi]$, meaning that it must work families.

For fixed Φ, each $O_{i,k}[\Phi]$ is simply a distribution on $\mathbb{C}^k$ with some coefficients. For any distribution α on $\mathbb{C}^k$, the expression $\int_\lambda D(\lambda)\rho^*_\lambda \alpha$ makes sense and is continuous in both α and the distribution$D$. Indeed, $\int_\lambda D(\lambda)\rho^*_\lambda \alpha$ is simply the push-forward map in distributions applied to the action map $S^1 \times \mathbb{C}^k \to \mathbb{C}^k$.

It follows that, for each distribution D on S^1, we can define

$$D * O_{i,k}[\Phi] := \int_{\lambda \in S^1} D(\lambda) \rho_\lambda^* O_{i,k}[\Phi].$$

As a function of D and $O_{i,k}[\Phi]$, this construction is smooth. Further, sending an observable $O[\Phi]$ to $D * O[\Phi]$ commutes with the renormalization group flow (between S^1-equivariant parametrices). It follows that we can define a new observable $D * O$ by

$$(D * O)_{i,k}[\Phi] = D * (O_{i,k}[\Phi]).$$

Now, a family of observables O^x (parametrized by $x \in M$, a smooth manifold) is smooth if and only if the family of functionals $O^x_{i,k}[\Phi]$ are smooth for all i, k and all Φ. In fact one need not check this for all Φ, but for any collection of parametrices which includes arbitrarily small parametrices. If follows that the map sending D and O to $D * O$ is smooth, that is, takes smooth families to smooth families.

Let us now check the remaining assumptions of theorem 5.2.2.1. Let $\mathcal{F}$ denote the factorization algebra of quantum observables of the theory and let $\mathcal{F}_k$ denote the kth eigenspace of the S^1 action. We first need to check that the inclusion

$$\operatorname{Gr}^m_k \mathcal{F}(D(0,r)) \to \operatorname{Gr}^m_k \mathcal{F}(D(0,r'))$$

is a quasi-isomorphism for $r < r'$. We need it to be a quasi-isomorphism of completed filtered differentiable vector spaces. The space $\operatorname{Gr}^m \mathcal{F}(D(0,r))$ is a finite direct sum of spaces of the form

$$\overline{\Omega}_c^{0,*}(D(0,r)^l, V^{\boxtimes l})_{S_l}.$$

It thus suffices to check that for the map

$$\overline{\Omega}_c^{0,*}(D(0,r)^m) \to \overline{\Omega}_c^{0,*}(D(0,r')^m)$$

is a quasi-isomorphism on each S^1-eigenspace. This is immediate.

The same holds to check that the cohomology of $\operatorname{Gr}^m_k \mathcal{F}(D(0,r))$ is zero for $k \gg 0$ and that it is finite-dimensional as a differentiable vector space. ☐

We have seen that any S^1-equivariant and holmorphically translation-invariant factorization algebra on $\mathbb{C}$ gives rise to a vertex algebra. We have also seen that the obstruction-theory method applies in this situation to construct holomorphically translation invariant factorization algebras from appropriate Lagrangians. In this way, we have a very general method for constructing vertex algebras.

16.3. Renormalizability and factorization algebras

A central concept in field theory is that of *renormalizability*. This is discussed in detail in [Cos11c]. The basic idea is the following.

The group $\mathbb{R}_{>0}$ acts on the collection of field theories on $\mathbb{R}^n$, where the action is induced from the scaling action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This action is implemented differently in different models for field theories. In the language if factorization algebras it is very simple, because any factorization algebra on $\mathbb{R}^n$ can be pushed forward under any diffeomorphism of $\mathbb{R}^n$ to yield a new factorization algebra on $\mathbb{R}^n$. Push-forward of factorization algebras under the map $x \mapsto \lambda^{-1}x$ (for $\lambda \in \mathbb{R}_{>0}$) defines the renormalization group flow on factorization algebra.

We will discuss how to implement this rescaling in the definition of field theory given in [**Cos11c**] shortly. The main result of this section is the statement that the map which assigns to a field theory the corresponding factorization algebra of observables intertwines the action of $\mathbb{R}_{>0}$.

Acting by elements $\lambda \in \mathbb{R}_{>0}$ on a fixed quantum field theory produces a one-parameter family of theories, depending on λ. Let F denote a fixed theory, either in the language of factorization algebras, the language of [**Cos11c**], or any other approach to quantum field theory. We will call this family of theories $\rho_\lambda(F)$. We will view the theory $\rho_\lambda(F)$ as being obtained from F by "zooming in" on $\mathbb{R}^n$ by an amount dicated by λ, if $\lambda < 1$, or by zooming out if $\lambda > 1$.

We should imagine the theory F as having some number of continuous parameters, called coupling constants. Classically, the coupling constants are simply constants appearing next to various terms in the Lagrangian. At the quantum level, we could think of the structure constants of the factorization algebra as being functions of the coupling constants (we will discuss this more precisely below).

Roughly speaking, a theory is *renormalizable* if, as $\lambda \to 0$, the family of theories $\rho_\lambda(F)$ converges to a limit. While this definition is a good one non-pertubatively, in perturbation theory it is not ideal. The reason is that often the coupling constants depend on the scale through quantities like $\lambda^\hbar$. If $\hbar$ was an actual real number, we could analyze the behaviour of $\lambda^\hbar$ for λ small. In perturbation theory, however, $\hbar$ is a formal parameter, and we must expand $\lambda^\hbar$ in a series of the form $1 + \hbar \log \lambda + \dots$. The coefficients of this series always grow as $\lambda \to 0$.

In other words, from a perturbative point of view, one can't tell the difference between a theory that has a limit as $\lambda \to 0$ and a theory whose coupling constants have logarithmic growth in λ.

This motivates us to define a theory to be *perturbatively renormalizable* if it has logarithmic growth as $\lambda \to 0$. We will introduce a formal definition of perturbative renormalizability shortly. Let us first indicate why this definition is important.

It is commonly stated (especially in older books) that perturbative renormalizability is a necessary condition for a theory to exist (in perturbation theory) at the quantum level.

This is *not* the case. Instead, renormalizability is a criterion which allows one to select a finite-dimensional space of well-behaved quantizations of a given classical field theory, from a possibly infinite dimensional space of all possible quantizations.

There are other criteria which one wants to impose on a quantum theory and which also help select a small space of quantizations: for instance, symmetry criteria. (In addition, one also requires that the quantum master equation holds, which is a strong constraint. This, however, is part of the definition of a field theory that we use). There are examples of non-renormalizable field theories for which nevertheless a unique quantization can be selected by other criteria. (An example of this nature is BCOV theory).

16.3.1. The renormalization group action on factorization algebras. Let us now discuss the concept of renormalizability more formally. We will define the action of the group $\mathbb{R}_{>0}$ on the set of theories in the definition used in [Cos11c], and on the set of factorization algebras on $\mathbb{R}^n$. We will see that the map which assigns a factorization algebra to a theory is $\mathbb{R}_{>0}$-equivariant.

Let us first define the action of $\mathbb{R}_{>0}$ on the set of factorization algebras on $\mathbb{R}^n$.

16.3.1.1 Definition. *If $\mathcal{F}$ is a factorization algebra on $\mathbb{R}^n$, and $\lambda \in \mathbb{R}_{>0}$, let $\rho_\lambda(\mathcal{F})$ denote the factorization algebra on $\mathbb{R}^n$ which is the push-forward of $\mathcal{F}$ under the diffeomorphism $\lambda^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given by multiplying by λ^{-1}. Thus,*

$$\rho_\lambda(\mathcal{F})(U) = \mathcal{F}(\lambda(U))$$

and the product maps in $\rho_\lambda(\mathcal{F})$ arise from those in $\mathcal{F}$. We will call this action of $\mathbb{R}_{>0}$ on the collection of factorization algebras on $\mathbb{R}^n$ the local renormalization group *action.*

Thus, the action of $\mathbb{R}_{>0}$ on factorization algebras on $\mathbb{R}^n$ is simply the obvious action of diffeomorphisms on $\mathbb{R}^n$ on factorization algebras on $\mathbb{R}^n$.

16.3.2. The renormalization group flow on classical theories. The action on field theories as defined in [Cos11c] is more subtle. Let us start by describing the action of $\mathbb{R}_{>0}$ on classical field theories. Suppose we have a translation-invariant classical field on $\mathbb{R}^n$, with space of fields $\mathscr{E}$. The space $\mathscr{E}$ is the space of sections of a trivial vector bundle on $\mathbb{R}^n$ with fibre E_0. The vector space E_0 is equipped with a degree -1 symplectic pairing valued in the line ω_0, the fibre of the bundle of top forms on $\mathbb{R}^n$ at 0. We also, of course, have a translation-invariant local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

Let us choose an action ρ_λ^0 of the group $\mathbb{R}_{>0}$ on the vector space E_0 with the property that the symplectic pairing on E_0 is $\mathbb{R}_{>0}$-equivariant, where the action of $\mathbb{R}_{>0}$ acts on the line ω_0 with weight $-n$. Let us further assume that this action is diagonalizable, and that the eigenvalues of ρ_λ^0 are rational integer powers of λ. (In practise, only integer or half-integer powers appear).

The choice of such an action, together with the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling, induces an action of $\mathbb{R}_{>0}$ on

$$\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$$

which sends

$$\phi \otimes e_0 \mapsto \phi(\lambda^{-1}x)\rho^0_\lambda(e_0),$$

where $\phi \in C^\infty(\mathbb{R}^n)$ and $e_0 \in E_0$. The convention that $x \mapsto \lambda^{-1}x$ means that for small λ, we are looking at small scales (for instance, as $\lambda \to 0$ the metric becomes large).

This action therefore induces an action on spaces associated to $\mathscr{E}$, such as the spaces $\mathscr{O}(\mathscr{E})$ of functionals and $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals. The compatibility between the action of $\mathbb{R}_{>0}$ and the symplectic pairing on E_0 implies that the Poisson bracket on the space $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals on $\mathscr{E}$ is preserved by the $\mathbb{R}_{>0}$ action. Let us denote the action of $\mathbb{R}_{>0}$ on $\mathscr{O}_{loc}(\mathscr{E})$ by ρ_λ.

16.3.2.1 Definition. *The local renormalization group flow on the space of translation-invariant classical field theories sends a classical action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ to $\rho_\lambda(I)$.*

This definition makes sense, because ρ_λ preserves the Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$. Note also that, if the action of ρ^0_λ on E_0 has eigenvalues in $\frac{1}{n}\mathbb{Z}$, then the action of ρ_λ on the space $\mathscr{O}_{loc}(\mathscr{E})$ is diagonal and has eigenvalues again in $\frac{1}{n}\mathbb{Z}$.

The action of $\mathbb{R}_{>0}$ on the space of classical field theories up to isomorphism is independent of the choice of action of $\mathbb{R}_{>0}$ on E_0. If we choose a different action, inducing a different action ρ'_λ of $\mathbb{R}_{>0}$ on everything, then $\rho_\lambda I$ and $\rho'_\lambda I$ are related by a linear and symplectic change of coordinates on the space of fields which covers the identity on $\mathbb{R}^n$. Field theories related by such a change of coordinates are equivalent.

It is often convenient to choose the action of $\mathbb{R}_{>0}$ on the space E_0 so that the quadratic part of the action is invariant. When we can do this, the local renormalization group flow acts only on the interactions (and on any small deformations of the quadratic part that one considers). Let us give some examples of the local renormalization group flow on classical field theories. Many more details are given in [Cos11c].

Consider the free massless scalar field theory on $\mathbb{R}^n$. The complex of fields is

$$C^\infty(\mathbb{R}^n) \xrightarrow{\mathrm{D}} C^\infty(\mathbb{R}^n).$$

We would like to choose an action of $\mathbb{R}_{>0}$ so that both the symplectic pairing and the action functional $\int \phi \mathrm{D} \phi$ are invariant. This action must, of course, cover the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling. If ϕ, ψ denote fields in the copies of $C^\infty(\mathbb{R}^n)$ in degrees 0 and 1 respectively, the desired action sends

$$\rho_\lambda(\phi(x)) = \lambda^{\frac{2-n}{2}}\phi(\lambda^{-1}x)$$
$$\rho_\lambda(\psi(x)) = \lambda^{\frac{-n-2}{2}}\psi(\lambda^{-1}x).$$

Let us then consider how ρ_λ acts on possible interactions. We find, for example, that if

$$I_k(\phi) = \int \phi^k$$

then

$$\rho_\lambda(I_k) = \lambda^{n-\frac{k(n-2)}{2}} I_k.$$

16.3.2.2 Definition. *A classical theory is* renormalizable *if, as $\lambda \to 0$, it flows to a fixed point under the local renormalization group flow.*

For instance, we see that in dimension 4, the most general renormalizable classical action for a scalar field theory which is invariant under the symmetry $\phi \mapsto -\phi$ is

$$\int \phi \,\mathrm{D}\, \phi + m^2\phi^2 + c\phi^4.$$

Indeed, the ϕ^4 term is fixed by the local renormalization group flow, whereas the ϕ^2 term is sent to zero as $\lambda \to 0$.

16.3.2.3 Definition. *A classical theory is* strictly renormalizable *if it is a fixed point under the local renormalization group flow.*

A theory which is renormalizable has good small-scale behaviour, in that the coupling constants (classically) become small at small scales. (At the quantum level there may also be logaritmic terms which we will discuss shortly). A renormalizable theory may, however, have bad large-scale behaviour: for instance, in four dimensions, a mass term $\int \phi^2$ becomes large at large scales. A strictly renormalizable theory is one which is classically scale invariant. At the quantum level, we will define a strictly renormalizable theory to be one which is scale invariant up to logarithmic corrections.

Again in four dimensions, the only strictly renormalizable interaction for the scalar field theory which is invariant under $\phi \mapsto -\phi$ is the ϕ^4 interaction. In six dimensions, the ϕ^3 interaction is strictly renormalizable, and in three dimensions the ϕ^6 interaction (together with finitely many other interactions involving derivatives) are strictly renormalizable.

As another example, recall that the graded vector space of fields of pure Yang-Mills theory (in the first order formalism) is

$$\left(\Omega^0[1] \oplus \Omega^1 \oplus \Omega^2_+ \oplus \Omega^2_+[-1] \oplus \Omega^3[-1] \oplus \Omega^4[-2]\right) \otimes \mathfrak{g}.$$

(Here Ω^i indicates forms on $\mathbb{R}^4$). The action of $\mathbb{R}_{>0}$ is the natural one, coming from pull-back of forms under the map $x \mapsto \lambda^{-1}x$. The Yang-Mills action functional

$$S(A,B) = \int F(A) \wedge B + B \wedge B$$

is obviously invariant under the action of $\mathbb{R}_{>0}$, since it only involves wedge product and integration, as well as projection to Ω^2_+. (Here $A \in \Omega^1 \otimes \mathfrak{g}$) and $B \in \Omega^2_+ \otimes \mathfrak{g}$). The other terms in the full BV action functional are also invariant, because the symplectic pairing on the space of fields and the action of the gauge group are both scale-invariant.

Something similar holds for Chern-Simons theory on $\mathbb{R}^3$, where the space of fields is $\Omega^*(\mathbb{R}^3) \otimes \mathfrak{g}[1]$. The action of $\mathbb{R}> 0$ is by pull-back by the map $x \mapsto \lambda^{-1}x$, and the Chern-Simons functional is obviously invariant.

16.3.2.4 Lemma. *The map which assigns to a translation-invariant classical field theory on $\mathbb{R}^n$ the associated P_0 factorization algebra commutes with the action of the local renormalization group flow.*

PROOF. The action of $\mathbb{R}_{>0}$ on the space of fields of the theory induces an action on the space $\mathrm{Obs}^{cl}(\mathbb{R}^n)$ of classical observables on $\mathbb{R}^n$, by sending an observable O (which is a functon on the space $\mathscr{E}(\mathbb{R}^n)$ of fields) to the observable

$$\rho_\lambda O : \phi \mapsto O(\rho_\lambda(\phi)).$$

This preserves the Poisson bracket on the subspace $\widetilde{\mathrm{Obs}}^{cl}(\mathbb{R}^n)$ of functionals with smooth first derivative, because by assumption the symplectic pairing on the space of fields is scale invariant. Further, it is immediate from the definition of the local renormalization group flow on classical field theories that

$$\rho_\lambda\{S, O\} = \{\rho_\lambda(I), \rho_\lambda(O)\}$$

where $S \in \mathscr{O}_{loc}(\mathscr{E})$ is a translation-invariant solution of the classical master equation (whose quadratic part is elliptic).

Let $\mathrm{Obs}^{cl}_\lambda$ denote the factorization algebra on $\mathbb{R}^n$ coming from the theory $\rho_\lambda(S)$ (where S is some fixed classical action). Then, we see that we have an isomorphism of cochain complexes

$$\rho_\lambda : \mathrm{Obs}^{cl}(\mathbb{R}^n) \cong \mathrm{Obs}^{cl}_\lambda(\mathbb{R}^n).$$

We next need to check what this isomorphism does to the support conditions. Let $U \subset \mathbb{R}^n$ and let $O \in \mathrm{Obs}^{cl}(U)$ be an observable supported on U. Then, one can check easily that $\rho_\lambda(O)$ is supported on $\lambda^{-1}(U)$. Thus, ρ_λ gives an isomorphism

$$\mathrm{Obs}^{cl}(U) \cong \mathrm{Obs}^{cl}_\lambda(\lambda^{-1}(U)).$$

and so,

$$\mathrm{Obs}^{cl}(\lambda U) \cong \mathrm{Obs}^{cl}_\lambda(U).$$

The factorization algebra

$$\rho_\lambda \operatorname{Obs}^{cl} = \lambda_* \operatorname{Obs}^{cl}$$

assigns to an open set $U \subset \mathbb{R}^n$ the value of Obs^{cl} on $\lambda(U)$. Thus, we have constructed an isomorphism of precosheaves on $\mathbb{R}^n$,

$$\rho_\lambda \operatorname{Obs}^{cl} \cong \operatorname{Obs}^{cl}_\lambda .$$

This isomorphism compatible with the commutative product and the (homotopy) Poisson bracket on both side, as well as the factorization product maps. □

16.3.3. The renormalization group flow on quantum field theories. The most interesting version of the renormalization group flow is, of course, that on quantum field theories. Let us fix a classical field theory on $\mathbb{R}^n$, with space of fields as above $\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$ where E_0 is a graded vector space. In this section we will define an action of the group $\mathbb{R}_{>0}$ on the simplicial set of quantum field theories with space of fields $\mathscr{E}$, quantizing the action on classical field theories that we constructed above. We will show that the map which assigns to a quantum field theory the corresponding factorization algebra commutes with this action.

Let us assume, for simplicity, that we have chosen the linear action of $\mathbb{R}_{>0}$ on E_0 so that it leaves invariant a quadratic action functional on $\mathscr{E}$ defining a free theory. Let $Q : \mathscr{E} \to \mathscr{E}$ be the corresponding cohomological differential, which, by assumption, is invariant under the $\mathbb{R}_{>0}$ action. (This step is not necessary, but will make the exposition simpler).

Let us also assume (again for simplicity) that there exists a gauge fixing operator $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the property that

$$\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^k Q^{GF}$$

for some $k \in \mathbb{Q}$. For example, for a massless scalar field theory on $\mathbb{R}^n$, we have seen that the action of $\mathbb{R}_{>0}$ on the space $C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n)[-1]$ of fields sends ϕ to $\lambda^{\frac{2-n}{2}} \phi(\lambda^{-1}x)$ and ψ to $\lambda^{\frac{-2-n}{2}} \psi(\lambda^{-1}x)$ (where ϕ is the field of cohomological degree 0 and ψ is the field of cohomological degree 1). The gauge fixing operator is the identity operator from $C^\infty(\mathbb{R}^n)[-1]$ to $C^\infty(\mathbb{R}^n)[0]$. In this case, we have $\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^2 Q^{GF}$.

As another example, consider pure Yang-Mills theory on $\mathbb{R}^4$. The fields, as we have described above, are built from forms on $\mathbb{R}^4$, equipped with the natural action of $\mathbb{R}_{>0}$. The gauge fixing operator is d^*. It is easy to see that $\rho_\lambda d^* \rho_{-\lambda} = \lambda^2 d^*$. The same holds for Chern-Simons theory, which also has a gauge fixing operator defined by d^* on forms.

A translation-invariant quantum field theory is defined by a family

$$\{I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})^{\mathbb{R}^n}[[\hbar]] \mid \Phi \text{ a translation-invariant parametrix}\}$$

which satisfies the renormalization group equation, quantum master equation, and the locality condition. We need to explain how scaling of $\mathbb{R}^n$ by $\mathbb{R}_{>0}$ acts on the (simplicial) set of quantum field theories. To do this, we first need to explain how this scaling action acts on the set of parametrices.

16.3.3.1 Lemma. *If Φ is a translation-invariant parametrix, then $\lambda^k \rho_\lambda(\Phi)$ is also a parametrix, where as above k measures the failure of Q^{GF} to commute with ρ_λ.*

PROOF. All of the axioms characterizing a parametrix are scale invariant, except the statement that

$$([Q, Q^{GF}] \otimes 1)\Phi = K_{id} - \text{ something smooth.}$$

We need to check that $\lambda^k \rho_\lambda \Phi$ also satisfies this. Note that

$$\rho_\lambda([Q, Q^{GF}] \otimes 1)\Phi = \lambda^k([Q, Q^{GF}] \otimes 1)\rho_\lambda(\Phi)$$

since ρ_λ commutes with Q but not with Q^{GF}. Also, ρ_λ preserves K_{id} and smooth kernels, so the desired identity holds. ☐

This lemma suggests a way to define the action of the group $\mathbb{R}_{>0}$ on the set of quantum field theories.

16.3.3.2 Lemma. *If $\{I[\Phi]\}$ is a theory, define $I_\lambda[\Phi]$ by*

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

Then, the collection of functionals $\{I_\lambda[\Phi]\}$ define a new theory.

On the right hand side of the equation in the lemma, we are using the natural action of ρ_λ on all spaces associated to $\mathscr{E}$, such as the space $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$ (to define $\rho_{-\lambda}(\Phi)$) and the space of functions on $\mathscr{E}$ (to define how ρ_λ acts on the function $I[\lambda^{-k}\rho_{-\lambda}(\Phi)]$).

Note that this lemma, as well as most things we discuss about renormalizability of field theories which do not involve factorization algebras, is discussed in more detail in [Cos11c], except that there the language of heat kernels is used. We will prove the lemma here anyway, because the proof is quite simple.

PROOF. We need to check that $I_\lambda[\Phi]$ satisfies the renormalization group equation, locality action, and quantum master equation. Let us first check the renormalization group flow. As a shorthand notation, let us write Φ_λ for the parametrix $\lambda^k\rho_\lambda(\Phi)$. Then, note that the propagator $P(\Phi_\lambda)$ is

$$P(\Phi_\lambda) = \rho_\lambda P(\Phi).$$

Indeed,

$$\begin{aligned}\rho_\lambda \tfrac{1}{2}(Q^{GF} \otimes 1 + 1 \otimes Q^{GF})\Phi &= \lambda^k \tfrac{1}{2}(Q^{GF} \otimes 1 + 1 \otimes Q^{GF})\rho_\lambda(\Phi)\\ &= P(\Phi_\lambda).\end{aligned}$$

It follows from this that, for all functionals $I \in \mathscr{O}_P^+(\mathscr{E})[[\hbar]]$,

$$\rho_\lambda(W(P(\Phi) - P(\Psi), I) = W(P(\Phi_\lambda) - P(\Psi_\lambda), \rho_\lambda(I).)$$

We need to verify the renormalization group equation, which states that

$$W(P(\Phi) - P(\Psi), I_\lambda[\Psi]) = I_\lambda[\Phi].$$

Because $I_\lambda[\Phi] = \rho_\lambda I[\Phi_{-\lambda}]$, this is equivalent to

$$\rho_{-\lambda} W(P(\Phi) - P(\Psi), \rho_\lambda(I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}].$$

Bringing $\rho_{-\lambda}$ inside the W reduces us to proving the identity

$$W(P(\Phi_{-\lambda} - P(\Psi_{-\lambda}, I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}]$$

which is the renormalization group identity for the functionals $I[\Phi]$.

The fact that $I_\lambda[\Phi]$ satisfies the quantum master equation is proved in a similar way, using the fact that

$$\rho_\lambda(\triangle_\Phi I) = \triangle_{\Phi_\lambda}\rho_\lambda(I)$$

where $\triangle_\Phi$ denotes the BV Laplacian associated to Φ and I is any functional.

Finally, the locality axiom is an immediate consequence of that for the original functionals $I[\Phi]$. ☐

16.3.3.3 Definition. *Define the* local renormalization group flow *to be the action of* $\mathbb{R}_{>0}$ *on the set of theories which sends, as in the previous lemma, a theory* $\{I[\Phi]\}$ *to the theory*

$$\{I_\lambda[\Phi]\} = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}\Phi]).$$

Note that this works in families, and so defines an action of $\mathbb{R}_{>0}$ *on the simplicial set of theories.*

Note that this definition simply means that we act by $\mathbb{R}_{>0}$ on everything involved in the definition of a theory, including the parametrices.

Let us now quote some results from [**Cos11c**], concerning the behaviour of this action. Let us recall that to begin with, we chose an action of $\mathbb{R}_{>0}$ on the space $\mathscr{E} = C^\infty(\mathbb{R}^4) \otimes E_0$ of fields, which arose from the natural rescaling action on $C^\infty(\mathbb{R}^4)$ and an action on the finite-dimensional vector space E_0. We assumed that the action on E_0 is diagonalizable, where on each eigenspace ρ_λ acts by λ^a for some $a \in \mathbb{Q}$. Let $m \in \mathbb{Z}$ be such that the exponents of each eigenvalue are in $\frac{1}{m}\mathbb{Z}$.

16.3.3.4 Theorem. *For any theory* $\{I[\Phi]\}$ *and any parametrix* Φ*, the family of functionals* $I_\lambda[\Phi]$ *depending on* λ *live in*

$$\mathscr{O}^+_{sm,P}(\mathscr{E})\left[\log\lambda, \lambda^{\frac{1}{m}}, \lambda^{-\frac{1}{m}}\right][[\hbar]].$$

In other words, the functionals $I_\lambda[\Phi]$ depend on λ only through polynomials in $\log \lambda$ and $\lambda^{\pm \frac{1}{m}}$. (More precisely, each functional $I_{\lambda,i,k}[\Phi]$ in the Taylor expansion of $I_\lambda[\Phi]$ has such polynomial dependence, but as we quantify over all i and k the degree of the polynomials may be arbitrarily large).

In [**Cos11c**], this result is only stated under the hypothesis that $m = 2$, which is the case that arises in most examples, but the proof in [**Cos11c**] works in general.

16.3.3.5 Lemma. *The action of* $\mathbb{R}_{>0}$ *on quantum field theories lifts that on classical field theories described earlier.*

This basic point is also discussed in [**Cos11c**]; it follows from the fact that at the classical level, the limit of $I[\Phi]$ as $\Phi \to 0$ exists and is the original classical interaction.

16.3.3.6 Definition. *A quantum theory is* renormalizable *if the functionals* $I_\lambda[\Phi]$ *depend on* λ *only by polynomials in* $\log \lambda$ *and* $\lambda^{\frac{1}{m}}$ *(where we assume that* $m > 0$*). A quantum theory is* strictly renormalizable *if it only depends on* λ *through polynomials in* $\log \lambda$*.*

Note that at the classical level, a strictly renormalizable theory must be scale-invariant, because logarithmic contributions to the dependence on λ only arise at the quantum level.

16.3.4. Quantization of renormalizable and strictly renormalizable theories.

Let us decompose $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$, the space of translation-invariant local functionals on $\mathscr{E}$, into eigenspaces for the action of $\mathbb{R}_{>0}$. For $k \in \frac{1}{m}\mathbb{Z}$, we let $\mathscr{O}_{loc}^{(k)}(\mathscr{E})^{\mathbb{R}^4}$ be the subspace on which ρ_λ acts by λ^k. Let $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ denote the direct sum of all the non-negative eigenspaces.

Let us suppose that we are interested in quantizing a classical theory, given by an interaction I, which is either strictly renormalizable or renormalizable. In the first case, I is in $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$, and in the second, it is in $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$.

By our initial assumptions, the Lie bracket on $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$ commutes with the action of $\mathbb{R}_{>0}$. Thus, if we have a strictly renormalizable classical theory, then $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex with differential $Q + \{I-,\}$. This is the cochain complex controlling first-order deformations of our classical theory as a strictly renormalizable theory. In physics terminology, this is the cochain complex of marginal deformations.

If we start with a classical theory which is simply renormalizable, then the space $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex under the differential $Q + \{I-,\}$. This is the cochain complex of renormalizable deformations.

Typically, the cochain complexes of marginal and renormalizable deformations are finite-dimensional. (This happens, for instance, for scalar field theories in dimensions greater than 2).

Here is the quantization theorem for renormalizable and strictly renormalizalble quantizations.

16.3.4.1 Theorem. *Fix a classical theory on $\mathbb{R}^n$ which is renormalizable with classical interaction I. Let $\mathcal{R}^{(n)}$ denote the set of renormalizable quantizations defined modulo $\hbar^{n+1}$. Then, given any element in $\mathcal{R}^{(n)}$, there is an obstruction to quantizing to the next order, which is an element*

$$O_{n+1} \in H^1\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}, Q + \{I, -\}\right).$$

If this obstruction vanishes, then the set of quantizations to the next order is a torsor for $H^0\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}\right)$.

This statement holds in the simplicial sense too: if $\mathcal{R}_{\triangle}^{(n)}$ denotes the simplicial set of renormalizable theories defined modulo $\hbar^{n+1}$ and quantizing a given classical theory, then there is a homotopy fibre diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{R}_{\triangle}^{(n+1)} \arrow[r] \arrow[d] & \mathcal{R}_{\triangle}^{(n)} \arrow[d] \\
0 \arrow[r] & \mathrm{DK}\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}[1], Q + \{I, -\}\right)
\end{tikzcd}$$

On the bottom right DK *indicates the Dold-Kan functor from cochain complexes to simplicial sets.*

All of these statements hold for the (simplicial) sets of strictly renormalizable theories quantizing a given strictly renormalizable classical theory, except that we should replace $\mathscr{O}_{loc}^{(\geq 0)}$ by $\mathscr{O}_{loc}^{(0)}$ everywhere. Further, all these results hold in families iwth evident modifications.

Finally, if $\mathscr{GF}$ denotes the simplicial set of translation-invariant gauge fixing conditions for our fixed classical theory (where we only consider gauge-fixing conditions which scale well with respect to ρ_λ as discussed earlier), then the simplicial sets of (strictly) renormalizable theories with a fixed gauge fixing condition are fibres of a simplicial set fibred over $\mathcal{GF}$. As before, this means that the simplicial set of theories is independent up to homotopy of the choice of gauge fixing condition.

This theorem is proved in [**Cos11c**], and is the analog of the quantization theorem for theories without the renormalizability criterion.

Let us give some examples of how this theorem allows us to construct small-dimensional families of quantizations of theories where without the renormalizability criterion there would be an infinite dimensional space of quantizations.

Consider, as above, the massless ϕ^4 theory on $\mathbb{R}^4$, with interaction $\int \phi D \phi + \phi^4$. At the classical level this theory is scale-invariant, and so strictly renormalizable. We have the following.

16.3.4.2 Lemma. *The space of strictly-renormalizable quantizations of the massless ϕ^4 theory in 4 dimensions which are also invariant under the $\mathbb{Z}/2$ action $\phi \mapsto -\phi$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

PROOF. We need to check that the obstruction group for this problem is zero, and the deformation group is one-dimensional. The obstruction group is zero for degree reasons, because for a theory without gauge symmetry the complex of local functionals is concentrated in degrees ≤ 0. To compute the deformation group, note that the space of local functionals which are scale invariant and invariant under $\phi \mapsto -\phi$ is two-dimensional, spanned by $\int \phi^4$ and $\int \phi D \phi$. The quotient of this space by the image of the differential $Q + \{I, -\}$ is one dimensional, because we can eliminate one of the two possible terms by a change of coordinates in ϕ. ☐

Let us give another, and more difficult, example.

16.3.4.3 Theorem. *The space of renormalizable (or strictly renormalizable) quantizations of pure Yang-Mills theory on $\mathbb{R}^4$ with simple gauge Lie algebra $\mathfrak{g}$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

PROOF. The relevant cohomology groups were computed in [Cos11c], where it was shown that the deformation group is one dimensional and that the obstruction group is $H^5(\mathfrak{g})$. The obstruction group is zero unless $\mathfrak{g} = \mathrm{sl}_n$ and $n \geq 3$. By considering the outer automorphisms of sl_n, it was argued in [Cos11c] that the obstruction must always vanish. ☐

This theorem then tells us that we have an essentially canonical quantization of pure Yang-Mills theory on $\mathbb{R}^4$, and hence a corresonding factorization algebra.

The following is the main new result of this section.

16.3.4.4 Theorem. *The map from translation-invariant quantum theories on $\mathbb{R}^n$ to factorization algebras on $\mathbb{R}^n$ commutes with the local renormalization group flow.*

PROOF. Suppose we have a translation-invariant quantum theory on $\mathbb{R}^n$ with space of fields $\mathscr{E}$ and family of effective actions $\{I[\Phi]\}$. Recall that the RG flow on theories sends this theory to the theory defined by

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

We let $\Phi_\lambda = \lambda^k \rho_\lambda \Phi$. As we have seen in the proof of lemma 16.3.3.2, we have

$$\begin{aligned} P(\Phi_\lambda) &= \rho_\lambda(P(\Phi)) \\ \triangle_{\Phi_\lambda} &= \rho_\lambda(\triangle_\Phi). \end{aligned}$$

Suppose that $\{O[\Phi]\}$ is an observable for the theory $\{I[\Phi]\}$. First, we need to show that

$$O_\lambda[\Phi] = \rho_\lambda(O[\Phi_{-\lambda}])$$

is an observable for the theory $O_\lambda[\Phi]$. The fact that $O_\lambda[\Phi]$ satisfies the renormalization group flow equation is proved along the same lines as the proof that $I_\lambda[\Phi]$ satisfies the renormalization group flow equation in lemma 16.3.3.2.

If Obs^q_λ denotes the factorization algebra for the theory I_λ, then we have constructed a map

$$\begin{aligned} \mathrm{Obs}^q(\mathbb{R}^n) &\to \mathrm{Obs}^q_\lambda(\mathbb{R}^n) \\ \{O[\Phi]\} &\mapsto \{O_\lambda[\Phi]\}. \end{aligned}$$

The fact that $\triangle_{\Phi_\lambda} = \rho_\lambda(\triangle_\Phi)$ implies that this is a cochain map. Further, it is clear that this is a smooth map, and so a map of differentiable cochain complexes.

Next we need to check is the support condition. We need to show that if $\{O[\Phi]\}$ is in $\mathrm{Obs}^q(U)$, where $U \subset \mathbb{R}^n$ is open, then $\{O_\lambda[\Phi]\}$ is in $\mathrm{Obs}^q(\lambda^{-1}(U))$. Recall that the support condition states that, for all i, k, there is some parametrix Φ_0 and a compact set $K \subset U$ such that $O_{i,k}[\Phi]$ is supported in K for all $\Phi \leq \Phi_0$.

By making Φ_0 smaller if necessary, we can assume that $O_{i,k}[\Phi_\lambda]$ is supported on K for $\Phi \leq \Phi_0$. (If Φ is supported within ε of the diagonal, then Φ_λ is supported within $\lambda^{-1}\varepsilon$.) Then, $\rho_\lambda O_{i,k}[\Phi_\lambda]$ will be supported on $\lambda^{-1}K$ for all $\Phi \leq \Phi_0$. This says that O_λ is supported on $\lambda^{-1}K$ as desired.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^q_\lambda(\lambda^{-1}(U)).$$

This isomorphism is compatible with inclusion maps and with the factorization product. Therefore, we have an isomorphism of factorization algebras

$$(\lambda^{-1})_* \mathrm{Obs}^q \cong \mathrm{Obs}^q_\lambda$$

where $(\lambda^{-1})_*$ indicates pushforward under the map given by multiplication by λ^{-1}. Since the action of the local renormalization group flow on factorization algebras on $\mathbb{R}^n$ sends $\mathcal{F}$ to $(\lambda^{-1})_*\mathcal{F}$, this proves the result. □

The advantage of the factorization algebra formulation of the local renormalization group flow is that it is very easy to define; it captures precisely the intuition that the renormalization group flow arises the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This theorem shows that the less-obvious definition of the renormalization group flow on theories, as defined in [**Cos11c**], coincides with the very clear definition in the language of factorization algebras. The advantage of the definition presented in [**Cos11c**] is that it is possible to compute with this definition, and that the relationship between this definition and how physicists define the β-function is more or less clear. For example, the one-loop β-function (one-loop contribution to the renormalization group flow) is calculated explicitly for the ϕ^4 theory in [**Cos11c**].

16.4. Cotangent theories and volume forms

In this section we will examine the case of a cotangent theory, in which our definition of a quantization of a classical field theory acquires a particularly nice interpretation. Suppose that $\mathcal{L}$ is an elliptic L_∞ algebra on a manifold M describing an elliptic moduli problem, which we denote by $B\mathcal{L}$. As we explained in Chapter **??**, section 11.6, we can construct a classical field theory from $\mathcal{L}$, whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$. The main observation of this section is that a quantization of this classical field theory can be interpreted as a kind of "volume form" on the elliptic moduli problem $B\mathcal{L}$. This point of view was developed in [**Cos11b**], and used in [**Cos11a**] to provide a geometric interpretation of the Witten genus.

The relationship between quantization of field theories and volume forms was discussed already at the very beginning of this book, in Chapter 2. There, we explained how to interpret (heuristically) the BV operator for a free field theory as the divergence operator for a volume form.

While this heuristic interpretation holds for many field theories, cotangent theories are a class of theories where this relationship becomes very clean. If we have a cotangent theory to an elliptic moduli problem $\mathcal{L}$ on a compact manifold, then the L_∞ algebra $\mathcal{L}(M)$ has finite-dimensional cohomology. Therefore, the formal moduli problem $B\mathcal{L}(M)$ is an honest finite-dimensional formal derived stack. We will find that a quantization of a cotangent theory leads to a volume form on $B\mathcal{L}(M)$ which is of a "local" nature.

Morally speaking, the partition function of a cotangent theory should be the volume of $B\mathcal{L}(M)$ with respect to this volume form. If, as we've been doing, we work in perturbation theory, then the integral giving this volume often does not converge. One has to replace $B\mathcal{L}(M)$ by a global derived moduli space of solutions to the equations of motion to have a chance at defining the volume. The volume form on a global moduli space is obtained by doing perturbation theory near every point and then gluing together the formal volume forms so obtained near each point.

This program has been successfully carried out in a number of examples, such as [**?**, **GG11**, **?**]. For example, in [**Cos11a**], the cotangent theory to the space of holomorphic maps from an elliptic curve to a complex manifold was studied, and it was shown that the partition function (defined in the way we sketched above) is the Witten elliptic genus.

16.4.1. A finite dimensional model. We first need to explain an algebraic interpretation of a volume form in finite dimensions. Let X be a manifold (or complex manifold or smooth algebraic variety; nothing we will say will depend on which geometric category we work in). Let $\mathscr{O}(X)$ denote the smooth functions on X, and let $\mathrm{Vect}(X)$ denote the vector fields on X.

If ω is a volume form on X, then it gives a divergence map

$$\mathrm{Div}_\omega : \mathrm{Vect}(X) \to \mathscr{O}(X)$$

defined via the Lie derivative:

$$\mathrm{Div}_\omega(V)\omega = \mathcal{L}_V \omega$$

for $V \in \mathrm{Vect}(X)$. Note that the divergence operator Div_ω satisfies the equations

$$\begin{aligned} \mathrm{Div}_\omega(fV) &= f\,\mathrm{Div}_\omega V + V(f). \\ \mathrm{Div}_\omega([V,W]) &= V\,\mathrm{Div}_\omega W - W\,\mathrm{Div}_\omega V. \end{aligned} \tag{$\dagger$}$$

The volume form ω is determined up to a constant by the divergence operator Div_ω.

Conversely, to give an operator $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ satisfying equations (†) is the same as to give a flat connection on the canonical bundle K_X of X, or, equivalently, to give a right D-module structure on the structure sheaf $\mathscr{O}(X)$.

16.4.1.1 Definition. *A* projective volume form *on a space X is an operator* $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ *satisfying equations (†).*

The advantage of this definition is that it makes sense in many contexts where more standard definitions of a volume form are hard to define. For example, if A is a quasi-free differential graded commutative algebra, then we can define a projective volume form on the dg scheme $\mathrm{Spec}\, A$ to be a cochain map $\mathrm{Der}(A) \to A$ satisfying equations (†). Similarly, if $\mathfrak{g}$ is a dg Lie or L_∞ algebra, then a projective volume form on the formal moduli problem $B\mathfrak{g}$ is a cochain map $C^*(\mathfrak{g}, \mathfrak{g}[1]) \to C^*(\mathfrak{g})$ satisfying equations (†).

16.4.2. There is a generalization of this notion that we will use where, instead of vector fields, we take any Lie algebroid.

16.4.2.1 Definition. *Let A be a commutative differential graded algebra over a base ring k. A Lie algebroid L over A is a dg A-module with the following extra data.*

1. *A Lie bracket on L making it into a dg Lie algebra over k. This Lie bracket will be typically not A-linear.*
2. *A homomorphism of dg Lie algebras $\alpha : L \to \mathrm{Der}^*(A)$, called the anchor map.*

(3) *These structures are related by the Leibniz rule*

$$[l_1, f l_2] = (\alpha(l_1)(f))\, l_2 + (-1)^{|l_1||f|} f [l_1, l_2]$$

for $f \in A$, $l_i \in L$.

In general, we should think of L as providing the derived version of a foliation. In ordinary as opposed to derived algebraic geometry, a foliation on a smooth affine scheme with algebra of functions A consists of a Lie algebroid L on A which is projective as an A-module and whose anchor map is fibrewise injective.

16.4.2.2 Definition. *If A is a commutative dg algebra and L is a Lie algebroid over A, then an L-projective volume form on A is a cochain map*

$$\mathrm{Div} : L \to A$$

satisfying

$$\begin{aligned} \mathrm{Div}(al) &= a \,\mathrm{Div}\, l + (-1)^{|l||a|} \alpha(l) a. \\ \mathrm{Div}([l_1, l_2]) &= l_1 \,\mathrm{Div}\, l_2 - (-1)^{|l_1||l_2| l_2} \,\mathrm{Div}\, l_1. \end{aligned}$$

Of course, if the anchor map is an isomorphism, then this structure is the same as a projective volume form on A. In the more general case, we should think of an L-projective volume form as giving a projective volume form on the leaves of the derived foliation.

16.4.3. Let us explain how this definition relates to the notion of quantization of P_0 algebras.

16.4.3.1 Definition. *Give the operad P_0 a $\mathbb{C}^\times$ action where the product has weight* 0 *and the Poisson bracket has weight* 1. *A graded P_0 algebra is a $\mathbb{C}^\times$-equivariant differential graded algebra over this dg operad.*

Note that, if X is a manifold, $\mathscr{O}(T^*[-1]X)$ has the structure of graded P_0 algebra, where the $\mathbb{C}^\times$ action on $\mathscr{O}(T^*[-1]X)$ is given by rescaling the cotangent fibers.

Similarly, if L is a Lie algebroid over a commutative dg algebra A, then $\mathrm{Sym}_A^* L[1]$ is a $\mathbb{C}^\times$-equivariant P_0 algebra. The P_0 bracket is defined by the bracket on L and the L-action on A; the $\mathbb{C}^\times$ action gives $\mathrm{Sym}^k L[1]$ weight $-k$.

16.4.3.2 Definition. *Give the operad BD over $\mathbb{C}[[\hbar]]$ a $\mathbb{C}^\times$ action, covering the $\mathbb{C}^\times$ action on $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, the product has weight* 0*, and the Poisson bracket has weight* 1.

Note that this $\mathbb{C}^\times$ action respects the differential on the operad BD, which is defined on generators by

$$\mathrm{d}(- * -) = \hbar\{-,-\}.$$

Note also that by describing the operad BD as a $\mathbb{C}^\times$-equivariant family of operads over $\mathbb{A}^1$, we have presented BD as a filtered operad whose associated graded operad is P_0.

16.4.3.3 Definition. *A filtered BD algebra is a BD algebra A with a $\mathbb{C}^\times$ action compatible with the $\mathbb{C}^\times$ action on the ground ring $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, and compatible with the $\mathbb{C}^\times$ action on BD.*

16.4.3.4 Lemma. *If L is Lie algebroid over a dg commutative algebra A, then every L-projective volume form yields a filtered BD algebra structure on $\mathrm{Sym}^*_A(L[1])[[\hbar]]$, quantizing the graded P_0 algebra $\mathrm{Sym}^*_A(L[1])$.*

PROOF. If $\mathrm{Div} : L \to A$ is an L-projective volume form, then it extends uniquely to an order two differential operator $\triangle$ on $\mathrm{Sym}^*_A(L[1])$ which maps

$$\mathrm{Sym}^i_A(L[1]) \to \mathrm{Sym}^{i-1}_A(L[1]).$$

Then $\mathrm{Sym}^*_A L[1][[\hbar]]$, with differential $\mathrm{d} + \hbar\triangle$, gives the desired filtered BD algebra.

□

16.4.4. Let $B\mathcal{L}$ be an elliptic moduli problem on a compact manifold M. The main result of this section is that there exists a special kind of quantization of the cotangent field theory for $B\mathcal{L}$ that gives a projective volume on this formal moduli problem $B\mathcal{L}$. Projective volume forms arising in this way have a special "locality" property, reflecting the locality appearing in our definition of a field theory.

Thus, let $\mathcal{L}$ be an elliptic L_∞ algebra on M. This gives rise to a classical field theory whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$, as described in Chapter **??**, section 11.6. Let us give the space $\mathscr{E}$ a $\mathbb{C}^\times$-action where $\mathcal{L}[1]$ has weight 0 and $\mathcal{L}^![-1]$ has weight 1. This induces a $\mathbb{C}^\times$ action on all associated spaces, such as $\mathscr{O}(\mathscr{E})$ and $\mathscr{O}_{loc}(\mathscr{E})$.

This $\mathbb{C}^\times$ action preserves the differential $Q + \{I, -\}$ on $\mathscr{O}(\mathscr{E})$, as well as the commutative product. Recall (Chapter **??**, section 12.2) that the subspace

$$\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E}) \subset \mathscr{O}(\mathscr{E})$$

of functionals with smooth first derivative has a Poisson bracket of cohomological degree 1, making it into a P_0 algebra. This Poisson bracket is of weight 1 with respect to the $\mathbb{C}^\times$ action on $\widetilde{\mathrm{Obs}}^{cl}(M)$, so $\widetilde{\mathrm{Obs}}^{cl}(M)$ is a graded P_0 algebra.

We are interested in quantizations of our field theory where the BD algebra $\mathrm{Obs}^q_\Phi(M)$ of (global) quantum observables (defined using a parametrix Φ) is a filtered BD algebra.

16.4.4.1 Definition. *A cotangent quantization of a cotangent theory is a quantization, given by effective interaction functionals* $I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ *for each parametrix* Φ*, such that* $I[\Phi]$ *is of weight* -1 *under the* $\mathbb{C}^\times$ *action on the space* $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ *of functionals.*

This $\mathbb{C}^\times$ *action gives* $\hbar$ *weight* -1*. Thus, this condition means that if we expand*

$$I[\Phi] = \sum \hbar^i I_i[\Phi],$$

then the functionals $I_i[\Phi]$ *are of weight* $i-1$*.*

Since the fields $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ *decompose into spaces of weights* 0 *and* 1 *under the* $\mathbb{C}^\times$ *action, we see that* $I_0[\Phi]$ *is linear as a function of* $\mathcal{L}^![-2]$*, that* $I_1[\Phi]$ *is a function only of* $\mathcal{L}[1]$*, and that* $I_i[\Phi] = 0$ *for* $i > 1$*.*

Remark:

1. The quantization $\{I[\Phi]\}$ is a cotangent quantization if and only if the differential $Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi$ preserves the $\mathbb{C}^\times$ action on the space $\mathscr{O}(\mathscr{E})[[\hbar]]$ of functionals. Thus, $\{I[\Phi]\}$ is a cotangent quantization if and only if the BD algebra $\mathrm{Obs}^q_\Phi(M)$ is a filtered BD algebra for each parametrix Φ.
2. The condition that $I_0[\Phi]$ is of weight -1 is automatic.
3. It is easy to see that the renormalization group flow

$$W\left(P(\Phi) - P(\Psi), -\right)$$

commutes with the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$.

◊

16.4.5. Let us now explain the volume-form interpretation of cotangent quantization. Let $\mathcal{L}$ be an elliptic L_∞ algebra on M, and let $\mathscr{O}(B\mathcal{L}) = C^*(\mathcal{L})$ be the Chevalley-Eilenberg cochain complex of M. The cochain complexes $\mathscr{O}(B\mathcal{L}(U))$ for open subsets $U \subset M$ define a commutative factorization algebra on M.

As we have seen in Chapter **??**, section **??**, we should interpret modules for an L_∞ algebra $\mathfrak{g}$ as sheaves on the formal moduli problem $B\mathfrak{g}$. The $\mathfrak{g}$-module $\mathfrak{g}[1]$ corresponds to the tangent bundle of $B\mathfrak{g}$, and so vector fields on $\mathfrak{g}$ correspond to the $\mathscr{O}(B\mathfrak{g})$-module $C^*(\mathfrak{g}, \mathfrak{g}[1])$.

Thus, we use the notation

$$\mathrm{Vect}(B\mathcal{L}) = C^*(\mathcal{L}, \mathcal{L}[1]);$$

this is a dg Lie algebra and acts on $C^*(\mathcal{L})$ by derivations (see Appendix B, section B.9, for details).

For any open subset $U \subset M$, the $\mathcal{L}(U)$-module $\mathcal{L}(U)[1]$ has a sub-module $\mathcal{L}_c(U)[1]$ given by compactly supported elements of $\mathcal{L}(U)[1]$. Thus, we have a sub-$\mathscr{O}(B\mathcal{L}(U))$-module

$$\mathrm{Vect}_c(B\mathcal{L}(U)) = C^*(\mathcal{L}(U), \mathcal{L}_c(U)[1]) \subset \mathrm{Vect}(B\mathcal{L}(U)).$$

This is in fact a sub-dg Lie algebra, and hence a Lie algebroid over the dg commutative algebra $\mathscr{O}(B\mathcal{L}(U))$. Thus, we should view the subspace $\mathcal{L}_c(U)[1] \subset \mathcal{L}_c(U)[1]$ as providing a foliation of the formal moduli problem $B\mathcal{L}(U)$, where two points of $B\mathcal{L}(U)$ are in the same leaf if they coincide outside a compact subset of U.

If $U \subset V$ are open subsets of M, there is a restriction map of L_∞ algebras $\mathcal{L}(V) \to \mathcal{L}(U)$. The natural extension map $\mathcal{L}_c(U)[1] \to \mathcal{L}_c(V)[1]$ is a map of $\mathcal{L}(V)$-modules. Thus, by taking cochains, we find a map

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \to \mathrm{Vect}_c(B\mathcal{L}(V)).$$

Geometrically, we should think of this map as follows. If we have an R-point α of $B\mathcal{L}(V)$ for some dg Artinian ring R, then any compactly-supported deformation of the restriction $\alpha \mid_U$ of α to U extends to a compactly supported deformation of α.

We want to say that a cotangent quantization of $\mathcal{L}$ leads to a "local" projective volume form on the formal moduli problem $B\mathcal{L}(M)$ if M is compact. If M is compact, then $\mathrm{Vect}_c(B\mathcal{L}(M))$ coincides with $\mathrm{Vect}(B\mathcal{L}(M))$. A local projective volume form on $B\mathcal{L}(M)$ should be something like a divergence operator

$$\mathrm{Div} : \mathrm{Vect}(B\mathcal{L}(M)) \to \mathscr{O}(B\mathcal{L}(M))$$

satisfying the equations (†), with the locality property that Div maps the subspace

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \subset \mathrm{Vect}(B\mathcal{L}(M))$$

to the subspace $\mathscr{O}(B\mathcal{L}(U)) \subset \mathscr{O}(B\mathcal{L}(M))$.

Note that a projective volume form for the Lie algebroid $\mathrm{Vect}_c(B\mathcal{L}(U))$ over $\mathscr{O}(B\mathcal{L}(U))$ is a projective volume form on the leaves of the foliation of $B\mathcal{L}(U)$ given by $\mathrm{Vect}_c(B\mathcal{L}(U))$. The leaf space for this foliation is described by the L_∞ algebra

$$\mathcal{L}_\infty(U) = \mathcal{L}(U)/\mathcal{L}_c(U) = \operatorname*{colim}_{K \subset U} \mathcal{L}(U \setminus K).$$

(Here the colimit is taken over all compact subsets of U.) Consider the one-point compactification U_∞ of U. Then the formal moduli problem $\mathcal{L}_\infty(U)$ describes the germs at ∞ on U_∞ of sections of the sheaf on U of formal moduli problems given by $\mathcal{L}$.

Thus, the structure we're looking for is a projective volume form on the fibers of the maps $B\mathcal{L}(U) \to B\mathcal{L}_\infty(U)$ for every open subset $U \subset M$, where the divergence operators describing these projective volume forms are all compatible in the sense described above.

What we actually find is something a little weaker. To state the result, recall (section 15.2) that we use the notation $\mathscr{P}$ for the contractible simplicial set of parametrices, and $\mathscr{C}\mathscr{P}$ for the cone on $\mathscr{P}$. The vertex of the cone $\mathscr{C}\mathscr{P}$ will denoted $\bar{0}$.

16.4.5.1 Theorem. *A cotangent quantization of the cotangent theory associated to the elliptic L_∞ algebra $\mathcal{L}$ leads to the following data.*

(1) *A commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$. The underlying graded algebra of this commutative dg algebra is $\mathscr{O}(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. The restriction of this commutative dg algebra to the vertex $\overline{0}$ of $\mathscr{C}\mathscr{P}$ is the commutative dg algebra $\mathscr{O}(B\mathcal{L})$.*

(2) *A dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$, whose underlying graded $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$-module is $\mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. At the vertex $\overline{0}$ of $\mathscr{C}\mathscr{P}$, the dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ coincides with the dg Lie algebroid $\mathrm{Vect}_c(B\mathcal{L})$.*

(3) *We let $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ be the restrictions of $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$. Then we have a divergence operator*

$$\mathrm{Div}_{\mathscr{P}} : \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \mathscr{O}_{\mathscr{P}}(B\mathcal{L})$$

defining the structure of a $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ projective volume form on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.

Further, when restricted to the sub-simplicial set $\mathscr{P}_U \subset \mathscr{P}$ of parametrices with support in a small neighborhood of the diagonal $U \subset M \times M$, all structures increase support by an arbitrarily small amount (more precisely, by an amount linear in U, in the sense explained in section 15.2).

PROOF. This follows almost immediately from theorem 15.2.2.1. Indeed, because we have a cotangent theory, we have a filtered BD algebra

$$\mathrm{Obs}^q_{\mathscr{P}}(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P}), \widehat{Q}_{\mathscr{P}}, \{-,-\}_{\mathscr{P}} \right).$$

Let us consider the sub-BD algebra $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$, which, as a graded vector space, is $\mathscr{O}_{sm}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P})$ (as usual, $\mathscr{O}_{sm}(\mathscr{E})$ indicates the space of functionals with smooth first derivative).

Because we have a filtered BD algebra, there is a $\mathbb{C}^\times$-action on this complex $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$. We let

$$\mathscr{O}_{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^0$$

be the weight 0 subspace. This is a commutative differential graded algebra over $\Omega^*(\mathscr{P})$, whose underlying graded algebra is $\mathscr{O}(B\mathcal{L})$; further, it extends (using again the results of 15.2.2.1) to a commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$, which when restricted to the vertex is $\mathscr{O}(B\mathcal{L})$.

Next, consider the weight -1 subspace. As a graded vector space, this is

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{P}) \oplus \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

We thus let

$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} / \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The Poisson bracket on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$ is of weight 1, and it makes the space $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ into a sub Lie algebra.

We have a natural decomposition of graded vector spaces

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \oplus \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The dg Lie algebra structure on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ gives us

1. The structure of a dg Lie algebra on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ (as the quotient of $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ by the differential Lie algebra ideal $\hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L})$).
2. An action of $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ by derivations; this defines the anchor map for the Lie algebroid structure on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.
3. A cochain map
$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
This defines the divergence operator.

It is easy to verify from the construction of theorem 15.2.2.1 that all the desired properties hold. □

16.4.6. The general results about quantization of [Cos11c] thus apply to this situation, to show that the following.

16.4.6.1 Theorem. *Consider the cotangent theory $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ to an elliptic moduli problem described by an elliptic L_∞ algebra $\mathcal{L}$ on a manifold M.*

The obstruction to constructing a cotangent quantization is an element in

$$H^1(\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^\times}) = H^1(\mathscr{O}_{loc}(B\mathcal{L})).$$

If this obstruction vanishes, then the simplicial set of cotangent quantizations is a torsor for the simplicial Abelian group arising from the cochain complex $\mathscr{O}_{loc}(B\mathcal{L})$ by the Dold-Kan correspondence.

As in Chapter **??**, section 10.5, we are using the notation $\mathscr{O}_{loc}(B\mathcal{L})$ to refer to a "local" Chevalley-Eilenberg cochain for the elliptic L_∞ algebra $\mathcal{L}$. If L is the vector bundle whose sections are $\mathcal{L}$, then as we explained in [Cos11c], the jet bundle $J(L)$ is a D_M L_∞ algebra and

$$\mathscr{O}_{loc}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

There is a de Rham differential (see section 11.3) mapping $\mathscr{O}_{loc}(B\mathcal{L})$ to the complex of local 1-forms,

$$\Omega^1_{loc}(B\mathcal{L}) = C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

The de Rham differential maps $\mathscr{O}_{loc}(B\mathcal{L})$ isomorphically to the subcomplex of $\Omega^1_{loc}B\mathcal{L})$ of closed local one-forms. Thus, the obstruction is a local closed 1-form on $B\mathcal{L}$ of cohomology degree 1: it is in

$$H^1(\Omega^1_{loc}(B\mathcal{L}).$$

Since the obstruction to quantizing the theory is the obstruction to finding a locally-defined volume form on $B\mathcal{L}$, we should view this obstruction as being the local first Chern class of $B\mathcal{L}$.

16.5. Correlation functions

So far in this chapter, we have proved the quantization theorem showing that from a field theory we can construct a factorization algebra. We like to think that this factorization algebra encodes most things one would want to with a quantum field theory in perturbation theory. To illustrate this, in this section, we will explain how to construct correlation frunctions form the factorization algebra, under certain additional hypothesis.

Suppose we have a field theory on a compact manifold M, with space of fields $\mathscr{E}$ and linearized differential Q on the space of fields. Let us suppose that

$$H^*(\mathscr{E}(M), Q) = 0.$$

This means the following: the complex $(\mathscr{E}(M), Q)$ is tangent complex to the formal moduli space of solutions to the equation of motion to our field theory, at the base point around which we are doing perturbation theory. The statement that this tangent complex has no cohomology means that there the trivial solution of the equation of motion has no deformations (up to whatever gauge symmetry we have). In other words, we are working with an isolated point in the moduli of solutions to the equations of motion.

As an example, consider a massive interacting scalar field theory on a compact manifold M, with action functional for example

$$\int_M \phi(\mathrm{D} + m^2)\phi + \phi^4$$

where $\phi \in C^\infty(M)$ and $m > 0$. Then, the complex $\mathscr{E}(M)$ of fields is the complex

$$C^\infty(M) \xrightarrow{\mathrm{D} + m^2} C^\infty(M).$$

Hodge theory tells us that this complex has no cohomology.

Let Obs^q denote the factorization algebra of quantum observables of a quantum field theory which satisfies this (classical) condition.

16.5.0.2 Lemma. *In this situation, there is a canonical isomorphism*

$$H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]].$$

(Note that we usually work, for simplicity, with complex vector spaces; this result holds where everything is real too, in which case we find $\mathbb{R}[[\hbar]]$ on the right hand side).

PROOF. There's a spectral sequence

$$H^*(\mathrm{Obs}^{cl}(M))[[\hbar]] \twoheadrightarrow H^*(\mathrm{Obs}^q(M)).$$

Further, $\mathrm{Obs}^{cl}(M)$ has a complete decreasing filtration whose associated graded is the complex

$$\mathrm{Gr}\,\mathrm{Obs}^{cl}(M) = \prod_n \mathrm{Sym}^n(\mathscr{E}(M)^\vee)$$

with differential arising from the linear differential Q on $\mathscr{E}(M)$. The condition that $H^*(\mathscr{E}(M), Q) = 0$ implies that the cohomology of $\mathrm{Sym}^n(\mathscr{E}(M)^\vee)$ is also zero, so that $H^*(\mathrm{Obs}^{cl}(M)) = \mathbb{C}$. This shows that there is an isomorphism of $\mathbb{C}[[\hbar]]$-modules from $H^*(\mathrm{Obs}^q(M))$ to $\mathbb{C}[[\hbar]]$. To make this isomorphism canonical, we declare that the vacuum observable $|\emptyset\rangle \in H^0(\mathrm{Obs}^q(M))$ (that is, the unit in the factorization algebra) gets sent to $1 \in \mathbb{C}[[\hbar]]$. ☐

16.5.0.3 Definition. *As above, let* Obs^q *denote the factorization algebra of obsevables of a QFT on* M *which satisfies* $H^*(\mathscr{E}(M), Q) = 0$.

Let $U_1, \ldots, U_n \subset M$ *be disjoint open sets, and let* $O_i \in \mathrm{Obs}^q(U_i)$*. Define the* expectation value *(or correlation function) of the obsevables* O_i*, denoted by*

$$\langle O_1, \ldots, O_n \rangle \in \mathbb{C}[[\hbar]],$$

to be the image of the product observable

$$O_1 * \cdots * O_n \in H^*(\mathrm{Obs}^q(M))$$

under the canonical isomorphism between $H^*(\mathrm{Obs}^q(M))$ *and* $\mathbb{C}[[\hbar]]$.

We have already encountered this definition when we discussed free theories (see definition 4.6.0.2 in Chapter 4). There we saw that this definition reproduced usual physics definitions of correlation functions for free field theories.

Part 5

Using the machine

CHAPTER 17

Noether's theorem in classical field theory

Noether's theorem is a central result in field theory, which states that there is a bijection between symmetries of a field theory and conserved currents. In this chapter we will develop a very general version of Noether's theorem for classical field theories in the language of factorization algebras. In the following chapter, we will develop the analogous theorem for quantum field theories.

The statement for classical field theories is the following. Suppose we have a classical field theory on a manifold M, and let $\widetilde{\mathrm{Obs}}^{cl}$ denote the P_0 factorization algebra of observables of the theory. Suppose that $\mathcal{L}$ is a local L_∞ algebra on M which acts on our classical field theory (we will define precisely what we mean by an action shortly). Let $\mathcal{L}_c$ denote the precosheaf of L_∞ algebras on M given by compactly supported section of $\mathcal{L}$. Note that the P_0 structure on $\widetilde{\mathrm{Obs}}^{cl}$ means that $\widetilde{\mathrm{Obs}}^{cl}[-1]$ is a precosheaf of dg Lie algebras.

The formulation of Noether's theorem we will prove involves shifted central extensions of hte cosheaf $\mathcal{L}_c$ of L_∞ algebras on M. Such central extensions were discussed insection 3.6.3; we are interested in -1-shifted central extensions, which fit into short exact sequences

$$0 \to \underline{\mathbb{C}}[-1] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0,$$

where $\underline{\mathbb{C}}$ is the constant precosheaf.

The theorem is the following.

Theorem. *Suppose that a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory with observables Obs^{cl}. Then, there is a -1-shifted central extension $\widetilde{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on M, and a map of precosheaves of L_∞ algebras*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

which, for every open subset U, sends the central element of $\widetilde{\mathcal{L}}_c$ to the observable $1 \in \mathrm{Obs}^{cl}(U)[-1]$.

This map is not arbitrary. Rather, it is compatible with the action of the cosheaf $\mathcal{L}_c$ on Obs^{cl} arising from the action $\mathcal{L}_c$ on the field theory. Let us explain the form this compatibility takes.

Note that the dg Lie algebra $\widetilde{\mathrm{Obs}}^{cl}(U)[-1]$ acts on $\mathrm{Obs}^{cl}(U)$ by the Poisson bracket, in such a way that the subspace spanned by the observable 1 acts by zero. The L_∞ map we just discussed therefore gives an action of $\widetilde{\mathcal{L}}_c(U)$ on $\mathrm{Obs}^{cl}(U)$, which descends to an action of $\mathcal{L}_c(U)$ because the central element acts by zero.

Theorem. *In this situation, the action of $\mathcal{L}_c(U)$ coming from the L_∞-map $\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}$ and the action coming from the action of $\mathcal{L}$ on the classical field theory coincide up to a homotopy.*

. Let us relate this formulation of Noether's theorem to familiar statements in classical field theory. Suppose we have a symplectic manifold X with an action of a Lie algebra $\mathfrak{g}$ by symplectic vector fields. Let us work locally on X, so that we can assume $H^1(X) = 0$. Then, there is a short exact sequence of Lie algebras

$$0 \to \mathbb{R} \to C^\infty(X) \to \mathrm{SympVect}(X) \to 0$$

where $\mathrm{SympVect}(X)$ is the Lie algebra of symplectic vector fields on X, and $C^\infty(X)$ is a Lie algebra under the Poisson bracket.

We can pull back this central extension under the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{SympVect}(X)$ to obtain a central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. This is the analog of the central extension $\widetilde{\mathcal{L}}_c$ that appeared in our formulation of Noether's theorem.

The Poisson algebra $C^\infty(X)$ is observables of the classical field theory. The map $\widetilde{\mathfrak{g}} \to C^\infty(X)$ sends the central element of $\widetilde{\mathfrak{g}}$ to $1 \in C^\infty(X)$. Further, the action of $\mathfrak{g}$ on $C^\infty(X)$ arising from the homomorphism $\mathfrak{g} \to C^\infty(X)$ coincides with the one arising from the original homomorphism $\mathfrak{g} \to \mathrm{SympVect}(X)$.

Thus, the map $\mathfrak{g} \to C^\infty(X)$ is entirely analogous to the map that appears in our formulation of classical Noether's theorem. Indeed, we defined a field theory to be a sheaf of formal moduli problems with a -1-shifted symplectic form. The P_0 Poisson bracket on the observables of a classical field theory is analogous to the Poisson bracket on observables in classical mechanics. Our formulation of Noether's theorem can be rephrased as saying that, after passing to a central extension, an action of a sheaf of Lie algebras by symplectic symmetries on a sheaf of formal moduli probblems is Hamiltonian.

This similarity is more than just an analogy. After some non-trivial work, one can show that our formulation of Noether's theorem, when applied to classical mechanics, yields the statement discussed above about actions of a Lie algebra on a symplectic manifold. The key result one needs in order to translate is a result of Nick Rozenblyum []. Observables of classical mechanics form a locally-constant P_0 factorization algebra on the real line, and so (by a theorem of Lurie discussed in section 6.2) an E_1 algebra in P_0 algebras. Rozenblyum shows E_1 algebras in P_0 algebras are the same as P_1, that is ordinary Poisson, algebras. This allows us to translate the shifted Poisson bracket on the factorization algebra on $\mathbb{R}$ of observable of classical mechanics into the ordinary unshifted Poisson bracket that is more familiar in classical mechanics, and to translate our formulation of Noether's theorem into the statement about Lie algebra actions on symplectic manifolds discussed above.

17.0.1.

17.0.1.1 Theorem. *Suppose we have a quantum field theory on a manifold M, which is acted on by a local dg Lie (or L_∞) algebra on M. Then, there is a map of factorization algebras on M from the twisted factorization envelope (3.6.3) of $\mathcal{L}$ to observables of the field theory.*

Of course, this is not an arbitrary map; rather, the action of $\mathcal{L}$ on observables can be recovered from this map together with the factorization product.

This theorem may seem quite different from Noether's theorem as it is usually stated. We explain the link between this result and the standard formulation in section **??**.

For us, the power of this result is that it gives us a very general method for understanding quantum observables. The factorization envelope of a local L_∞ algebra is a very explicit and easily-understood object. By contrast, the factorization algebra of quantum observables of an interacting field theory is a complicated object which resists explicit description. Our formulation of Noether's theorem shows us that, if we have a field theory which has many symmetries, we can understand explicitly a large part of the factorization algebra of quantum observables.

17.1. Symmetries of a classical field theory

We will start our discussion of Noether's theorem by examining what it means for a homotopy Lie algebra to act on a field theory. We are particularly interested in what it means for a *local L_∞ algebra* to act on a classical field theory. Recall **??** that a local L_∞ algebra $\mathcal{L}$ is a sheaf of L_∞ algebras which is the sheaf of sections of a graded vector bundle L, and where the L_∞-structure maps are poly-differential operators.

We know from chapter **??** that a perturbative classical field theory is described by an elliptic moduli problem on M with a degree -1 symplectic form. Equivalently, it is described by a local L_∞ algebra $\mathcal{M}$ on M equipped with an invariant pairing of degree -3. Therefore, an action of $\mathcal{L}$ on $\mathcal{M}$ should be an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$. Thus, the first thing we need to understand is what it means for one L_∞ algebra to act on the other.

17.1.1. Actions of L_∞ algebras. If $\mathfrak{g}, \mathfrak{h}$ are ordinary Lie algebras, then it is straightforward to say what it means for $\mathfrak{g}$ to act on $\mathfrak{h}$. If $\mathfrak{g}$ does act on $\mathfrak{h}$, then we can define the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$. This semi-direct product lives in a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0.$$

Further, we can recover the action of $\mathfrak{g}$ on $\mathfrak{h}$ from the data of a short exact sequence of Lie algebras like this.

We will take this as a model for the action of one L_∞ algebra $\mathfrak{g}$ on another L_∞ algebra $\mathfrak{h}$.

17.1.1.1 Definition. *An action of an L_∞ algebra $\mathfrak{g}$ on an L_∞ algebra $\mathfrak{h}$ is, by definition, an L_∞-algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$ with the property that the (linear) maps in the exact sequence*

$$0 \to \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g} \to 0$$

are maps of L_∞ algebras.

Remark:

(1) The set of actions of $\mathfrak{g}$ on $\mathfrak{h}$ enriches to a simplicial set, whose n-simplices are families of actions over the dg algebra $\Omega^*(\triangle^n)$.

(2) There are other possible notions of action of $\mathfrak{g}$ on $\mathfrak{h}$ which might seem more natural to some readers. For instance, an abstract notion is to say that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is an L_∞ algebra $\widetilde{\mathfrak{h}}$ with a map $\phi : \widetilde{\mathfrak{h}} \to \mathfrak{g}$ and an isomorphism of L_∞ algebras between the homotopy fibre $\phi^{-1}(0)$ and $\mathfrak{h}$. One can show that this more fancy definition is equivalent to the concrete one proposed above, in the sense that the two ∞-groupoids of possible actions are equivalent.

If $\mathfrak{h}$ is finite dimensional, then we can identify the dg Lie algebra of derivations of $C^*(\mathfrak{h})$ with $C^*(\mathfrak{h}, \mathfrak{h}[1])$ with a certain dg Lie bracket. We can thus view $C^*(\mathfrak{h}, \mathfrak{h}[1])$ as the dg Lie algebra of vector fields on the formal moduli problem $B\mathfrak{h}$.

17.1.1.2 Lemma. *Actions of $\mathfrak{g}$ on $\mathfrak{h}$ are the same as L_∞-algebra maps $\mathfrak{g} \to C^*(\mathfrak{h}, \mathfrak{h}[1])$.*

PROOF. This is straightforward. □

This lemma shows that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is the same as an action of $\mathfrak{g}$ on the formal moduli problem $B\mathfrak{h}$ which *may not* preserve the base-point of $B\mathfrak{h}$.

17.1.2. Actions of local L_∞ algebras.

Now let us return to the setting of local L_∞ algebras, and define what it means for one local L_∞ algebra to act on another.

17.1.2.1 Definition. *Let $\mathcal{L}$, $\mathcal{M}$ be local L_∞ algebras on M. Then an $\mathcal{L}$ action on $\mathcal{M}$ is given by a local L_∞ structure on $\mathcal{L} \oplus \mathcal{M}$, such that the exact sequence*

$$0 \to \mathcal{M} \to \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \to 0$$

is a sequence of L_∞ algebras.

More explicitly, this says that $\mathcal{M}$ (with its original L_∞ structure) is a sub-L_∞-algebra of $\mathcal{M} \oplus \mathcal{L}$, and is also an L_∞-ideal: all operations which take as input at least one element of $\mathcal{M}$ land in $\mathcal{M}$. We will refer to the L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$ with the L_∞ structure defining the action as $\mathcal{L} \ltimes \mathcal{M}$.

17.1.2.2 Definition. *Suppose that $\mathcal{M}$ has an invariant pairing. An action of $\mathcal{L}$ on $\mathcal{M}$ preserves the pairing if, for local compactly supported sections α_i, β_j of $\mathcal{L}$ and $\mathcal{M}$ the tensor*

$$\langle l_{r+s}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s), \beta_{s+1} \rangle$$

is totally symmetric if $s+1$ is even (or antisymmetric if $s+1$ is odd) under permutation of the β_i.

17.1.2.3 Definition. *An action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory defined by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3 is, as above, an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$ which preserves the pairing.*

As an example, we have the following.

17.1.2.4 Lemma. *Suppose that $\mathcal{L}$ acts on an elliptic L_∞ algebra $\mathcal{M}$. Then $\mathcal{L}$ acts on the cotangent theory for $\mathcal{M}$.*

PROOF. This is immediate by naturality, but we can also write down explicitly the semi-direct product L_∞ algebra describing the action. Note that $\mathcal{L} \ltimes \mathcal{M}$ acts linearly on

$$(\mathcal{L} \ltimes \mathcal{M})^! [-3] = \mathcal{L}^! [-3] \oplus \mathcal{M}^! [-3].$$

Further, $\mathcal{L}^![-3]$ is a submodule for this action, so that we can form the quotient $\mathcal{M}^![-3]$. Then,

$$(\mathcal{L} \ltimes \mathcal{M}) \ltimes \mathcal{M}^! [-3]$$

is the desired semi-direct product. ☐

Remark: Note that this construction is simply giving the -1-shifted relative cotangent bundle to the map

$$B(\mathcal{L} \ltimes \mathcal{M}) \to B\mathcal{L}.$$

The definition we gave above of an action of a local L_∞ algebra on a classical field theory is a little abstract. We can make it more concrete as follows.

Recall that the space of fields of the classical field theory associated to $\mathcal{M}$ is $\mathcal{M}[1]$, and that the L_∞ structure on $\mathcal{M}$ is entirely encoded in the action functional

$$S \in \mathscr{O}_{loc}(\mathcal{M}[1])$$

which satisfies the classical master equation $\{S, S\} = 0$. (The notation $\mathscr{O}_{loc}$ always indicates local functionals modulo constants).

An action of a local L_∞ algebra $\mathcal{L}$ on $\mathcal{M}$ can also be encoded in a certain local functional, which depends on $\mathcal{L}$. We need to describe the precise space of functionals that arise in this interpretation.

If X denotes the space-time manifold on which $\mathcal{L}$ and $\mathcal{M}$ are sheaves, then $\mathcal{L}(X)$ is an L_∞ algebra. Thus, we can form the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L}(X)) = (\mathscr{O}(\mathcal{L}(X)[1]), \mathrm{d}_{\mathcal{L}})$$

as well as it's reduced version $C^*_{red}(\mathcal{L}(X))$.

We can form the completed tensor product of this dg algebra with the shifted Lie algebra $\mathscr{O}_{loc}(\mathcal{M}[1])$, to form a new shifted dg Lie algebra $C^*_{red}(\mathcal{L}(X)) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$.

Inside this is the subspace

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right) \subset C^*_{red}(\mathcal{L}(X)) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$$

of functionals which are local as a function of $\mathcal{L}[1]$. Note that we are working with functionals which must depend on *both* $\mathcal{L}[1]$ and $\mathcal{M}[1]$: we discard those functionals which depend only on one or the other.

One can check that this graded subspace is preserved both by the Lie bracket $\{-,-\}$, the differential $\mathrm{d}_{\mathcal{L}}$ and the differential $\mathrm{d}_{\mathcal{M}}$ (coming from the L_∞ structure on $\mathcal{L}$ and $\mathcal{M}$). This space thus becomes a shifted dg Lie algebra, with the differential $\mathrm{d}_{\mathcal{L}} \oplus \mathrm{d}_{\mathcal{M}}$ and with the degree $+1$ bracket $\{-,-\}$.

17.1.2.5 Lemma. *To give an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory corresponding to a local L_∞ algebra $\mathcal{M}$ with invariant pairing, is the same as to give an action functional*

$$S^{\mathcal{L}} \in \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right)$$

which is of cohomological degree 0*, and satisfied the Maurer-Cartan equation*

$$(\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}})S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

PROOF. Given such an $S^{\mathcal{L}}$, then

$$\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} + \{S^{\mathcal{L}}, -\}$$

defines a differential on $\mathscr{O}(\mathcal{L}(X)[1] \oplus \mathcal{M}(X)[1])$. The classical master equation implies that this differential is of square zero, so that it defines an L_∞ structure on $\mathcal{L}(X) \oplus \mathcal{M}(X)$. The locality condition on $S^{\mathcal{L}}$ guarantees that this is a local L_∞ algebra structure. A simple analysis shows that this L_∞ structure respects the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \to 0$$

and the invariant pairing on $\mathcal{M}$. □

This lemma suggests that we should look at a classical field theory with an action of $\mathcal{L}$ as a family of classical field theories over the sheaf of formal moduli problems $B\mathcal{L}$. Further justification for this idea will be offered in proposition **??**.

17.1.3. Let $\mathfrak{g}$ be an ordinary L_∞ algebra (not a sheaf of such), which we assume to be finite-dimensional for simplicity. Let $C^*(\mathfrak{g})$ be it's Chevalley-Eilenberg cochain algebra, viewed as a pro-nilpotent commutative dga. Suppose we have a classical field theory, represented as an elliptic L_∞ algebra $\mathcal{M}$ with an invariant pairing. Then we can define the notion of a $\mathfrak{g}$-action on $\mathcal{M}$ as follows.

17.1.3.1 Definition. *A $\mathfrak{g}$-action on $\mathcal{M}$ is any of the following equivalent data.*

(1) An L_∞ structure on $\mathfrak{g} \oplus \mathcal{M}(X)$ such that the exact sequence

$$0 \to \mathcal{M}(X) \to \mathfrak{g} \oplus \mathcal{M}(X) \to \mathfrak{g}$$

is a sequence of L_∞-algebras, and such that the structure maps

$$\mathfrak{g}^{\otimes n} \otimes \mathcal{M}(X)^{\otimes m} \to \mathcal{M}(X)$$

are poly-differential operators in the $\mathcal{M}$-variables.

(2) An L_∞-homomorphism

$$\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{M})[-1]$$

(the shift is so that $\mathscr{O}_{loc}(B\mathcal{M})[-1]$ is an ordinary, and not shifted, dg Lie algebra).

(3) An element

$$S^{\mathfrak{g}} \in C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{M})$$

which satisfies the Maurer-Cartan

$$\mathrm{d}_{\mathfrak{g}} S^{\mathfrak{g}} \mathrm{d}_{\mathcal{M}} S^{\mathfrak{g}} + \tfrac{1}{2}\{S^{\mathfrak{g}}, S^{\mathfrak{g}}\} = 0.$$

It is straightforward to verify that these three notions are identical. The third version of the definition can be viewed as saying that a $\mathfrak{g}$-action on a classical field theory is a family of classical field theories over the dg ring $C^*(\mathfrak{g})$ which reduces to the original classical field theory modulo the maximal ideal $C^{>0}(\mathfrak{g})$. This version of the definition generalizes to the quantum level.

Our formulation of Noether's theorem will be phrased in terms of the action of a local L_∞ algebra on a field theory. However, we are often presented with the action of an ordinary, finite-dimensional L_∞-algebra on a theory, and we would like to apply Noether's theorem to this situation. Thus, we need to be able to formulate this kind of action as an action of a local L_∞ algebra.

The following lemma shows that we can do this.

17.1.3.2 Lemma. *Let $\mathfrak{g}$ be an L_∞-algebra. Then, the simplicial sets describing the following are canonically homotopy equivalent:*

(1) *Actions of* $\mathfrak{g}$ *on a fixed classical field theory on a space-time manifold* X.
(2) *Actions of the local* L_∞ *algebra* $\Omega^*_X \otimes \mathfrak{g}$ *on the same classical field theory.*

Note that the sheaf $\Omega^*_X \otimes \mathfrak{g}$ is a fine resolution of the constant sheaf of L_∞ algebras with value $\mathfrak{g}$. The lemma can be generalized to show that, given any locally-constant sheaf of L_∞ algebras $\mathfrak{g}$, an action of $\mathfrak{g}$ on a theory is the same thing as an action of a fine resolution of $\mathfrak{g}$.

PROOF. Suppose that $\mathcal{M}$ is a classical field theory, and suppose that we have an action of the local L_∞ algebra $\Omega^*_X \otimes \mathfrak{g}$ on $\mathcal{M}$.

Actions of $\mathfrak{g}$ on $\mathcal{M}$ are Maurer-Cartan elements of the pro-nilpotent dg Lie algebra

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) \overset{\text{def}}{=} C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$$

, with dg Lie structure described above. Actions of $\Omega^*_X \otimes \mathfrak{g}$ are Maurer-Cartan elements of the pro-nilpotent dg Lie algebra

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) \overset{\text{def}}{=} \mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right),$$

again with the dg Lie algebra structure defined above.

Recall that there is an inclusion of dg Lie algebras

$$\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]) \subset C^*(\Omega^*(X) \otimes \mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1]).$$

Further, there is an inclusion of L_∞ algebras

$$\mathfrak{g} \hookrightarrow \Omega^*(X) \otimes \mathfrak{g}$$

(by tensoring with the constant functions). Composing these maps gives a map of dg Lie algebras

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) \to \mathrm{Act}(\mathfrak{g}, \mathcal{M}).$$

It suffices to show that this map is an equivalence.

We will do this by using the D_X-module interpretation of the left hand side. Let $J(\mathcal{M})$ and $J(\Omega^*_X)$ refer to the D_X-modules of jets of sections of $\mathcal{M}$ and of the de Rham complex, respectively. Note that the natural map of D_X-modules

$$C^\infty_X \to J(\Omega^*_X)$$

is a quasi-isomorphism (this is the Poincaré lemma.).

Recall that

$$\mathscr{O}_{loc}(\mathcal{M}[1]) = \omega_X \otimes_{D_X} C^*_{red}(J(\mathcal{M})).$$

The cochain complex underlying the dg Lie algebra $\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M})$ has the following interpretation in the language of D_X-modules:

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) = \omega_X \otimes_{D_X} \left(C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

Under the other hand, the complex $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$ has the D_X-module interpretation

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) = \omega_X \otimes_{D_X} \left(C^*_{red}(\mathfrak{g} \otimes C^\infty_X) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

Because the map $C^\infty_X \to J(\Omega^*_X)$ is a quasi-isomorphism of D_X-modules, the natural map

$$C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \\ \to C^*_{red}(\mathfrak{g} \otimes C^\infty_X) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M}))$$

is a quasi-isomorphism of D_X-modules. Now, both sides of this equation are flat as left D_X-modules; this follows from the fact that $C^*_{red}(J(\mathcal{M}))$ is a flat D_X-module. If follows that this map is still a quasi-isomorphism after tensoring over D_X with ω_X. ☐

17.2. Examples of classical field theories with an action of a local L_∞ algebra

One is often interested in particular classes of field theories: for example, conformal field theories, holomorphic field theories, or field theories defined on Riemannian manifolds. It turns out that these ideas can be formalized by saying that a theory is acted on by a particular local L_∞ algebra, corresponding to holomorphic, Riemannian, or conformal geometry. This generalizes to any geometric structure on a manifold which can be described by a combination of differential equations and symmetries.

In this section, we will describe the local L_∞ algebras corresponding to holomorphic, conformal, and Riemannian geometry, and give examples of classical field theories acted on by these L_∞ algebras.

We will first discuss the holomorphic case. Let X be a complex manifold, and define a local dg Lie algebra algebra $\mathcal{L}^{hol}$ by setting

$$\mathcal{L}^{hol}(X) = \Omega^{0,*}(X, TX),$$

equipped with the Dolbeault differential and the Lie bracket of vector fields. A holomorphic classical field theory will be acted on by $\mathcal{L}^{hol}(X)$.

Remark: A stronger notion of holomorphicity might require the field theory to be acted on by the group of holomorphic symmetries of X, and that the derivative of this action extends to an action of the local dg Lie algebra $\mathcal{L}^{hol}$.

Let us now give some examples of field theories acted on by $\mathcal{L}^{hol}$.

Example: Let X be a complex manifold of complex dimension d, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie group G. Then, $\Omega^{0,*}(X, \mathfrak{g})$ describes the formal moduli space of principle G-bundles on X. We can form the cotangent theory to this, which is a classical field theory, by letting

$$\mathcal{M} = \Omega^{0,*}(X, \mathfrak{g}) \oplus \Omega^{d,*}(X, \mathfrak{g}^\vee)[d-3].$$

As discussed in [**Cos11b**], this example is important in physics. If $d = 2$ it describes a holomorphic twist of $N = 1$ supersymmetric gauge theory. In addition, one can use the formalism of L_∞ spaces [**Cos11a**, **Cos11b**] to write twisted supersymmetric σ-models in these terms (when $d = 1$ and $\mathfrak{g}$ is a certain sheaf of L_∞ algebras on the target space).

The dg Lie algebra $\mathcal{L}^{hol}(X)$ acts by Lie derivative on $\Omega^{k,*}(X)$ for any k. One can make this action explicit as follows: the contraction map

$$\begin{aligned}\Omega^{0,*}(X, TX) \times \Omega^{k,*}(X) &\to \Omega^{k-1,*}(X)\\ (V, \omega) &\mapsto \iota_V \omega\end{aligned}$$

is $\Omega^{0,*}(X)$-linear and defined on $\Omega^{0,0}(X, TX)$ in the standard way. The Lie derivative is defined by the Cartan homotopy formula

$$\mathcal{L}_V \omega = [\iota_V, \partial]\omega.$$

In this way, $\mathcal{L}$ acts on $\mathcal{M}$. This action preserves the invariant pairing.

We can write this in terms of an $\mathcal{L}$-dependent action functional, as follows. If $\alpha \in \Omega^{0,*}(X, \mathfrak{g})[1]$, $\beta \in \Omega^{d,*}(X, \mathfrak{g}^\vee)[d-2]$ and $V \in \Omega^{0,*}(X, TX)[1]$, we define

$$S^{\mathcal{L}}(\alpha, \beta, V) = \int \left\langle \beta, (\overline{\partial} + \mathcal{L}_V)\alpha \right\rangle + \tfrac{1}{2} \left\langle \beta, [\alpha, \alpha] \right\rangle .$$

(The fields α, β, V can be of mixed degree).

Note that if $V \in \Omega^{0,*}(X, TX)$ is of cohomological degree 1, it defines a deformation of complex structure of X, and the $\overline{\partial}$-operator for this deformed complex structure is $\overline{\partial} + \mathcal{L}_V$. The action functional $S^{\mathcal{L}}$ therefore describes the variation of the original action functinal S as we vary the complex structure on X. Other terms in $S^{\mathcal{L}}$ encode the fact that S is invariant under holomorphic symmetries of X.

We will return to this example throughout our discussion of Noether's theorem. We will see that, in dimension $d = 1$ and with $\mathfrak{g}$ Abelian, it leads to a version of the Segal-Sugawara construction: a map from the Virasoro vertex algebra to the vertex algebra associated to a free $\beta - \gamma$ system.

Example: Next, let us discuss the situation of field theories defined on a complex manifold X together with a holomorphic principal G-bundle. In the case that X is a Riemann surface, field theories of this form play an important role in the mathematics of chiral conformal field theory.

For this example, we define a local dg Lie algebra $\mathcal{L}$ on a complex manifold X by

$$\mathcal{L}(X) = \Omega^{0,*}(X, TX) \ltimes \Omega^{0,*}(X, \mathfrak{g})$$

so that $\mathcal{L}(X)$ is the semi-direct product of the Dolbeault resolution of holomorphic vector fields with the Dolbeault complex with coefficients in $\mathfrak{g}$. Thus, $\mathcal{L}(X)$ is the dg Lie algebra controlling deformations of X as a complex manifold equipped with a holomorphic G-bundle, near the trivial bundle. (The dg Lie algebra controlling deformations of the pair (X, P) where P is a non-trivial principal G-bundle on X is $\Omega^{0,*}(X, \mathrm{At}_P)$ where At_P is the Atiyah algebroid of P, and everything that follows works in the more general case when P is non-trivial and we use $\Omega^{0,*}(X, \mathrm{At}_P)$ in place of $\mathcal{L}$).

Let V be a representation of G. We can form the cotangent theory to the elliptic moduli problem of sections of V, defined by the Abelian elliptic L_∞ algebra

$$\mathcal{M}(X) = \Omega^{0,*}(X, V)[-1] \oplus \Omega^{0,*}(X, V^\vee)[d-2].$$

This is acted on by the local L_∞ algebra $\mathcal{L}$ we described above.

More generally, we could replace V by a complex manifold M with a G-action and consider the cotangent theory to the moduli of holomorphic maps to M.

Example: In this example we will introduce the local dg Lie algebra $\mathcal{L}^{Riem}$ on a Riemannian manifold X which controls deformations of X as a Riemannian manifold. This local dg Lie algebra acts on field theories which are defined on Riemannian manifolds; we will show this explicitly in the case of scalar field theories.

Let (X, g_0) be a Riemannian manifold of dimension d, which for simplicity we assume to be oriented.

Consider the local dg Lie algebra

$$\mathcal{L}^{Riem}(X) = \mathrm{Vect}(X) \oplus \Gamma(X, \mathrm{Sym}^2 TX)[-1].$$

The differential is $\mathrm{d}V = \mathscr{L}_V g_0$ where $\mathscr{L}_V$ indicates Lie derivative. The Lie bracket is defined by saying that the bracket of a vector field V with anything is given by Lie derivative.

Note that $\mathcal{L}^{Riem}(X)$ is the dg Lie algebra describing the formal neighbourhood of X in the moduli space of Riemannian manifolds.

Consider the free scalar field theory on X, defined by the abelian elliptic dg Lie algebra

$$\mathcal{M}(X) = C^\infty(X)[-1] \xrightarrow{\triangle_{g_0}} \Omega^d(X)[-2]$$

where the superscript indicates cohomological degree, and

$$\triangle_{g_0} = \mathrm{d} * \mathrm{d}$$

is the Laplacian for the metric g_0, landing in top-forms.

We define the action of $\mathcal{L}^{Riem}(X)$ on $\mathcal{M}(X)$ by defining an action functional $S^{\mathcal{L}}$ which couples the fields in $\mathcal{L}^{Riem}(X)$ to those in $\mathcal{M}(X)$. If $\phi, \psi \in \mathcal{M}(X)[1]$ are fields of cohomological degree 0 and 1, and $V \in \operatorname{Vect}(X)$, $\alpha \in \Gamma(X, \operatorname{Sym}^2 TX)$, then we define $S^{\mathcal{L}}$ by

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) = \int \phi(\triangle_{g_0+\alpha} - \triangle_{g_0})\phi + \int (V\phi)\psi.$$

On the right hand side we interpret $\triangle_{g_0+\alpha}$ as a formal power series in the field α. The fact that this satisfies the master equation follows from the fact that the Laplacian $\triangle_{g_0+\alpha}$ is covariant under infinitesimal diffeomorphisms:

$$\triangle_{g_0+\alpha} + \varepsilon[V, \triangle_{g_0+\alpha}] = \triangle_{g_0+\alpha+\varepsilon\mathcal{L}_V g_0 + \varepsilon\mathscr{L}_V \alpha}.$$

One can rewrite this in the language of L_∞ algebras by Taylor expanding $\triangle_{g_0+\alpha}$ in powers of α. The resulting semi-direct product L_∞-algebra $\mathcal{L}^{Riem}(X) \ltimes \mathcal{M}(X)$ is the describes the formal moduli space of Riemannian manifolds together with a harmonic function ϕ.

Example: Let us modify the previous example by considering a scalar field theory with a polynomial interaction, so that the action functional is of the form

$$\int \phi\triangle_{g_0}\phi + \sum_{n\geq 2} \lambda_n \tfrac{1}{n!}\phi^n \mathrm{d}Vol_{g_0}.$$

In this case, $\mathcal{M}$ is deformed into a non-abelian L_∞ algebra, with maps l_n defined by

$$\begin{aligned} l_n &: C^\infty(X)^{\otimes n} \to \Omega^d(X) \\ l_n(\phi_1, \dots, \phi_n) &= \lambda_n \phi_1 \cdots \phi_n \mathrm{d}Vol_{g_0}. \end{aligned}$$

The action of $\mathcal{L}^{Riem}$ on $\mathcal{M}$ is defined, as above, by declaring that the action functional $S^{\mathcal{L}}$ coupling the two theories is

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) + S(\phi, \psi) = \int \phi\triangle_{g_0+\alpha}\phi + \sum_{n\geq 2} \lambda_n \tfrac{1}{n!}\phi^n \mathrm{d}Vol_{g_0+\alpha} + \int (V\phi)\psi.$$

Example: Next let us discuss the classical conformal field theories.

As above, let (X, g_0) be a Riemannian manifold. Define a local dg Lie algebra $\mathcal{L}^{conf}$ on X by setting

$$\mathcal{L}^{conf}(X) = \operatorname{Vect}(X) \oplus C^\infty(X) \oplus \Gamma(X, \operatorname{Sym}^2 TX)[-1].$$

The copy of $C^\infty(X)$ corresponds to Weyl rescalings.

The differential on $\mathcal{L}^{conf}(X)$ is

$$\mathrm{d}(V, f) = \mathscr{L}_V g_0 + f g_0$$

where $V \in \operatorname{Vect}(X)$ and $f \in C^\infty(X)$. The Lie bracket is defined by saying that $\operatorname{Vect}(X)$ acts on everything by Lie derivative, and that if $f \in C^\infty(X)$ and $\alpha \in \Gamma(X, \operatorname{Sym}^2 TX)$, $[f, \alpha] = f\alpha$.

It is easy to verify that $H^0(\mathcal{L}^{conf}(X))$ is the Lie algebra of conformal symmetries of X, and that $H^1(\mathcal{L}^{conf}(X))$ is the space of first-order conformal deformations of X. The local dg Lie algebra $\mathcal{L}^{conf}$ will act on any classical conformal field theory.

We will see this explicitly in the case of the free scalar field theory in dimension 2. Let $\mathcal{M}$ be the elliptic dg Lie algebra corresponding to the free scalar field theory on a Riemannian 2-manifold X, as described in the previous example.

The action of $\mathcal{L}^{conf}$ on $\mathcal{M}$ is defined such that the sub-algebra $\mathcal{L}^{Riem}$ acts in the same way as before, and that $C^\infty(X)$ acts by zero.

This does not define an action for the two-dimensional theory with polynomial interaction, because the polynomial interaction is not conformally invariant.

There are many other, more complicated, examples of this nature. If X is a conformal 4-manifold, then Yang-Mills theory on X is conformally invariant at the classical level. The same goes for self-dual Yang-Mills theory. One can explicitly write an action of $\mathcal{L}^{conf}$ on the elliptic L_∞-algebra on X describing either self-dual or full Yang-Mills theory.

Example: In this example, we will see how we can describe sources for local operators in the language of local dg Lie algebras.

Let X be a Riemannian manifold, and consider a scalar field theory on X with a ϕ^3 interaction, whose associated elliptic L_∞ algebra $\mathcal{M}$ has been described above. Recall that $\mathcal{M}(X)$ consists of $C^\infty(X)$ in degree 1 and of $\Omega^d(X)$ in degree 2, where $d = \dim X$.

The action functional encoding the L_∞ structure on $\mathcal{M}$ is the functional on $\mathcal{M}[1]$ defined by

$$S(\phi,\psi) = \int \phi \triangle \phi + \int \phi^3.$$

where ϕ is a degree 0 element of $\mathcal{M}(X)[1]$, so that ϕ is a smooth function.

Let us view the sheaf $C^\infty_X[-1]$ as an Abelian local L_∞-algebra on X, situated in degree 1 with zero differential and bracket.

Let us define an action of $\mathcal{L}$ on $\mathcal{M}$ by giving an action functional

$$S^{\mathcal{L}}(\alpha,\phi,\psi) = \int \phi \triangle \phi + \int \phi \alpha.$$

Here,

$$\alpha \in \mathcal{L}[1] = C^\infty(M)$$

and ϕ, ψ are elements of degrees 0 and 1 of $\mathcal{M}(X)[1]$. This gives a semi-direct product L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$, whose underlying cochain complex is

$$\begin{tikzcd}
C^\infty(X)^1 \arrow[dr, "\mathrm{Id}"] & \\
C^\infty(X)^1 \arrow[r, "\triangle"] & C^\infty(X)^2
\end{tikzcd}$$

where the first row is $\mathcal{L}$ and the second row is $\mathcal{M}$. The only non-trivial Lie bracket is the original Lie bracket on $\mathcal{M}$.

17.3. The factorization algebra of equivariant observable

17.3.0.3 Proposition. *Suppose that $\mathcal{M}$ is a classical field theory with an action of $\mathcal{L}$. Then, there is a P_0 factorization algebra $\mathrm{Obs}^{cl}_{\mathcal{L}}$ of equivariant observables, which is a factorization algebra in modules for the factorization algebra in commutative dg algebras $C^*(\mathcal{L})$, which assigns to an open subset U the commutative dga $C^*(\mathcal{L}(U))$.*

PROOF. Since $\mathcal{L}$ acts on $\mathcal{M}$, we can construct the semi-direct product local L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$. We define the equivariant classical observables

$$\mathrm{Obs}^{cl}_{\mathcal{L}} = C^*(\mathcal{L} \ltimes \mathcal{M})$$

to be the Chevalley-Eilenberg cochain factorization algebra associated to this semi-direct product.

As in section 12.4, we will construct a sub-factorization algebra on which the Poisson bracket is defined and which is quasi-isomorphic. We simply let

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathcal{L}}(U) \subset \mathrm{Obs}^{cl}_{\mathcal{L}}(U)$$

be the subcomplex consisting of those functionals which have smooth first derivative but only in the $\mathcal{M}$-directions. As in section 12.4, there is a P_0 structure on this subcomplex, which on generators is defined by the dual of the non-degenerate invariant pairing on $\mathcal{M}$. Those functionals which lie in $C^*(\mathcal{L}(U))$ are central for this Poisson bracket.

It is clear that this constructs a P_0-factorization algebra over the factorization algebra $C^*(\mathcal{L})$. □

17.4. Inner actions

A stronger notion of action of a local L_∞ algebra on a classical field theory will be important for Noeother's theorem. We will call this stronger notion an *inner* action of a local L_∞ algebra on a classical field theory. For classical field theories, every action can be lifted canonically to an inner action, but at the quantum level this is no longer the case.

We defined an action of a local L_∞ algebra $\mathcal{L}$ on a field theory $\mathcal{M}$ (both on a manifold X) to be a Maurer-Cartan element in the differential graded Lie algebra $\operatorname{Act}(\mathcal{L},\mathcal{M})$ whose underlying cochain complex is

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right)$$

with the Chevalley-Eilenberg differential for the direct sum L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$.

An inner action will be defined as a Maurer-Cartan element in a larger dg Lie algebra which is a central extension of $\operatorname{Act}(\mathcal{L},\mathcal{M})$ by $\mathscr{O}_{loc}(\mathcal{L}[1])$. Note that

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) \subset C^*_{red}(\mathcal{L}(X) \oplus \mathcal{M})$$

has the structure of dg Lie algebra, where the differential is the Chevalley-Eilenberg differential for the direct sum dg Lie algebra, and the bracket arises, as usual, from the invariant pairing on $\mathcal{M}$.

Further, there's a natural map of dg Lie algebras from this to $\mathscr{O}_{loc}(\mathcal{M}[1])$, which arises by applying the functor of Lie algebra cochains to the inclusion $\mathcal{M} \hookrightarrow \mathcal{M} \oplus \mathcal{L}$ of L_∞ algebras.

We let

$$\operatorname{InnerAct}(\mathcal{L},\mathcal{M}) \subset \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1])$$

be the kernel of this map. Thus, as a cochain complex,

$$\operatorname{InnerAct}(\mathcal{L},\mathcal{M}) = \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{loc}(\mathcal{M}[1])$$

with differential the Chevalley-Eilenberg differential for the direct sum L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$. Note that the Lie bracket on $\operatorname{InnerAct}(\mathcal{L},\mathcal{M})$ is of cohomological degree $+1$.

17.4.0.4 Definition. *An* inner *action of $\mathcal{L}$ on $\mathcal{M}$ is a Maurer-Cartan element*

$$S^{\mathcal{L}} \in \operatorname{InnerAct}(\mathcal{L},\mathcal{M}).$$

Thus, $S^{\mathcal{L}}$ is of cohomological degree 0, and satisfies the master equation

$$\mathrm{d}S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}.$$

17.4.0.5 Lemma. *Suppose we have an action of $\mathcal{L}$ on a field theory $\mathcal{M}$. Then there is an obstruction class in $H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))$ such that the action extends to an inner action if and only if this class vanishes.*

PROOF. There is a short exact sequence of dg Lie algebras

$$0 \to \mathscr{O}_{loc}(\mathcal{L}[1]) \to \operatorname{InnerAct}(\mathcal{L},\mathcal{M}) \to \operatorname{Act}(\mathcal{L},\mathcal{M}) \to 0$$

and $\mathscr{O}_{loc}(\mathcal{L}[1])$ is central. The result follows from general facts about Maurer-Cartan simplicial sets.

More explicitly, the obstruction is calculated as follows. Suppose we have an action functional

$$S^{\mathcal{L}} \in \mathrm{Act}(\mathcal{L}, \mathcal{M}) \subset C^*_{red}(\mathcal{L}(X)) \otimes C^*_{red}(\mathcal{M}(X)).$$

Then, let us view $S^{\mathcal{L}}$ as a functional in

$$\widetilde{S}^{\mathcal{L}}\,\mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \subset C^*_{red}(\mathcal{L}(X)) \otimes C^*(\mathcal{M}(X))$$

using the natural inclusion $C^*_{red}(\mathcal{M}(X)) \hookrightarrow C^*(\mathcal{M}(X))$. The obstruction is simply the failure of $\widetilde{S}^{\mathcal{L}}$ to satisfy the Maurer-Cartan equation in $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$. ☐

Let us now briefly remark on some refinements of this lemma, which give some more control over obstruction class.

Recall that we sometimes use the notation $C^*_{red,loc}(\mathcal{L})$ for the complex $\mathscr{O}_{loc}(\mathcal{L}[1])$. Thus, $C^*_{red,loc}(\mathcal{L})$ is a subcomplex of $C^*_{red}(\mathcal{L}(X))$. We let $C^{\geq 2}_{,loc}(\mathcal{L})$ be the kernel of the natural map

$$C^*_{red,loc}(\mathcal{L}) \to \mathcal{L}^!(X)[-1].$$

Thus, $C^{\geq 2}_{red,loc}(\mathcal{L})$ is a subcomplex of $C^{\geq 2}_{red}(\mathcal{L}(X))$.

17.4.0.6 Lemma. *If a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory $\mathcal{M}$, then the obstruction to extending $\mathcal{L}$ to an inner action lifts naturally to an element of the subcomplex*

$$C^{\geq 2}_{red,loc}(\mathcal{L}) \subset C^*_{red,loc}(\mathcal{L}).$$

PROOF. Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$ is encoded by an action functional $S^{\mathcal{L}}$, as before. The obstruction is

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}\right) |_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

Here, $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ are the Chevalley-Eilenberg differentials for the two L_∞ algebras.

We need to verify that no terms in this expression can be linear in $\mathcal{L}$. Recall that the functional $S^{\mathcal{L}}$ has no linear terms. Further, the differentials $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ respect the filtration on everything by polynomial degree, so that they can not produce a functional with a linear term from a functional which does not have a linear term. ☐

Remark: There is a somewhat more general situation when this lemma is false. When one works with families of classical field theories over some dg ring R with a nilpotent ideal I, one allows the L_∞ algebra $\mathcal{M}$ describing the field theory to be curved, as long as the curving vanishes modulo I. This situation is encountered in the study of σ-models: see [Cos11a]. When $\mathcal{M}$ is curved, the differential $\mathrm{d}_{\mathcal{M}}$ does not preserve the filtration by polynomial degree, so that this argument fails.

Let us briefly discuss a special case when the obstruction vanishes.

17.4.0.7 Lemma. *Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$, when viewed as an action of $\mathcal{L}$ on the sheaf of formal moduli problems $B\mathcal{M}$, preserves the base point of $B\mathcal{M}$. In the language of L_∞ algebras, this means that the L_∞ structure on $\mathcal{L} \oplus \mathcal{M}$ defining the action has no terms mapping*

$$\mathcal{L}^{\otimes n} \to \mathcal{M}$$

for some $n > 0$.

Then, the action of $\mathcal{L}$ extends canonically to an inner action.

PROOF. We need to verify that the obstruction

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} \right) |_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

is identically zero. Our assumptions on $S^{\mathcal{L}}$ mean that it is at least quadratic as a function on $\mathcal{M}[1]$. It follows that the obstruction is also at least quadratic as a function of $\mathcal{M}[1]$, so that it is zero when restricted to being a function of just $\mathcal{L}[1]$. ☐

17.5. Classical Noether's theorem

As we showed in lemma **??**, there is a bijection between classes in $H^1(C^*_{red,loc}(\mathcal{L}))$ and local central extensions of $\mathcal{L}_c$ shifted by -1.

17.5.0.8 Theorem. *Let $\mathcal{M}$ be a classical field theory with an action of a local L_∞ algebra $\mathcal{L}$. Let $\widetilde{\mathcal{L}}_c$ be the central extension corresponding to the obstruction class $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ for lifting $\mathcal{L}$ to an inner action. Let $\widetilde{\mathrm{Obs}}^{cl}$ be the classical observables of the field theory $\mathcal{M}$, equipped with its P_0 structure. Then, there is an L_∞-map of precosheaves of L_∞-algebras*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

which sends the central element c to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}[-1]$ (note that, after the shift, the unit 1 is in cohomological degree 1, as is the central element c).

Remark: The linear term in the L_∞-morphism is a map of precosheaves of cochain complexes from $\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1]$. The fact that we have such a map of precosheaves implies that we have a map of commutative dg factorization algebras

$$\widehat{\mathrm{Sym}}^*(\widetilde{\mathcal{L}}_c[1]) \to \widetilde{\mathrm{Obs}}^{cl}.$$

which, as above, sends the central element to 1. This formulation is the one that will quantize: we will find a map from a certain Chevalley-Eilenberg chain complex of $\widetilde{\mathcal{L}}_c[1]$ to quantum observables.

Remark: Lemma 17.4.0.6 implies that the central extension $\widetilde{\mathcal{L}}_c$ is split canonically as a presheaf of cochain complexes:

$$\widetilde{\mathcal{L}}_c(U) = \mathbb{C}[-1] \oplus \mathcal{L}_c(U).$$

Thus, we have a map of precosheaves of cochain complexes

$$\mathcal{L}_c \to \mathrm{Obs}^{cl}.$$

The same proof will show that that this cochain map to a continuous map from the distributional completion $\overline{\mathcal{L}}_c(U)$ to Obs^{cl}.

PROOF. Let us first consider a finite-dimensional version of this statement, in the case when the central extension splits. Let $\mathfrak{g}, \mathfrak{h}$ be L_∞ algebras, and suppose that $\mathfrak{h}$ is equipped with an invariant pairing of degree -3. Then, $C^*(\mathfrak{h})$ is a P_0 algebra. Suppose we are given an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of cohomological degree 0, satisfying the Maurer-Cartan

$$\mathrm{d}G + \tfrac{1}{2}\{G, G\} = 0$$

where $\mathrm{d}_\mathfrak{g}, \mathrm{d}_\mathfrak{h}$ are the Chevalley-Eilenberg differentials for $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and $\{-,-\}$ denotes the Poisson bracket coming from the P_0 structure on $C^*(\mathfrak{h})$.

Then, G is precisely the data of an L_∞ map

$$\mathfrak{g} \to C^*(\mathfrak{h})[-1].$$

Indeed, for any L_∞ algebra $\mathfrak{j}$, to give a Maurer-Cartan element in $C^*_{red}(\mathfrak{g}) \otimes \mathfrak{j}$ is the same as to give an L_∞ map $\mathfrak{g} \to \mathfrak{j}$. Further, the simplicial set of L_∞-maps and homotopies between them is homotopy equivalent to the Maurer-Cartan simplicial set.

Let us now consider the case when we have a central extension. Suppose that we have an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of degree 0, and an obstruction element

$$\alpha \in C^*_{red}(\mathfrak{g})$$

of degree 1, such that

$$\mathrm{d}G + \tfrac{1}{2}\{G, G\} = \alpha \otimes 1.$$

Let $\widetilde{\mathfrak{g}}$ be the -1-shifted central extension determined by α, so that there is a short exact sequence

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Then, the data of G and α is the same as a map of L_∞ algebras $\widetilde{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$ which sends the central element of $\widetilde{\mathfrak{g}}$ to $1 \in C^*(\mathfrak{h})$.

To see this, let us choose a splitting $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \cdot c$ where the central element c is of degree 1. Let $c^\vee$ be the linear functional on $\widetilde{\mathfrak{g}}$ which is zero on $\mathfrak{g}$ and sends c to 1.

Then, the image of α under the natural map $C^*(\mathfrak{g}) \to C^*(\widetilde{\mathfrak{g}})$ is made exact by $c^\vee$, viewed as a zero-cochain in $C^*(\widetilde{\mathfrak{g}})$. It follows that

$$G + c^\vee \otimes 1 \in C^*_{red}(\widetilde{\mathfrak{g}}) \otimes C^*(\mathfrak{h})$$

satisfies the Maurer-Cartan equation, and therefore defines (as above) an L_∞-map $\widetilde{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$. This L_∞-map sends $c \to 1$: this is because G only depends on c by the term $c^\vee \otimes 1$.

Let us apply these remarks to the setting of factorization algebras. First, let us remark a little on the notation: we normally use the notation $\mathscr{O}_{loc}(\mathcal{L}[1])$ to refer to the complex of local functionals on $\mathcal{L}[1]$, with the Chevalley-Eilenberg differential. However, we can also refer to this object as $C^*_{red,loc}(\mathcal{L})$, the reduced, local cochains of $\mathcal{L}$. It is the subcomplex of $C^*_{red}(\mathcal{L}(X))$ of cochains which are local.

Suppose we have an action of a local L_∞-algebra $\mathcal{L}$ on a classical field theory $\mathcal{M}$. Let

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1]) = C^*_{red,loc}(\mathcal{L})$$

be a 1-cocycle representing the obstruction to lifting to an inner action on $\mathcal{M}$. Let

$$\widetilde{\mathcal{L}}_c = \mathcal{L}_c \oplus \underline{\mathbb{C}}[-1]$$

be the corresponding central extension.

By the definition of α, we have a functional

$$S^{\mathcal{L}} \in C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M})$$

of cohomological degree 0 satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = \alpha.$$

For every open subset $U \subset M$, we have an injective cochain map

$$\Phi : C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M}) \to C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U)),$$

where $\widehat{\otimes}$ refers to the completed tensor product and $\widetilde{C}^*(\mathcal{M}(U))$ refers to the subcomplex of $C^*(\mathcal{M}(U))$ consisting of functionals with smooth first derivative. The reason we have such a map is simply that a local functional on U is defined when at least if its inputs is compactly supported.

The cochain map Φ is in fact a map of dg Lie algebras, where the Lie bracket arises as usual from the pairing on $\mathcal{M}$. Thus, for every U, we have an element

$$S^{\mathcal{L}}(U) \in C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U))$$

satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}}(U) + \tfrac{1}{2}\{S^{\mathcal{L}}(U), S^{\mathcal{L}}(U)\} = \alpha(U).$$

It follows, as in the finite-dimensional case discussed above, that $S^{\mathcal{L}}(U)$ gives rise to a map of L_∞ algebras

$$\widetilde{\mathcal{L}}_c(U) \to \widetilde{C}^*(\mathcal{M}(U))[-1] = \widetilde{\mathrm{Obs}}^{cl}(U)[-1]$$

sending the central element c in $\widetilde{\mathcal{L}}_c(U)$ to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}(U)$. The fact that $S^{\mathcal{L}}$ is local implies immediately that this is a map of precosheaves. ☐

17.6. Conserved currents

Traditionally, Noether's theorem states that there is a conserved current associated to every symmetry. Let us explain why the version of (classical) Noether's theorem presented above leads to this more traditional statement. Similar remarks will hold for the quantum version of Noether's theorem.

In the usual treatment, a *current* is taken to be an $d-1$-form valued in Lagrangians (if we're dealing with a field theory on a manifold X of dimension d). In our formalism, we make the following definition (which will be valid at the quantum level as well).

17.6.0.9 Definition. *A conserved current in a field theory is a map of precosheaves*

$$J : \overline{\Omega}_c^*[1] \to \mathrm{Obs}^{cl}$$

to the factorization algebra of classical observables.

Dually, J can be viewed as a closed, degree 0 element of

$$J(U)\Omega^*(U)[n-1] \widehat{\otimes} \mathrm{Obs}^{cl}(U)$$

defined for every open subset U, and which is compatible with inclusions of open subsets in the obviious way.

In particular, we can take the component of $J(U)^{n-1,0}$ which is an element

$$J(U)^{n-1,0} \in \Omega^{n-1}(U) \widehat{\otimes} \mathrm{Obs}^{cl}(U)^0.$$

The superscript in $\mathrm{Obs}^{cl}(U)^0$ indicates cohomological degree 0. Thus, $J(U)^{n-1,0}$ is an $n-1$-form valued in observables, which is precisely what is traditionally called a current.

Let us now explain why our definition means that this current is conserved (up to homotopy). We will need to introduce a little notation to explain this point. If $N \subset X$ is a closed subset, we let

$$\mathrm{Obs}^{cl}(N) = \mathrm{holim}_{N \subset U} \mathrm{Obs}^{cl}(U)$$

be the homotopy limit of observables on open neighbourhoods of N. Thus, an element of $\mathrm{Obs}^{cl}(N)$ is an observable defined on every open neighbourhood of N, in a way compatible (up to homotopy) with inclusions of open sets. The fact that we are taking a homotopy limit instead of an ordinary limit is not so important for this discussion, it's to ensure that the answer doesn't depend on arbitrary choices.

For example, if $p \in X$ is a point, then $\mathrm{Obs}^{cl}(p)$ should be thought of as the space of local observables at p.

Suppose we have a conserved current (in the sense of the definition above). Then, for every compact codimension 1 oriented submanifold $N \subset X$, the delta-distribution on N is an element

$$[N] \in \overline{\Omega}^1(U)$$

defined for every open neighbourhood U of N. Applying the map defining the closed current, we get an element

$$J[N] \in \mathrm{Obs}^{cl}(U)$$

for every neighbourhood U of N. This element is compatible with inclusions $U \hookrightarrow U'$, so defines an element of $\mathrm{Obs}^{cl}(N)$.

Let $M \subset X$ be a top-dimensional submanifold with boundary $\partial M = N \amalg N'$. Then,

$$\mathrm{d}J[M] = J[N] - J[N'] \in \mathrm{Obs}^{cl}(M).$$

It follows that the cohomology class $[J[N]]$ of $J[N]$ doesn't change if N is changed by a cobordism.

In particular, let us suppose that our space-time manifold X is a product

$$X = N \times \mathbb{R}.$$

Then, the observable $[J[N_t]]$ associated to the submanifold $N \times \{t\}$ is independent of t.

This is precisely the condition (in the traditional formulation) for a current to be conserved.

Now let us explain why our version of Noether's theorem, as explained above, produces a conserved current from a symmetry.

17.6.0.10 Lemma. *Suppose we have a classical field theory on a manifold X which has an infinitesimal symmetry. To this data, our formulation of Noether's theorem produces a conserved current.*

PROOF. A theory with an infinitesimal symmetry is acted on by the abelian Lie algebra $\mathbb{R}$ (or $\mathbb{C}$). Lemma **??** shows us that such an action is equivalent to the action of the Abelian local dg Lie algebra Ω_X^*. Lemma 17.4.0.6 implies that the central extension $\widetilde{\mathcal{L}}_c$ is split as a cochain complex (*except* in the case that we work in families and the classical field theory $\mathcal{M}$ has curving). We thus get a map

$$\Omega^*_{X,c}[1] \to \mathrm{Obs}^{cl}.$$

The remark following theorem 17.5.0.8 tells us that this map extends to a continuous cochain map

$$\overline{\Omega}^*_{X,c}[1] \to \mathrm{Obs}^{cl}$$

which is our defintion of a conserved current.

□

17.7. Examples of classical Noether's theorem

Let's give some simple examples of this construction. All of the examples we will consider here will satisfy the criterion of lemma **??** which implies that the central extension of the local L_∞ algebra of symmetries is trivial.

Example: Suppose that a field theory on a manifold X of dimension d has an inner action of the Abelian local L_∞ algebra $\Omega^d_X[-1]$. Then, we get a map of presheaves of cochain complexes

$$\overline{\Omega}^d_X \to \mathrm{Obs}^{cl}.$$

Since, for every point $p \in X$, the delta-function δ_p is an element of $\overline{\Omega}^d(X)$, in this way we get a local observable in $\mathrm{Obs}^{cl}(p)$ for every point. This varies smoothly with p.

For example, consider the free scalar field theory on X. We can define an action of $\Omega^d_X[-1]$ on the free scalar field theory as follows. If $\phi \in C^\infty(X)$ and $\psi \in \Omega^d(X)[-1]$ are fields of the free scalar field theory, and $\gamma \in \Omega^d(X)$ is an element of the Abelian local L_∞ algebra we want to act, then the action is described by the action functional describing how $\mathcal{L}$

$$S^{\mathcal{L}}(\phi, \psi, \gamma) = \int \phi\gamma.$$

The corresponding map L_∞ map

$$(\dagger) \qquad \Omega^d_c(U) \to \mathrm{Obs}^{cl}(U)$$

is linear, and sends $\gamma \in \Omega^d_c(U)$ to the observable $\int \phi\gamma$.

Example: Let's consider the example of a scalar field theory on a Riemannian manifold X, described by the action functional

$$\int_X \phi \triangle \phi + \phi^3 \mathrm{d}Vol.$$

This is acted on by the dg Lie algebra $\mathcal{L}^{Riem}$ describing deformations of X as a Riemannian manifold.

If U is an open subset of X, then $\mathcal{L}_c^{Riem}(U)$ consists, in degree 0, of the compactly-supported first-order deformations of the Riemannian metric g_0 on $U \subset X$. If

$$\alpha \in \Gamma_c(U, \mathrm{Sym}^2 TX)$$

is such a deformation, let us Taylor expand the Laplace-Beltrami operator $\phi \triangle_{g_0+\alpha} \phi$ as a sum

$$\triangle_{g_0+\alpha} = \triangle_{g_0} \sum_{n \geq 1} \tfrac{1}{n!} D_n(\alpha, \dots, \alpha)$$

where D_n are poly-differential operators from $\Gamma(X, TX)^{\otimes n}$ to the space of order ≤ 2 differential operators on X. Explicit formula for the operators D_n can be derived from the formula for the Laplace-Beltrami operator in terms of the metric.

Note that if α has compact support in U, then

$$\int \phi D_n(\alpha, \dots, \alpha) \phi$$

defines an observable in $\mathrm{Obs}^{cl}(U)$, and in fact in $\widetilde{\mathrm{Obs}}^{cl}(U)$ (as it has smooth first derivative in ϕ.

The L_∞ map

$$\mathcal{L}_c^{Riem}(U) \to \mathrm{Obs}^{cl}(U)[-1]$$

has Taylor terms

$$\Phi_n : \mathcal{L}_c^{Riem}(U)^{\otimes n} \to \mathrm{Obs}^{cl}(U)$$

defined by the observables

$$\Phi_n((\alpha_1, V_1), \dots, (\alpha_n, V_n))(\phi, \psi) = \begin{cases} \int \phi D_n(\alpha_1, \dots, \alpha_n) \phi & \text{if } n > 1 \\ \int \phi D_1(\alpha) \phi + \int (V\phi) \psi & \text{if } n = 1. \end{cases}$$

One is often just interested in the cochain map

$$\mathcal{L}_c^{Riem}(U) \to \mathrm{Obs}^{cl}(U),$$

corresponding to ϕ_1 above, and not in the higher terms. This cochain map has two terms: one given by the observable describing the first-order variation of the metric, and one given by the observable $\int (V\phi)\psi$ describing the action of vector fields on the fields of the theory.

A similar analysis describes the map from $\mathcal{L}_c^{Riem}$ to the observables of a scalar field theory with polynomial interaction.

Example: Let us consider the $\beta\gamma$ system in one complex dimension, on $\mathbb{C}$. The dg Lie algebra $\mathcal{M}$ describing this theory is

$$\mathcal{M}(\mathbb{C}) = \Omega^{0,*}(\mathbb{C}, V)[-1] \oplus \Omega^{1,*}(\mathbb{C}, V^*)[-1].$$

Let

$$\mathcal{L} = \Omega^{0,*}(\mathbb{C}, T\mathbb{C})$$

be the Dolbeault resolution of holomorphic vector fields on $\mathbb{C}$. $\mathcal{L}$ acts on $\mathcal{M}$ by Lie derivative. We can write down the action functional encoding this action by

$$S^{\mathcal{L}}(\beta,\gamma,V) = \int (\mathscr{L}_V\beta)\gamma.$$

Here, $\beta \in \Omega^{0,*}(\mathbb{C}, V)$, $\gamma \in \Omega^{1,*}(\mathbb{C}, V^*)$ and $V \in \Omega^{0,*}(\mathbb{C}, T\mathbb{C})$.

Lemma **??** implies that in this case there is no central extension. Therefore, we have a map

$$\Phi : \mathcal{L}_c[1] \to \mathrm{Obs}^{cl}$$

of precosheaves of cochain complexes. At the cochain level, this map is very easy to describe: it simply sends a compactly supported vector field $V \in \Omega^{0,*}_c(U, TU)[1]$ to the observable

$$\Phi(V)(\beta,\gamma) = \int_U (\mathscr{L}_V\beta)\,\gamma.$$

We are interested in what this does at the level of cohomology. Let us work on an open annulus $A \subset \mathbb{C}$. We have seen (section **??**??) that the cohomology of $\mathrm{Obs}^{cl}(A)$ can be expressed in terms of the dual of the space of holomorphic functions on A:

$$H^0(\mathrm{Obs}^{cl}(A)) = \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V\right).$$

Higher cohomology of $\mathrm{Obs}^{cl}(A)$ vanishes.

Here, $\mathrm{Hol}(A)$ denotes holomorphic functions on A, $\Omega^1_{hol}(A)$ denotes holomorphic 1-forms, and we are taking the continuous linear duals of these spaces. Further, we use, as always, the completed tensor product when defining the symmetric algebra.

In a similar way, we can identify

$$H^*(\Omega^{0,*}_c(A, TA)) = H^*(\Omega^{0,*}(A, K_A^{\otimes 2})^\vee[-1].$$

The residue pairing gives a dense embedding

$$\mathbb{C}[t,t^{-1}]\mathrm{d}t \subset \mathrm{Hol}(A)^\vee.$$

A concrete map

$$\mathbb{C}[t,t^{-1}][-1] \to \Omega^{0,*}_c(A)$$

which realizes this map is defined as follows. Choose a smooth function f on the annulus which takes value 1 near the outer boundary and value 0 near the inner boundary. Then, $\overline{\partial} f$ has compact support. The map sends a polynomial $P(t)$ to $\overline{\partial}(fP)$. One can check, using Stokes' theorem, that this is compatible with the residue pairing: if $Q(t)\mathrm{d}t$ is a holomorphic one-form on the annulus,

$$\oint P(t)Q(t)\mathrm{d}t = \int_A \overline{\partial}(f(t,\overline{t})P(t))Q(t)\mathrm{d}t.$$

In particular, the residue pairing tells us that a dense subspace of $H^1(\mathcal{L}_c(A))$ is

$$\mathbb{C}[t, t^{-1}]\partial_t \subset H^1(\Omega_c^{0,*}(A, TA)).$$

We therefore need to describe a map

$$\Phi : \mathbb{C}[t, t^{-1}]\partial_t \to \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V\right).$$

In other words, given an element $P(t)\partial_t \in \mathbb{C}[t, t^{-1}]\mathrm{d}t$, we need to describe a functional $\Phi(P(t)\partial_t)$ on the space of pairs

$$(\beta, \gamma) \in \mathrm{Hol}(A) \otimes V \oplus \Omega^1_{hol}(A) \otimes V^\vee.$$

From what we have explained so far, it is easy to calculate that this functional is

$$\Phi(P(t)\partial_t)(\beta, \gamma) = \oint (P(t)\partial_t\beta(t))\, \gamma(t).$$

The reader familiar with the theory of vertex algebras will see that this is the classical limit of a standard formula for the Virasoro current.

CHAPTER 18

Noether's theorem in quantum field theory

18.1. Quantum Noether's theorem

So far, we have explained the classical version of Noether's theorem, which states that given an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory, we have a central extension $\widetilde{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞-algebras, and a map of precosheaves of L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1].$$

Our quantum Noether's theorem provides a version of this at the quantum level. Before we explain this theorem, we need to introduce some algebraic ideas about enveloping algebras of homotopy Lie algebras.

Given any dg Lie algebra $\mathfrak{g}$, one can construct its P_0 envelope, which is the universal P_0 algebra containing $\mathfrak{g}$. This functor is the homotopy left adjoint of the forgetful functor from P_0 algebras to dg Lie algebras. Explicitly, the P_0 envelope is

$$U^{P_0}(\mathfrak{g}) = \mathrm{Sym}^* \mathfrak{g}[1]$$

with the obvious product. The Poisson bracket is the unique bi-derivation which on the generators $\mathfrak{g}$ is the given Lie bracket on $\mathfrak{g}$.

Further, if we have a shifted central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbb{C}[-1]$, determined by a class $\alpha \in H^1(\mathfrak{g})$, we can define the twisted P_0 envelope

$$U^{P_0}_\alpha(\mathfrak{g}) = U^{P_0}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

obtained from the P_0 envelope of $\widetilde{\mathfrak{g}}$ by specializing the central parameter to 1.

We will reformulate the classical Noether theorem using the factorization P_0 envelope of a sheaf of L_∞ algebras on a manifold. The quantum Noether theorem will then be formulated in terms of the factorization BD envelope, which is the quantum version of the factorization P_0 envelope. The factorization BD envelope is a close relative of the factorization envelope of a sheaf of L_∞ algebras that we discussed in Section 3.6 of Chapter 3.

For formal reasons, a version of these construction holds in the world of L_∞ algebras. One can show that a commutative dg algebra together with a 1-shifted L_∞ structure with the property that all higher brackets are multi-derivations for the product structure defines a homotopy P_0 algebra. (The point is that the operad describing such gadgets is naturally quasi-isomorphic to the operad P_0).

If $\mathfrak{g}$ is an L_∞ algebra, one can construct a homotopy P_0 algebra which has underlying commutative algebra $\operatorname{Sym}^* \mathfrak{g}[1]$, and which has the unique shifted L_∞ structure where $\mathfrak{g}[1]$ is a sub-L_∞ algebra and all higher brackets are derivations in each variable. This L_∞ structure makes $\operatorname{Sym}^* \mathfrak{g}[1]$ into a homotopy P_0 algebra, and one can show that it is the homotopy P_0 envelope of $\mathfrak{g}$.

We can rephrase the classical version of Noether's theorem as follows.

18.1.0.11 Theorem. *Suppose that a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory, and that the obstruction to lifting this to an inner action is a local cochain α. Then there is a map of homotopy P_0 factorization algebras*

$$U_\alpha^{P_0}(\mathcal{L}_c) \to \operatorname{Obs}^{cl}.$$

Here $U_\alpha^{P_0}(\mathcal{L}_c)$ is the twisted homotopy P_0 factorization envelope, which is defined by saying that on each open subset $U \subset M$ it is $U_\alpha^{P_0}(\mathcal{L}_c(U))$.

The universal property of $U_\alpha^{P_0}(\mathcal{L}_c)$ means that this theorem is a formal consequence of the version of Noether's theorem that we have already proved. At the level of commutative factorization algebras, this map is obtained just by taking the cochain map $\widetilde{\mathcal{L}}_c(U) \to \operatorname{Obs}^{cl}(U)$ and extending it in the unique way to a map of commutative dg algebras

$$\operatorname{Sym}^*(\widetilde{\mathcal{L}}_c(U)) \to \operatorname{Obs}^{cl}(U),$$

before specializing by setting the central parameter to be 1. There are higher homotopies making this into a map of homotopy P_0 algebras, but we will not write them down explicitly (they come from the higher homotopies making the map $\widetilde{\mathcal{L}}_c(U) \to \operatorname{Obs}^{cl}(U)$ into a map of L_∞ algebras).

This formulation of classical Noether's theorem is clearly ripe for quantization. We must simply replace classical observables by quantum observables, and the P_0 envelope by the BD envelope.

Recall that the BD operad is an operad over $\mathbb{C}[[\hbar]]$ which quantizes the P_0 operad. A BD algebra cochain complex with a Poisson bracket of degree 1 and a commutative product, such that the failure of the differential to be a derivation for the commutative product is measured by $\hbar$ times the Poisson bracket.

There is a map of operads over $\mathbb{C}[[\hbar]]$ from the Lie operad to the BD operad. At the level of algebras, this map takes a BD algebra A to the dg Lie algebra $A[-1]$ over $\mathbb{C}[[\hbar]]$, with the Lie bracket given by the Poisson bracket on A. The BD envelope of a dg Lie algebra $\mathfrak{g}$ is defined to be homotopy-universal BD algebra $U^{BD}(\mathfrak{g})$ with a map of dg Lie algebras from $\mathfrak{g}[[\hbar]]$ to $U^{BD}(\mathfrak{g})[-1]$.

One can show that for any dg Lie algebra $\mathfrak{g}$, the homotopy BD envelope of $\mathfrak{g}$ is the Rees module for the Chevalley chain complex $C_*(\mathfrak{g}) = \mathrm{Sym}^*(\mathfrak{g}[-1])$, which is equipped with the increasing filtration defined by the symmetric powers of $\mathfrak{g}$. Concretely,

$$U^{BD}(\mathfrak{g}) = C_*(\mathfrak{g})[[\hbar]] = \mathrm{Sym}^*(\mathfrak{g}[-1])[[\hbar]]$$

with differential $\mathrm{d}_{\mathfrak{g}} + \hbar \mathrm{d}_{CE}$, where $\mathrm{d}_{\mathfrak{g}}$ is the internal differential on $\mathfrak{g}$ and d_{CE} is the Chevalley-Eilenberg differential. The commutative product and Lie bracket are the $\hbar$-linear extensions of those on the P_0 envelope we discussed above. A similar statement holds for L_∞ algebras.

This discussion holds at the level of factorization algebras too: the BD envelope of a local L_∞ algebra $\mathcal{L}$ is defined to be the factorization algebra which assigns to an open subset U the BD envelope of $\mathcal{L}_c(U)$. Thus, it is the Rees factorization algebra associated to the factorization envelope of U. (We will describe this object in more detail in section 18.5 of this chapter).

Now we can state the quantum Noether theorem.

18.1.0.12 Theorem. *Suppose we have a quantum field theory on a manifold M acted on by a local L_∞ algebra $\mathcal{L}$. Let Obs^q be the factorization algebra of quantum observables of this field theory.*

In this situation, there is a $\hbar$-dependent local cocycle

$$\alpha \in H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))[[\hbar]]$$

and a homomorphism of factorization algebras from the twisted BD envelope

$$U^{BD}_{\alpha}(\mathcal{L}_c) \to \mathrm{Obs}^q .$$

The relationship between this formulation of quantum Noether's theorem and the traditional point of view on Noether's theorem was discussed (in the classical case) in section **??**.Let us explain, however, some aspects of this story which are slightly different in the quantum and classical settings.

Suppose that we have an action of an ordinary Lie algebra $\mathfrak{g}$ on a quantum field theory on a manifold M. Then the quantum analogue of the result of lemma **??** (which we will prove below) shows that we have an action of the local dg Lie algebra $\Omega^*_X \otimes \mathfrak{g}$ on the field theory. It follows that we have a central extention of $\Omega^*_X \otimes \mathfrak{g}$, given by a class $\alpha \in H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))$, and a map from the twisted BD envelope of this central extension to observables of our field theory.

Suppose that $N \subset X$ is an oriented codimension 1 submanifold. (We assume for simplicity that X is also oriented). Let us choose an identification of a tubular neighbourhood of N with $N \times \mathbb{R}$. Let $\pi_N : N \times \mathbb{R} \to \mathbb{R}$ denote the projection map to $\mathbb{R}$. The push forward of the factorization algebra $U^{BD}(\Omega^*_X \otimes \mathfrak{g})$ along the projection $\pi_{\mathbb{R}}$ defines a locally-constant facotrization algebra on $\mathbb{R}$, and so an associative algebra.

Let us assume, for the moment, that the central extension vanishes. Then a variant of lemma **??** shows that there is an isomorphism of associative algebras

$$H^*\left(\pi_N U^{BD}(\Omega^*_X \otimes \mathfrak{g})\right) \cong \operatorname{Rees}(U(H^*(N) \otimes \mathfrak{g})).$$

The algebra on the right hand side is the Rees algebra for the universal enveloping algebra of $H^*(N) \otimes \mathfrak{g}$. This algebra is a $\mathbb{C}[[\hbar]]$-algebra which specializes at $\hbar = 0$ to the completed symmetric algebra of $H^*(N) \otimes \mathfrak{g}$, but is generically non-commutative.

In this way, we see that Noether's theorem gives us a map of factorization algebras on $\mathbb{R}$

$$\operatorname{Rees}(U(\mathfrak{g})) \to H^0(\pi_N \operatorname{Obs}^q)$$

where on the right hand side we have quantum observables of our theory, projected to $\mathbb{R}$.

Clearly this is closely related to the traditional formulation of Noether's theorem: we are saying that every symmetry (i.e. element of $\mathfrak{g}$) gives rise to an observable on every codimension 1 manifold (that is, a current). The operator product between these observables is the product in the universal enveloping algebra.

Now let us consider the case when the central extension is non-zero. A small calculation shows that the group containing possible central extensions can be identified as

$$H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))[[\hbar]] = H^{d+1}(X, C^*_{red}(\mathfrak{g})) = \oplus_{i+j=d+1} H^i(X) \otimes H^j_{red}(\mathfrak{g})[[\hbar]],$$

where d is the real dimension of X, and $C^*_{red}(\mathfrak{g})$ is viewed as a constant sheaf of cochain complexes on X.

Let us assume that X is of the form $N \times \mathbb{R}$, where as above N is compact and oriented. Then the cocycle above can be integrated over N to yield an element in $H^2_{red}(\mathfrak{g})$, which can be viewed as an ordinary, unshifted central extension of the Lie algebra $\mathfrak{g}$ (which depends on $\hbar$). We can form the twisted universal enveloping algebra $U_\alpha(\mathfrak{g})$, obtained as usual by taking the universal enveloping algebra of the central extension of $\mathfrak{g}$ and then setting the central parameter to 1. This twisted enveloping algebra admits a filtration, so that we can form the Rees algebra. Our formulation of Noether's theorem then produces a map of factorization algebras on $\mathbb{R}$

$$\operatorname{Rees}(U_\alpha(\mathfrak{g})) \to H^0(\pi_N^* \operatorname{Obs}^q).$$

18.2. Actions of a local L_∞-algebra on a quantum field theory

Let us now turn to the proof of the quantum version of Noether's theorem. As in the discussion of the classical theory, the first thing we need to pin down is what it means for a local L_∞ algebra to act on a quantum field theory.

As in the setting of classical field theories, there are two variants of the definition we need to consider: one defining a field theory with an $\mathcal{L}$ action, and one a field theory with an *inner* $\mathcal{L}$-action. Just as in the classical story, the central extension that appears in our formulation of Noether's theorem appears as the obstruction to lifting a field theory with an action to a field theory with an inner action.

We have used throughout the definition of quantum field theory given in [**Cos11c**]. The concept of field theory with an action of a local L_∞-algebra $\mathcal{L}$ relies on a refined definition of field theory, also given in [**Cos11c**]: the concept of a field theory with background fields. Let us explain this definition.

Let us fix a classical field theory, defined by a local L_∞ algebra $\mathcal{M}$ on X with an invariant pairing of cohomological degree -3. Let us choose a gauge fixing operator Q^{GF} on $\mathcal{M}$, as discussed in section **??**. Then as before, we have an elliptic differential operator $[Q, Q^{GF}]$ (where Q refers to the linear differential on $\mathcal{M}$). As explained in section **??**, this leads to the following data.

(1) A propagator $P(\Phi) \in \overline{\mathcal{M}}[1]^{\otimes 2}$, defined for every parametrix Φ. If Φ, Ψ are parametrices, then $P(\Phi) - P(\Psi)$ is smooth.
(2) A kernel $K_\Phi \in \mathcal{M}[1]^{\otimes 2}$ for every parametrix Φ, satisfying

$$Q(P(\Phi) - P(\Psi)) = K_\Psi - K_\Phi.$$

These kernels lead, in turn, to the definition of the RG flow operator and of the BV Laplacian

$$\begin{aligned} W(P(\Phi) - P(\Psi), -) : \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] &\to \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] \\ \triangle_\Phi : \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] &\to \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] \end{aligned}$$

associated to parametrices Φ and Ψ. There is also a BV bracket $\{-,-\}_\Phi$ which satisfies the usual relation with the BV Laplacian $\triangle_\Phi$. The space $\mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$ is the space of functionals with proper support and smooth first derivative which are at least cubic modulo $\hbar$.

The homological interpretation of these objects are as follows.

(1) For every parametrix Φ, we have the structure of 1-shifted differential graded Lie algebra on $\mathscr{O}(\mathcal{M}[1])[[\hbar]]$. The Lie bracket is $\{-,-\}_\Phi$, and the differential is
$$Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi.$$
The subspace $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ is a nilpotent sub-dgla. The Maurer-Cartan equation in this space is called the *quantum master equation*.

(2) The map $W(P(\Phi) - P(\Psi), -)$ takes solutions to the QME with parametrix Ψ to solutions with parametrix Φ. Equivalently, the Taylor terms of this map define an L_∞ isomorphism between the dglas associated to the parametrices Ψ and Φ.

If $\mathcal{L}$ is a local L_∞ algebra, then $\mathscr{O}(\mathcal{L}[1])$, with its Chevalley-Eilenberg differential, is a commutative dg algebra. We can identify the space $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ of functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$ with the completed tensor product
$$\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1]) = \mathscr{O}(\mathcal{L}[1]) \widehat{\otimes}_\pi \mathscr{O}(\mathcal{M}[1]).$$
The operations $\triangle_\Phi$, $\{-,-\}_\Phi$ and $\partial_{P(\Phi)}$ associated to a parametrix on $\mathcal{M}$ extend, by $\mathscr{O}(\mathcal{L}[1])$-linearity, to operations on the space $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. For instance, the operator $\partial_{P(\Phi)}$ is associated to the kernel
$$P(\Phi) \in (\mathcal{M}[1])^{\otimes 2} \subset (\mathcal{M}[1] \oplus \mathcal{L}[1])^{\otimes 2}.$$
If $\mathrm{d}_{\mathcal{L}}$ denotes the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, then we can form an operator $\mathrm{d}_{\mathcal{L}} \otimes 1$ on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. Similarly, the linear differential Q on $\mathcal{M}$ induces a derivation of $\mathscr{O}(\mathcal{M}[1])$ which we also denote by Q; we can for a derivation $1 \otimes Q$ of $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$.

The operators $\triangle_\Phi$ and ∂_Φ both commute with $\mathrm{d}_{\mathcal{L}} \otimes 1$ and satisfy the same relation described above with the operator $1 \otimes Q$.

We will let
$$\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]] \subset \mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$$
denote the space of those functionals which satisfy the following conditions.

(1) They are at least cubic modulo $\hbar$ when restricted to be functions just on $\mathcal{M}[1]$. That is, we allow functionals which are quadratic as long as they are either quadratic in $\mathcal{L}[1]$ or linear in both $\mathcal{L}[1]$ and in $\mathcal{M}[1]$, and we allow linear functionals as long as they are independent of $\mathcal{M}[1]$. Further, we work modulo the constants $\mathbb{C}[[\hbar]]$. (This clause is related to the superscript $+$ in the notation).

(2) We require our functionals to have proper support, in the usual sense (as functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$).

(3) We require our functionals to have smooth first derivative, again in the sense we discussed before. Note that this condition involves differentiation by elements of both $\mathcal{L}[1]$ and $\mathcal{M}[1]$.

The renormalization group flow operator $W(P(\Phi) - P(\Psi), -)$ on the space $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ extends to an $\mathscr{O}(\mathcal{L})$-linear operator on the space

$$\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]].$$

It is defined by the equation, as usual,

$$W(P(\Phi) - P(\Psi), I) = \hbar \log \exp(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}) \exp(I/\hbar).$$

We say that an element

$$I \in \mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

satisfies the quantum master equation for the parametrix Φ if it satisfies the equation

$$\mathrm{d}_{\mathcal{L}} I + QI + \{I, I\}_\Phi + \hbar\triangle_\Phi I = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ indicates the Chevalley differential on $\mathscr{O}(\mathcal{L}[1])$, extended by tensoring with 1 to an operator on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$, and Q is the extension of the linear differential on $\mathcal{M}[1]$.

The renormalization group equation takes solutions to the quantum master equation for the parametrix Φ to those for the parametrix Ψ.

There are two different versions of quantum field theory with an action of a Lie algebra that we consider: an actionn and an inner action. For theories with just an action, the functionals we consider are in the quotient

$$\mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]] = \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{P,sm}(\mathcal{L}[1])[[\hbar]]$$

of our space of functionals by those which only depend on $\mathcal{L}$.

Now we can define our notion of a quantum field theory acted on by the local L_∞ algebra $\mathcal{L}$.

18.2.0.13 Definition. *Suppose we have a quantum field theory on M, with space of fields $\mathcal{M}[1]$. Thus, we have a collection of effective interactions*

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$$

satisfying the renormalization group equation, BV master equation, and locality axiom, as detailed in subsection 14.2.9.1.

An action of $\mathcal{L}$ on this field theory is a collection of functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]]$$

satisfying the following properties.

(1) The renormalization group equation

$$W(P(\Phi) - P(\Psi), I^{\mathcal{L}}[\Psi]) = I^{\mathcal{L}}[\Phi].$$

(2) *Each $I[\Phi]$ must satisfy the quantum master equation (or Maurer-Cartan equation) for the dgla structure associated to the parametrix Φ. We can explicitly write out the various terms in the quantum master equation as follows:*

$$\mathrm{d}_{\mathcal{L}} I^{\mathcal{L}}[\Phi] + Q I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar \triangle_{\Phi} I^{\mathcal{L}}[\Phi] = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ refers to the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, and Q to the linear differential on $\mathcal{M}[1]$. As above, $\{-,-\}_{\Phi}$ is the Lie bracket on $\mathscr{O}(\mathcal{M}[1])$ which is extended in the natural way to a Lie bracket on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$.

(3) *The locality axiom, as explained in subsection 14.2.9.1, holds: saying that the support of $I^{\mathcal{L}}[\Phi]$ converges to the diagonal as the support of Φ tends to zero, with the same bounds explained in section 14.2.9.1.*

(4) *The image of $I^{\mathcal{L}}[\Phi]$ under the natural map*

$$\mathscr{O}^{+}_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]] \to \mathscr{O}^{+}_{sm,P}(\mathcal{M}[1])[[\hbar]]$$

(given by restricting to functions just of $\mathcal{M}[1]$) must be the original action functional $I[\Phi]$ defining the original theory.

An inner action is defined in exactly the same way, except that the functionals $I^{\mathcal{L}}[\Phi]$ are elements

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]].$$

That is, we don't quotient our space of functionals by functionals just of $\mathcal{L}[1]$. We require that axioms $1-5$ hold in this context as well.

Remark: One should interpret this definition as a variant of the definition of a family of theories over a pro-nilpotent base ring $\mathcal{A}$. Indeed, if we have an $\mathcal{L}$-action on a theory on M, then the functionals $I^{\mathcal{L}}[\Phi]$ define a family of theories over the dg base ring $C^*(\mathcal{L}(M))$ of cochains on the L_∞ algebra $\mathcal{L}(M)$ of global sections of $\mathcal{L}$. In the case that M is compact, the L_∞ algebra $\mathcal{L}(M)$ often has finite-dimensional cohomology, so that we have a family of theories over a finitely-generated pro-nilpotent dg algebra.

Standard yoga from homotopy theory tells us that a $\mathfrak{g}$-action on any mathematical object (if $\mathfrak{g}$ is a homotopy Lie algebra) is the same as a family of such objects over the base ring $C^*(\mathfrak{g})$ which restrict to the given object at the central fibre. Thus, our definition of an action of the sheaf $\mathcal{L}$ of L_∞ algebras on a field theory on M gives rise to an action (in this homotopical sense) of the L_∞ algebra $\mathcal{L}(M)$ on the field theory.

However, our definition of action is stronger than this. The locality axiom we impose on the action functionals $I^{\mathcal{L}}[\Phi]$ involves both fields in $\mathcal{L}$ and in $\mathcal{M}$. As we will see later, this means that we have a homotopy action of $\mathcal{L}(U)$ on observables of our theory on U, for every open subset $U \subset M$, in a compatible way. $\lozenge$

18.3. Obstruction theory for quantizing equivariant theories

The main result of [**Cos11c**] as explained in section **??** states that we can construct quantum field theories from classical ones by obstruction theory. If we start with a classical field theory described by an elliptic L_∞ algebra $\mathcal{M}$, the obstruction-deformation complex is the reduced local Chevalley-Eilenberg cochain complex $C^*_{red,loc}(\mathcal{M})$, which by definition is the complex of local functionals on $\mathcal{M}[1]$ equipped with the Chevalley-Eilenberg differential.

A similar result holds in the equivariant context. Suppose we have a classical field theory with an action of a local L_∞ algebra $\mathcal{L}$. In particular, the elliptic L_∞ algebra $\mathcal{M}$ is acted on by $\mathcal{L}$, so we can form the semi-direct product $\mathcal{L} \ltimes \mathcal{M}$. Thus, we can form the local Chevalley-Eilenberg cochain complex

$$\mathscr{O}_{loc}((\mathcal{L} \ltimes \mathcal{M})[1]) = C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}).$$

The obstruction-deformation complex for quantizing a classical field theory with an action of $\mathcal{L}$ into a quantum field theory with an action of $\mathcal{L}$ is the same as the deformation complex of the original classical field theory with an action of $\mathcal{L}$. This is the complex $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})$, the quotient of $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$ by $C^*_{red,loc}(\mathcal{L})$.

One can also study the complex controlling deformations of the action of $\mathcal{L}$ on $\mathcal{M}$, while fixing the classical theory. This is the complex we denoted by $\mathrm{Act}(\mathcal{L},\mathcal{M})$ earlier: it fits into an exact sequence of cochain complexes

$$0 \to \mathrm{Act}(\mathcal{L},\mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L}) \to C^*_{red,loc}(\mathcal{M}) \to 0.$$

There is a similar remark at the quantum level. Suppose we fix a non-equivariant quantization of our original $\mathcal{L}$-equivariant classical theory $\mathcal{M}$. Then, one can ask to lift this quantization to an $\mathcal{L}$-equivariant quantization. The obstruction/deformation complex for this problem is the group $\mathrm{Act}(\mathcal{L},\mathcal{M})$.

We can analyze, in a similar way, the problem of quantizing a classical field theory with an inner $\mathcal{L}$-action into a quantum field theory with an inner $\mathcal{L}$-acion. The relevant obstruction/deformation complex for this problem is $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$. If, instead, we fix a non-equivariant quantization of the original classical theory $\mathcal{M}$, we can ask for the obstruction/deformation complex for lifting this to a quantization with an inner $\mathcal{L}$-action. The relevant obstruction-deformation complex is the complex denoted $\mathrm{InnerAct}(\mathcal{L},\mathcal{M})$ in section 17.4. Recall that $\mathrm{InnerAct}(\mathcal{L},\mathcal{M})$ fits into a short exact sequence of cochain complexes (of sheaves on X)

$$0 \to \mathrm{InnerAct}(\mathcal{L},\mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}) \to C^*_{red,loc}(\mathcal{M}) \to 0.$$

A more formal statement of these results about the obstruction-deformation complexes is the following.

Fix a classical field theory $\mathcal{M}$ with an action of a local L_∞ algebra $\mathcal{L}$. Let $\mathcal{T}_{\mathcal{L}}^{(n)}$ denote the simplicial set of $\mathcal{L}$-equivariant quantizations of this field theory defined modulo $\hbar^{n+1}$. The simplicial structure is defined exactly as in chapter 14.2: an n-simplex is a family of theories over the base ring $\Omega^*(\triangle^n)$ of forms on the n-simplex. Let $\mathcal{T}^{(n)}$ denote the simplicial set of quantizations without any $\mathcal{L}$-equivariance condition.

Theorem. *The simplicial sets $\mathcal{T}_{\mathcal{L}}^{(n)}$ are Kan complexes. Further, the main results of obstruction theory hold. That is, there is an obstruction map of simplicial sets*

$$\mathcal{T}_{\mathcal{L}}^{(n)} \to DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right).$$

(Here DK denotes the Dold-Kan functor). Further, there is a homotopy fibre diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathcal{L}}^{(n+1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathscr{T}_{\mathcal{L}}^{(n)} \arrow[r] & DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right).
\end{tikzcd}$$

Further, the natural map

$$\mathscr{T}_{\mathcal{L}}^{(n)} \to \mathscr{T}^{(n)}$$

(obtained by forgetting the $\mathcal{L}$-equivariance data in the quantization) is a fibration of simplicial sets.

Finally, there is a homotopy fibre diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathcal{L}}^{(n+1)} \arrow[r] \arrow[d] & \mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}_{\mathcal{L}}^{(n)} \arrow[d] \\
\mathscr{T}_{\mathcal{L}}^{(n)} \arrow[r] & DK\left(\operatorname{Act}(\mathcal{L}, \mathcal{M})[1]\right).
\end{tikzcd}$$

We should interpret the second fibre diagram as follows. The simplicial set $\mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}_{\mathcal{L}}^{(n)}$ describes pairs consisting of an $\mathcal{L}$-equivariant quantization modulo $\hbar^{n+1}$ and a non-equivariant quantization modulo $\hbar^{n+2}$, which agree as non-equivariant quantizations modulo $\hbar^{n+1}$. The deformation-obstruction group to lifting such a pair to an equivariant quantization modulo $\hbar^{n+2}$ is the group $\operatorname{Act}(\mathcal{L}, \mathcal{M})$. That is, a lift exists if the obstruction class in $H^1(\operatorname{Act}(\mathcal{L}, \mathcal{M}))$ is zero, and the simplicial set of such lifts is a torsor for the simplicial Abelian group associated to the cochain complex $\operatorname{Act}(\mathcal{L}, \mathcal{M})$). At the level of zero-simplices, the set of lifts is a torsor for $H^0(\operatorname{Act}(\mathcal{L}, \mathcal{M}))$.

This implies, for instance, that if we fix a non-equivariant quantization to all orders, then the obstruction-deformation complex for making this into an equivariant quantization is $\mathrm{Act}(\mathcal{L}, \mathcal{M})$).

Further elaborations, as detailed in chapter 14.2, continue to hold in this context. For example, we can work with families of theories over a dg base ring, and everything is fibred over the (typically contractible) simplicial set of gauge fixing conditions. In addition, all of these results hold when we work with translation-invariant objects on $\mathbb{R}^n$ and impose "renormalizability" conditions, as discussed in section **??**.

The proof of this theorem in this generality is contained in [**Cos11c**], and is essentially the same as the proof of the corresponding non-equivariant theorem. In [**Cos11c**], the term "field theory with background fields" is used instead of talking about a field theory with an action of a local L_∞ algebra.

For theories with an inner action, the same result continues to hold, except that the obstruction-deformation complex for the first statement is $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$, and in the second case is $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$.

18.3.1. Lifting actions to inner actions.

Given a field theory with an action of $\mathcal{L}$, we can try to lift it to one with an inner action. For classical field theories, we have seen that the obstruction to doing this is a class in $H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))$ (with, of course, the Chevalley-Eilenberg differential).

A similar result holds in the quantum setting.

18.3.1.1 Proposition. *Suppose we have a quantum field theory with an action of $\mathcal{L}$. Then there is a cochain*

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]] = C^*_{red,loc}(\mathcal{L})$$

of cohomological degree 1 which is closed under the Chevalley-Eilenberg differential, such that trivializing α is the same as lifting $\mathcal{L}$ to an inner action.

PROOF. This follows immediately from the obstruction-deformation complexes for constructing the two kinds of $\mathcal{L}$-equivariant field theories. However, let us explain explicitly how to calculate this obstruction class (because this will be useful later). Indeed, let us fix a theory with an action of $\mathcal{L}$, defined by functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]].$$

It is always possible to lift $I[\Phi]$ to a collection of functionals

$$\widetilde{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

which satisfy the RG flow and locality axioms, but may not satisfy the quantum master equation. The space of ways of lifting is a torsor for the graded abelian group $\mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]]$

of local functionals on $\mathcal{L}$. The failure of the lift $\widetilde{I}^{\mathcal{L}}[\Phi]$ to satisfy the quantum master equation is, as explained in [Cos11c], independent of Φ, and therefore is a local functional $\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])$. That is, we have

$$\alpha = \mathrm{d}_{\mathcal{L}}\widetilde{I}^{\mathcal{L}}[\Phi] + Q\widetilde{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widetilde{I}^{\mathcal{L}}[\Phi], \widetilde{I}^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar\triangle_{\Phi}\widetilde{I}^{\mathcal{L}}[\Phi].$$

Note that functionals just of $\mathcal{L}$ are in the centre of the Poisson bracket $\{-,-\}_{\Phi}$, and are also acted on trivially by the BV operator $\triangle_{\Phi}$.

We automatically have $\mathrm{d}_{\mathcal{L}}\alpha = 0$. It is clear that to lift $I^{\mathcal{L}}[\Phi]$ to a functional $\widetilde{I}^{\mathcal{L}}[\Phi]$ which satisfies the quantum master equation is equivalent to making α exact in $C^*_{red,loc}(\mathcal{L})[[\hbar]]$. ☐

18.4. The factorization algebra associated to an equivariant quantum field theory

In this section, we will explain what structure one has on observables of an equivariant quantum field theory. As above, let $\mathcal{M}$ denote the elliptic L_∞ algebra on a manifold M describing a classical field theory, which is acted on by a local L_∞-algebra $\mathcal{L}$. Let us define a factorization algebra $C^*_{fact}(\mathcal{L})$ by saying that to an open subset $U \subset M$ it assigns $C^*(\mathcal{L}(U))$. (As usual, we use the appropriate completion of cochains). Note that $C^*_{fact}(\mathcal{L})$ is a factorization algebra valued in (complete filtered differentiable) commutative dg algebras on $\mathcal{M}$.

In this section we will give a brief sketch of the following result.

18.4.0.2 Proposition. *Suppose we have a quantum field theory equipped with an action of a local Lie algebra $\mathcal{L}$; let $\mathcal{M}$ denote the elliptic L_∞ algebra associated to the corresponding classical field theory. Then there is a factorization algebra of equivariant quantum observables which is a factorization algebra in modules for the factorization algebra $C^*(\mathcal{L})$ of cochains on $\mathcal{L}$. This quantizes the classical factorization algebra of equivariant observables constructed in proposition 17.3.0.3.*

PROOF. The construction is exactly parallel to the non-equivariant version which was explained in chapter 14.2, so we will only sketch the details. We define an element of $\mathrm{Obs}^q_{\mathcal{L}}(U)$ of cohomological degree k to be a family of functionals $O[\Phi]$, of cohomological degree k one for every parametrix, on the space $\mathcal{L}(M)[1] \oplus \mathcal{M}(M)[1]$ of fields of the theory. We require that, if ε is a parameter of cohomological degree $-k$ and square zero, that $I^{\mathcal{L}}[\Phi] + \varepsilon O[\Phi]$ satisfies the renormalization group equation

$$W(P(\Phi) - P(\Psi), I^{\mathcal{L}}[\Psi] + \varepsilon O[\Psi]) = I^{\mathcal{L}}[\Phi] + \varepsilon O[\Phi].$$

Further, we require the same locality axiom that was detailed in section 15.4, saying roughly that $O[\Phi]$ is supported on U for sufficiently small parametrices U.

The differential on the complex $\mathrm{Obs}^q_{\mathcal{L}}(U)$ is defined by

$$(\mathrm{d}O)[\Phi] = \mathrm{d}_{\mathcal{L}}O[\Phi] + QO[\Phi] + \{I^{\mathcal{L}}[\Phi], O[\Phi]\}_{\Phi} + \hbar\triangle_{\Phi}O[\Phi],$$

where Q is the linear differential on $\mathcal{M}[1]$, and $\mathrm{d}_{\mathcal{L}}$ corresponds to the Chevalley-Eilenberg differential on $C^*(\mathcal{L})$.

We can make $\mathrm{Obs}^q_{\mathcal{L}}(U)$ into a module over $C^*(\mathcal{L}(U))$ as follows. If $O \in \mathrm{Obs}^q_{\mathcal{L}}(U)$ and $\alpha \in C^*(\mathcal{L}(U))$, we can define a new observable $\alpha \cdot O$ defined by

$$(\alpha \cdot O)[\Phi] = \alpha \cdot (O[\Phi]).$$

This makes sense, because α is a functional on $\mathcal{L}(U)[1]$ and so can be made a functional on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$. The multiplication on the right hand side is simply multiplication of functionals on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$.

It is easy to verify that $\alpha \cdot O$ satisfies the renormalization group equation; indeed, the infinitesimal renormalization group operator is given by differentiating with respect to a kernel in $\mathcal{M}[1]^{\otimes 2}$, and so commutes with multiplication by functionals of $\mathcal{L}[1]$. Similarly, we have

$$\mathrm{d}(\alpha \cdot O) = (\mathrm{d}\alpha) \cdot O + \alpha \cdot \mathrm{d}O$$

where $\mathrm{d}O$ is the differential discussed above, and $\mathrm{d}\alpha$ is the Chevalley-Eilenberg differential applied to $\alpha \in C^*(\mathcal{L}(U)[1])$.

As is usual, at the classical level we can discuss observables at scale 0. The differential at the classical level is $\mathrm{d}_{\mathcal{L}} + Q + \{I^{\mathcal{L}}, -\}$ where $I^{\mathcal{L}} \in \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ is the classical equivariant Lagrangian. This differential is the same as the differential on the Chevalley-Eilenberg differential on the cochains of the semi-direct product L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$. Thus, it is quasi-isomorphic, at the classical level, to the one discussed in proposition 17.3.0.3. □

18.5. Quantum Noether's theorem

Finally, we can explain Noether's theorem at the quantum level. As above, suppose we have a quantum field theory on a manifold M with space of fields $\mathcal{M}[1]$. Let $\mathcal{L}$ be a local L_∞ algebra which acts on this field theory. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ denote the obstruction to lifting this action to an inner action.

Recall that the factorization envelope of the local L_∞ algebra $\mathcal{L}$ is the factorization algebra whose value on an open subset $U \subset M$ is the Chevalley chain complex $C_*(\mathcal{L}_c(U))$. Given a cocycle $\beta \in H^1(C^*_{red,loc}(\mathcal{L}))$, we can form a shifted central extension

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on M. Central extensions of this form have already been discussed in section **??**.

We can then form the *twisted* factorization envelope $U^{\beta}(\mathcal{L})$, which is a factorization algebra on M. The twisted factorization envelope is defined by saying that it's value on an open subset $V \subset M$ is

$$U^{\beta}(\mathcal{L})(V) = \mathbb{C}_{c=1} \otimes_{\mathbb{C}[c]} C_*(\widetilde{\mathcal{L}}_c(V)).$$

Here the complex $C_*(\widetilde{\mathcal{L}}_c(V))$ is made into a $\mathbb{C}[c]$-module by multiplying by the central element.

We have already seen **??** that the Kac-Moody vertex algebra arises as an example of a twisted factorization envelope.

There is a $\mathbb{C}[[\hbar]]$-linear version of the twisted factorization envelope construction too: if our cocycle α is in $H^1(\mathbb{C}^*_{red,loc}(\mathcal{L}))[[\hbar]]$, then we can form a central extension of the form

$$0 \to \mathbb{C}[[\hbar]][-1] \to \widetilde{\mathcal{L}}_c[[\hbar]] \to \mathcal{L}_c[[\hbar]] \to 0.$$

This is an exact sequence of precosheaves of L_∞ algebras on M in the category of $\mathbb{C}[[\hbar]]$-modules. By performing the $\mathbb{C}[[\hbar]]$-linear version of the construction above, one finds the twisted factorization envelope $U^{\alpha}(\mathcal{L})$. This is a factorization algebra on M in the category of $\mathbb{C}[[\hbar]]$-modules, whose value on an open subset $V \subset M$ is

$$U^{\alpha}(\mathcal{L})(V) = \mathbb{C}[[\hbar]]_{c=1} \otimes_{\mathbb{C}[[\hbar]][c]} C_*(\widetilde{\mathcal{L}}_c[[\hbar]]).$$

Here Chevalley chains are taken in the $\mathbb{C}[[\hbar]]$-linear sense.

Our version of Noether's theorem will relate the factorization envelope of $\mathcal{L}_c$, twisted by the cocycle α, to the factorization algebra of quantum observables of the field theory on M. The main theorem is the following.

18.5.0.3 Theorem. *Suppose that the local $\mathcal{L}_\infty$-algebra $\mathcal{L}$ acts on a field theory on M, and that the obstruction to lifting this to an inner action is a cocycle $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$. Then, there is a $\mathbb{C}((\hbar))$-linear homomorphism of factorization algebras*

$$U^{\alpha}(\mathcal{L})[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}].$$

(Note that on both sides we have inverted $\hbar$).

One can ask how this relates to Noether's theorem for classical field theories. In order to provide such a relationship, we need to state a version of quantum Noether's theorem which holds without inverting $\hbar$. For every open subset $V \subset M$, we define the Rees module

$$\mathrm{Rees}\, U^{\alpha}(\mathcal{L})(V) \subset U^{\alpha}(\mathcal{L})(V)$$

to be the submodule spanned by elements of the form $\hbar^k \gamma$ where $\gamma \in \mathrm{Sym}^{\leq k}(\mathcal{L}_c(V))$. This is a sub-$\mathbb{C}[[\hbar]]$-module, and also forms a sub-factorization algebra. The reason for the terminology is that in the case $\alpha = 0$, or more generally α is independent of $\hbar$, Rees $U^{\alpha}(\mathcal{L})(V)$ is the Rees module for the filtered chain complex $C^{\alpha}_*(\mathcal{L}_c(V))$.

One can check that $\operatorname{Rees} U^\alpha(V)$ is a free $\mathbb{C}[[\hbar]]$-module and that, upon inverting $\hbar$, we find

$$(\operatorname{Rees} U^\alpha(V))\,[\hbar^{-1}] = U^\alpha(V).$$

18.5.0.4 Theorem. *The Noether map of factorization algebras*

$$U^\alpha(\mathcal{L})[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}]$$

over $\mathbb{C}((\hbar))$ *refines to a map*

$$\operatorname{Rees} U^\alpha(\mathcal{L}) \to \mathrm{Obs}^q$$

of factorization algebras over $\mathbb{C}[[\hbar]]$.

We would like to compare this statement to the classical version of Noether's theorem. Let α^0 denote the reduction of α modulo $\hbar$. Let $\widetilde{\mathcal{L}}_c$ denote the central extension of $\mathcal{L}_c$ arising from α^0. We have seen that the classical Noether's theorem states that there is a map of precosheaves L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

where on the right hand side, $\widetilde{\mathrm{Obs}}^{cl}[-1]$ is endowed with the structure of dg Lie algebra coming from the shifted Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}$. Further, this map sends the central element in $\widetilde{\mathcal{L}}_c$ to the unit element in $\widetilde{\mathrm{Obs}}^{cl}[-1]$.

In particular, the classical Noether map gives rise to a map of precosheaves of cochain complexes

$$\widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}.$$

We will not use the fact that this arises from an L_∞ map in what follows. Because Obs^{cl} is a commutative factorization algebra, we automatically get a map of commutative prefactorization algebras

$$\mathrm{Sym}^* \widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}.$$

Further, because the Noether map sends the central element to the unit observable, we get a map of commutative factorization algebras

$$\mathbb{C}_{c=1} \otimes_{\mathbb{C}[c]} \mathrm{Sym}^* \widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}. \tag{$\dagger$}$$

Now we have set up the classical Noether map in a way which is similar to the quantum Noether map. Recall that the quantum Noether map with $\hbar$ not inverted is expressed in terms of the Rees module $\operatorname{Rees} U^\alpha(\mathcal{L})$. When we set $\hbar = 0$, we can identify

$$\operatorname{Rees} U^\alpha(\mathcal{L})(V) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_{\hbar=0} = \mathrm{Sym}^*(\widetilde{\mathcal{L}}_c(V)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}.$$

18.5.0.5 Lemma. *The quantum Noether map*

$$\operatorname{Rees} U^\alpha(\mathcal{L}) \to \mathrm{Obs}^q$$

of factorization algebras becomes, upon setting $\hbar = 0$, *the map in equation* ($\dagger$).

18.5.1. Proof of the quantum Noether theorem. Before we being our proof of quantum Noether's theorem, it will be helpful to discuss the meaning (in geometric terms) of the chains and cochains of an L_∞ algebra twisted by a cocycle.

If $\mathfrak{g}$ is an L_∞ algebra, then $C^*(\mathfrak{g})$ should be thought of as functions on the formal moduli problem $B\mathfrak{g}$ associated to $\mathfrak{g}$. Similarly, $C_*(\mathfrak{g})$ is the space of distributions on $B\mathfrak{g}$. If $\alpha \in H^1(C^*(\mathfrak{g}))$, then α defines a line bundle on $B\mathfrak{g}$, or equivalently, a rank 1 homotopy representation of $\mathfrak{g}$. Sections of this line bundle are $C^*(\mathfrak{g})$ with the differential $\mathrm{d}_{\mathfrak{g}} - \alpha$, i.e. we change the differential by adding a term given by multiplication by $-\alpha$. Since α is closed and of odd degree, it is automatic that this differential squares to zero. We will sometimes refer to this complex as $C^*_\alpha(\mathfrak{g})$.

Similarly, we can define $C_{*,\alpha}(\mathfrak{g})$ to be $C_*(\mathfrak{g})$ with a differential given by adding the operator of contracting with $-\alpha$ to the usual differential. We should think of $C_{*,\alpha}(\mathfrak{g})$ as the distributions on $B\mathfrak{g}$ twisted by the line bundle associated to α; i.e. distributions which pair with sections of this line bundle.

Let $\widetilde{\mathfrak{g}}$ be the shifted central extension of $\mathfrak{g}$ associated to α. Then $C_*(\widetilde{\mathfrak{g}})$ is a module over $\mathbb{C}[c]$, where c is the central parameter. Then we can identify

$$C_*(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1} = C_{*,\alpha}(\mathfrak{g}).$$

A similar remark holds for cochains.

In particular, if $\mathcal{L}$ is a local L_∞ algebra on a manifold M and $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ is a local cochain, then

$$U^\alpha(\mathcal{L})(V) = C_{*,\alpha}(\mathcal{L}_c(V))$$

for an open subset $V \subset M$.

Now we will turn to the proof of theorems 18.5.0.3 and 18.5.0.4 and lemma 18.5.0.5, all stated in the previous section.

The first thing we need to do is to produce, for every open subset $V \subset M$, a chain map

$$C^\alpha_*(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}].$$

A linear map

$$f : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathrm{Obs}^q(V)[\hbar^{-1}]$$

is the same as a collection of linear maps

$$f[\Phi] : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathscr{O}(\mathcal{M}(M)[1])((\hbar))$$

one for every parametrix Φ, which satisfy the renormalization group equation and the locality axiom. This, in turn, is the same as a collection of functionals

$$O[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M)[1])((\hbar))$$

satisfying the renormalization group equation and the locality axiom. We are using the natural pairing between the symmetric algebra of $\mathcal{L}_c(V)[1]$ and the space of functionals on $\mathcal{L}_c(V)[1]$ to identify a linear map $f[\Phi]$ with a functional $O[\Phi]$.

We will write down such a collection of functionals. Recall that, because we have an action of the local L_∞ algebra $\mathcal{L}$ on our theory, we have a collection of functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1])[[\hbar]]$$

which satisfy the renormalization group equation and the following quantum master equation:

$$(\mathrm{d}_{\mathcal{L}} + Q)I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_\Phi + \hbar\triangle_\Phi\left(I^{\mathcal{L}}[\Phi]\right) = \alpha.$$

We will use the projections from $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$ to $\mathcal{L}_c(M)[1]$ and $\mathcal{M}_c(M)[1]$ to lift functionals on these smaller spaces to functionals on $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$. In particular, if as usual $I[\Phi]$ denotes the effective action of our quantum field theory, which is a function of the fields in $\mathcal{M}_c(M)[1]$, we will use the same notation to denote the lift of $I[\Phi]$ to a function of the fields in $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$.

Let

$$\widehat{I}^{\mathcal{L}}[\Phi] = I^{\mathcal{L}}[\Phi] - I[\Phi] \in \mathscr{O}(\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1])[[\hbar]].$$

This functional satisfies the following master equation:

$$(\mathrm{d}_{\mathcal{L}} + Q)\widehat{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widehat{I}^{\mathcal{L}}[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_\Phi + \{I[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_\Phi + \hbar\triangle_\Phi\left(\widehat{I}^{\mathcal{L}}[\Phi]\right) = \alpha.$$

The renormalization group equation for the functionals $\widehat{I}^{\mathcal{L}}[\Phi]$ states that

$$\exp\left(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}\right)\exp\left(I[\Psi]/\hbar\right)\exp\left(\widehat{I}^{\mathcal{L}}[\Psi]/\hbar\right) = \exp\left(I[\Phi]/\hbar\right)\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).$$

This should be compared with the renormalization group equation that an observable $\{O[\Phi]\}$ in $\mathrm{Obs}^q(M)$ satisfies:

$$\exp\left(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}\right)\exp\left(I[\Psi]/\hbar\right)O[\Psi] = \exp\left(I[\Phi]/\hbar\right)O[\Phi].$$

Note also that

$$\widehat{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M))[[\hbar]].$$

The point is the following. Let

$$\widehat{I}^{\mathcal{L}}_{i,k,m}[\Phi] : \mathcal{L}_c(M)^{\otimes k} \times \mathcal{M}_c(M)^{\otimes m} \to \mathbb{C}$$

denote the coefficients of $\hbar^i$ in the Taylor terms of this functional. This Taylor term is zero unless $k > 0$, and further it has proper support (which can be made as close as we like to the diagonal by making Φ small). The proper support condition implies that this Taylor term extends to a functional

$$\mathcal{L}_c(M)^{\otimes k} \times \mathcal{M}(M)^{\otimes m} \to \mathbb{C},$$

that is, only one of the inputs has to have compact support and we can choose this to be an $\mathcal{L}$-input.

From this, it follows that

$$\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M)[1])((\hbar)).$$

Although there is a $\hbar^{-1}$ in the exponent on the left hand side, each Taylor term of this functional only involves finitely many negative powers of $\hbar$, which is what is required to be in the space on the right hand side of this equation.

Further, the renormalization group equation satisfied by $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$ is precisely the one necessary to define (as Φ varies) an element which we denote

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_\alpha(\mathcal{L}_c(V), \mathrm{Obs}^q(M))[\hbar^{-1}].$$

The locality property for the functionals $\widehat{I}^{\mathcal{L}}[\Phi]$ tells us that for Φ small these functionals are supported arbitrarily close to the diagonal. This locality axiom immediately implies that

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_\alpha(\mathcal{L}_c(V), \mathrm{Obs}^q(V))[\hbar^{-1}].$$

Thus, we have produced the desired linear map

$$F : C^\alpha_*(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}].$$

Explicitly, this linear map is given by the formula

$$F(l)[\Phi] = \left\langle l, \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\rangle$$

where $\langle -, - \rangle$ indicates the duality pairing between $C^\alpha_*(\mathcal{L}_c(V))$ and $C^*_\alpha(\mathcal{L}_c(V))$.

Next, we need to verify that F is a cochain map. Since the duality pairing between changes and cochains of $\mathcal{L}_c(V)$ (twisted by α) is a cochain map, it suffices to check that the element $\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right)$ is closed. This is equivalent to saying that, for each parametrix Φ, the following equation holds:

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi)\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) = 0.$$

Here, $\mathrm{d}_{\mathcal{L}}$ indicates the Chevalley-Eilenberg differential on $C^*(\mathcal{L}_c(V))$ and α indicates the operation of multiplying by the cochain α in $C^1(\mathcal{L}_c(V))$.

This equation is equivalent to the statement that

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi)\exp\left(I[\Phi]/\hbar + \widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) = 0$$

which is equivalent to the quantum master equation satisfied by $I^{\mathcal{L}}[\Phi]$.

Thus, we have produced a cochain map from $C^{\alpha}_{*}(\mathcal{L}_c(V))$ to $\mathrm{Obs}^q(V)[\hbar^{-1}]$. It remains to show that this cochain map defines a map of factorization algebras.

It is clear from the construction that the map we have constructed is a map of precosheaves, that is, it is compatible with the maps coming from inclusions of open sets $V \subset W$. It remains to check that it is compatible with products.

Let V_1, V_2 be two disjoint subsets of M, but contained in W. We need to verify that the following diagram commutes:

$$\begin{tikzcd}
C^{\alpha}_{*}(\mathcal{L}_c(V_1)) \times C^{\alpha}_{*}(\mathcal{L}_c(V_2)) \arrow[r] \arrow[d] & \mathrm{Obs}^q(V_1)[\hbar^{-1}] \times \mathrm{Obs}^q(V_2)[\hbar^{-1}] \arrow[d] \\
C^{\alpha}_{*}(\mathcal{L}_c(W)) \arrow[r] & \mathrm{Obs}^q(W)[\hbar^{-1}].
\end{tikzcd}$$

Let $l_i \in C^{\alpha}_{*}(\mathcal{L}_c(V_i))$ for $i = 1, 2$. Let $\cdot$ denote the factorization product on the factorization algebra $C^{\alpha}_{*}(\mathcal{L}_c)$. This is simply the product in the symmetric algebra on each open set, coupled with the maps coming from the inclusions of open sets.

Recall that if O_i are observables in the open sets V_i, then the factorization product$O_1 O_2 \in \mathrm{Obs}^q(W)$ of these observables is defined by

$$(O_1 O_2)[\Phi] = O_1[\Phi] \cdot O_2[\Phi]$$

for Φ sufficiently small, where $\cdot$ indicates the obvious product on the space of functions on $\mathcal{M}(M)[1]$. (Strictly speaking, we need to check that for each Taylor term this identity holds for sufficiently small parametrices, but we have discussed this technicality many times before and will not belabour it now).

We need to verify that, for Φ sufficiently small,

$$F(l_1)[\Phi] \cdot F(l_2)[\Phi] = F(l_1 \cdot l_2)[\Phi] \in \mathscr{O}(\mathcal{M}(M)[1])((\hbar)).$$

By choosing a sufficiently small parametrix, we can assume that $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close to the diagonal as we like. We can further assume, without loss of generality, that each l_i is a product of elements in $\mathcal{L}_c(V_i)$. Let us write $l_i = m_{1i} \dots m_{k_i i}$ for $i = 1, 2$ and each $m_{ji} \in \mathcal{L}_c(V_i)$. (To extend from this special case to the case of general l_i requires a small functional analysis argument using the fact that F is a smooth map, which it is. Since we restrict attention to this special case only for notation convenience, we won't give more details on this point).

Then, we can explicitly write the map F applied to the elements l_i by the formula

$$F(l_i)[\Phi] = \left\{ \frac{\partial}{\partial m_{1i}} \cdots \frac{\partial}{\partial m_{k_i i}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar \right) \right\} |_{0 \times \mathcal{M}(M)[1]} \, .$$

In other words, we apply the product of all partial derivatives by the elements $m_{ji} \in \mathcal{L}_c(V_i)$ to the function $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$ (which is a function on $\mathcal{L}_c(M)[1] \oplus \mathcal{M}(M)[1]$) and then restrict all the $\mathcal{L}_c(V_i)$ variables to zero.

To show that

$$F(l_1 \cdot l_2)[\Phi] = F(l_1)[\Phi] \cdot F(l_2)[\Phi]$$

for sufficiently small Φ, it suffices to verify that

$$\left\{\frac{\partial}{\partial m_{11}} \cdots \frac{\partial}{\partial m_{k_1 1}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\left\{\frac{\partial}{\partial m_{12}} \cdots \frac{\partial}{\partial m_{k_2 2}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}$$
$$= \frac{\partial}{\partial m_{11}} \cdots \frac{\partial}{\partial m_{k_1 1}} \frac{\partial}{\partial m_{12}} \cdots \frac{\partial}{\partial m_{k_2 2}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).$$

Each side can be expanded, in an obvious way, as a sum of terms each of which is a product of factors of the form

$$(\dagger) \qquad \frac{\partial}{\partial m_{j_1 i_1}} \cdots \frac{\partial}{\partial m_{j_r i_r}} \widehat{I}^{\mathcal{L}}[\Phi]$$

together with an overall factor of $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$. In the difference between the two sides, all terms cancel except for those which contain a factor of the form expressed in equation ($\dagger$) where $i_1 = 1$ and $i_2 = 2$. Now, for sufficiently small parametrices,

$$\frac{\partial}{\partial m_{j_1 1}} \frac{\partial}{\partial m_{j_2 2}} \widehat{I}^{L}[\Phi] = 0$$

because $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close as we like to the diagonal and $m_{j_1 1} \in \mathcal{L}_c(V_1)$ and $m_{j_2 2} \in \mathcal{L}_c(V_2)$ have disjoint support.

Thus, we have constructed a map of factorization algebras

$$F : U^{\alpha}(\mathcal{L}) \to \mathrm{Obs}^q[\hbar^{-1}].$$

It remains to check the content of theorem 18.5.0.4 and of lemma 18.5.0.5. For theorem 18.5.0.4, we need to verify that if

$$l \in \mathrm{Sym}^k(\mathcal{L}_c(V))$$

(for some open subset $V \subset M$) then

$$F(l) \in \hbar^{-k} \mathrm{Obs}^q(V).$$

That is, we need to check that for each Φ, we have

$$F(l)[\Phi] \in \hbar^{-k} \mathscr{O}(\mathcal{M}(M)[1])[[\hbar]].$$

Let us assume, for simplicty, that $l = m_1 \dots m_k$ where $m_i \in \mathcal{L}_c(V)$. Then the explicit formula

$$F(l)[\Phi] = \left\{\partial_{m_1} \dots \partial_{m_k} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\} |_{\mathcal{M}(M)[1]}$$

makes it clear that the largest negative power of $\hbar$ that appears is $\hbar^{-k}$. (Note that $\widehat{I}^{\mathcal{L}}[\Phi]$ is zero when restricted to a function of just $\mathcal{M}(M)[1]$.)

Finally, we need to check lemma 18.5.0.5. This states that the classical limit of our quantum Noether map is the classical Noether map we constructed earlier. Let $l \in \mathcal{L}_c(V)$. Then the classical limit of our quantum Noether map sends l to the classical observable

$$\begin{aligned}\lim_{\hbar\to 0} \hbar F(l) &= \lim_{\Phi\to 0}\lim_{\hbar\to 0}\left\{\hbar\partial_l \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\mid_{\mathcal{M}(M)[1]}\\ &= \lim_{\Phi\to 0}\left\{\partial_l I^{\mathcal{L}}_{classical}[\Phi]\right\}\mid_{\mathcal{M}(M)[1]}\\ &= \left\{\partial_l I^{\mathcal{L}}_{classical}\right\}\mid_{\mathcal{M}(M)[1]}.\end{aligned}$$

Note that by $I^{\mathcal{L}}_{classical}[\Phi]$ we mean the scale Φ version of the functional on $\mathcal{L}[1] \oplus \mathcal{M}[1]$ which defines the inner action (at the classical level) of $\mathcal{L}$ on our classical theory, and by $I^{\mathcal{L}}_{classical}$ we mean the scale zero version.

Now, our classical Noether map is the map appearing in the last line of the above displayed equation.

This completes the proof of theorems 18.5.0.3 and 18.5.0.4 and lemma 18.5.0.5.

18.6. The quantum Noether theorem and equivariant observables

So far in this chapter, we have explained that if we have a quantum theory with an action of the local L_∞ algebra $\mathcal{L}$, then one finds a homotopical action of $\mathcal{L}$ on the quantum observables of the theory. We have also stated and proved our quantum Noether theorem: in the same situation, there is a homomorphism from the twisted factorization envelope of $\mathcal{L}$ to the quantum observables. It is natural to expect that these two constructions are closely related. In this section, we will explain the precise relationship. Along the way, we will prove a somewhat stronger version of the quantum Noether theorem. The theorems we prove in this section will allow us to formulate later a definition of the *local index* of an elliptic complex in the language of factorization algebras.

Let us now give an informal statement of the main theorem in this section. Quantum observables on U have a homotopy action of the sheaf $\mathcal{L}(U)$ of L_∞ algebras. By restricting to compactly-supported sections, we find that $\mathrm{Obs}^q(U)$ has a homotopy action of $\mathcal{L}_c(U)$. This action is compatible with the factorization structure, in the sense that the product map

$$\mathrm{Obs}^q(U) \otimes \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(W)$$

(defined when $U \amalg V \subset W$) is a map of $\mathcal{L}_c(U) \oplus \mathcal{L}_c(V)$-modules, where $\mathrm{Obs}^q(W)$ is made into an $\mathcal{L}_c(U) \oplus \mathcal{L}_c(V)$ module via the natural inclusion map from this L_∞ algebra to $\mathcal{L}_c(W)$.

We can say that the factorization algebra Obs^q is an $\mathcal{L}_c$-equivariant factorization algebra.

It can be a little tricky formulating correctly all the homotopy coherences that go into such an action. When we state our theorem precisely, we will formulate this kind of action slightly differently in a way which captures all the coherences we need.

In general, for an L_∞ algebra $\mathfrak{g}$, an element $\alpha \in H^1(C^*(\mathfrak{g}))$ defines an L_∞-homomorphism $\mathfrak{g} \to \mathbb{C}$ (where $\mathbb{C}$ is given the trivial L_∞ structure). In other words, we can view such a cohomology class as a character of $\mathfrak{g}$. (More precisely, the choice of a cochain representative of α leads to such an L_∞ homomorphism, and different cochain representatives give L_∞-equivalent L_∞-homomorphisms).

Suppose that $\mathcal{L}$ is a local L_∞ algebra on M and that we have an element $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$. This means, in particular, that for every open subset $V \subset M$ we have a character of $\mathcal{L}_c(V)$, and so an action of $\mathcal{L}_c(V)$ on $\mathbb{C}$. Let $\underline{\mathbb{C}}$ denote the trivial factorization algebra, which assigns the vector space $\mathbb{C}$ to each open set. The fact that α is local guarantees that the action of each $\mathcal{L}_c(V)$ on $\mathbb{C}$ makes $\underline{\mathbb{C}}$ into an $\mathcal{L}_c$-equivariant factorization algebra. Let us denote this $\mathcal{L}_c$-equivariant factorization algebra by $\underline{\mathbb{C}}_\alpha$.

More generally, given any $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}$ on M, we can form a new $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}_\alpha$, defined to be the tensor product of $\mathcal{F}$ and $\underline{\mathbb{C}}_\alpha$ in the category of factorization algebras with multiplicative $\mathcal{L}_c$-actions.

Suppose that we have a field theory on M with an action of a local L_∞ algebra $\mathcal{L}$, and with factorization algebra of quantum observables Obs^q. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ be the obstruction to lifting this to an inner action. Let $\underline{\mathbb{C}}_\alpha[[\hbar]]$ denote the trivial factorization algebra $\underline{\mathbb{C}}[[\hbar]]$ viewed as an $\mathcal{L}_c$-equivariant factorization algebra using the character α. As we have seen above, the factorization algebra Obs^q is an $\mathcal{L}_c$-equivariant factorization algebra. We can tensor Obs^q with $\underline{\mathbb{C}}_\alpha[[\hbar]]$ to form a new $\mathcal{L}_c$-equivariant factorization algebra Obs^q_α (the tensor product is of course taken over the base ring $\mathbb{C}[[\hbar]]$). As a factorization algebra, Obs^q_α is the same as Obs^q. Only the $\mathcal{L}_c$-action has changed.

The main theorem is the following.

18.6.0.1 Theorem. *The $\mathcal{L}_c$-action on $\mathrm{Obs}^q_\alpha[\hbar^{-1}]$ is homotopically trivial.*

In other words, after twisting the action of $\mathcal{L}_c$ by the character α, and inverting $\hbar$, the action of $\mathcal{L}_c$ on observables is homotopically trivial. The trivialization of the action respects the fact that its multiplicative.

An alternative way to state the theorem is that the action of $\mathcal{L}_c$ on $\mathrm{Obs}^q[\hbar^{-1}]$ is homotopically equivalent to the α-twist of the trivial action. That is, on each open set $V \subset M$ the action of $\mathcal{L}_c(V)$ on $\mathrm{Obs}^q(V)[\hbar^{-1}]$ is by the identity times the character α.

Let us now explain how this relates to Noether's theorem. If we have an $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}$ (valued in convenient or pro-convenient vector spaces) then on every open subset $V \subset M$ we can form the Chevalley chain complex $C_*(\mathcal{L}_c(V), \mathcal{F}(V))$ of $\mathcal{L}_c(V)$ with coefficients in $\mathcal{F}(V)$. This is defined by taking the (bornological) tensor product of $C_*(\mathcal{L}_c(V))$ with $\mathcal{F}(V)$ on every open set, with a differential which incorporates the usual Chevalley differential as well as the action of $\mathcal{L}_c(V)$ on $\mathcal{F}(V)$. The cochain complexes $C_*(\mathcal{L}_c(V), \mathcal{F}(V))$ form a new factorization algebra which we call $C_*(\mathcal{L}_c, \mathcal{F})$. (As we will see shortly in our more technical statement of the theorem, the factorization algebra $C_*(\mathcal{L}_c, \mathcal{F})$, with a certain structure of $C_*(\mathcal{L}_c)$-comodule, encodes the $\mathcal{F}$ as an $\mathcal{L}_c$-equivariant factorization algebra).

In particular, when we have an action of $\mathcal{L}$ on a quantum field theory on a manifold M, we can form the factorization algebra $C_*(\mathcal{L}_c, \mathrm{Obs}^q)$, and also the version of this twisted by α, namely $C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha)$. We can also consider the chains of $\mathcal{L}_c$ with coefficients on Obs^q with the trivial action: this is simply $U(\mathcal{L}_c) \otimes \mathrm{Obs}^q$ (where we complete the tensor product to the bornological tensor product).

Then, the theorem above implies that we have an isomorphism of factorization algebras

$$\Phi : C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha)[\hbar^{-1}] \cong U(\mathcal{L}_c) \otimes \mathrm{Obs}^q[\hbar^{-1}].$$

(Recall that $U(\mathcal{L}_c)$ is another name for $C_*(\mathcal{L}_c)$ with trivial coefficients, and that the tensor product on the right hand side is the completed bornological one).

Now, the action of $\mathcal{L}_c$ on Obs^q preserves the unit observable. This means that the unit map

$$\mathbb{1} : \underline{\mathbb{C}}[[\hbar]] \to \mathrm{Obs}^q$$

of factorization algebras is $\mathcal{L}_c$-equivariant. Taking Chevalley chains and twisting by α, we get a map of fatorization algebras

$$\mathbb{1} : C_*(\mathcal{L}_c, \underline{\mathbb{C}}_\alpha[[\hbar]]) = U^\alpha(\mathcal{L}_c) \to C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha).$$

Further, there is a natural map of factorization algebras

$$\varepsilon : U(\mathcal{L}_c) \to \underline{\mathbb{C}},$$

which on every open subset is the map $C_*(\mathcal{L}_c(V)) \to \mathbb{C}$ which projects onto $\mathrm{Sym}^0 \mathcal{L}_c(V)$. (This map is the counit for a natural cocommutative coalgebra structure on $C_*(\mathcal{L}_c(V))$. Tensoring this with the identity map gives a map

$$\varepsilon \otimes \mathrm{Id} : U(\mathcal{L}_c) \otimes \mathrm{Obs}^q \to \mathrm{Obs}^q.$$

The compatibility of the theorem stated in this section with the Noether map is the following.

Theorem. *The composed map of factorization algebras*

$$U^{\alpha}(\mathcal{L}_c) \xrightarrow{\mathbb{1}} C_*(\mathcal{L}_c, \mathrm{Obs}^q_{\alpha})[\hbar^{-1}] \xrightarrow{\cong} U(\mathcal{L}_c) \otimes \mathrm{Obs}^q[\hbar^{-1}] \xrightarrow{\varepsilon \otimes \mathrm{Id}} \mathrm{Obs}^q[\hbar^{-1}]$$

is the Noether map from theorem 18.5.0.3.

This theorem therefore gives a compatibility between the action of $\mathcal{L}_c$ on observables and the Noether map.

18.6.1. Let us now turn to a more precise statement, and proof, of theorem 18.6.0.1. We first need to give a more careful statement of what it means to have a factorization algebra with a multiplicative action of $\mathcal{L}_c$, where, as usual, $\mathcal{L}$ indicates a local L_∞ algebra on a manifold M.

The first fact we need is that the factorization algebra $U(\mathcal{L})$, which assigns to every open subset $V \subset M$ the complex $C_*(\mathcal{L}_c(V))$, is a factorization algebra valued in commutative coalgebras (in the symmetric monoidal category of convenient cochain complexes).

To see this, first observe that for any L_∞ algebra $\mathfrak{g}$, the cochain complex $C_*(\mathfrak{g})$ is a cocommutative dg coalgebra. The coproduct

$$C_*(\mathfrak{g}) \to C_*(\mathfrak{g}) \otimes C_*(\mathfrak{g})$$

is the map induced from the diagonal map of Lie algebras $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, combined with the Chevalley-Eilenberg chain complex functor. Here we are using the fact that there is a natural isomorphism

$$C_*(\mathfrak{g} \oplus \mathfrak{h}) \cong C_*(\mathfrak{g}) \otimes C_*(\mathfrak{h}).$$

We want, in the same way, to show that $C_*(\mathcal{L}_c(V))$ is a cocommutative coalgebra, for every open subset $V \subset M$. The diagonal map $\mathcal{L}_c(V) \to \mathcal{L}_c(V) \oplus \mathcal{L}_c(V)$ gives us, as above, a putative coproduct map

$$C_*(\mathcal{L}_c(V)) \to C_*(\mathcal{L}_c(V) \oplus \mathcal{L}_c(V)).$$

The only point which is non-trivial is to verify that the natural map

$$C_*(\mathcal{L}_c(V)) \widehat{\otimes}_{\beta} C_*(\mathcal{L}_c(V)) \to C_*(\mathcal{L}_c(V) \oplus \mathcal{L}_c(V))$$

is an isomorphism, where on the right hand side we use the completed bornological tensor product of convenient vector spaces.

The fact that this map is an isomorphism follows from the fact that for any two manifolds X and Y, we have a natural isomorphism

$$C^\infty_c(X) \widehat{\otimes}_{\beta} C^\infty_c(Y) \cong C^\infty_c(X \times Y).$$

Thus, for every open subset $V \subset M$, $U(\mathcal{L})(V)$ is a cocommutative coalgebra. The fact that the assignment of the cocommutative coalgebra $C_*(\mathfrak{g})$ to an L_∞ algebra $\mathfrak{g}$ is functorial immediately implies that $U(\mathcal{L})(V)$ is a prefactorization algebra in the category of cocommutative coalgebras. (It is a factorization algebra, and not just a pre-algebra, because the forgetful functor from cocommutative coalgebras to cochain complexes preserves colimits).

Now we can give a formal definition of a factorization algebra with a multiplicative $\mathcal{L}_c$-action.

18.6.1.1 Definition. *Let $\mathcal{F}$ be a factorization algebra on a manifold M valued in convenient vector spaces, and let $\mathcal{L}$ be a local L_∞ algebra. Then, a multiplicative $\mathcal{L}_c$-action on $\mathcal{F}$ is is the following:*

1. *A factorization algebra $\mathcal{F}^{\mathcal{L}}$ in the category of (convenient) comodules for $U(\mathcal{L})$.*
2. *Let us give $\mathcal{F}$ the trivial coaction of $U(\mathcal{L})$. Then, we have a map of dg $U(\mathcal{L})$-comodule factorization algebras $\mathcal{F} \to \mathcal{F}^{\mathcal{L}}$.*
3. *For every open subset $V \subset M$, $\mathcal{F}^{\mathcal{L}}$ is quasi-cofree: this means that there is an isomorphism*
$$\mathcal{F}^{\mathcal{L}} \cong U(\mathcal{L}_c)(V) \widehat{\otimes}_\beta \mathcal{F}(V).$$
of graded, but not dg, $U(\mathcal{L}_c)(V)$ comodules, such that the given map from $\mathcal{F}(V)$ is obtained by tensoring the identity on $\mathcal{F}(V)$ with the coaugmentation map $\mathbb{C} \to U(\mathcal{L}_c)(V)$. (The coaugmentation map is simply the natural inclusion of $\mathbb{C}$ into $C_*(\mathcal{L}_c(V)) = \mathrm{Sym}^* \mathcal{L}_c(V)[1]$).

More generally, suppose that $\mathcal{F}$ is a convenient factorization algebra with a complete decreasing filtration. We give $U(\mathcal{L})$ a complete decreasing filtration by saying that $F^i(U(\mathcal{L}))(V) = 0$ for $i > 0$. In this situation, a multiplicative $\mathcal{L}_c$-action on $\mathcal{F}$ is a complete filtered convenient $U(\mathcal{L})$ comodule $\mathcal{F}^{\mathcal{L}_c}$ with the same extra data and properties as above, except that the tensor product is the one in the category of complete filtered convenient vector spaces.

One reason that this is a good definition is the following.

18.6.1.2 Lemma. *Suppose that $\mathcal{F}$ is a (complete filtered) convenient factorization algebra with a multiplicative $\mathcal{L}_c$ action in the sense above. Then, for every open subset $V \subset M$, there is an L_∞ action of $\mathcal{L}_c(V)$ on $\mathcal{F}(V)$ and an isomorphism of dg $C_*(\mathcal{L}_c(V))$-comodules*
$$C_*(\mathcal{L}_c(V), \mathcal{F}(V)) \cong \mathcal{F}^{\mathcal{L}_c}(V),$$
where on the left hand side we take chains with coefficients in the L_∞-module $\mathcal{F}(V)$.

PROOF. Let $\mathfrak{g}$ be any L_∞ algebra. There is a standard way to translate between L_∞ $\mathfrak{g}$-modules and $C_*(\mathfrak{g})$-comodules: if W is an L_∞ $\mathfrak{g}$-module, then $C_*(\mathfrak{g}, W)$ is a $C_*(\mathfrak{g})$-comodule. Conversely, to give a differential on $C_*(\mathfrak{g}) \otimes W$ making it into a $C_*(\mathfrak{g})$-comodule with the property that the map $W \to C_*(\mathfrak{g}) \otimes W$ is a cochain map, is the same as to give an L_∞ action of $\mathfrak{g}$ on W. Isomorphisms of comodules (which are the identity on the copy of W contained in $C_*(\mathfrak{g}, W)$) are the same as L_∞ equivalences.

This lemma just applies these standard statements to the symmetric monoidal category of convenient cochain complexes. ☐

To justify the usefulness of our definition to the situation of field theories, we need to show that if we have a field theory with an action of $\mathcal{L}$ then we get a factorization algebra with a multiplicative $\mathcal{L}_c$ action in the sense we have explained.

18.6.1.3 Proposition. *Suppose we have a field theory on a manifold M with an action of a local L_∞ algebra $\mathcal{L}$. Then quantum observables Obs^q of the field theory have a multiplicative $\mathcal{L}_c$-action in the sense we described above.*

PROOF. We need to define the space of elements of $\mathrm{Obs}^{q,\mathcal{L}_c}$. ☐

Now we can give the precise statement, and proof, of the main theorem of this section. Suppose that we have a quantum field theory on a manifold M, with an action of a local L_∞ algebra $\mathcal{L}$. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ be the obstruction to lifting this to an inner action.

18.7. Noether's theorem and the local index

In this section we will explain how Noether's theorem – in the stronger form formulated in the previous section – gives rise to a definition of the *local index* of an elliptic complex with an action of a local L_∞ algebra.

Let us explain what we mean by the local index. Suppose we have an elliptic complex on a compact manifold M. We will let $\mathscr{E}(U)$ denote the cochain complex of sections of this elliptic complex on an open subset $U \subset M$.

Then, the cohomology of $\mathscr{E}(M)$ is finite dimensional, and the index of our elliptic complex is defined to be the Euler characteristic of the cohomology. We can write this as

$$\mathrm{Ind}(\mathscr{E}(M)) = \mathrm{STr}_{H^*(\mathscr{E}(M))} \mathrm{Id}\,.$$

That is, the index is the super-trace (or graded trace) of the identity operator on cohomology.

More generally, if $\mathfrak{g}$ is a Lie algebra acting on global sections of our elliptic complex $\mathscr{E}(M)$, then we can consider the character of $\mathfrak{g}$ on $H^*(\mathscr{E}(M))$. If $X \in \mathfrak{g}$ is any element, the character can be written as

$$\mathrm{Ind}(X, \mathscr{E}(M)) = \mathrm{STr}_{H^*(\mathscr{E}(M))X.}$$

Obviously, the usual index is the special case when $\mathfrak{g}$ is the one-dimensional Lie algebra acting on $\mathscr{E}(M)$ by scaling.

We can rewrite the index as follows. For any endomorphism X of $H * (\mathscr{E}(M))$, the trace of X is the same as the trace of X acting on the determinant of $H^*(\mathscr{E}(M))$. Note that for this to work, we need $H^*(\mathscr{E}(M))$ to be treated as a super-line: it is even or odd depending on whether the Euler characteristic of $H^*(\mathscr{E}(M))$ is even or odd.

It follows that the character of the action of a Lie algebra $\mathfrak{g}$ on $\mathscr{E}(M))$ can be encoded entirely in the natural action of $\mathfrak{g}$ on the determinant of $H^*(\mathscr{E}(M))$. In other words, the character of the $\mathfrak{g}$ action is the same data as the one-dimensional $\mathfrak{g}$-representation $\det H^*(\mathscr{E}(M))$.

Now suppose that $\mathfrak{g}$ is global sections of a sheaf $\mathcal{L}$ of dg Lie algebras (or L_∞ algebras) on M. We will further assume that $\mathcal{L}$ is a local L_∞ algebra. Let us also assume that the action of $\mathfrak{g} = \mathcal{L}(M)$ on $\mathscr{E}(M)$ arises from a local action of the sheaf $\mathcal{L}$ of L_∞ algebras on the sheaf $\mathscr{E}$ of cochain complexes.

Then, one can ask the following question: is there some way in which the character of the $\mathcal{L}(M)$ action on $\mathscr{E}(M)$ can be expressed in a local way on the manifold? Since, as we have seen, the character of the $\mathcal{L}(M)$ action is entirely expressed in the homotopy $\mathcal{L}(M)$ action on the determinant of the cohomology of $\mathscr{E}(M)$, this question is equivalent to the following one: is it possible to express the determinant of the cohomology of $\mathscr{E}(M)$ in a way local on the manifold M, in an $\mathcal{L}$-equivariant way?

Now, $\mathscr{E}(M)$ is a sheaf, so that we can certainly describe $\mathscr{E}(M)$ in a way local on M. Informally, we can imagine $\mathscr{E}(M)$ as being a direct sum of its fibres at various points in M. More formally if we choose a cover $\mathfrak{U}$ of M, then the Čech double complex for $\mathfrak{U}$ with coefficients in the sheaf $\mathscr{E}$ produces for us a complex quasi-isomorphic to $\mathscr{E}(M)$. This double complex is an additive expression describing $\mathscr{E}(M)$ in terms of sections of $\mathscr{E}$ in the open cover $\mathfrak{U}$ of M.

Heuristically, the Čech double complex gives a formula of the form

$$\mathscr{E}(M) \sim \sum_i \mathscr{E}(U_i) - \sum_{i,j} \mathscr{E}(U_i \cap U_j) + \sum_{i,j,k} \mathscr{E}(U_i \cap U_j \cap U_k) - \dots$$

which we should imagine as the analog of the inclusion-exclusion formula from combinatorics. If $\mathfrak{U}$ is a finite cover and each $\mathscr{E}(U)$ has finite-dimensional cohomology, this formula becomes an identity upon taking Euler characteristics.

Since M is compact, one can also view $\mathscr{E}(M)$ as the global sections of the cosheaf of compactly supported sections of $\mathscr{E}$, and then Čech homology gives us a similar expression.

The determinant functor from vector spaces to itself takes sums to tensor products. We thus could imagine that the determinant of the cohomology of $\mathscr{E}(M)$ can be expressed in a local way on the manifold M, but where the direct sums that appear in sheaf theory are replaced by tensor products.

Factorization algebras have the feature that the value on a disjoint union is a tensor product (rather than a direct sum as appears in sheaf theory). That is, factorization algebras are multiplicative versions of cosheaves.

It is therefore natural to express that the determinant of the cohomology of $\mathscr{E}(M)$ can be realized as global sections of a factorization algebra, just as $\mathscr{E}(M)$ is global sections of a cosheaf.

It turns out that this is the case.

18.7.0.4 Lemma. *Let $\mathscr{E}$ be any elliptic complex on a compact manifold M. Let us form the free cotangent theory to the Abelian elliptic Lie algebra $\mathscr{E}[-1]$. This cotangent theory has elliptic complex of fields $\mathscr{E} \oplus \mathscr{E}^![-1]$.*

Let $\mathrm{Obs}^q_{\mathscr{E}}$ denote the factorization algebra of observables of this theory. Then, there is a quasi-isomorphism

$$H^*(\mathrm{Obs}^q_{\mathscr{E}}(M)) = \det H^*(\mathscr{E}(M))[d]$$

where d is equal to the Euler characteristic of $H^*(\mathscr{E}(M))$ modulo 2.

Recall that by $\det H^*(\mathscr{E}(M))$ we mean

$$\det H^*(\mathscr{E}(M)) = \otimes_i \left\{ \det H^i(\mathscr{E}(M)) \right\}^{(-1)^i}.$$

This lemma therefore states that the cohomology of global observables of the theory is the determinant of the cohomology of $\mathscr{E}(M)$, with its natural $\mathbb{Z}/2$ grading. The proof of this lemma, although easy, will be given at the end of this section.

This lemma shows that the factorization algebra $\mathrm{Obs}^q_{\mathscr{E}}$ is a local version of the determinant of the cohomology of $\mathscr{E}(M)$. One can then ask for a local version of the index. Suppose that $\mathcal{L}$ is a local L_∞ algebra on M which acts linearly on $\mathscr{E}$. Then, as we have seen, the precosheaf of L_∞-algebras given by compactly supported sections $\mathcal{L}_c$ of $\mathcal{L}$ acts on the factorization algebra Obs^q. We have also seen that, up to coherent homotopies which respect the factorization algebra structure, the action of $\mathcal{L}_c(U)$ on $\mathrm{Obs}^q(U)$ is by a character α times the identity matrix.

18.7.0.5 Definition. *In this situation, the* local index *is the multiplicative $\mathcal{L}_c$-equivariant factorization algebra $\mathrm{Obs}^q_{\mathscr{E}}$.*

This makes sense, because as we have seen, the action of $\mathcal{L}(M)$ on $\mathrm{Obs}^q_{\mathscr{E}}(M)$ is the same data as the character of the $\mathcal{L}(M)$ action on $\mathscr{E}(M)$, that is, the index.

Theorem 18.6.0.1 tells us that the multiplicative action of $\mathcal{L}_c$ on $\mathrm{Obs}^q_{\mathscr{E}}$ is through the character α of $\mathcal{L}_c$, which is also the obstruction to lifting the action to an inner action.

18.7.1. Proof of lemma 18.7.0.4. Before we give the (simple) proof, we should clarify some small points. Recall that for a free theory, there are two different versions of quantum observables we can consider. We can take our observables to be polynomial functions on the space of fields, and not introduce the formal parameter $\hbar$; or we can take our observables to be formal power series on the space of fields, in which case one needs to introduce the parameter $\hbar$. These two objects encode the same information: the second construction is obtained by applying the Rees construction to the first construction. We will give the proof for the first (polynomial) version of quantum observables. A similar statement holds for the second (power series) version, but one needs to invert $\hbar$ and tensor the determinant of cohomology by $\mathbb{C}((\hbar))$.

Globally, polynomial quantum observables can be viewed as the space $P(\mathscr{E}(M))$ of polynomial functions on $\mathscr{E}(M)$, with a differential which is a sum of the linear differential Q on $\mathscr{E}(M)$ with the BV operator. Let us compute the cohomology by a spectral sequence associated to a filtration of $\mathrm{Obs}^q_{\mathscr{E}}(M)$. The filtration is the obvious increasing filtration obtained by declaring that

$$F^i \,\mathrm{Obs}^q_{\mathscr{E}}(M) = \mathrm{Sym}^{\leq i}(\mathscr{E}(M) \oplus \mathscr{E}^!(M)[-1])^\vee.$$

The first page of this spectral sequence is cohomology of the associated graded. The associated graded is simply the symmmetric algebra

$$H^* \,\mathrm{Gr}\,\mathrm{Obs}^q_{\mathscr{E}}(M) = \mathrm{Sym}^* \left(H^*(\mathscr{E}(M))^\vee \oplus H^*(\mathscr{E}^!(M)[-1])^\vee \right).$$

The differential on this page of the spectral sequence comes from the BV operator associated to the non-degenerate pairing between $H^*(\mathscr{E}(M))$ and $H^*(\mathscr{E}^!(M))[-1]$. Note that $H^*(\mathscr{E}^!(M))$ is the dual to $H^*(\mathscr{E}(M))$.

It remains to show that the cohomology of this secondary differential yields the determinant of $H^*(\mathscr{E}(M))$, with a shift.

We can examine a more general problem. Given any finite-dimensional graded vector space V, we can give the algebra $P(V \oplus V^*[-1])$ of polynomial functions on $V \oplus V^*[-1]$ a BV operator $\triangle$ arising from the pairing between V and $V^*[-1]$. Then, we need to produce an isomorphism

$$H^*(P(V \oplus V^*[-1]), \triangle) \cong \det(V)[d]$$

where the shift d is equal modulo 2 to the Euler characteristic of V.

Sending V to $H^*(P(V \oplus V^*[-1]), \triangle)$ is a functor from the groupoid of finite-dimensional graded vector spaces and isomorphisms between them, to the category of graded vector spaces. It sends direct sums to tensor products. It follows that to check whether or not it returns the determinant, one needs to check that it does in the case that V is a graded line.

Thus, let us assume that $V = \mathbb{C}[k]$ for some $k \in \mathbb{Z}$. We will check that our functor returns $V[1]$ if k is even and V^* if k is odd. Thus, viewed as a $\mathbb{Z}/2$ graded line, our functor returns $\det V$ with a shift by the Euler characteristic of V.

To check this, note that

$$P(V \oplus V^*[-1]) = \mathbb{C}[x, y]$$

where x is of cohomological degree k and y is of degree $-1-k$. The BV operator is

$$\triangle = \frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$

A simple calculation shows that the cohomology of this complex is 1 dimensional, spanned by x if k is odd and by y if k is even. Since x is a basis of V^* and y is a basis of V, this completes the proof.

18.8. The partition function and the quantum Noether theorem

Our formulation of the quantum Noether theorem goes beyond a statement just about symmetries (in the classical sense of the word). It also involves deformations, which are symmetries of cohomological degree 1, as well as symmetries of other cohomological degree. Thus, it has important applications when we consider families of field theories.

The first application we will explain is that the quantum Noether theorem leads to a definition of the *partition function* of a perturbative field theory.

Suppose we have a family of field theories which depends on a formal parameter c, the coupling constant. (Everything we will say will work when the family depends on a number of formal parameters, or indeed on a pro-nilpotent dg algebra). For example, we could start with a free theory and deform it to an interacting theory. An example of such a family of scalar field theories is given by the action functional

$$S(\phi) = \int \phi(\triangle + m^2)\phi + c\phi^4.$$

We can view such a family of theories as being a *single* theory – in this case the free scalar field theory – with an action of the Abelian L_∞ algebra $\mathbb{C}[1]$. Indeed, by definition, an action of an L_∞ algebra $\mathfrak{g}$ on a theory is a family of theories over the dg ring $C^*(\mathfrak{g})$ which specializes to the original theory upon reduction by the maximal ideal $C^{>0}(\mathfrak{g})$.

We have seen (lemma **??**) that actions of $\mathfrak{g}$ on a theory are the same thing as actions of the local Lie algebra $\Omega^*_X \otimes \mathfrak{g}$. In this way, we see that a family of theories over the base ring $\mathbb{C}[[c]]$ is the same thing as a single field theory with an action of the local abelian L_∞ algebra $\Omega_X^*[-1]$.

Here is our definition of the partition function. We will give the definition in a general context, for a field theory acted on by a local L_∞ algebra; afterwards, we will analyze what it means for a family of field theories depending on a formal parameter c.

The partition function is only defined for field theories with some special properties. Suppose we have a theory on a compact manifold M, described classically by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3. Suppose that $H^*(\mathcal{M}(M)) = 0$; geometrically, this means we are perturbing around an solated solution to the equations of motion on the compact manifold M. This happens, for instance, with a massive scalar field theory.

This assumption implies that $H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]]$. There is a preferred $\mathbb{C}[[\hbar]]$-linear isomorphism which sends the observable $1 \in H^0(\mathrm{Obs}^q(M))$ to the basis vector of $\mathbb{C}$.

Suppose that this field theory is equipped with an inner action of a local L_∞ algebra $\mathcal{L}$. Then, proposition **??** tells us that for any open subset $U \subset M$, the action of $\mathcal{L}_c(U)$ on $\mathrm{Obs}^q(U)$ is homotopically trivialized. In particular, since M is compact, the action of $\mathcal{L}(M)$ on $\mathrm{Obs}^q(M)$ is homotopically trivialized.

A theory with an $\mathcal{L}$-action is the same as a family of theories over $B\mathcal{L}$. The complex $\mathrm{Obs}^q(M)^{\mathcal{L}(M)}$ of $\mathcal{L}(M)$-equivariant observables should be interpreted as the $C^*(\mathcal{L}(M))$-module of sections of the family of observables over $B\mathcal{L}(M)$.

Since the action of $\mathcal{L}(M)$ is trivialized, we have a quasi-isomorphism of $C^*(\mathcal{L})[[\hbar]]$-modules

$$\mathrm{Obs}^q(M)^{\mathcal{L}(M)} \simeq \mathrm{Obs}^q(M) \otimes C^*(\mathcal{L}).$$

Since $\mathrm{Obs}^q(M)$ is canonically quasi-isomorphic to $\mathbb{C}[[\hbar]]$, we get a quasi-isomorphism

$$\mathrm{Obs}^q(M)^{\mathcal{L}(M)} \simeq C^*(\mathcal{L})[[\hbar]].$$

18.8.0.1 Definition. *The partition function is the element in $C^*(\mathcal{L})[[\hbar]]$ which is the image of the observable $1 \in \mathrm{Obs}^q(M)^{\mathcal{L}(M)}$.*

Another way to interpret this is as follows. The fact that $\mathcal{L}$ acts linearly on our theory implies that the family of global observables over $B\mathcal{L}(M)$ is equipped with a flat connection. Since we have also trivialized the central fibre, via the quasi-isomorphism $\mathrm{Obs}^q(M) \simeq \mathbb{C}[[\hbar]]$, this whole family is trivialized. The observable 1 then becomes a section of the trivial line bundle on $B\mathcal{L}(M)$ with fibre $\mathbb{C}[[\hbar]]$, that is, an element of $C^*(\mathcal{L}(M))[[\hbar]]$.

Let us now analyze some examples of this definition.

Example: Let us first see what this definition amounts to when we work with one-dimensional topological field theories, which (in our formalism) are encoded by associative algebras.

Thus, suppose that we have a one-dimensional topological field theory, whose factorization algebra is described by an associative algebra A. We will view A as a field theory on S^1. For any interval $I \subset S^1$, observables of theory are A, but observables on S^1 are the Hochschild homology $HH_*(A)$.

Suppose that $\mathfrak{g}$ is a Lie algebra with an inner action on A, given by a Lie algebra homomorphism $\mathfrak{g} \to A$. The analog of proposition **??** in this situation is the statement that that the action of $\mathfrak{g}$ on the Hochschild homology of A is trivialized. At the level of HH_0, this is clear, as $HH_0(A) = A/[A, A]$ and bracketing with any element of A clearly acts by zero on $A/[A, A]$.

The obstruction to extending an action of $\Omega^*_X[-1]$ to an inner action is an element in $H^1(C^*_{red,loc}(\Omega^*_X[-1]))$. The D-module formulation **??** of the complex of local cochains of a local Lie algebra gives us a quasi-isomorphism

$$C^*_{red,loc}(\Omega^*_X[-1]) \cong c\Omega^*(X)[[c]][d]$$

where $d = \dim X$ and c is a formal parameter. Therefore, there are no obstructions to lifting an action of $\Omega^*_X[-1]$ to an inner action. However, there is an ambiguity in giving such a lift, coming from the space $cH^d(X)[[c]]$.

Let us suppose that we have a field theory with an inner action of $\Omega^*_X[-1]$. Let Obs^q denote observables of this theory. Then, Noether's theorem gives us a map of factorization algebras

$$\widehat{U}^{BD}(\Omega^*_{X,c}) \to \mathrm{Obs}^q$$

where Obs^q denotes the factorization algebra of observables of our theory, and $\widehat{U}^{BD}(\Omega^*_X[-1])$ is the completed BD envelope factorization algebra of Ω^*_X. Since $\Omega^*_X[-1]$ is abelian, then the BD envelope is simply a completed symmetric algebra:

$$\widehat{U}^{BD}(\Omega^*_c(U)[-1]) = \widehat{\mathrm{Sym}}^*(\Omega^*_c(U))[[\hbar]].$$

In particular, if M is a compact manifold, then we have a natural isomorphism

$$\widehat{\mathrm{Sym}}^*(H^*(M))[[\hbar]] = H^*\left(\widehat{U}^{BD}(\Omega^*_c(M)[-1])\right).$$

Applying Noether's theorem, we get a map

$$\widehat{\mathrm{Sym}}^{*(H^0(M))[[\hbar]] \to H^0(\mathrm{Obs}^q(M)).}$$

Part 6

Appendices

APPENDIX A

Background

We use techniques from disparate areas of mathematics throughout this book and not all of these techniques appear in the standard graduate curriculum, so here we provide a terse introduction to

- simplicial sets and simplicial techniques;
- operads, colored operads (or multicategories), and algebras over colored operads;
- differential graded (dg) Lie algebras, L_∞ algebras, their (co)homology, and their relation to deformation theory;
- sheaves, cosheaves, and their homotopical generalizations;
- elliptic complexes, formal Hodge theory, and parametrices,

along with pointers to more thorough treatments. By no means does the reader need to be expert in all these areas to use our results or follow our arguments. She just needs a working knowledge of this background machinery, and this appendix aims to provide the basic definitions, to state the relevant results for us, and to explain the essential intuition.

We do assume that the reader is familiar with basic homological algebra and basic category theory. For homological algebra, there are numerous excellent sources, in books and online, among which we recommend the complementary texts by Weibel [**Wei94**] and Gelfand-Manin [**GM03**]. For category theory, the standard reference [**ML98**] is more than adequate to our needs; we also recommend the series by Borceux [**Bor94**].

Remark: Our references are not meant to be complete, and we apologize in advance for the omission of many important works. We simply point out sources that we found pedagogically oriented or particularly accessible. ◊

A.1. Simplicial techniques

Simplicial sets are a combinatorial substitute for topological spaces, so it should be no surprise that they can be quite useful. On the one hand, we can borrow intuition for them from algebraic topology; on the other, simplicial sets are extremely concrete to work with because of their combinatorial nature. In fact, many constructions in homological algebra are best understood via their simplicial origins. Analogs of simplicial sets (i.e., simplicial objects in other categories) are useful as well.

In this book, we use simplicial sets in two ways:

- when we want to talk about a family or space of QFTs or parametrices, we will usually construct a simplicial set of such objects instead; and
- we accomplish some homological constructions by passing through simplicial sets (e.g., in constructing the extension of a factorization algebra from a basis).

After giving the essential definitions, we state the main theorems we use.

A.1.0.2 Definition. *Let Δ denote the category whose objects are totally ordered finite sets and whose morphisms are non-decreasing maps between ordered sets. We usually work with the skeletal subcategory whose objects are*

$$[n] = \{0 < 1 < \cdots < n\}.$$

A morphism $f : [m] \to [n]$ then satisfies $f(i) \leq f(j)$ if $i < j$.

We will relate these objects to geometry below, but it helps to bear in mind the following picture. The set $[n]$ corresponds to the n-simplex $\triangle^n$ equipped with an ordering of its vertices, as follows. View the n-simplex $\triangle^n$ as living in $\mathbb{R}^{n+1}$ as the solution to

$$\left\{ (x_0, x_1, \ldots, x_n) \;\middle|\; \sum_j x_j = 1 \text{ and } 0 \leq x_j \leq 1 \; \forall j \right\}.$$

Identify the element $0 \in [n]$ with the zeroth basis vector $e_0 = (1, 0, \ldots, 0)$, $1 \in [n]$ with $e_1 = (0, 1, 0, \ldots, 0)$, and $k \in [n]$ with the kth basis vector e_k. The ordering on $[n]$ then prescribes a path along the edges of $\triangle^n$, starting at e_0, then going to e_1, and on till the path ends at e_n.

Every map $f : [m] \to [n]$ induces a linear map $f_* : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ by setting $f_*(e_k) = e_{f(k)}$. This linear map induces a map of simplices $f_* : \triangle^m \to \triangle^n$.

There are particularly simple maps that play an important role throughout the subject. Note that every map f factors into a surjection followed by an injection. We can then factor every injection into a sequence of *coface maps*, namely maps of the form

$$\begin{array}{cccc} f_k : & [n] & \to & [n+1] \\ & i & \mapsto & \left\{ \begin{array}{ll} i, & i \leq k \\ i+1, & i > k \end{array} \right. \end{array}.$$

Similarly, we can factor every surjection into a sequence of *codegeneracy maps*, namely maps of the form

$$\begin{array}{rccc} d_k: & [n] & \to & [n-1] \\ & i & \mapsto & \left\{ \begin{array}{cc} i, & i \leq k \\ i-1, & i > k \end{array} \right. . \end{array}$$

The names *face* and *degeneracy* fit nicely with the picture from above: a coface map corresponds to a choice of n-simplex in the boundary of the $n+1$-simplex, and a codegeneracy map corresponds to "collapsing" an edge of the n-simplex to project the n-simplex onto an $n-1$-simplex.

We now introduce the main character.

A.1.0.3 Definition. *A* simplicial set *is a functor $X : \Delta^{op} \to Sets$, often denoted $X_\bullet$. The set $X([n])$ (often denoted X_n) is called the "set of n-simplices of X." A map of simplicial sets $F : X \to Y$ is a natural transformation of functors.*

Let's quickly examine what the coface maps tell us about a simplicial set X. For example, by definition, a map $f_k : [n] \to [n+1]$ in Δ goes to $X(f) : X_{n+1} \to X_n$. We interpret this map $X(f)$ as describing the kth n-simplex sitting as a "face on the boundary" of an $n+1$-simplex of X. A similar interpretation applies to the d_k.

A.1.1. Simplicial sets and topological spaces. When working in a homotopical setting, simplicial sets often provide a more tractable approach than topological spaces themselves. In this book, for instance, we describe "spaces of field theories" as simplicial sets. Below, we sketch how to relate these two kinds of objects.

One can use a simplicial set $X_\bullet$ as the "construction data" for a topological space: each element of X_n labels a distinct n-simplex $\triangle^n$, and the structure maps of $X_\bullet$ indicate how to glue the simplices together. In detail, the *geometric realization* is the quotient topological space

$$|X_\bullet| = \left(\coprod_n X_n \times \triangle^n \right) / \sim$$

under the equivalence relation $\sim$ where $(x, s) \in X_m \times \triangle^m$ is equivalent to $(y, t) \in X_n \times \triangle^n$ if there is a map $f : [m] \to [n]$ such that $X(f) : X_n \to X_m$ sends y to x and $f_*(s) = t$.

A.1.1.1 Lemma. *Under the Yoneda embedding, $[n]$ defines a simplicial set*

$$\Delta[n] : [m] \in \Delta^{op} \mapsto \Delta([m], [n]) \in Sets.$$

The geometric realization of $\Delta[n]$ is the n-simplex $\triangle^n$ (more accurately, it is homeomorphic to the n-simplex).

In general, every (geometric) simplicial complex can be obtained by the geometric realization of some simplicial set. Thus, simplicial sets provide an efficient way to study combinatorial topology.

One can go the other way, from topological spaces to simplicial sets: given a topological space X, there is a simplicial set $\mathrm{Sing}\, X$, known as the "singular simplicial set of S." The set of n-simplices $(\mathrm{Sing}\, X)_n$ is simply $Top(\triangle^n, X)$, the set of continuous maps from the n-simplex $\triangle^n$ into X. The structure maps arise from pulling back along the natural maps of simplices.

A.1.1.2 Theorem. *Geometric realization and the singular functor form an adjunction*

$$|-| : sSets \rightleftarrows Top : \mathrm{Sing}$$

between the category of simplicial sets and the category of topological spaces.

This relationship suggests, for instance, how to transport notions of homotopy to simplicial sets: the homotopy groups of a simplicial set $X_\bullet$ are the homotopy groups of its realization $|X_\bullet|$, and maps of simplicial sets are homotopic if their realizations are. We would then like to say that these functors $|-|$ and Sing make the categories of simplicial sets and topological spaces equivalent, *in some sense*, particularly when it comes to questions about homotopy type. One sees immediately that some care must be taken, since there are topological spaces very different in nature from simplicial or cell complexes and for which no simplicial set could provide an accurate description. The key is only to think about topological spaces and simplicial sets up to the appropriate notion of equivalence.

Remark: It is more satisfactory to define these homotopy notions directly in terms of simplicial sets and then to verify that these match up with the topological notions. We direct the reader to the references for this story, as the details are not relevant to our work in the book. ◊

We say a continuous map $f : X \to Y$ of topological spaces is a *weak equivalence* if it induces a bijection between connected components and an isomorphism $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ for every $n > 0$ and every $x \in X$. Let $Ho(Top)$, the *homotopy category* of Top, denote the category of topological spaces where we localize at the weak equivalences. ("Localizing" means that we formally invert the weak equivalences.) There is a concrete way to think about this homotopy category. For every topological space, there is some CW complex weakly equivalent to it, under a zig zag of weak equivalences; and by the Whitehead theorem, a weak equivalence between CW complexes is in fact a homotopy equivalence. Thus, $Ho(Top)$ is equivalent to the category of CW complexes with morphisms given by continuous maps modulo homotopy equivalence.

Likewise, let $Ho(sSets)$ denote the homotopy category of simplicial sets, where we localize at the appropriate notion of simplicial homotopy. Then Quillen proved the following wonderful theorem.

A.1.1.3 Theorem. *The adjunction induces an equivalence*

$$|-| : Ho(sSets) \rightleftarrows Ho(Top) : \mathrm{Sing}$$

between the homotopy categories. (In particular, they provide a Quillen equivalence *between the standard model category structures on these categories.)*

This theorem justifies the assertion that, from the perspective of homotopy type, we are free to work with simplicial sets in place of topological spaces. In addition, it helpful to know that algebraic topologists typically work with a better behaved categories of topological spaces, such as compactly generated spaces.

Among simplicial sets, those that behave like topological spaces are known as *Kan complexes* or *fibrant simplicial sets*. Their defining property is a simplicial analogue of the homotopy lifting property, which we now describe explicitly. The *horn* for the kth face of the n-simplex, denoted $\Lambda_k[n]$, is the subsimplicial set of $\Delta[n]$ given by the union of the all the faces $\Delta[n-1] \hookrightarrow \Delta[n]$ except the kth. (As a functor on Δ, the horn takes the $[m]$ to monotonic maps $[m] \to [n]$ that do not have k in the image.) A simplicial set X is a *Kan complex* if for every map of a horn $\Lambda_k[n]$ into X, we can extend the map to the n-simplex $\Delta[n]$. Diagrammatically, we can fill in the dotted arrow

$$\begin{tikzcd}
\Lambda_k[n] \arrow[r] \arrow[d] & X \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}$$

to get a commuting diagram. In general, one can always find a “fibrant replacement” of a simplicial set $X_\bullet$ (e.g., by taking Sing $|X_\bullet|$ or via Kan's Ex^∞ functor) that is weakly equivalent and a Kan complex.

A.1.2. Simplicial sets and homological algebra. Our other use for simplicial sets relates to homological algebra. We always work with *co*chain complexes, so our conventions will differ from those who prefer chain complexes. For instance, the chain complex computing the homology of a topological space is concentrated in non-negative degrees. We work instead with the cochain complex concentrated in non-positive degrees.

A.1.2.1 Definition. *A* simplicial abelian group *is a simplicial object $A_\bullet$ in the category of abelian groups, i.e., a functor $A : \Delta^{op} \to AbGps$.*

By composing with the forgetful functor $U : AbGps \to Sets$ that sends a group to its underlying set, we see that a simplicial abelian group has an underlying simplicial set. To define simplicial vector spaces or simplicial R-modules, one simply replaces *abelian group* by *vector space* or *R-module* everywhere. All the work below will carry over to these settings in a natural way.

There are *two* natural ways to obtain a cochain complex of abelian groups (respectively, vector spaces) from a simplicial abelian group. The *unnormalized chains* $\mathbf{C}A$ is the cochain complex

$$(\mathbf{C}A)^m = \begin{cases} A_{|m|}, & m \leq 0 \\ 0, & m > 0 \end{cases}$$

with differential

$$\begin{array}{cccc} d: & (\mathbf{C}A)^m & \to & (\mathbf{C}A)^{m+1} \\ & a & \mapsto & \sum_{k=0}^{|m|}(-1)^k A(f_k)(a), \end{array}$$

where the f_k run over the coface maps from $[|m|-1]$ to $[|m|]$. The *normalized chains* $\mathbf{N}A$ is the cochain complex

$$(\mathbf{N}A)^m = \bigcap_{k=0}^{|m|-1} \ker A(f_k),$$

where $m \leq 0$ and where the f_k run over the coface maps from $[|m|-1]$ to $[|m|]$. The differential is $A(f_{|m|})$, the remaining coface map. One can check that the inclusion $\mathbf{N}A \hookrightarrow \mathbf{C}A$ is a quasi-isomorphism (in fact, a chain homotopy equivalence).

Example: Given a topological space X, its singular chain complex $C_*(X)$ arises as a composition of three functors in this simplicial world. (Note that because we prefer cochain complexes, the singular chain complex is, in fact, a cochain complex concentrated in *nonpositive* degrees.) First, we make the simplicial set $\operatorname{Sing} X$, which knows about all the ways of mapping a simplex into X. Then we apply the free abelian group functor $\mathbb{Z}- : \mathit{Sets} \to \mathit{AbGps}$ levelwise to obtain the simplicial abelian group

$$\mathbb{Z}\operatorname{Sing} X : [n] \mapsto \mathbb{Z}(\mathit{Top}(\triangle^n, X)).$$

Then we apply the unnormalized chains functor to obtain the singular chain complex

$$C_*(X) = \mathbf{C}\mathbb{Z}\operatorname{Sing} X.$$

In other words, the simplicial language lets us decompose the usual construction into its atomic components. ◊

It is clear from the constructions that we only ever obtain cochain complexes concentrated in *nonpositive* degrees from simplicial abelian groups. In fact, the *Dold-Kan correspondence* tells us that we are free to work with either kind of object — simplicial abelian group or such a cochain complex — as we prefer.

Let $Ch^{\leq 0}(AbGps)$ denote the category of cochain complexes concentrated in nonpositive degrees, and let $sAbGps$ denot the category of simplicial abelian groups.

A.1.2.2 Theorem (Dold-Kan correspondence). *The normalized chains functor*

$$\mathbf{N} : sAbGps \to Ch^{\leq}(AbGps)$$

is an equivalence of categories. Under this correspondence,

$$\pi_n(A) \cong H^{-n}(\mathbf{N}A)$$

for all $n \geq 0$, *and simplicial homotopies go to chain homotopies.*

Throughout the book, we will often work with cochain complexes equipped with algebraic structures (e.g., commutative dg algebras or dg Lie algebras). Thankfully, it is well-understood how the Dold-Kan correspondence intertwines with the tensor structures on $sAbGps$ and $Ch^{\leq 0}(AbGps)$.

Let A and B be simplicial abelian groups. Then their *tensor product* $A \otimes B$ is the simplicial abelian group with n-simplices

$$(A \otimes B)_n = A_n \otimes_{\mathbb{Z}} B_n.$$

There is a natural transformation

$$\nabla_{A,B} : \mathbf{C}A \otimes \mathbf{C}B \to \mathbf{C}(A \otimes B),$$

known as the *Eilenberg-Zilber map* or *shuffle map*, which relates the usual tensor product of complexes with the tensor product of simplicial abelian groups. It is a quasi-isomorphism.

A.1.2.3 Theorem. *The unnormalized chains functor and the normalized chains functor are both lax monoidal functors via the Eilenberg-Zilber map.*

Thus, with a little care, we can relate algebra in the setting of simplicial modules with algebra in the setting of cochain complexes.

A.1.3. References. Friedman's paper [Fri12] is a very accessible and concrete introduction to simplicial sets, with lots of intuition and pictures. Weibel [Wei94] explains clearly how simplicial sets appear in homological algebra, notably for us, the Dold-Kan correspondence and the Eilenberg-Zilber map. As usual, Gelfand-Manin [GM03] provides a nice complement to Weibel. The expository article [GS07] provides a lucid and quick discussion of how simplicial methods relate to model categories and related issues. For the standard, modern reference on simplicial sets and homotopy theory, see the thorough and clear Goerss-Jardine [GJ09].

A.2. Operads and algebras

An operad is a way of describing the essential structure underlying some class of algebraic objects. For instance, there is an operad *Ass* that captures the essence of associative algebras, and there is an operad *Lie* that captures the essence of Lie algebras. An algebra over an operad is an algebraic object with that kind of structure: a Lie algebra is an algebra over the operad *Lie*. Although their definition can seem abstract and unwieldy at first, operads provide an efficient language for thinking about algebra and proving theorems about large classes of algebraic objects. As a result, they have become nearly ubiquitous in mathematics.

A colored operad is a way of describing a collection of objects that interact algebraically. For example, there is a colored operad that describes a pair consisting of an algebra and a module over that algebra. Another name for a colored operad is a symmetric multicategory, which emphasizes a different intuition: it is a generalization of the notion of a category in which we allow maps with n inputs and one output.

In the book, we use these notions in several ways:

- we capture the algebraic essence of the Batalin-Vilkovisky notion of a quantization via the *Beilinson-Drinfeld operad*;
- the notion of a prefactorization algebra — perhaps the central notion in the book — is an algebra over a colored operad made from the open sets of a topological space; and
- we define colored operads that describe how observables vary under translation and that generalize the notion of a vertex algebra.

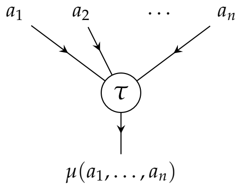
The first use has a different flavor than the others, so we begin here by focusing on operads with a linear flavor before we introduce the general formalism of colored operads. We hope that by being concrete in the first part, the abstractions of the second part will not seem arid.

A.2.1. Operads. In the loosest sense — encompassing Lie, associative, commutative, and more — an algebra is a vector space A with some way of combining elements multilinearly. Typically, we learn first about examples determined by a binary operation $\mu : A \otimes A \to A$ such that

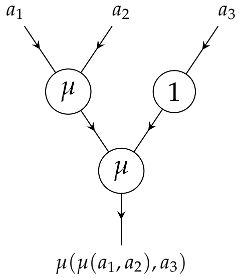
- μ has some symmetry under permutation of the inputs (e.g., $\mu(a,b) = -\mu(b,a)$ for a Lie algebra), and
- the induced 3-ary operations $\mu \circ (\mu \otimes 1)$ and $\mu \circ (1 \otimes \mu)$ satisfy a linear relation (e.g., associativity or the Jacobi identity).

But we recognize that there should be more elaborate notions involving many different n-ary operations required to satisfy complicated relations. As a basic example, a Poisson algebra has two binary operations.

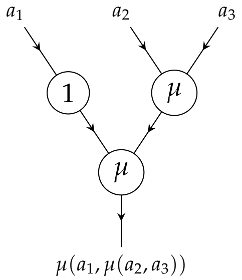
Before we give the general definition of an operad in (dg) vector spaces, we explain how to visualize such algebraic structures. An n-ary operation $\tau : A^{\otimes n} \to A$ is pictured as a rooted tree with n labeled leaves and one root.



For us, operations move down the page. To compose operations, we need to specify where to insert the output of each operation. We picture this as stacking rooted trees. For example, given a binary operation μ, the composition $\mu \circ (\mu \otimes 1)$ corresponds to the tree



whereas $\mu \circ (1 \otimes \mu)$ is the tree



with the first μ on the other side. We allow permutations of the inputs, which rearranges the inputs. Thus the vector space of n-ary operations $\{\tau : A^{\otimes n} \to A\}$ has an action of the permutation group S_n. We also want the permutations to interact in the natural way with composition.

We now give the formal definition. We will describe an operad in vector spaces over a field $\mathbb{K}$ of characteristic zero (e.g., $\mathbb{C}$ or $\mathbb{R}$) and use $\otimes$ to denote $\otimes_{\mathbb{K}}$. It is straightforward to give a general definition of an operad in an arbitrary symmetric monoidal category. It should be clear, for instance, how to replace vector spaces with cochain complexes.

A.2.1.1 Definition. *An* operad $\mathcal{O}$ *in vector spaces consists of* *(i) a sequence of vector spaces* $\{\mathcal{O}(n) \mid n \in \mathbb{N}\}$*, called the* operations,

(ii) a collection of multilinear maps

$$\circ_{n;m_1,\ldots,m_n} : \mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \to \mathcal{O}\left(\sum_{j=1}^{n} m_j\right),$$

called the composition maps,

(iii) a unit element $\eta : \mathbb{K} \to \mathcal{O}(1)$.

This data is equivariant, associative, and unital in the following way.

(1) The n-ary operations $\mathcal{O}(n)$ *have a right action of* S_n.

(2) The composition maps are equivariant in the sense that the diagram below commutes,

$$\begin{tikzcd}
\mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\circ"'] & \mathcal{O}(n) \otimes (\mathcal{O}(m_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(m_{\sigma(n)})) \arrow[d, "\circ"] \\
\mathcal{O}\left(\sum_{j=1}^{n} m_j\right) \arrow[r, "\sigma(m_{\sigma(1)},\ldots,m_{\sigma(n)})"] & \mathcal{O}\left(\sum_{j=1}^{n} m_j\right)
\end{tikzcd}$$

where $\sigma \in S_n$ *acts as a block permutation on the* $\sum_{j=1}^{n} m_j$ *inputs, and the diagram below also commutes,*

$$\begin{tikzcd}
\mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \arrow[r, bend left, "1\otimes(\tau_1\otimes\cdots\otimes\tau_n))"] \arrow[d, "\circ"'] & \mathcal{O}(n) \otimes (\mathcal{O}(m_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(m_{\sigma(n)})) \arrow[d, "\circ"] \\
\mathcal{O}\left(\sum_{j=1}^{n} m_j\right) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_n"] & \mathcal{O}\left(\sum_{j=1}^{n} m_j\right)
\end{tikzcd}$$

where each τ_j *is in* S_{m_j} *and* $\tau_1 \oplus \cdots \oplus \tau_n$ *denotes the blockwise permutation in* $S_{\sum_{j=1}^{n} m_j}$.

(3) The composition maps are associative in the following sense. Let $n, m_1, \ldots, m_n, \ell_{1,1}, \ldots, \ell_{1,m_1}, \ell_{2,1}, \ldots, \ell_{n,m_n}$ *be positive integers, and set* $M = \sum_{j=1}^{n} m_j$, $L_j = \sum_{i=1}^{m_j} \ell_{j,i}$, *and*

$$N = \sum_{i=1}^{n} L_j = \sum_{(j,k)\in M} \ell_{j,k}.$$

Then the diagram

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \left(\bigotimes_{j=1}^{n} \left(\mathcal{O}(m_j) \otimes \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k}) \right) \right) \arrow[d, "1\otimes(\otimes_j \circ)"'] \arrow[dr, "\mathit{shuffle}"] & \\
& \left(\mathcal{O}(n) \otimes \bigotimes_{j=1}^{n} \mathcal{O}(m_j) \right) \otimes \left(\bigotimes_{j=1}^{n} \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k}) \right) \arrow[dd, "\circ\otimes 1"] \\
\mathcal{O}(n) \otimes \left(\bigotimes_{j=1}^{n} \mathcal{O}(L_j) \right) \arrow[dd, "\circ"'] & \\
& \mathcal{O}(M) \otimes \left(\bigotimes_{j=1}^{n} \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k}) \right) \arrow[dl, "\circ"] \\
\mathcal{O}(N) &
\end{tikzcd}$$

commutes.

(4) *The unit diagrams commute:*

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathbb{K}^{\otimes n} \arrow[r, "1\otimes\eta^{\otimes n}"'] \arrow[rr, bend left, "\cong"] & \mathcal{O}(n) \otimes \mathcal{O}(1)^{\otimes n} \arrow[r, "\circ"'] & \mathcal{O}(n)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \otimes \mathcal{O}(n) \arrow[r, "\eta\otimes 1"'] \arrow[rr, bend left, "\cong"] & \mathcal{O}(1) \otimes \mathcal{O}(n) \arrow[r, "\circ"'] & \mathcal{O}(n)\ .
\end{tikzcd}$$

A *map of operads* $f : \mathcal{O} \to \mathcal{P}$ is a sequence of linear maps $\{f(n) : \mathcal{O}(n) \to \mathcal{P}(n)\}$ interwining in the natural way with all the structure of the operads.

Example: The operad *Com* that describes commutative algebras has $Com(n) \cong \mathbb{K}$, with the trivial S_n action, for all n. This is because there is only one way to multiply n elements: even if we permute the inputs, we have the same output. ◊

Example: The operad *Ass* that describes associative algebras has $Ass(n) = \mathbb{K}[S_n]$, the regular representation of S_n, for all n. This is because the product of n elements only depends on their left-to-right ordering, not on a choice of parantheses. We should have a distinct n-ary product for each ordering of n elements. ◊

Remark: One can describe operads via generators and relations. The two examples above are generated by a single binary operation. In Ass, there is a relation between the 3-ary operations generated by that binary operation — the associativity relation — as already discussed. For a careful treatment of this style of description, we direct the reader to the references. ◊

We now explain the notion of an *algebra over an operad*. Our approach is modeled on defining a representation of a group G on a vector space V as a group homomorphism $\rho : G \to GL(V)$. Given a vector space V, there is an operad End_V that contains all imaginable multilinear operations on V and how they compose, just like $GL(V)$ contains all linear automorphisms of V.

A.2.1.2 Definition. *The* endomorphism operad End_V *of a vector space V has n-ary operations* $End_V(n) = Hom(V^{\otimes n}, V)$ *and compositions*

$$\begin{array}{rccc} \circ_{n;m_1,\ldots,m_n} : & \mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) & \to & \mathcal{O}\left(\sum_{j=1}^{n} m_j\right) \\ & \mu_n \otimes (\mu_{m_1} \otimes \cdots \otimes \mu_{m_n}) & \mapsto & \mu_n \circ (\mu_{m_1} \otimes \cdots \otimes \mu_{m_n}) \end{array}$$

are simply composition.

A.2.1.3 Definition. *For $\mathcal{O}$ an operad and V a vector space, we make V an* algebra over $\mathcal{O}$ *by choosing a map of operads* $\rho : \mathcal{O} \to End_V$.

For example, an associative algebra A is given by specifying a vector space A and a linear map $\mu : A^{\otimes 2} \to A$ satisfying the associativity relation. This data is equivalent to specifying an operad map $Ass \to End_A$.

There is one final notion from the theory of linear operads that we use: a *Hopf operad*. Algebras over a Hopf operad are closed under tensor product, just as commutative algebras can be tensored to form a new commutative algebra.

A.2.1.4 Definition. *A* Hopf operad *is a reduced operad $\mathcal{P}$ (i.e., $\mathcal{P}(0) = 0$) equipped with a counit, a map of operads*

$$\varepsilon_{\mathcal{P}} : \mathcal{P} \to Com,$$

and coproduct, a map of operads

$$\Delta_{\mathcal{P}} : \mathcal{P} \to \mathcal{P} \otimes_H \mathcal{P},$$

that is counital and coassociative.

Here $\mathcal{P} \otimes_H \mathcal{Q}$ denotes the *Hadamard product* of operads. The n-ary operations are

$$\mathcal{P} \otimes_H \mathcal{Q}(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n),$$

and the composition maps are obtained by tensoring the compositions in $\mathcal{P}$ and $\mathcal{Q}$ "side by side." If A is a $\mathcal{P}$-algebra and B is a $\mathcal{Q}$-algebra, then $A \otimes B$ is naturally a $\mathcal{P} \otimes_H \mathcal{Q}$-algebra. Building on this observation, we obtain the following.

A.2.1.5 Proposition. *For $\mathcal{P}$ a Hopf operad, the tensor product $A \otimes B$ of $\mathcal{P}$- algebra A and B is again a $\mathcal{P}$-algebra. Moreover, there is a natural isomorphism of $\mathcal{P}$-algebras*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

for C another $\mathcal{P}$-algebra.

A.2.1.1. *References.* For a very brief introduction to operads, see Stasheff's "What is" column [**Sta04**]. The recent article by Vallette [**Val**] provides a nice motivation and overview for linear operads and their relation to homotopical algebra. For a systematic treatment with an emphasis on Koszul duality, see Loday-Vallette [**LV12**]. In [**Cos04**], there is a description of the basics emphasizing a diagrammatic approach: an operad is a functor on a category of rooted trees. Finally, Fresse's book-in-progress is wonderful [**Fre**].

A.2.2. Colored operads *aka* multicategories.

We now introduce a natural generalization of the notions of a category and of an operad. The essential idea is to have a collection of objects that we can "combine" or "multiply." If there is only one object, we recover the notion of an operad. If there are no ways to combine multiple objects, then we recover the notion of a category.

We will give the *symmetric* version of this concept, just as we did with operads.

Let $(\mathcal{C}, \boxtimes)$ be a symmetric monoidal category, such as $(Sets, \times)$ or $(Vect_{\mathbb{K}}, \otimes_{\mathbb{K}})$. We require $\mathcal{C}$ to have all reasonable colimits.

A.2.2.1 Definition. *A* multicategory *(or* colored operad*) $\mathcal{M}$ over $\mathcal{C}$ consists of*

- *(i) a collection of* objects *(or* colors*) $Ob\mathcal{M}$,*
- *(ii) for every $n+1$-tuple of objects $(x_1, \dots, x_n \mid y)$, an object*

$$\mathcal{M}(x_1, \dots, x_n \mid y)$$

  in $\mathcal{C}$ called the maps *from the x_j to y,*
- *(iii) a* unit element *$\eta_x : 1_{\mathcal{C}} \to \mathcal{M}(x \mid x)$ for every object x.*
- *(iv) a collection of* composition maps *in $\mathcal{C}$*

$$\mathcal{M}(x_1, \dots, x_n \mid y) \boxtimes \left(\mathcal{M}(x_1^1, \dots, x_1^{m_1} \mid x_1) \boxtimes \cdots \boxtimes \mathcal{M}(x_n^1, \dots, x_n^{m_n} \mid x_n)\right) \to \mathcal{M}(x_1^1, \dots, x_n^{m_n} \mid y),$$

- *(v) for every $n+1$-tuple $(x_1, \dots, x_n \mid y)$ and every permutation $\sigma \in S_n$ a morphism*

$$\sigma^* : \mathcal{M}(x_1, \dots, x_n \mid y) \to \mathcal{M}(x_{\sigma(1)}, \dots, x_{\sigma(n)} \mid y)$$

  in $\mathcal{C}$.

This data satisfies conditions of associativity, units, and equivariance directly analogous to that of operads. For instance, given $\sigma, \tau \in S_n$, we require

$$\sigma^* \tau^* = (\tau\sigma)^*,$$

so we have an analog of a right S_n action on maps out of n objects. Each unit η_x is a two-sided unit for composition in $\mathcal{M}(x \mid x)$.

A.2.2.2 Definition. *A* map of multicategories *(or* functor *between multicategories) $F : \mathcal{M} \to \mathcal{N}$ consists of*

(i) an object $F(x)$ in $\mathcal{N}$ for each object x in $\mathcal{M}$, and
(ii) a morphism

$$F(x_1, \ldots, x_n \mid y) : \mathcal{M}(x_1, \ldots, x_n \mid y) \to \mathcal{N}(F(x_1), \ldots, F(x_n) \mid F(y))$$

in the category $(\mathcal{C}, \boxtimes)$ for every tuple $(x_1, \ldots, x_n \mid y)$ of objects in $\mathcal{M}$

such that the structure of a multicategory is preserved (namely, the units, associativity, and equivariance).

There are many familiar examples.

Example: Let $\mathcal{B}$ be an ordinary category, so that each collection of morphisms $\mathcal{B}(x, y)$ is a set. Then $\mathcal{B}$ is a multicategory over the symmetric monoidal category $(Sets, \times)$ where $\mathcal{B}(x \mid y) = \mathcal{B}(x, y)$ and $\mathcal{B}(x_1, \ldots, x_n \mid y) = \emptyset$ for $n > 1$. ♢

Similarly, a $\mathbb{K}$-linear category is a multicategory over the symmetric monoidal category $(Vect_{\mathbb{K}}, \otimes_{\mathbb{K}})$ with no compositions between two or more elements.

Example: An operad $\mathcal{O}$, in the sense of the preceding subsection, is a multicategory over the symmetric monoidal category $(Vect_{\mathbb{K}}, \otimes_{\mathbb{K}})$ with a single object $*$ and

$$\mathcal{O}(\underbrace{*, \ldots, *}_{n} \mid *) = \mathcal{O}(n).$$

Replacing $Vect_{\mathbb{K}}$ with another symmetric monoidal category $\mathcal{C}$, we obtain a definition for operad in $\mathcal{C}$. ♢

Example: Every symmetric monoidal category $(\mathcal{C}, \boxtimes)$ has an underlying multicategory $\underline{\mathcal{C}}$ with the same objects and with maps

$$\underline{\mathcal{C}}(x_1, \ldots, x_n \mid y) = \mathcal{C}(x_1 \boxtimes \cdots \boxtimes x_n, y).$$

When $\mathcal{C} = Vect_{\mathbb{K}}$, these are precisely all the multilinear maps. ♢

For every multicategory $\mathcal{M}$, one can construct a *symmetric monoidal envelope* $\mathbf{S}\mathcal{M}$ by forming the left adjoint to the forgetful functor from symmetric monoidal categories to multicategories. An object of $\mathbf{S}\mathcal{M}$ is a formal finite sequence $[x_1, \ldots, x_m]$ of colors x_i, and a morphism $f : [x_1, \ldots, x_m] \to [y_1, \ldots, y_n]$ consists of a surjection $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ and a morphism $f_j \in \mathcal{M}(\{x_i\}_{i \in \phi^{-1}(j)} \mid y_j)$ for every $1 \leq j \leq n$. The symmetric monoidal product in $\mathbf{S}\mathcal{M}$ is simply concatenation of formal sequences.

Finally, an *algebra over a colored operad* $\mathcal{M}$ with values in $\mathcal{N}$ is simply a functor of multicategories $F : \mathcal{M} \to \mathcal{N}$. When we view $\mathcal{O}$ as a multicategory and use the underlying multicategory $\underline{Vect}_{\mathbb{K}}$, then $F : \mathcal{O} \to \underline{Vect}_{\mathbb{K}}$ reduces to an algebra over the operad $\mathcal{O}$ as in the preceding subsection.

A.2.2.1. *References.* For a readable discussion of operads, multicategories, and different approaches to them, see Leinster [**Lei04**]. Beilinson and Drinfeld [**?**] develop *pseudo-tensor categories* — yet another name for this concept — for exactly the same reasons as we do in this book. In [**Lurb**], Lurie provides a thorough treatment of colored operads compatible with ∞-categories. Another approach to higher operads is provided by the dendroidal sets of Moerdijk, Weiss, and collaborators [**MW07**].

A.3. Lie algebras and L_∞ algebras

Lie algebras, and their homotopical generalization L_∞ algebras, appear throughout this book in a variety of contexts. It might surprise the reader that we never use their representation theory or almost any aspects emphasized in textbooks on Lie theory. Instead, we primarily use dg Lie algebras as a convenient language for formal derived geometry. In this section, we overview the language, and in the following section, we overview the relationship with derived geometry.

We use these ideas in the following settings.

- We use the Chevalley-Eilenberg complex to construct a large class of factorization algebras, via the *factorization envelope* of a sheaf of dg Lie algebras. This class includes the observables of free field theories and the Kac-Moody vertex algebras.
- We use the Lie-theoretic approach to deformation functors to motivate our approach to classical field theory.
- We introduce the notion of a *local* Lie algebra to capture the symmetries of a field theory and prove generalizations of Noether's theorem.

We also use Lie algebras in the construction of gauge theories in the usual way.

A.3.1. A quick review of homological algebra with ordinary Lie algebras. Let $\mathbb{K}$ be a field of characteristic zero. (We always have in mind $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.) A *Lie algebra* over $\mathbb{K}$ is a vector space $\mathfrak{g}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g} \to \mathfrak{g}$ such that

- $[x, y] = -[y, x]$ (antisymmetry) and

- $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ (Jacobi rule).

The Jacobi rule is the statement that $[x,-]$ acts as a derivation. A simple example is the space of $n \times n$ matrices $M_n(\mathbb{K})$, usually written as $\mathfrak{gl}_n$ in this context, where the bracket is

$$[A,B] = AB - BA,$$

using the commutator with the usual matrix multiplication on the right hand side. Another classic example is given by $Vect(M)$, the vector fields on a smooth manifold M, via the commutator bracket.

A *module* over $\mathfrak{g}$, or *representation* of $\mathfrak{g}$, is a vector space M with a bilinear map $\rho : \mathfrak{g} \otimes_{\mathbb{K}} M \to M$ such that

$$\rho(x \otimes \rho(y \otimes m)) - \rho(y \otimes \rho(x \otimes m)) = \rho([x,y] \otimes m).$$

Usually, we will suppress the notation ρ and simply write $x \cdot m$ or $[x,m]$. Continuing with the examples from above, the matrices $\mathfrak{gl}_n$ acts on $\mathbb{K}^n$ by left multiplication, so $\mathbb{K}^n$ is naturally a $\mathfrak{gl}_n$-module. Analogously, vector fields $Vect(M)$ act on smooth functions $C^\infty(M)$ as derivations, and so $C^\infty(M)$ is a $Vect(M)$-module.

There is a category $\mathfrak{g} - mod$ whose objects are $\mathfrak{g}$-modules and whose morphisms are the natural structure-preserving maps. To be explicit, a map $f \in \mathfrak{g} - mod(M,N)$ of $\mathfrak{g}$-modules is a linear map $f : M \to N$ such that $[x, f(m)] = f([x,m])$ for every $x \in \mathfrak{g}$ and every $m \in M$.

Lie algebra homology and cohomology arise as the derived functors of two natural functors on the category of $\mathfrak{g}$-modules. We define the *invariants* as the functor

$$\begin{array}{ccc} \mathfrak{g} - mod & \to & Vect_{\mathbb{K}} \\ M & \mapsto & M^{\mathfrak{g}} \end{array}$$

where $M^{\mathfrak{g}} = \{m \mid [x,m] = 0 \ \forall x \in \mathfrak{g}\}$. A nonlinear analog is taking the fixed points of a group action on a set. The *coinvariants* is the functor

$$\begin{array}{ccc} \mathfrak{g} - mod & \to & Vect_{\mathbb{K}} \\ M & \mapsto & M_{\mathfrak{g}} \end{array}$$

where $M_{\mathfrak{g}} = M/\mathfrak{g}M = M/\{[x,m] \mid x \in \mathfrak{g}, \ m \in M\}$. A nonlinear analog is taking the quotient, or orbit space, of a group action on a set (i.e., the collection of orbits).

To define the derived functors, we rework our constructions into the setting of modules over *associative* algebras so that we can borrow the *Tor* and *Ext* functors. The *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ is

$$U\mathfrak{g} = Tens(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y])$$

where $Tens(\mathfrak{g}) = \sum_{n \geq 0} \mathfrak{g}^{\otimes n}$ denotes the tensor algebra of $\mathfrak{g}$. Note that the ideal by which we quotient ensures that the commutator in $U\mathfrak{g}$ agrees with the bracket in $\mathfrak{g}$: for all $x, y \in$

$\mathfrak{g}$,

$$x \cdot y - y \cdot x = [x,y],$$

where $\cdot$ denotes multiplication in $U\mathfrak{g}$. It is straightforward to verify that there is an adjunction

$$U : LieAlg_{\mathbb{K}} \leftrightarrows AssAlg_{\mathbb{K}} : Forget,$$

where $Forget(A)$ views an associative algebra A over $\mathbb{K}$ as a vector space with bracket given by the commutator of its product. As a consequence, we can view $\mathfrak{g} - mod$ as the category of left $U\mathfrak{g}$-modules, which we'll denote $U\mathfrak{g} - mod$, without harm.

Now observe that $\mathbb{K}$ is a *trivial* $\mathfrak{g}$-module for any Lie algebra $\mathfrak{g}$: $x \cdot k = 0$ for all $x \in \mathfrak{g}$ and $k \in \mathbb{K}$. Moreover, $\mathbb{K}$ is the quotient of $U\mathfrak{g}$ by the ideal $(\mathfrak{g})$ generated by $\mathfrak{g}$ itself, so that $\mathbb{K}$ is a *bi*module over $U\mathfrak{g}$. It is then straightforward to verify that

$$M_{\mathfrak{g}} = \mathbb{K} \otimes_{U\mathfrak{g}} M \quad \text{and} \quad M^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(\mathbb{K}, M)$$

for every module M.

A.3.1.1 Definition. *For M a $\mathfrak{g}$-module, the* Lie algebra homology *of M is*

$$H_*(\mathfrak{g}, M) = \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{K}, M),$$

and the Lie algebra cohomology *of M is*

$$H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U\mathfrak{g}}(\mathbb{K}, M).$$

As usual, there are concrete interpretations of the lower (co)homology groups, in terms of derivations and extensions. Notice that H^0 and H_0 recover invariants and coinvariants, respectively.

There are standard cochain complexes for computing Lie algebra (co)homology, and their generalizations will play a large role throughout the book. The key is to find an efficient, tractable resolution of $\mathbb{K}$ as a $U\mathfrak{g}$ module. We use again that $\mathbb{K}$ is a quotient of $U\mathfrak{g}$:

$$(\cdots \to \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \to \cdots \to \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \to U\mathfrak{g}) \xrightarrow{\simeq} \mathbb{K},$$

where the final map is given by the quotient and the remaining maps are of the form

$$\begin{aligned}(y_1 \wedge \cdots \wedge y_n) \otimes (x_1 \cdots x_m) \mapsto & \sum_{k=1}^{n} (-1)^{n-k} (y_1 \wedge \cdots \widehat{y_k} \cdots \wedge y_n) \otimes (y_k \cdot x_1 \cdots x_m) \\ & - \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} ([y_j, y_k] \wedge y_1 \wedge \cdots \widehat{y_j} \cdots \widehat{y_k} \cdots \wedge y_n) \otimes (x_1 \cdots x_m),\end{aligned}$$

where the hat $\widehat{y_k}$ indicates removal. As this is a free resolution of $\mathbb{K}$, we use it to compute the relevant Tor and Ext groups: for coinvariants we have

$$\mathbb{K} \otimes^{\mathbb{L}}_{U\mathfrak{g}} M \simeq (\cdots \to \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} M \to \cdots \to \mathfrak{g} \otimes_{\mathbb{K}} M \to M)$$

and for invariants we have

$$\mathbb{R}\operatorname{Hom}_{U\mathfrak{g}}(\mathbb{K}, M) \simeq (M \to \mathfrak{g}^\vee \otimes_{\mathbb{K}} M \to \cdots \to \wedge^n \mathfrak{g}^\vee \otimes_{\mathbb{K}} M \to \cdots),$$

where $\mathfrak{g}^\vee = \operatorname{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathbb{K})$ is the linear dual. These resolutions were introduced by Chevalley and Eilenberg and so their names are attached.

A.3.1.2 Definition. *The* Chevalley-Eilenberg complex for Lie algebra homology *of the $\mathfrak{g}$-module M is*

$$C_*(\mathfrak{g}, M) = (\operatorname{Sym}_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the bracket of $\mathfrak{g}$ on itself and on M. Explicitly, we have

$$d(x_1 \wedge \cdots \wedge x_n \otimes m) = \sum_{1 \leq j < k \leq n} (-1)^{j+k} [x_j, x_k] \wedge x_1 \wedge \cdots \widehat{x_j} \cdots \widehat{x_k} \cdots \wedge x_n \otimes m$$
$$+ \sum_{j=1}^{n} (-1)^{n-j} x_1 \wedge \cdots \widehat{x_j} \cdots \wedge x_n \otimes [x_j, m].$$

We often call this complex the Chevalley-Eilenberg chains.

The Chevalley-Eilenberg complex for Lie algebra cohomology *of the $\mathfrak{g}$-module M is*

$$C^*(\mathfrak{g}, M) = (\operatorname{Sym}_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the linear dual to the bracket of $\mathfrak{g}$ on itself and on M. Fixing a linear basis $\{e_k\}$ for $\mathfrak{g}$ and hence a dual basis $\{e^k\}$ for $\mathfrak{g}^\vee$, we have

$$d(e^k \otimes m) = -\sum_{i<j} e^k([e_i, e_j]) e^i \wedge e^j \otimes m + \sum_l e^k \wedge e^l \otimes [e_l, m]$$

and we extend d to the rest of the complex as a derivation of cohomological degree 1 (i.e., use the Leibniz rule repeatedly to reduce to the case above). We often call this complex the Chevalley-Eilenberg cochains.

When M is the trivial module $\mathbb{K}$, we simply write $C_*(\mathfrak{g})$ and $C^*(\mathfrak{g})$. It is important for us that $C^*(\mathfrak{g})$ is a commutative dg algebra and that $C_*(\mathfrak{g})$ is a cocommutative dg coalgebra, as a little work shows. This property suggests a geometric interpretation of the Chevalley-Eilenberg complexes: under the philosophy that every commutative algebra should be interpreted as the functions on some "space," we view $C^*(\mathfrak{g})$ as "functions on a space $B\mathfrak{g}$" and $C_*(\mathfrak{g})$ as "distributions on $B\mathfrak{g}$." Here we interpret the natural pairing between the two complexes as providing the pairing between functions and distributions. This geometric perspective on the Chevalley-Eilenberg complexes motivates the role of Lie algebras in deformation theory, as we explain in the following section.

A.3.1.1. *References.* Weibel [Wei94] contains a chapter on the homological algebra of ordinary Lie algebras, of which we have given a gloss. In [Lura], Lurie gives an efficient treatment of this homological algebra in the language of model and infinity categories.

A.3.2. Dg Lie algebras and L_∞ algebras. We now quickly extend and generalize homologically the notion of a Lie algebra. Our base ring will now be a commutative algebra R over a characteristic zero field $\mathbb{K}$, and we encourage the reader to keep in mind the simplest case: where $R = \mathbb{R}$ or $\mathbb{C}$. Of course, one can generalize the setting considerably, with a little care, by working in a symmetric monoidal category (with a linear flavor); the cleanest approach is to use operads.

Before introducing L_∞ algebras, we treat the simplest homological generalization.

A.3.2.1 Definition. *A* dg Lie algebra *over R is a $\mathbb{Z}$-graded R-module $\mathfrak{g}$ such that*

(1) *there is a differential*

$$\cdots \xrightarrow{d} \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0 \xrightarrow{d} \mathfrak{g}^1 \to \cdots$$

making $(\mathfrak{g}, d)$ into a dg R-module;

(2) *there is a bilinear bracket $[-,-] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that*
   - $[x,y] = -(-1)^{|x||y|}[y,x]$ *(graded antisymmetry),*
   - $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$ *(graded Leibniz rule),*
   - $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$ *(graded Jacobi rule),*

   where $|x|$ denotes the cohomological degree of $x \in \mathfrak{g}$.

In other words, a dg Lie algebra is an algebra over the operad *Lie* in the category of dg R-modules. In practice — and for the rest of the section — we require the graded pieces $\mathfrak{g}^k$ to be projective R-modules so that we do not need to worry about the tensor product or taking duals.

Here are several examples.

(a) Let (V, d_V) be a cochain complex over $\mathbb{K}$. Then $\mathrm{End}(V) = \oplus_n \mathrm{Hom}^n(V,V)$, the graded vector space where Hom^n denotes the linear maps that shift degree by n, becomes a cochain complex via the differential

$$d_{\mathrm{End}\, V} = [d_V, -] : f \mapsto d_V \circ f - (-1)^{|f|} f \circ d_V.$$

The commutator bracket makes $\mathrm{End}(V)$ a dg Lie algebra over $\mathbb{K}$. When $V = \mathbb{K}^n$, this example simply reduces to $\mathfrak{gl}_n$.

(b) For M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra (such as $su(2)$), the tensor product $\Omega^*(M) \otimes_{\mathbb{R}} \mathfrak{g}$ is a dg Lie algebra where the differential is simply the exterior derivative and the bracket is

$$[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x,y].$$

We can view this dg Lie algebra as living over $\mathbb{K}$ or over the commutative dg algebra $\Omega^*(M)$. This example appears naturally in the context of gauge theory.

(c) For X a simply-connected topological space, let $\mathfrak{g}_X^{-n} = \pi_{1+n}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and use the Whitehead product (or bracket) to provide the bracket. Then $\mathfrak{g}_X$ is a dg Lie algebra with zero differential. This example appears naturally in rational homotopy theory.

We now introduce a generalization where we weaken the Jacobi rule on the brackets in a systematic way. After providing the (rather convoluted) definition, we sketch some motivations.

A.3.2.2 Definition. *An L_∞ algebra over R is a $\mathbb{Z}$-graded, projective R-module $\mathfrak{g}$ equipped with a sequence of multilinear maps of cohomological degree $2-n$*

$$\ell_n : \underbrace{\mathfrak{g} \otimes_R \cdots \otimes_R \mathfrak{g}}_{n \ \text{times}} \to \mathfrak{g},$$

with $n = 1, 2, \ldots$, satisfying the following properties.

(1) Each bracket ℓ_n is graded-antisymmetric, so that

$$\ell_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -(-1)^{|x_i||x_{i+1}|}\ell_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

for every n-tuple of elements and for every i between 1 and $n-1$.

(2) Each bracket ℓ_n satisfies the n-Jacobi rule, so that

$$0 = \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1 < \cdots < i_k \\ j_{k+1} < \cdots < j_n \\ \{i_1, \ldots, j_n\} = \{1, \ldots, n\}}} (-1)^{\varepsilon}\ell_{n-k+1}(\ell_k(x_{i_1}, \ldots, x_{i_k}), x_{j_{k+1}}, \ldots, x_{j_n}).$$

Here $(-1)^{\varepsilon}$ denotes the following sign for the permutation

$$\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_{k+1} & \cdots & j_n \end{pmatrix}$$

acting on the element $x_1 \otimes \cdots \otimes x_n$: the sign arises from the alternating-Koszul sign rule, where the transposition $ab \mapsto ba$ acquires sign $-(-1)^{|a||b|}$.

For small values of n, we recover familiar relations. For example, the 1-Jacobi rule says that $\ell_1 \circ \ell_1 = 0$. In other words, ℓ_1 is a differential! Momentarily, let's denote ℓ_1 by d and ℓ_2 by the bracket $[-,-]$. The 2-Jacobi rule then says that

$$-[dx_1, x_2] + [dx_2, x_1] + d[x_1, x_2] = 0,$$

which encodes the graded Leibniz rule. Finally, the 3-Jacobi rule rearranges to

$$\begin{aligned} &[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] \\ &\qquad = d\ell_3(x_1, x_2, x_3) + \ell_3(dx_1, x_2, x_3) + \ell_3(dx_2, x_3, x_1) + \ell_3(dx_3, x_1, x_2). \end{aligned}$$

In other words, $\mathfrak{g}$ does not satisfy the usual Jacobi rule *on the nose* but the failure is described by the other brackets. In particular, at the level of cohomology, the usual Jacobi rule *is* satisfied.

Example: There are numerous examples of L_∞ algebras throughout the book, but many are simply dg Lie algebras spiced with analysis. We describe here a small, algebraic example of interest in topology and elsewhere (see, for instance, [**?**], [**?**], [**?**]). The *String Lie 2-algebra* $string(n)$ is the graded vector space $so(n) \oplus \mathbb{R}\beta$, where β has degree 1 — thus $string(n)$ is concentrated in degrees 0 and 1 — equipped with two nontrivial brackets:

$$\ell_2(x,y) = \begin{cases} [x,y], & x,y \in so(n) \\ 0, & x = \beta \end{cases}$$

$$\ell_3(x,y,z) = \mu(x,y,z)\beta \quad x,y,z \in so(n),$$

where μ denotes $\langle -, [-,-]\rangle$, the canonical (up to scale) 3-cocycle on $so(n)$ arising from the Killing form. This L_∞ algebra arises as a model for the "Lie algebra" of $String(n)$, which itself appears in various guises (as a topological group, as a smooth 2-group, or as a more sophisticated object in derived geometry). ◊

There are two important cochain complexes associated to an L_∞ algebra, which generalize the two Chevalley-Eilenberg complexes we defined earlier.

A.3.2.3 Definition. *For* $\mathfrak{g}$ *an* L_∞ *algebra, the* Chevalley-Eilenberg complex for homology $C_*\mathfrak{g}$ *is the dg cocommutative coalgebra*

$$\mathrm{Sym}_R(\mathfrak{g}[1]) = \bigoplus_{n=0}^{\infty} \left((\mathfrak{g}[1])^{\otimes n}\right)^{S_n}$$

equipped with the coderivation d *whose restriction to cogenerators* $d_n : \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ *are precisely the higher brackets* ℓ_n*.*

Remark: The coproduct $\Delta : C_*\mathfrak{g} \to C_*\mathfrak{g} \otimes_R C_*\mathfrak{g}$ is just the natural way one can "break a monomial into two smaller monomials." Namely,

$$\Delta(x_1 \cdots x_n) = \sum_{\sigma \in S_n} \sum_{1 \le k \le n-1} (x_{\sigma(1)} \cdots x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots x_{\sigma(n)}).$$

A coderivation respects the coalgebra analog of the Leibniz property, and so it is determined by its behavior on cogenerators. ◊

This coalgebra $C_*\mathfrak{g}$ conveniently encodes all the data of the L_∞ algebra $\mathfrak{g}$. The coderivation d puts all the brackets together into one operator, and the fact that $d^2 = 0$ encodes all the higher Jacobi relations. It also allows for a concise definition of a map between L_∞ algebras.

A.3.2.4 Definition. *A* map of L_∞ algebras $F : \mathfrak{g} \to \mathfrak{h}$ *is given by a map of dg cocommutative coalgebras* $F : C_*\mathfrak{g} \to C_*\mathfrak{h}$*.*

Note that a map of L_∞ algebras is *not* determined just by its behavior on $\mathfrak{g}$. Unwinding the definition above, one discovers that it is necessary to specify a linear map $\mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{h}$ for each n, satisfying some compatibility conditions.

To define the other Chevalley-Eilenberg complex $C^*\mathfrak{g}$, we use the graded linear dual of $\mathfrak{g}$,

$$\mathfrak{g}^\vee = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_R(\mathfrak{g}^n, R)[n],$$

which is the natural notion of dual in this context.

A.3.2.5 Definition. *For* $\mathfrak{g}$ *an* L_∞ *algebra, the* Chevalley-Eilenberg complex for cohomology $C^*\mathfrak{g}$ *is the dg commutative algebra*

$$\widehat{\mathrm{Sym}}_R(\mathfrak{g}[1]^\vee) = \prod_{n=0}^{\infty} \left((\mathfrak{g}[1]^\vee)^{\otimes n}\right)_{S_n}$$

equipped with the derivation d *whose Taylor coefficients* $d_n : \mathfrak{g}[1]^\vee \to \mathrm{Sym}^n(\mathfrak{g}[1]^\vee)$ *are dual to the higher brackets* ℓ_n*.*

We emphasize that this dg algebra is *completed* with respect to the filtration by powers of the ideal generated by $\mathfrak{g}[1]^\vee$. This filtration will play a crucial role in the setting of deformation theory.

A.3.3. References. We highly recommend **[?]** for an elegant and efficient treatment of L_∞ algebras (over $\mathbb{K}$) (and simplicial sets and also how these constructions fit together with deformation theory). The book of Kontsevich and Soibelman [**KS**] provides a wealth of examples, motivation, and context.

A.4. Derived deformation theory

In physics, one often studies very small perturbations of a well-understood system, wiggling an input infinitesimally or deforming an operator by a small amount. Asking questions about how a system behaves under small changes is ubiquitous in mathematics, too, and there is an elegant formalism for such problems in the setting of algebraic geometry, known as *deformation theory*. Here we will give a very brief sketch of *derived* deformation theory, where homological ideas are mixed with classical deformation theory.

A major theme of this book is that perturbative aspects of field theory — both classical and quantum — are expressed cleanly and naturally in the language of derived deformation theory. In particular, many constructions from physics, like the the Batalin-Vilkovisky formalism, obtain straightforward interpretations. Moreover, derived deformation theory suggests how to rephrase standard results in concise, algebraic terms and also suggests how to generalize these results substantially (see, for instance, the discussion on Noether's theorem).

In this section, we begin with a quick overview of formal deformation theory in algebraic geometry. We then discuss its generalization in derived algebraic geometry. Finally, we explain the powerful relationship between deformation theory and L_∞ algebras, which we exploit throughout the book.

A.4.1. The formal neighborhood of a point. Let $\mathcal{S}$ denote some category of spaces, such as smooth manifolds or complex manifolds or schemes. The Yoneda lemma implies we can understand any particular space $X \in \mathcal{S}$ by understanding how other spaces $Y \in \mathcal{S}$ map into X. That is, the functor represented by X,

$$\begin{array}{rccc} h_X : & \mathcal{S}^{op} & \to & Sets \\ & Y & \mapsto & \mathcal{S}(Y,X) \end{array} ,$$

knows everything about X as an $\mathcal{S}$ type of space. We call h_X the *functor of points of* X, and this functorial perspective on geometry will guide our work below. Although abstract at first acquaintance, this perspective is especially useful for thinking about general features of geometry.

Suppose we want to describe what X looks like near some point $p \in X$. Motivated by the perspective of functor of points, we might imagine describing "X near p" by some kind of functor. The input category ought to capture all possible "small neighborhoods of a point" permitted in $\mathcal{S}$, so that we can see how such models map into X near p. We now make this idea precise in the setting of algebraic geometry.

Let $\mathcal{S} = Sch_{\mathbb{C}}$ denote the category of schemes over $\mathbb{C}$. Every such scheme X consists of a topological space X_{top} equipped with a sheaf of commutative $\mathbb{C}$-algebras $\mathscr{O}_X$ (satisfying various conditions we will not specify). We interpret the algebra $\mathscr{O}_X(U)$ on the open set U as the "algebra of functions on U." Every commutative $\mathbb{C}$-algebra R determines a scheme Spec R where the prime ideals of R provide the set for underlying topological space and where the stalk of $\mathscr{O}$ at a prime ideal $\mathfrak{P}$ is precisely the localization of R with respect to $R - \mathfrak{P}$. We call such a scheme Spec R an *affine scheme*. By definition, every scheme admits an open cover by affine schemes.

It is a useful fact that the functor of points h_X of a scheme X is determined by its behavior on the subcategory $Aff_{\mathbb{C}}$ of affine schemes. By construction, $Aff_{\mathbb{C}}$ is the opposite category to $CAlg_{\mathbb{C}}$, the category of commutative $\mathbb{C}$-algebras. Putting these facts together, we know that every scheme X provides a functor from $CAlg_{\mathbb{C}}$ to $Sets$.

Example: Consider the polynomial $q(x,y) = x^2 + y^2 - 1$. The functor

$$\begin{array}{rccc} h_X : & CAlg_{\mathbb{C}} & \to & Sets \\ & R & \mapsto & \{(a,b) \in R^2 \mid 0 = q(a,b) = a^2 + b^2 - 1\} \end{array}$$

corresponds to the affine scheme $\operatorname{Spec} S$ for the algebra $S = \mathbb{C}[x,y]/(q)$. This functor simply picks out solutions to q in the algebra R, which we might call the "unit circle" in R^2. It should be clear how any system of polynomials (or ideal in an algebra) defines a similar functor of "solutions to the system of equations." ♢

Example: Consider the scheme SL_2, viewed as the functor

$$\begin{array}{rccc} SL_2 : & CAlg_{\mathbb{C}} & \to & Sets \\ & R & \mapsto & \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in R \text{ such that } 1 = ad - bc \right\} \end{array}$$

where $CAlg_{\mathbb{C}}$ denotes the category of the commutative $\mathbb{C}$-algebras. Note that $SL_2(\mathbb{C})$ is precisely the set that we usually mean. One can check as well that this functor factors through the category of groups. ♢

The notion of "point" in this category is given by $\operatorname{Spec}\mathbb{C}$, which is the locally ringed space given by a one-point space $\{*\}$ equipped with $\mathbb{C}$ as its algebra of functions. A *point in the scheme* X is then a map $p : \operatorname{Spec}\mathbb{C} \to X$.[1] Every point is contained in some affine patch $U \cong \operatorname{Spec} R \subset X$, so it suffices to understand points in affine schemes. It is now possible to provide an answer to the question, "What are the affine schemes that look like small thickenings of a point?"

A.4.1.1 Definition. *A commutative* $\mathbb{C}$*-algebra* A *is* artinian *if* A *is finite-dimensional as a* $\mathbb{C}$*-vector space. A* local algebra A *with unique maximal ideal* $\mathfrak{m}$ *is artinian if and only if there is some integer* n *such that* $\mathfrak{m}^n = 0$.

Any local artinian algebra $(A, \mathfrak{m})$ provides a scheme $\operatorname{Spec} A$ whose underlying topological space is a point but whose scheme structure has "infinitesimal directions" in the sense that every function $f \in \mathfrak{m}$ is "small" because $f^n = 0$. Let $Art_{\mathbb{C}}$ denote the category of local artinian algebras, which we will view as the category encoding "small neighborhoods of a point."[2]

A point $p : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} R$ corresponds to a map of algebras $P : R \to \mathbb{C}$. Every local artinian algebra $(A, \mathfrak{m})$ has a distinguished map $Q : A \to A/\mathfrak{m} \cong \mathbb{C}$. Given a point p in $\operatorname{Spec} R$, we obtain a functor

$$\begin{array}{rccc} h_p : & Art_{\mathbb{C}} & \to & Sets \\ & (A, \mathfrak{m}) & \mapsto & \{F : R \to A \mid P = Q \circ F\} \end{array} .$$

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[1] As practice in translation, try phrasing this notion using the functor of points perspective.

[2] Hopefully it seems reasonable to choose $Art_{\mathbb{C}}$ as a model for "small neighborhoods of a point." There are other approaches imaginable but this choice is quite useful. In particular, the most obvious topology for schemes — the Zariski topology — is quite coarse, so that open sets are large and hence do not reflect the idea of "zooming in near the point." Instead, we use schemes whose space is just a point but have interesting but tractable algebra.

Geometrically, this condition on ϕ means p is the composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A \overset{\operatorname{Spec} F}{\to} \operatorname{Spec} R$. The map F thus describes some way to "extend infinitesimally" away from the point p in X.

A concrete example is in order. Our favorite point in SL_2 is given by the identity element $\mathbb{1}$. Let $h_{\mathbb{1}}$ denote the associated functor of artinian algebras. We can describe the tangent space $T_{\mathbb{1}}SL_2$ using it, as follows. Consider the artinian algebra $\mathbb{D} = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, often called the *dual numbers*. Then

$$\begin{aligned} h_{\mathbb{1}}(\mathbb{D}) &= \left\{ M = \begin{pmatrix} 1+s\varepsilon & t\varepsilon \\ u\varepsilon & 1+v\varepsilon \end{pmatrix} \;\middle|\; \begin{matrix} s,t,u,v \in \mathbb{C} \text{ and} \\ 1 = (1+s\varepsilon)(1+v\varepsilon) - tu\varepsilon^2 = 1+(s+v)\varepsilon \end{matrix} \right\} \\ &\cong \{N \in M_2(\mathbb{C}) \mid \operatorname{Tr} N = 0\} \\ &= \mathfrak{sl}_2(\mathbb{C}), \end{aligned}$$

where the isomorphism is given by $M = \mathbb{1} + \varepsilon N$. Thus, we have recovered the underlying set of the Lie algebra.

For any point p in a scheme X, the set $h_p(\mathbb{D})$ is the tangent space to p in X. By considering more complicated artinian algebras, one can study the higher order jets at p. We say that h_p describes the *formal neighborhood* of p in X. The following proposition motivates this terminology.

A.4.1.2 Proposition. *Let $P : R \to \mathbb{C}$ be a map of algebras (i.e., we have a point $p : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} R$). Then the functor $h_p : Art_{\mathbb{C}} \to Sets$ is given by*

$$A \mapsto CAlg_{\mathbb{C}}(\widehat{R}_p, A),$$

where

$$\widehat{R}_p = \varprojlim R/\mathfrak{m}_p^n$$

is the completed local ring given by the inverse limit over powers of $\mathfrak{m}_p = \ker P$, the maximal ideal given by the functions vanishing at p.

In other words, the functor h_p is not represented by an artinian algebra (unless R is artinian), but it is represented inside the larger category $CAlg_{\mathbb{C}}$. When R is noetherian, the ring $\widehat{R}_p$ is given by an inverse system of artinian algebras, so we say h_p is *pro*-represented. When R is a regular ring (such as a polynomial ring over $\mathbb{C}$), $\widehat{R}_p$ is isomorphic to formal power series. This important example motivates the terminology of *formal neighborhood*.

There are several properties of such a functor h_p we would like to emphasize, as they guide our generalization in the next section. First, by definition, $h_p(\mathbb{C})$ is simply a point, namely the point p. Second, we can study h_p in stages, by a process we call *artinian induction*. Observe that every artinian algebra $(A, \mathfrak{m})$ is equipped with a natural filtration

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n = 0.$$

Thus, every artinian algebra can be constructed iteratively by a sequence of *small* extensions, namely a short exact sequence of vector spaces

$$I \hookrightarrow B \xrightarrow{f} A$$

where $f : B \to A$ is a map of algebras and I is an ideal in B such that $\mathfrak{m}_B I = 0$. We can thus focus on understanding the maps $h_p(f) : h_p(B) \to h_p(A)$, which are simpler to analyze. In summary, h_p is completely determined by how it behaves with respect to small extensions.

A third property is categorical in nature. Consider a pullback of artinian algebras

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

and note that $B \times_A C$ is artinian as well. Then the natural map

$$h_p(B \times_A C) \to h_p(B) \times_{h_p(A)} h_p(C)$$

is surjective — in fact, it is an isomorphism. (This property will guide us in the next subsection.)

As an example, we describe how to study small extensions for the model case. Let $(R, \mathfrak{m}_R)$ be a complete local ring with residue field $R/\mathfrak{m}_R \cong \mathbb{C}$ and with finite-dimensional *tangent space* $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$. Consider the functor $h_R : A \mapsto CAlg(R, A)$, which describes the formal neighborhood of the closed point in Spec R. The following proposition provides a tool for understanding the behavior of h_R on small extensions.

A.4.1.3 Proposition. *For every small extension*

$$I \hookrightarrow B \xrightarrow{f} A,$$

there is a natural exact sequence of sets

$$0 \to T_R \otimes_{\mathbb{C}} I \to h_R(B) \xrightarrow{f\circ -} h_R(A) \xrightarrow{ob} O_R \otimes I,$$

where exact *means that a map $\phi \in h_R(A)$ lifts to a map $\tilde{\phi} \in h_R(B)$ if and only if $ob(\phi) = 0$ and the space of liftings is an affine space for the vector space $T_R \otimes_{\mathbb{C}} I$.*

Here *ob* denotes the *obstruction* to lifting maps, and O_R is a set where an obstruction lives. An obstruction space O_R only depends on the algebra R, not on the small extension. One can construct an obstruction space as follows. If $d = \dim_{\mathbb{C}} T_R$, there is a surjection of algebras

$$r : S = \mathbb{C}[[x_1, \ldots, x_d]] \to R$$

such that $J = \ker r$ satisfies $J \subset \mathfrak{m}_S^2$, where $\mathfrak{m}_S = (x_1, \ldots, x_d)$ is the maximal ideal of S. In other words, Spec R can be embedded into the formal neighborhood of the origin in $\mathbb{A}^d$, and minimally, in some sense. Then O_R is $(J/\mathfrak{m}_S J)^\vee$. For a proof of the proposition, see chapter 6 of [FGI+05].

This proposition hints that something homotopical lurks behind the scenes, and that the exact sequence of sets is the truncation of a longer sequence. For a discussion of these ideas and the modern approach to deformation theory, we highly recommend the 2010 ICM talk of Lurie [Lur10].

A.4.1.1. *References.* The textbook of Eisenbud and Harris [EH00] is a lovely introduction to the theory of schemes, full of examples and motivation. There is an extensive discussion of the functor of points approach to geometry, carefully compared to the locally ringed space approach. For an introduction to deformation theory, we recommend the article of Fantechi and Göttsche in [FGI+05]. Both texts provide extensive references to the literature.

A.4.2. Formal moduli spaces.

The functorial perspective on algebraic geometry suggests natural generalizations of the notion of a scheme by changing the source and target categories. For instance, stacks arise as functors from $CAlg_{\mathbb{C}}$ to the category of groupoids, allowing one to capture the idea of a space "with internal symmetries." It is fruitful to generalize even further, by enhancing the source category from commutative algebras to dg commutative algebras (or simplicial commutative algebras) and by enhancing the target category from sets to simplicial sets. (Of course, one needs to simultaneously adopt a more sophisticated version of category theory, namely ∞-category theory.) This generalization is the subject of derived algebraic geometry, and much of its power arises from the fact that it conceptually integrates geometry, commutative algebra, and homotopical algebra. As we try to show in this book, the viewpoint of derived geometry provides conceptual interpretations of constructions like Batalin-Vilkovisky quantization.

We now outline the derived geometry version of studying the formal neighborhood of a point. Our aim to pick out a class of functors that capture our notion of a formal derived neighborhood.

A.4.2.1 Definition. *An* artinian dg algebra *A is a dg commutative algebra over $\mathbb{C}$ such that*

1. *each component A^k is finite-dimensional,* $\dim_{\mathbb{C}} A^k = 0$ *for $k << 0$ and for $k > 0$, and*
2. *A has a unique maximal ideal $\mathfrak{m}$, closed under the differential, and $A/\mathfrak{m} = \mathbb{C}$.*

Let $dgArt_{\mathbb{C}}$ denote the category of artinian algebras, where morphisms are simply maps of dg commutative algebras.

Note that, as we only want to work with local rings, we simply included it as part of the definition. Note as well that we require A to be concentrated in nonpositive degrees. (This second condition is related to the Dold-Kan correspondence: we want A to correspond to a simplicial commutative algebra.)

We now provide an abstract characterization of a functor that behaves like the formal neighborhood of a point, motivated by our earlier discussion of functors h_p.

A.4.2.2 Definition. *A formal moduli problem is a functor*

$$F : dgArt_{\mathbb{C}} \to sSets$$

such that

(1) $F(\mathbb{C})$ is contractible,
(2) F sends a surjection of dg artinian algebras to a fibration of simplicial sets, and
(3) for every pullback diagram in $dgArt$

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

the map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is a weak homotopy equivalence.

Note that since surjections go to fibrations, the strict pullback $F(B) \times_{F(A)} F(C)$ agrees with the homotopy pullback $F(B) \times^h_{F(A)} F(C)$.

We now describe a large class of examples. Let R be a commutative dg algebra over $\mathbb{C}$ whose underlying graded algebra is $\widehat{\mathrm{Sym}} V$, where V is a $\mathbb{Z}$-graded vector space, and whose differential d_R is a degree 1 derivation. It has a unique maximal ideal generated by V. Let h_R denote the functor into simplicial sets whose n-simplices are

$$h_R(A)_n = \{f : R \to A \otimes \Omega^*(\triangle^n) \mid f \text{ a map of unital dg commutative algebras}\}$$

and whose structure maps arise from those between the de Rham complexes of simplices.

A.4.2.1. *References.* We are modeling our approach on Lurie's, as explained in his ICM talk [Lur10] and his paper on deformation theory [Lura]. For a discussion of these ideas in our context of field theory, see [Cos].

A.4.3. The role of L_∞ algebras in deformation theory. There is another algebraic source of formal moduli functors — L_∞ algebras — and, perhaps surprisingly, formal moduli functors arising in geometry often manifest themselves in this form. We begin by introducing the Maurer-Cartan equation for an L_∞ algebra $\mathfrak{g}$ and how it provides a formal moduli functor. *This construction is at the heart of our approach to classical field theory.* We then describe several examples from geometry and algebra.

A.4.3.1 Definition. *Let $\mathfrak{g}$ be an L_∞ algebra. The* Maurer-Cartan equation *(or MC equation) is*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0,$$

where α denotes a degree 1 element of $\mathfrak{g}$.

Note that when we consider the dg Lie algebra $\Omega^*(M) \otimes \mathfrak{g}$, with M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra, the MC equation becomes the equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

for a flat $\mathfrak{g}$-connection $\alpha \in \Omega^1 \otimes \mathfrak{g}$ on the trivial principal G-bundle on M. (This is the source of the name Maurer-Cartan.)

There are two other perspectives on the MC equation. First, observe that a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to \mathbb{C}$ is determined by its behavior on the generators $\mathfrak{g}^\vee[-1]$ of the algebra $C^*\mathfrak{g}$. Hence $\underline{\alpha}$ is a linear functional of degree 0 on $\mathfrak{g}^\vee[-1]$ — or, equivalently, a degree 1 element α of $\mathfrak{g}$ — that commutes with differentials. This condition $\underline{\alpha} \circ d = 0$ is precisely the MC equation for α. The second perspective uses the coalgebra $C_*\mathfrak{g}$, rather than the algebra $C^*\mathfrak{g}$. A solution to the MC equation α is equivalent to giving a map of cocommutative dg coalgebras $\tilde{\alpha} : \mathbb{C} \to C_*\mathfrak{g}$.

Now observe that L_∞ algebras behave nicely under base change: if $\mathfrak{g}$ is an L_∞ algebra over $\mathbb{C}$ and A is a commutative dg algebra over $\mathbb{C}$, then $\mathfrak{g} \otimes A$ is an L_∞ algebra (over A and, of course, $\mathbb{C}$). Solutions to the MC equation go along for the ride as well. For instance, a solution α to the MC equation of $\mathfrak{g} \otimes A$ is equivalent to both a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to A$ and a map of cocommutative dg coalgebras $\tilde{\alpha} : A^\vee \to C_*\mathfrak{g}$. Again, we simply unravel the conditions of such a map restricted to (co)generators. As maps of algebras compose, for instance, solutions play nicely with base change. Thus, we can construct a functor out of the MC solutions.

A.4.3.2 Definition. *The* Maurer-Cartan functor *of an L_∞ algebra $\mathfrak{g}$*

$$\mathrm{MC}_\mathfrak{g} : dgArt_\mathbb{C} \to sSets$$

sends $(A, \mathfrak{m})$ to the simplicial set whose n-simplices are solutions to the MC equation in $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\triangle^n)$.

We remark that tensoring with the nilpotent ideal $\mathfrak{m}$ makes $\mathfrak{g} \otimes \mathfrak{m}$ is nilpotent. This condition then ensures that the simplicial set $\mathrm{MC}_\mathfrak{g}(A)$ is a Kan complex **[?]** **[?]**. In fact, their work shows the following.

A.4.3.3 Theorem. *The Maurer-Cartan functor* $\mathrm{MC}_{\mathfrak{g}}$ *is a formal moduli problem.*

In fact, every formal moduli problem is represented — up to a natural notion of weak equivalence — by the MC functor of an L_∞ algebra [**?**].

A.4.3.1. *References.* For a clear, systematic introduction with an expository emphasis, we highly recommend Manetti's lecture [**Man09**], which carefully explains how dg Lie algebras relate to deformation theory and how to use them in algebraic geometry. The unpublished book [**KS**] contains a wealth of ideas and examples; it also connects these ideas to many other facets of mathematics. The article of Hinich [**?**] is the original published treatment of derived deformation theory, and it provides one approach to necessary higher category theory. For the relation with L_∞ algebras, we recommend [**?**], which contains elegant arguments for many of the ingredients too. Finally, see Lurie's [**Lura**] for a proof that every formal moduli functor is described by a dg Lie algebra.

A.5. Sheaves, cosheaves, and their homotopical generalizations

Sheaves appear throughout geometry and topology because they capture the idea of gluing together local data to obtain something global. *Cosheaves* are an equally natural construction that are nonetheless used less frequently. We will give a very brief discussion of these ideas. As we always work with sheaves and cosheaves of a linear nature, we give definitions in that setting.

A.5.1. Sheaves.

A.5.1.1 Definition. *A* presheaf *of vector spaces on a space X is a functor* $\mathcal{F} : Opens_X^{op} \to Vect_{\mathbb{K}}$ *where* $Opens_X$ *is the category encoding the partially ordered set of open sets in X (i.e., the objects are open sets in X and there is a map from U to V exactly when* $U \subset V$*).*

In other words, a presheaf $\mathcal{F}$ assigns a vector space $\mathcal{F}(U)$ to each open U and a *restriction map* $res_{V \supset U} : \mathcal{F}(V) \to \mathcal{F}(U)$ whenever $U \subset V$. Bear in mind the following two standard examples. The *constant presheaf* $\mathcal{F} = \underline{\mathbb{K}}$ has $\underline{\mathbb{K}}(U) = \mathbb{K}$ (hence it assigns the same vector space to every open) and its restriction map is always the obvious isomorphism. The *presheaf of continuous functions* C_X^0 assigns the vector space $C_X^0(U)$ of continuous functions from U to $\mathbb{R}$ (or $\mathbb{C}$, as one prefers) and the restriction map $res_{V \supset U}$ consists precisely of restricting a continuous function from V to a smaller open U.

A sheaf is a presheaf whose value on a big open is determined by its behavior on small opens.

A.5.1.2 Definition. *A* sheaf *of vector spaces on a space X is a presheaf $\mathcal{F}$ such that for every open U and every cover $\mathfrak{U} = \{V_i\}_{i\in I}$ of U, we have*

$$\mathcal{F}(U) \xrightarrow{\cong} \lim\left(\prod_{i\in I} \mathcal{F}(V_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(V_i \cap V_j)\right),$$

where the map out of $\mathcal{F}(U)$ is the product of the restriction maps from U to the V_i and where, in the limit diagram, the top arrow is restriction from V_i to $V_i \cap V_j$ and the bottom arrow is restriction from V_j to $V_i \cap V_j$.

This *gluing condition* says precisely that an element $s \in \mathcal{F}(U)$, called a *section of $\mathcal{F}$ on U*, is given by sections on the cover, $(s_i \in \mathcal{F}(V_i))_{i\in I}$, that agree on the overlapping opens $V_i \cap V_j$. It captures in a precise way how to reconstruct $\mathcal{F}$ on big opens in terms of data on a cover.

It is a good exercise to verify that C^0_X is always a sheaf and that $\underline{\mathbb{K}}$ is *not* a sheaf on a disconnected space.

In this book, our spaces are nearly always smooth manifolds, and most of our sheaves arise in the following way. Let $E \to X$ be a vector bundle on a smooth manifold. Let $\mathscr{E}$ denote the presheaf where $\mathscr{E}(U)$ is the vector space of smooth sections $E|_U \to U$ on the open set U. It is quick to show that $\mathscr{E}$ is a sheaf.

A.5.2. Cosheaves.

We now discuss the dual notion of a *cosheaf*.

A.5.2.1 Definition. *A* precosheaf *of vector spaces on a space X is a functor $\mathcal{G} : Opens_X \to Vect$. A* cosheaf *is a precosheaf such that for every open U and every cover $\mathfrak{U} = \{V_i\}_{i\in I}$ of U, we have*

$$\operatorname{colim}\left(\coprod_{i,j\in I} \mathcal{G}(V_i \cap V_j) \rightrightarrows \coprod_{i\in I} \mathcal{G}(V_i)\right) \xrightarrow{\cong} \mathcal{G}(U),$$

where the map into $\mathcal{G}(U)$ is the coproduct of the extension *maps from the V_i to U and where, in the colimit diagram, the top arrow is extension from $V_i \cap V_j$ to V_i and the bottom arrow is extension from $V_i \cap V_j$ to V_j.*

The crucial example of a cosheaf (for us) is the functor $\mathscr{E}_c$ that assigns to the open U, the vector space $\mathscr{E}_c(U)$ of *compactly-supported* smooth sections of E on U. If $U \subset V$, we can *extend* a section $s \in \mathscr{E}_c(U)$ to a section $ext_{U\subset V}(s) \in \mathscr{E}_c(V)$ on V by setting it equal to zero on $V \setminus U$.

A closely related example is the cosheaf of compactly-supported distributions dual to the sheaf $\mathscr{E}$ of smooth sections of a bundle. When $\mathscr{E}$ denotes the sheaf of fields for a field theory, this cosheaf describes the linear observables on the fields and, importantly, organizes them by their support.

A.5.3. Homotopical versions. Often, we want our sheaves or cosheaves to take values in categories of a homotopical nature. For instance, we might assign a cochain complex to each open set, rather than a mere vector space. In this homotopical setting, one typically works with a modified version of the gluing axioms above. The modifications are twofold:

(1) we work with all finite intersections of the opens in the cover (i.e., not just overlaps $V_i \cap V_j$ but also $V_{i_1} \cap \cdots \cap V_{i_n}$), and
(2) we use the *homotopy* limit (or colimit).

The first modification is straightforward — and familiar to anyone who has seen a Čech complex — but the second is more subtle. (We highly recommend [Dug] for an introduction and development of these notions.)

In our main examples, we give explicit complexes that encode the relevant information.

For completeness' sake, a *homotopy cosheaf* with values in dg vector spaces is a functor $\mathcal{G} : Opens_X \to dgVect$ such that for every open U and every cover $\mathfrak{U} = \{V_i\}_{i\in I}$ of U, we have a quasi-isomorphism

$$\mathrm{Tot}^{oplus}\left(\bigoplus_{n=0}^{\infty} \bigoplus_{\vec{i}\in I^{n+1}} \mathcal{G}(\cap_{j=0}^{n} V_{i_j})[n]\right) \xrightarrow{\simeq} \mathcal{G}(U),$$

where the total differential on the left hand side is the sum of the internal differentials of $\mathcal{G}(V_{\vec{i}})$ and a differential that takes the alternating sum of the structure maps. We say that $\mathcal{G}$ satisfies Čech (co)descent.

A.5.3.1. *References.* Nearly any modern book on differential or algebraic geometry contains an introduction to sheaves. See, for instance, [GH94] or [Ram05]. For a clear exposition (and more) of cosheaves, see [Cur].

A.6. Elliptic complexes

Classical field theory involves the study of systems of partial differential equations (or generalizations), which is an enormously rich and sophisticated subject. We focus in this book on a tractable and well-understood class of PDE that appear throughout differential geometry: elliptic complexes. Here we will spell out the basic definitions and some examples, as these suffice for our work in this book. (In the development of substantial examples, deeper aspects of the theory of elliptic complexes will undoubtedly appear.)

We start with a local description of differential operators before homing in on elliptic operators. Let U be an open set in $\mathbb{R}^n$. A linear *differential operator* is an $\mathbb{R}$-linear map

$L : C^\infty(U) \to C^\infty(U)$ of the form

$$L(f) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha f(x),$$

where we use the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ to efficiently denote

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

where the coefficients a_α are smooth functions, and where only finitely many of the a_α are nonzero functions. We call $|\alpha| = \sum_j \alpha_j$ the *order* of the index, and thus the *order of* L is the maximum order $|\alpha|$ among the nonzero coefficients a_α. (For example, the order of the Laplacian $\sum_j \partial^2/\partial x_j^2$ is two.) The *principal symbol* (or *leading symbol*) of a kth order differential operator L is the "fiberwise polynomial"

$$\sigma_L(\xi) = \sum_{|\alpha|=k} a_\alpha(x) i^k \xi^\alpha$$

obtained by summing over the indices of order k and replacing the partial derivative $\partial/\partial x_j$ by variables $i\xi_j$, where i is the usual square root of one. (It is the standard convention to include the factor of i, due to the role of Fourier transforms in motivating many of these constructions and definitions.) It is natural to view the principal symbol as a function on the cotangent bundle T^*U that is a homogeneous polynomial of degree k along the cotangent fibers, where ξ_j is the linear functional dual to dx_j. The principal symbol controls the qualitative behavior of L. It also behaves nicely under changes of coordinates, transforming as a section of the bundle $\mathrm{Sym}^k(T_U) \to U$.

Remark: Elsewhere in the text, we talk about differential operators in a more abstract, homological setting. For instance, we say the BV Laplacian is a second-order differential operator. Our use of the differential operators in this homological setting is inspired by their use in analysis, but the definitions are modified, of course. For instance, we view the odd directions as geometric in the BV formalism, so the BV Laplacian is second-order.

It is straightforward to extend these notions to a smooth manifold and to smooth sections of vector bundles on that manifold. Let $E \to X$ be a rank m vector bundle and let $F \to X$ be a rank n vector bundle. Let $\mathscr{E}$ denote the smooth global sections of E and let $\mathscr{F}$ denote the smooth global sections of F. A *differential operator* from E to F is an $\mathbb{R}$-linear map $L : \mathscr{E} \to \mathscr{F}$ such that for a local choice of coordinates on X and trivializations of E and F,

$$L(s)_i = \sum_{\alpha \in \mathbb{N}^n} a_\alpha^{ij}(x) \partial^\alpha s_j(x),$$

where $s = (s_1, \ldots, s_m)$ is a section of $\mathscr{E}(U) \cong C^\infty(U)^m$ on a small neighborhood U in X, $Ls = ((Ls)_1, \ldots, (Ls)_n)$ is a section of $\mathscr{F}(U) \cong C^\infty(U)^n$ on that small neighborhood, and where the $a_\alpha^{ij}(x)$ are smooth functions. In other words, we have a matrix-valued differential operator locally. The *principal symbol* σ_L of a kth order differential operator L

defines a section of the bundle $\mathrm{Sym}^k(T_X) \otimes E^\vee \otimes F \to X$. Thus, by pulling back along the canonical projection $\pi : T^*X \to X$, we can view the principal symbol as a map of *vector bundles* $\sigma_L : \pi^*E \to \pi^*F$ over the cotangent bundle T^*X.

A.6.0.1 Definition. *An* elliptic operator $L : \mathscr{E} \to \mathscr{F}$ *is a differential operator whose principal symbol is* $\sigma_L : \pi^*E \to \pi^*F$ *is an isomorphism of vector bundles on* $T^*X \setminus X$*, the cotangent bundle with the zero section removed.*

This definition says that the principal symbol is an *invertible* linear operator after evaluating at every nonzero covector. As an example, consider the Laplacian on $\mathbb{R}^n$: its principal symbol is $\sum \xi_j^2$, which only vanishes when all the ξ_j are zero.

Ellipticity is purely local and is an easy property to check in practice. Globally, it has powerful consequences, of which the following is the most famous.

A.6.0.2 Theorem. *For X a closed manifold (i.e., compact and boundaryless), an elliptic operator $Q : \mathscr{E} \to \mathscr{F}$ is Fredholm, so that its kernel and cokernel are finite-dimensional vector spaces.*

Thus, an elliptic operator on a closed manifold is invertible up to a finite-dimensional "error." Moreover, there is a rich body of techniques for constructing these partial inverses, especially for classical operators such as the Laplacian.

The notions from above extend naturally to cochain complexes. A *differential complex* on a manifold X is a $\mathbb{Z}$-graded vector bundle $\oplus_n E^n \to X$ (with finite total rank) and a differential operator $Q^n : \mathscr{E}^n \to \mathscr{E}^{n+1}$ for each integer n such that $Q^{n+1} \circ Q^n = 0$. We typically denote this by $(\mathscr{E}, Q)$. There is an associated *principal symbol complex* (π^*E, σ_Q) on T^*X by taking the principal symbol of each operator Q^n.

A.6.0.3 Definition. *An* elliptic complex *is a differential complex whose principal symbol complex is exact on* $T^*X \setminus X$ *(i.e., the cohomology of the symbol complex vanishes away from the zero section of the cotangent bundle).*

Every elliptic operator $Q : \mathscr{E} \to \mathscr{F}$ defines a two-term elliptic complex

$$\cdots \to 0 \to \mathscr{E} \xrightarrow{Q} \mathscr{F} \to 0 \to \cdots .$$

The other standard examples of elliptic complexes are the de Rham complex $(\Omega^*(X), d)$ of a smooth manifold and the Dolbeault complex $(\Omega^{0,*}(X), \bar{\partial})$ of a complex manifold.

The analog of the Fredholm result above is the following, sometimes known as the formal Hodge theorem (by analogy to the Hodge theorem, which is usually focused on the de Rham complex).

A.6.0.4 Theorem. *Let $(\mathscr{E}, Q)$ be an elliptic complex on a closed manifold X. Then the cohomology groups $H^k(\mathscr{E}, Q)$ are finite-dimensional vector spaces.*

In proving these theorems, one constructs a partial inverse with nice, geometric properties. We will not give a full definition here (because we do not want to delve into pseudodifferential operators) but will state the important properties.

Recall that every continuous linear operator $F : \mathscr{E} \to \mathscr{F}$ between smooth sections of a vector bundles possesses a Schwartz kernel, K_F, a section of the bundle $(E^\vee \otimes \mathrm{Dens}) \boxtimes F$ on $X \times X$ that is distributional along the first copy of X (i.e., in the E direction) and smooth along the second copy of X (i.e., in the F direction). Then

$$F(s)(x) = \int_{y \in X} s(y) K_F(y, x).$$

If the kernel K_F is a smooth over all of $X \times X$, then F is called a *smoothing* operator. For X a closed manifold, a smoothing operator F is compact (viewed as an operator between Sobolev space completions of the smooth sections).

The Fredholm operators are stable under compact perturbations: if T is Fredholm and C is compact, then $T + C$ is also Fredholm. In particular, $1 + C$ is always Fredholm.

For $Q : \mathscr{E} \to \mathscr{F}$ an elliptic operator, a *parametrix* is an operator $P : \mathscr{F} \to \mathscr{E}$ satisfying

(1) $\mathbb{1}_{\mathscr{E}} - PQ = S$ for some smoothing operator S,
(2) $\mathbb{1}_{\mathscr{F}} - QP = T$ for some smoothing operator T, and
(3) the Schwartz kernel of P is smooth away from the diagonal $X \subset X \times X$.

Thus, a parametrix P for Q is a partial or approximate inverse to Q. Armed with a parametrix P, we know that Q is Fredholm. The theory of pseudodifferential operators provides a construction of such a parametrix for any elliptic operator.

We want to explain how to generalize the parametrix approach to elliptic *complexes*. The essential idea to make a contraction of the elliptic complex $\mathscr{E}$ onto its cohomology $H^*\mathscr{E}$ (viewed as a complex with zero differential),

$$\begin{tikzcd} \mathscr{E} \arrow[loop left, "\eta"] \arrow[r, shift left, "\pi"] & H^*\mathscr{E} \arrow[l, shift left, "\iota"] \end{tikzcd},$$

with some special properties. Recall that a retraction means that ι includes the cohomology as a subcomplex, π projects away everything but cohomology, and η is a degree -1 map on $\mathscr{E}$, such that these maps satisfy

$$\mathbb{1}_{H^*\mathscr{E}} = \pi \circ \iota \quad \text{and} \quad \mathbb{1}_{\mathscr{E}} - \iota \circ \pi = [Q, \eta] = Q\eta + \eta Q.$$

Thus, a contraction is a homological generalization of a partial inverse.

We want the contractions for elliptic complexes to have two important properties. First, we require all the maps above to be continuous with respect to the usual Fréchet topologies. Second, the operator $C = \iota \circ \pi$ should be a cochain map that is smoothing. This property ensures that $\mathbb{1}_{\mathscr{E}} - C$ is Fredholm.

The existence of such a contraction implies the theorem. The operator $C : \mathscr{E} \to \mathscr{E}$ sends any cocycle s to a distinguished representative $C(s)$ of its cohomology class. In fact, $C^2 = C$, so it is a projection operator whose image is isomorphic to the cohomology of $\mathscr{E}$. Moreover, C annihilates exact cocycles (since π does) and elements that are not cocycles. As $\mathbb{1}_{\mathscr{E}} - C$ is a projection operator, we know the kernel of this operator is finite-dimensional, and so the image of C is finite-dimensional.

Finding the necessary contraction η exploits the existence of parametrices for elliptic operators. In the situations relevant to this book, the general idea is simple. First, one finds a differential operator Q^* of degree -1 such that the commutator $D = [Q, Q*]$ is a generalized Laplacian (i.e., its principal symbol looks like that of a Laplacian).[3] In our setting, the existence of such a Q^* is a hypothesis. This operator D is Fredholm by the earlier theorem, so we know it has finite-dimensional kernel.

Second, let G denote the parametrix for D (where G is for "Green's function"). Set $\eta = Q^* G$. We need to show it a contracting homotopy with the desired properties. Let

$$\mathbb{1} - DG = S \quad \text{and} \quad \mathbb{1} - GD = T,$$

where S and T are smoothing endomorphisms of cohomological degree 0. Note that

$$QD = QQ^*Q = DQ,$$

so

$$G(QD)G = G(DQ)G \Longrightarrow (GQ)(\mathbb{1} - S) = (\mathbb{1} - T)(QG).$$

Hence we find $GQ = QG - U$, where $U = QS - TQG$ is smoothing because Q is a differential operator. In consequence,

$$\begin{aligned} [Q, \eta] &= QQ^*G + Q^*GQ \\ &= QQ^*G + Q^*QG - Q^*U \\ &= DG - Q^*U \\ &= \mathbb{1} - (S + Q^*U). \end{aligned}$$

The final term in parantheses is smoothing. We have verified that η has the desired properties.

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[3] To do this, one can pick inner products on the bundles E^j (hermitian, if complex bundles) and a Riemannian metric on X. This provides an adjoint to Q, i.e., a differential operator Q^* of degree -1.

A.6.1. References. As usual, Atiyah and Bott [**?**] explain beautifully the essential ideas of elliptic complexes and pseudodifferential techniques and show how to use them efficiently. For an accessible development of the analytic methods in the geometric setting, we recommend [**Wel08**]. The full story and much more is available in the classic works of Hörmander [**Hör03**].

APPENDIX B

Homological algebra with differentiable vector spaces

The factorization algebras we consider take values in vector spaces of an analytical nature, like the space of smooth functions on a manifold. We would thus like to perform homological algebra in this setting. The standard approach to working with objects of this nature is to treat them as topological vector spaces. However, it is not completely obvious how one should set up homological algebra when using topological vector spaces. It is also not straightforward to construct the topology on vector spaces which appear in our most important examples of factorization algebras: the observables of a quantum field theory.

Thus we will work with a weaker and more flexible concept, that of *differentaible vector space*. This Appendix develops homological algebra in the category of differentiable vector spaces. Related approaches to functional analysis are developed in [**Pau10**] and in [**KM97**].

B.1. Diffeological vector spaces

Let us remind the reader of the concept of diffeological space [**Sta11**].

B.1.0.1 Definition. *The site of smooth manifolds is the site whose objects are smooth manifolds, morphisms are smooth maps, and where a map $M \to N$ is an open covering if it is a surjective local diffeomorphism.*

A diffeological space X *is a sheaf of sets on the site of smooth manifolds with the property that, for all smooth manifolds M, the map*

$$X(M) \to \mathrm{Hom}_{Sets}(M, X(*))$$

is injective. (On the right hand side, $X(*)$ is the value of X on a point.)

A map of diffeological spaces *is a map of sheaves of sets on the smooth site.*

We will sometimes refer to maps of diffeological spaces as smooth maps, to distinguish them from maps of the underlying sets.

We can rewrite the axioms of a diffeological space in more explicit terms, as follows. A diffeological space is determined by a set $X = X(*)$, and for each smooth manifold M, a subset $X(M) \subset \operatorname{Maps}(M, X)$ of *smooth maps* from M to X. These subsets must satisfy the following conditions. If $f : M \to X$ is a smooth map and if $g : N \to M$ is a smooth map of ordinary manifolds, then $f \circ g : N \to X$ is a smooth map. Further, a map $f : M \to X$ is smooth if and only if it is smooth locally on M. Finally, all constant maps to X are smooth. We call this collection of smooth maps the diffeology of X.

Note that if X, Y are diffeological spaces, then so is $X \times Y$: a map $M \to X \times Y$ is smooth if the composition with both projection maps is smooth.

B.1.0.2 Definition. *A* diffeological vector space *is a vector space V together with a diffeology compatible with the vector space structure. Thus, the sum map $V \times V \to V$ and the scalar multiplication map $\mathbb{R} \times V \to V$ are maps of diffeological spaces (where $\mathbb{R}$ is given the standard diffeology).*

A diffeological vector space V has enough structure to talk about smooth maps from a manifold M to V. We also want to be able to differentiate such maps. This requires extra structure.

We use the notation $C^\infty(M, V)$ to denote the $C^\infty(M)$-module of smooth maps from M to V. Thus, $C^\infty(M, V)$ is, as a vector space, just $V(M)$, equipped with the natural structure of module over the (discrete) algebra $C^\infty(M)$.

Similarly, we let

$$\Omega^k(M, V) = \Omega^k(M) \otimes_{C^\infty(M)} C^\infty(M, V)$$

denote the space of k-forms with values in V. This is just the *algebraic* tensor product. This is a reasonable thing to do because $\Omega^k(M)$ is a finitely-generated projective $C^\infty(M)$ module: it is a direct summand of a free finite rank $C^\infty(M)$-module. This implies that $\Omega^k(M, V)$ is a direct summand of $C^\infty(M, V)^{\oplus l}$ for some l.

B.1.0.3 Definition. *A* differentiable vector space *is a diffeological vector space together with, for each smooth manifold M, a flat connection*

$$\nabla_{M,V} : C^\infty(M, V) \to \Omega^1(M, V)$$

such that, for all smooth maps $f : N \to M$,

$$f^* \nabla_{M,V} = \nabla_{N,V}.$$

To say that $\nabla_{M,V}$ is a flat connection means, of course, that it satisfies the Leibniz rule,

$$\nabla_{M,V}(f \cdot s) = (\mathrm{d}f)s + f \nabla_{M,V} s,$$

and that the curvature

$$F(\nabla_{M,V}) = (\nabla_{M,V})^2 : C^\infty(M,V) \to \Omega^2(M,V)$$

vanishes.

The flat connection $\nabla_{M,V}$ allows us to differentiate smooth maps $M \to V$. If $f : M \to V$ is a smooth map and if $X \in \mathrm{Vect}(M)$ is a vector field on M, we define

$$X(f) = \langle X, \nabla_{M,V} f \rangle \in C^\infty(M,V),$$

where $\langle -, - \rangle$ indicates the $C^\infty(M)$-linear pairing

$$\mathrm{Vect}(M) \times \Omega^1(M,V) \to C^\infty(M,V).$$

Differentiable vector spaces form a category that we denote DVS. An object is a differentiable vector space V. A morphism $\phi : V \to W$ is a linear map such that for every smooth map $f : M \to V$, the map $\phi \circ f$ is smooth, and which is compatible with connections in the sense that, for all smooth manifolds M,

$$\phi \circ \nabla_{M,V} = \nabla_{M,V} \circ \phi.$$

We will often refer to morphisms of differentiable vector spaces as smooth linear maps.

Differentiable vector spaces appear naturally in geometry. In section B.2 below, we show that for M a manifold and E is a vector bundle on M, the space $C^\infty(N,E)$ of smooth sections of E has a natural structure of differentiable vector space. The same holds for the space of compactly supported or distributional sections of E. Most of our examples of differentiable vector spaces arise in this way.

Below, we will examine various properties and examples of differentiable vector spaces. Many of these constructions work equally well for diffeological vector spaces.

Remark: One of the main reasons we consider differentiable vector spaces instead of topological vector spaces is that homological algebra for sheaves of vector spaces on a site is relatively standard, whereas homological algebra for topological vector spaces is trickier.

Another reason is that, when we consider our construction of factorization algebras in families, we will only use families where the base is a smooth manifold. Differentiable vector spaces have just enough structure to talk about such families.

B.1.1. Limits and colimits.

Let V be a differentiable vector space, and let $i : W \subset V$ be a sub-vector space. The *subspace diffeology* on W is defined by saying that a map $f : M \to W$ is smooth if the composed map $i \circ f : M \to V$ is smooth. We say that the diffeological subspace W is a *differentiable subspace* if, for all smooth manifolds M, the connection $\nabla_{M,V}$ maps $C^\infty(M,W) \subset C^\infty(M,V)$ to $\Omega^1(M,W) \subset \Omega^1(M,V)$.

Differentiable subspaces have the usual universal property: if A is another differentiable vector space, a linear map $A \to W$ is smooth if and only if the composed map $A \to V$ is.

If $W \subset V$ is a differentiable subspace, then we can form the quotient V/W. A map from M to V/W is smooth if, locally on M, it lifts to a smooth map to V. The connection on V/W is uniquely determined by the requirement that the map $V \to V/W$ is compatible with connections (and so a map of differentiable spaces). We call V/W a differentiable quotient of V.

Again, this has the usual universal property: if A is another differentiable space, a linear map $V/W \to A$ is smooth if and only if the composed map $V \to A$ is smooth.

B.1.1.1 Lemma. *The category of differentiable spaces admits all products and coproducts.*

These can be described explicitly as follows. Let $\{V_i \mid i \in I\}$ be some family of differentiable spaces indexed by a set I. The product $\prod_i V_i$ differentiable space has, as underlying vector space, the product vector space $\prod V_i$. A map $M \to \prod_i V_i$ is smooth if and only if the composed maps $M \to V_i$ are smooth for all i. The connection map

$$\nabla_{M,\prod V_i} : C^\infty(M, \prod V_i) = \prod C^\infty(M, V_i) \to \Omega^1(M, \prod V_i) = \prod \Omega^1(M, V_i)$$

is the product of the connections ∇_{M,V_i}.

Similarly, the differentiable coproduct of the V_i has, as underlying vector space, the ordinary direct sum $\oplus V_i$. A map $f : M \to \oplus V_i$ is smooth if, locally on M, f can be written as a finite sum of smooth maps to some $V_{i_1}, \dots, V_{i_k}$. The connection

$$\nabla_{M,\oplus V_i} : C^\infty(M, \oplus V_i) \to \Omega^1(M, \oplus V_i)$$

is the unique connection which restricts to ∇_{M,V_i} on the subspace $C^\infty(M, V_i)$.

PROOF. We need to verify that the product and coproduct as described above have the desired universal properties. For the product, this is immediate. Let's verify it for the coproduct. Let A be another differentiable vector space. Let $f : \oplus V_i \to A$ be a linear map. Suppose that the maps $f_i : V_i \to A$ are all smooth. We need to show that f is smooth. Let $\phi : M \to \oplus V_i$ be a smooth map. To show that $f \circ \phi$ is smooth, it suffices to do so locally on M. Thus, we can assume that ϕ can be written as a finite sum of smooth maps $\phi_i : M \to V_i$. Then, $f \circ \phi$ is a finite sum of $f \circ \phi_i$, and by assumption, $f \circ \phi_i : M \to A$ are smooth. It is straightforward to verify that the fact that the maps f_i are compatible with the connections on V_i and A imply that f is compatible with connections. □

B.1.1.2 Corollary. *The category of differentiable vector spaces admits all limits (and so is complete).*

PROOF. Arbitrary limits are obtained from products and kernels.Thus, we need to verify that the category of differentiable vector spaces admits kernels.

Let $f : V \to W$ be a map of differentiable vector spaces. Let us consider the kernel $\operatorname{Ker} f \subset V$, just as an ordinary vector space. We say that a map $M \to \operatorname{Ker} f$ is smooth if and only if the composed map to V is smooth: this gives $\operatorname{Ker} f$ the subspace diffeology. Then, the sequence

$$0 \to C^\infty(M, \operatorname{Ker} f) \to C^\infty(M, V) \to C^\infty(M, W)$$

is exact.

We need to give $\operatorname{Ker} f$ a connection. Since the map $C^\infty(M, V) \to C^\infty(M, W)$ is compatible with connections, the connection $\nabla_{M,V}$ on $C^\infty(M, V)$ must map $C^\infty(M, \operatorname{Ker} f)$ to $\Omega^1(M, \operatorname{Ker} f)$.

It is easy to verify that $\operatorname{Ker} f$ satisfies the universal property of a kernel. □

Note that the forgetful functor DVS → Vect preserves all limits.

B.1.2. Cokernels and exact sequences. The category of differentiable spaces *does not* admit all cokernels. Here is the prime example. Let W be a differentiable space, and let $V \subset W$ be an arbitrary linear subspace. We equip V with the initial diffeology, by saying that the space of smooth maps $M \to V$ is the algebraic tensor product $C^\infty(M) \otimes_{alg} V$, rather than the subspace diffeology. Suppose these two diffeologies differ. The quotient W/V has a natural diffeology, by saying that a map $M \to W/V$ is smooth if locally it lifts to a smooth map to W. The fact that the sequence

$$C^\infty(M) \otimes_{alg} V = C^\infty(M, V) \to C^\infty(M, W) \to C^\infty(M, W/V) \to 0$$

is not exact means that the connection on $C^\infty(M, W)$ need not descend to one on $C^\infty(M, V/W)$.

Happily, this example is the only way that things can go wrong.

B.1.2.1 Definition. *A map $f : V \to W$ of differentiable spaces is* admissible *if, for all manifolds M and all maps $\phi : M \to \operatorname{Im} f$, the composed map $M \to W$ is smooth if and only if ϕ lifts locally to a smooth map to V.*

In other words, f is admissible if the two natural pre-diffeologies on $\operatorname{Im} f$ *(where we view it as a quotient of V or a subspace of W) coincide.*

B.1.2.2 Lemma. *Cokernels of admissible maps exist.*

PROOF. If $f : V \to W$ is an admissible map, we give $\operatorname{Coker} f$ the quotient diffeology: a map $M \to \operatorname{Coker} f$ is smooth if locally it lifts to a smooth map to W. Let $C^\infty_M(V)$ denote the sheaf on M which sends $U \subset M$ to $C^\infty(U, V)$. Then, the sequence of sheaves

$$C^\infty_M(V) \to C^\infty_M(W) \to C^\infty_M(\operatorname{Coker} f) \to 0$$

is exact. This implies that the connection on $C^\infty(M, W)$ descends uniquely to one on $C^\infty(M, \operatorname{Coker} f)$. □

Another simple class of colimits that exist is the following.

B.1.2.3 Lemma. *The category of differentiable vector spaces is closed under taking sequential colimits of injective maps.*

By injective, we just mean that the map on the underlying vector space is injective.

PROOF. Let V_i for $i \in \mathbb{Z}_{\geq 0}$ be a sequence of differentiable vector spaces, and let $f_{ij} : V_i \to V_j$ be injective maps with $f_{jk} f_{ij} = f_{ik}$. Let V denote the ordinary vector space

$$V = \operatorname{colim} V_i = \cup V_i.$$

We say a map from a smooth manifold M to V is smooth if, locally on M, it comes from a smooth map to one of the V_i. Let $C^\infty_M(V)$ denote the sheaf on M of smooth maps to V; then

$$C^\infty_M(V) = \operatorname{colim} C^\infty_M(V_i).$$

This identification uses the fact that the maps in our directed system are injective. Recall also that the colimit in the category of sheaves is defined to be the sheafification of the colimit in the category of presheaves.

Now, we define the flat connection $\nabla_{M,V}$ to be the map of sheaves

$$\nabla_{M,V} : C^\infty_M(V) \to \Omega^1_M(V)$$

which arises as the colimit of the maps of sheaves

$$C^\infty_M(V_i) \to \Omega^1_M(V_i).$$

□

B.1.2.4 Definition. *A sequence* $0 \to A \to B \to C \to 0$ *of differentiable spaces is* exact *if it is exact as a sequence of ordinary vector spaces, $A \subset B$ is a differentiable subspace, and C is a differentiable quotient.*

Equivalently, the sequence is exact if A is the kernel of the map $B \to C$ and C is the cokernel of the map $A \to B$.

Let V be a differentiable vector space. By evaluating V on open subsets of $\mathbb{R}^n$, V becomes a sheaf on $\mathbb{R}^n$. We can thus define the *stalk*

$$\operatorname{Stalk}_n(V) = \operatorname{colim}_{0 \in U \subset \mathbb{R}^n} V(U)$$

of V at the origin in $\mathbb{R}^n$. The colimit above is taken over open subsets of $\mathbb{R}^n$ containing the origin.

Note that the stalk of V at a point in any manifold can be defined in the same way, but the stalk at a point in a n-dimensional manifold is the same as the stalk at the origin in $\mathbb{R}^n$.

B.1.2.5 Lemma. *A sequence of differentiable vector spaces* $0 \to A \to B \to C \to 0$ *is exact if and only if, for all* n*, the sequence*

$$0 \to \mathrm{Stalk}_n(A) \to \mathrm{Stalk}_n(B) \to \mathrm{Stalk}_n(C) \to 0$$

of vector spaces is exact.

PROOF. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of differentiable vector spaces. Then, for any manifold M, the sequence $0 \to C^\infty(M, A) \to C^\infty(M, B) \to C^\infty(M, C)$ is exact. Further, a map $M \to C$ is smooth if locally on M it lifts to a smooth map to B. Thus, if $C^\infty_M(B)$ denotes the sheaf on M of smooth maps to B, the sequence

$$0 \to C^\infty_M(A) \to C^\infty_M(B) \to C^\infty_M(C) \to 0 \tag{†}$$

of sheaves on M is exact. This implies that the corresponding sequence on stalks is exact.

Conversely, if the sequence of stalks is exact for all n, then the sequence (†) of sheaves is exact for all manifolds M. This implies that the sequence

$$0 \to \Omega^1_M(A) \to \Omega^1_M(B) \to \Omega^1_M(C) \to 0$$

is also exact, where we define

$$\Omega^1_M(A) = \Omega^1_M \otimes_{C^\infty_M} C^\infty_M(A).$$

It follows that the connection $\nabla_{M,C}$ on C is the unique connection which descends from that on B, and that the connection on A is the restriction of the connection on B to A. These are the conditions we imposed for A to be a differentiable subspace of B and for C to be a differentiable quotient. ☐

B.1.3. The multicategory structure. We will consider the category of differentiable vector spaces as a multicategory, instead of a symmetric monoidal category.

B.1.3.1 Definition. *If* $V_1, \dots, V_k, W$ *are differentiable vector spaces, then a* smooth multilinear map

$$\phi : V_1 \times \cdots \times V_k \to W$$

is a multilinear map with the following properties.

(1) It is smooth: if $f_i : M \to V_i$ *are smooth maps from a manifold* M*, then*

$$\phi(f_1, \dots, f_k) : M \to W$$

is a smooth map.

(2) *It is compatible with flat connections: if* $f_i : M \to V$ *are smooth and if* X *is a vector field on* M*, then*

$$\nabla_X \phi(f_1, \dots, f_k) = \sum_i \phi(f_1, \dots, \nabla_X f_i, \dots, f_k).$$

Differentiable vector spaces form a multicategory where the multi-morphisms $\mathrm{Hom}(V_1, \dots V_k; W)$ *are smooth multilinear maps.*

B.1.4. In general, we can not tensor differentiable vector spaces together. Nonetheless, certain tensor products arise naturally and we will use them repeatedly.

First, we can tensor a differentiable vector space V with the algebra of smooth functions on a manifold: we use the notation

$$C^\infty(M) \otimes V = C^\infty(M, V)$$

when V is a differentiable vector space.

Similarly, if E is a vector bundle on M, then $C^\infty(M, E)$ is a projective module over $C^\infty(M)$. Thus, it's reasonable to form the algebraic tensor product

$$C^\infty(M, E) \otimes_{C^\infty(M)} C^\infty(M, V).$$

We interpret the output as a kind of completed tensor product of V with $C^\infty(M, E)$. We will use the notation $C^\infty(M, E \otimes V)$ to denote this tensor product. This notation is natural: we can tensor a differentiable vector space with a finite-dimensional vector space, so that $E \otimes V$ can be thought of as a bundle of differentiable vector spaces on M.

B.1.4.1 Lemma. *Let* E, F *be vector bundles on* M*. Let* $D : C^\infty(M, E) \to C^\infty(M, F)$ *be a differential operator. Let* V *be a differentiable vector space. Then, the map*

$$D \otimes 1 : C^\infty(M, E) \otimes_{alg} V \to C^\infty(M, F) \otimes_{alg} V$$

extends canonically to a map

$$C^\infty(M, E \otimes V) \to C^\infty(M, F \otimes V).$$

PROOF. Let's start with the case when E and F are trivial of rank 1. Then we are asserting that differential operators on M act naturally on $C^\infty(M, V)$. This action arises in the standard way from the connection $\nabla_{M,V} : C^\infty(M, V) \to \Omega^1(M, V)$ which we are given as part of the structure of a differentiable vector space.

In the case when E and F are non-trivial, one constructs the desired map in local trivializations and then verifies that it is independent of the choice of local trivializations. This is standard. □

Here is a formal way to describe these structures. Let $\mathcal{C}$ denote the following symmetric monoidal category: the objects are smooth manifolds M equipped with a vector bundle E; a morphism $(M, E) \to (N, F)$ is a smooth map $f : M \to N$ together with a differential operator $C^\infty(M, f^*F) \to C^\infty(M, E)$; and the tensor product is defined by

$$(M, E) \otimes (N, F) = (M \times N, E \boxtimes F).$$

Then, the category of differentiable vector spaces DVS is tensored over $\mathcal{C}^{op}$.

B.2. Differentiable vector spaces from sections of a vector bundle

In this section, we will describe various classes of sections of a vector bundle on a manifold M and show that each is equipped with a natural diffeology and flat connection. There are various ways to express this construction; we begin in a more geometric language and then rephrase in the language of topological vector spaces.

The most basic example is as follows. Let M be a manifold, and let E be a vector bundle on M. Then the space

$$\mathscr{E} = \Gamma(M, E)$$

of smooth sections of E is a differentiable vector space. We let

$$\mathscr{E}_c = \Gamma_c(M, E)$$

be the space of compactly supported smooth sections of E on M.

B.2.0.2 Definition. *Equip $\mathscr{E}$ with the diffeology where, for N a smooth manifold, a smooth map $f : N \to \mathscr{E}$ is a section of the pullback bundle $\pi_M^* E$ on $N \times M$ arising from the projection map $\pi_M : N \times M \to M$.*

Similarly, give $\mathscr{E}_c$ a diffeology by saying that a smooth map $N \to \mathscr{E}_c$ is a section s of $\pi_M^* E$ on $N \times M$ with the property that the map $\operatorname{Supp}(s) \to N$ is proper (where $\operatorname{Supp}(s)$ is the closure of the locus on which s is non-vanishing).

Note that the spaces $\mathscr{E}$ and $\mathscr{E}_c$ are complete locally-convex topological vector spaces, using the standard topologies for these spaces. As is explained in [KM97], for example, one has a notion of smooth map $N \to V$ for any manifold N and for any such topological vector space V. Thus, V defines a diffeological space. The diffeologies described above on $\mathscr{E}$ and $\mathscr{E}_c$ arise from the standard topologies on these spaces.

Notice as well that $C^\infty(N, \mathscr{E})$ is the vector space given by the completed projective tensor product $C^\infty(N) \otimes \mathscr{E}$, which is their natural tensor product as nuclear spaces.

Next, we explain the flat connections on the spaces $\mathscr{E}$ and $\mathscr{E}_c$.

B.2.0.3 Definition. *Let N be a smooth manifold. Equip the pullback bundle $\pi_M^* E$ on $N \times M$ with the natural flat connection along the fibers of the projection map $\pi_M : N \times M \to M$. We thus obtain a map*

$$\nabla_{N,\mathscr{E}} : \Gamma(N \times M, \pi_M^* E) \to \Gamma(N \times M, T^*N \boxtimes E)$$

or, equivalently, a map

$$\nabla_{N,\mathscr{E}} : C^\infty(N, \mathscr{E}) \to \Omega^1(N, \mathscr{E}).$$

This defines the flat connection on $C^\infty(N, \mathscr{E})$ and so gives $\mathscr{E}$ the structure of a differentiable vector space.

This flat connection preserves the subspace $C^\infty(N, \mathscr{E}_c)$, giving $\mathscr{E}_c$ the structure of a differentiable vector space.

B.2.1. We are also interested in distributional sections of a vector bundle E. Let $\mathcal{D}(M)$ denote the space of distributions on M, that is, the continuous dual of the space $C_c^\infty(M)$. Let $\mathcal{D}_c(M)$ denote the space of compactly supported distributions on M, which is the continuous dual of $C^\infty(M)$.

We let

$$\overline{\mathscr{E}}(M) = \mathscr{E}(M) \otimes_{C^\infty(M)} \mathcal{D}(M)$$

be the space of distributional sections of E. (These are sections whose coefficients are distributions rather than functions.) We let

$$\overline{\mathscr{E}}_c(M) = \mathscr{E}_c(M) \otimes_{C_c^\infty(M)} \mathcal{D}_c(M)$$

be the space of compactly supported distributional sections of E.

B.2.1.1 Definition. *Equip the space $\mathcal{D}(M)$ of distributions on M with the diffeology where a smooth map $N \to \mathcal{D}(M)$ is a continuous linear map $C_c^\infty(M) \to C^\infty(N)$. Similarly, give $\mathcal{D}_c(M)$ a diffeology by saying that a smooth map $N \to \mathcal{D}_c(M)$ is a continuous linear map $C^\infty(M) \to C^\infty(N)$.*

Equip $\overline{\mathscr{E}}(M)$ with the diffeology in which the vector space of smooth maps $N \to \overline{\mathscr{E}}(M)$ is

$$C^\infty(N, \overline{\mathscr{E}}(M)) = C^\infty(N, \mathscr{E}(M)) \otimes_{C^\infty(M \times N)} C^\infty(N, \mathcal{D}(M))$$

(where we use the notation $C^\infty(N, V)$ to indicate the space of smooth maps from N to a diffeological vector space V).

Similarly, give $\overline{\mathscr{E}}_c(M)$ a diffeology by saying that

$$C^\infty(N, \overline{\mathscr{E}}_c(M)) = C^\infty(N, \mathscr{E}_c(M)) \otimes_{C^\infty(N, C_c^\infty(M))} C^\infty(N, \mathcal{D}_c(M)).$$

Remark: The diffeologies we have defined on these spaces of distributions again arise from the standard topology on these spaces. In particular, note that

$$C^\infty(N, \mathcal{D}(M)) = C^\infty(N) \otimes \mathcal{D}(M)$$

as vector spaces, where $\otimes$ denotes the completed projective tensor product. Compare this to the analogous definition of $C^\infty(N, C^\infty(M))$ from earlier.

We need to equip these diffeological vector spaces with flat connections to make them into differentiable vector spaces. There is a natural choice.

B.2.1.2 Definition. *We extend the diffeological vector space $\mathcal{D}(M)$ to a differentiable vector space as follows. There is a natural inclusion*

$$C^\infty(N, \mathcal{D}(M)) \hookrightarrow \mathcal{D}(N \times M).$$

The Lie algebra $\mathrm{Vect}(N)$ *of vector fields on N acts naturally on $\mathcal{D}(N \times M)$; this action preserves the subspace $C^\infty(N, \mathcal{D}(M))$, giving this subspace the desired flat connection.*

The action of $\mathrm{Vect}(N)$ *also preserves the smaller subspace*

$$C^\infty(N, \mathcal{D}_c(M)) \subset \mathcal{D}(N \times M)$$

and so gives $C^\infty(N, \mathcal{D}_c(M))$ the structure of differentiable vector space.

Similarly, the space $\mathcal{D}(N \times M, \pi_M^* E)$ of distributional sections on $N \times M$ of the vector bundle $\pi_M^* E$ has a natural action of vector fields on M. This preserves the subspaces $C^\infty(N, \overline{E}(M))$ and $C^\infty(N, \overline{E}_c(M))$, and gives those the structure of differentiable vector spaces.

Let $E^!$ denote the vector bundle $E \otimes \mathrm{Dens}_M$, where Dens_M denotes the bundle of densities on M. Then, as above, we can define vector spaces

$$\begin{aligned} \mathscr{E}^! &= \Gamma(M, E^!) \\ \mathscr{E}_c^! &= \Gamma_c(M, E^!). \end{aligned}$$

The vector spaces have natural diffeologies and topologies. There are natural identifications

$$\begin{aligned} \overline{\mathscr{E}}(N) &= \mathrm{Hom}_{cont}(\mathscr{E}_c^!, C^\infty(N)) \\ \overline{\mathscr{E}}_c(N) &= \mathrm{Hom}_{cont}(\mathscr{E}^!, C^\infty(N)) \end{aligned}$$

where Hom_{cont} denotes the vector space of continuous linear maps.

B.3. Differentiable vector spaces from holomorphic sections of a holomorphic vector bundle

For a complex manifold X and a holomorphic vector bundle E on X, the space of global holomorphic sections $\mathscr{O}_X(E)$ provides another important and natural example of a topological vector space arising from geometry. We now show how such spaces and their continuous duals yield differentiable vector spaces.

We explain the simplest example, as the general case is completely parallel. Let X be a complex manifold. Let $\mathscr{O}(X)$ denote the vector space of holomorphic functions on X. Recall that it has a natural topology inherited from smooth functions on X (this topology is also given by uniform convergence on compact sets).

B.3.0.3 Definition. *Equip $\mathscr{O}(X)$ with the diffeology where, for N a smooth manifold, a smooth map $f : N \to \mathscr{O}(X)$ is a smooth function on $N \times X$ such that $f_n(x) := f(n, x)$ is a holomorphic function on X for every $n \in N$.*

Alternatively, we could simply use the construction from [KM97], turning a complete locally-convex vector space into a diffeological vector space.

Note that $\mathscr{O}(X)$ is a diffeological subspace of the differentiable vector space $C^\infty(X)$. Moreover, it is preserved by the flat connection on $C^\infty(X)$ and hence inherits a differentiable structure.

Let $\mathscr{O}(X)^\vee$ denote the vector space of continuous linear functionals on $\mathscr{O}(X)$.

B.3.0.4 Definition. *Equip $\mathscr{O}(X)^\vee$ with the diffeology where, for N a smooth manifold, a smooth map $f : N \to \mathscr{O}(X)$ is a continuous linear map from $\mathscr{O}(X)$ to $C^\infty(N)$.*

This definition states that $C^\infty(N, \mathscr{O}(X))$ is the vector space $C^\infty(N) \otimes \mathscr{O}(X)$, where $\otimes$ here denotes the completed projective tensor product. The flat connection is then simply $d_N \otimes 1_{\mathscr{O}(X)}$, where d_N is the exterior derivative on N.

B.4. Differentiable cochain complexes

B.4.0.5 Definition. *A* differentiable cochain complex *is a cochain complex V where each V^i is a differentiable vector space and each differential* $\mathrm{d} : V^i \to V^{i+1}$ *is a smooth map.*

A map of differentiable cochain complexes *is simply a cochain map $V \to W$ whose constituent maps $V^i \to W^i$ are all smooth.*

A cochain homotopy *of such maps is a cochain homotopy whose constituent maps $V^i \to W^{i-1}$ are all smooth.*

We want to perform standard constructions from homological algebra with differentiable cochain complexes. Of course, we need to make sure the definitions take into account the diffeological structure.

B.4.0.6 Definition. *(1) A sequence $0 \to A \to B \to C \to 0$ of differentiable cochain complexes is* exact *if the component sequences $0 \to A^i \to B^i \to C^i \to 0$ are exact.*

(2) *A map $f : A \to B$ of differentiable cochain complexes is a* cofibration *if it fits into an exact sequences* $0 \to A \xrightarrow{f} B \to C \to 0$.
(3) *Similarly, a map $f : A \to B$ is a* fibration *if it fits into an exact sequence* $0 \to C \to A \xrightarrow{f} B \to 0$.

Note that $A \to B$ is a cofibration (respectively, fibration) if and only if, for all n, the map $\mathrm{Stalk}_n(A) \to \mathrm{Stalk}_n(B)$ is an injective (respectively, surjective) map of cochain complexes. Equivalently, the map $A \to B$ is a cofibration if in each cohomological degree, A^i is a differentiable subspace of B^i. This means that $A^i \to B^i$ is injective and that a map $M \to A^i$ is smooth if and only if the composed map to B^i is smooth. Similarly, $A \to B$ is a fibration if and only if every $A^i \to B^i$ is surjective and a map $M \to B^i$ is smooth if and only if it locally lifts to a map to A^i.

We have seen above that one can take kernels and cokernels in the category of differentiable vector spaces. Thus, we can define the cohomology groups $H^i(A)$ of any differentiable cochain complex. These are differentiable vector spaces.

B.4.0.7 Definition. *A map $A \to B$ is a* weak equivalence *if and only if, for all n, the map of cochain complexes* $\mathrm{Stalk}_n A \to \mathrm{Stalk}_n B$ *is a quasi-isomorphism.*

Standard constructions and lemmas from ordinary homological algebra hold in this setting. For example, if $f : V \to W$ is a map of differentiable cochain complexes, we can form the cone $\mathrm{Cone}(f)$, whose underlying graded differentiable space is $V[1] \oplus W$, but equipped with differential

$$\begin{pmatrix} \mathrm{d}_{V[1]} & f \\ 0 & \mathrm{d}_W \end{pmatrix}.$$

If f is a fibration, then the map

$$\mathrm{Ker}\, f[1] \to \mathrm{Cone}(f)$$

is an equivalence. If f is a cofibration, then the map

$$\mathrm{Cone}(f) \to \mathrm{Coker}(f)$$

is an equivalence.

B.4.1. We will often use versions of spectral sequence arguments in the category of differentiable complexes.

Suppose that V_i is a directed system of differentiable cochain complexes indexed by $i \in \mathbb{Z}_{\geq 0}$. Thus, we have maps $f_i : V_i \to V_{i+1}$.

Let us suppose that the maps f_i are cofibrations. Then, since the category of differentiable vector spaces is closed under colimits of cofibrant maps, we can form the differentiable cochain complex $\operatorname{colim}_i V$, which in cohomological degree k is $\operatorname{colim}_i V_i^k$.

B.4.1.1 Lemma. *Let V_*, W_* be sequential directed systems where the maps $V_i \to V_{i+1}$, $W_i \to W_{i+1}$ are cofibrations. Let $V_* \to W_*$ be a map of directed systems.*

Suppose that the maps $V_i / V_{i-1} \to W_i / W_{i-1}$ are all weak equivalences.

Then the map

$$\operatorname{colim} V_i \to \operatorname{colim} W_j$$

is a weak equivalence.

Proof. We need to verify that the maps are equivalences at the level of stalks. The forgetful functors

$$\mathrm{Stalk}_n : \mathrm{DVS} \to \mathrm{Vect}$$

commute with all colimits of cofibrations. It follows that, in the situation above,

$$\operatorname{colim} \mathrm{Stalk}_n V_i = \mathrm{Stalk}_n \operatorname{colim} V_i.$$
$$\mathrm{Stalk}_n (V_i / V_{i-1}) = \mathrm{Stalk}_n V_i / \mathrm{Stalk}_n V_{i-1}.$$

Now, $\mathrm{Stalk}_n V_i$ is a directed system of cochain complexes where the maps are injective, and likewise for $\mathrm{Stalk}_n W_i$. Therefore, by the usual spectral sequence argument, if the map

$$\mathrm{Stalk}_n V_i / \mathrm{Stalk}_n V_{i-1} \to \mathrm{Stalk}_n W_i \, \mathrm{Stalk}_n W_{i-1}$$

is a quasi-isomorphism for all n, the map

$$\operatorname{colim} \mathrm{Stalk}_n V_i \to \operatorname{colim} \mathrm{Stalk}_n W_i$$

is also a weak equivalence, giving the desired result. ☐

Similarly, we have the following.

B.4.1.2 Lemma. *Let V_*, W_* be sequential directed systems of differentiable cochain complexes, where the maps $V_i \to V_{i+1}$, $W_i \to W_{i+1}$ are cofibrations. Let $V_* \to W_*$ be a map of systems, such that the constituent maps $V_i \to W_i$ are quasi-isomorphisms. Then the map* $\operatorname{colim} V_i \to \operatorname{colim} W_i$ *is a quasi-isomorphism.*

Proof. The proof is almost identical to that of the previous lemma. ☐

We have a similar statement for inverse systems, but only under some stronger hypothesis.

B.4.1.3 Lemma. *Let V_*, W_* be sequential inverse systems of differentiable cochain complexes. Thus, there are maps $f_i : V_i \to V_{i-1}$ and $g_i : W_i \to W_{i-1}$. Suppose that these maps are fibrations and that the systems V_i, W_i are eventually constant. In other words, the maps f_i, g_i are isomorphisms for i sufficiently large.*

Let $V_* \to W_*$ be a map of inverse systems, which induces a quasi-isomorphism

$$\operatorname{Ker} f_i \to \operatorname{Ker} g_i$$

for each i.

Then the map $\varprojlim V \to \varprojlim W$ is a quasi-isomorphism.

PROOF. The functor of stalks commutes with finite limits of fibrations. The fact that the maps $V_i \to V_{i-1}$ are isomorphisms for sufficiently large $i \geq N$ thus implies that

$$\operatorname{Stalk}_n \varprojlim V_i = \varprojlim \operatorname{Stalk}_n V_i.$$

Because the maps $V_i \to V_{i-1}$ are fibrations, the maps $\operatorname{Stalk}_n V_i \to \operatorname{Stalk}_n V_j$ are surjective maps of cochain complexes. The spectral sequence argument implies that the map

$$\varprojlim \operatorname{Stalk}_n V_i \to \varprojlim \operatorname{Stalk}_n W_i$$

is a quasi-isomorphism as desired. □

B.4.2. With these definitions, we can define a factorization algebra valued in the multicategory of differentiable cochain complexes. Indeed, we have already given a general definition of factorization algebra valued in a multicategory. The definition presented here is just an exegesis of the general definition.

B.4.2.1 Definition. *A* prefactorization algebra *on a manifold M valued in the multicategory of differentiable cochain complexes is the assignment of a differentiable cochain complex $\mathcal{F}(U)$ to every open subset $U \subset M$, together with smooth multilinear cochain maps*

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

if $U_1, \ldots, U_n$ are disjoint open subsets of V, and satisfying the coherence axioms explained earlier.

Given any such prefactorization algebra $\mathcal{F}$ and any Weiss cover

$$\mathfrak{U} = \{U_i \mid i \in I\}$$

of an open set $V \subset M$, we can form the Čech complex

$$\check{C}(\mathfrak{U}, \mathcal{F}).$$

As usual, this is the direct sum

$$\check{C}(\mathfrak{U}, \mathcal{F}) = \oplus_{i_1,\ldots,i_k \subset I} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_n})[k-1]$$

of all finite intersections of elements of the open cover.

Since differentiable cochain complexes admit all coproducts, this Čech complex is again a differentiable cochain complex.

B.4.2.2 Definition. *A* differentiable factorization algebra *on M is a differentiable prefactorization algebra $\mathcal{F}$ on M with the property that, for every factorizing cover $\mathfrak{U}$ of an open subset $V \subset M$, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(V)$$

is a weak equivalence of differentiable cochain complex (as defined above).

B.5. Pro-cochain complexes

This section involves ordinary vector spaces, not differentiable vector spaces, but it prepares us for an important notion we use throughout the book.

Most of the examples of factorization algebras we construct will take values not in the category of ordinary cochain complexes but in the category of pro-cochain complexes or, equivalently, of complete filtered cochain complexes.

B.5.0.3 Definition. *A* complete filtered cochain complex *is a cochain complex V equipped with a decreasing filtration $F^i V \subset V$ by sub-cochain complexes indexed by $i \in \mathbb{Z}_{\geq 0}$, such that $F^0 V = V$ and*

$$V = \varprojlim V / F^i V.$$

A map of complete filtered cochain complexes *is a map $V \to W$ which preserves the filtration.*

Such a map is a weak equivalence *if the map $\operatorname{Gr}^i V \to \operatorname{Gr}^i W$ is a quasi-isomorphism for all i. (Note that this implies that the map $V \to W$ is a quasi-isomorphism.)*

Some care is needed when defining colimits of complete filtered cochain complexes.

B.5.0.4 Definition. *Let $\{V_\alpha \mid \alpha \in A\}$ be a collection of complete filtered cochain complexes indexed by some set A. Then the* direct sum $\bigoplus_{\alpha \in A} V_\alpha$ *is defined by*

$$\bigoplus_{\alpha \in A} V_\alpha = \varprojlim_{i \in \mathbb{Z}_{\geq 0}} \left(\bigoplus_{\alpha} V_\alpha / F^i V_\alpha \right).$$

(On the right hand side of this equation, $\bigoplus V_\alpha / F^i V_\alpha$ indicates the ordinary direct sum of cochain complexes.)

The reason for making this definition is that the filtration on the naive direct sum of the V_α is not complete. It is easy to verify that the direct sum defined above is a coproduct in the category of complete filtered cochain complexes.

Similarly, the tensor product of complete filtered cochain complexes needs to be completed.

B.5.0.5 Definition. *Let V, W be complete filtered cochain complexes. The* tensor product $V \otimes W$ *is defined as the limit*

$$V \otimes W = \varprojlim_{i,j} (V/F^i V) \otimes (V/F^j W).$$

The filtration on $V \otimes W$ is defined by

$$F^k(V \otimes W) = \operatorname*{colim}_{i+j \geq k} F^i V \otimes F^j W,$$

where the tensor product $F^i V \otimes F^j W$ is defined as the limit of $F^i V/F^r V \otimes F^j W/F^s V$.

Again, the reason for this definition is that the filtration on the naive tensor product of $V \otimes W$ is not complete.

With this definition of completed direct sum and tensor product, it is straightforward to modify our definition of factorization algebra to take values in the category of complete filtered cochain complexes.

B.5.0.6 Definition. *A* complete filtered factorization algebra *$\mathcal{F}$ on X is a prefactorization algebra $\mathcal{F}$ taking values in the symmetric monoidal category of complete filtered cochain complexes, using the tensor product described above, such that, for every factorizing open cover $\mathfrak{U}$ of an open subset U of X, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is an equivalence. The direct sums and tensor products appearing in the definition of the Čech complex are completed, as above.

B.6. Differentiable pro-cochain complexes

As a final elaboration on the concept of cochain complex, we will put together the two ideas described above.

B.6.0.7 Definition. *A* differentiable pro-cochain complex *is a differentiable cochain complex V equipped with a decreasing filtration by differentiable subcomplexes $F^i V$ with the following properties.*

1. *$F^0 V = V$.*
2. *The maps $F^i V \to F^j V$ if $i > j$ are cofibrations. This means that they are injective and that in each cohomological degree, the map $F^i V^k \to F^j V^k$ has the property that a map $M \to F^j V^k$ is smooth if it lifts to a smooth map to $F^i V^k$.*

This implies that we can form the quotient differentiable vector space V/F^iV, and that the maps

$$V/F^iV \to V/F^jV$$

are fibrations (again for $i > j$).

(3) *We require that*

$$V = \varprojlim V/F^iV.$$

A map of differentiable pro-cochain complexes *is a filtration-preserving map $V \to W$. Such a map is a weak equivalence if the maps $\mathrm{Gr}^i V \to \mathrm{Gr}^i W$ are weak equivalences of differentiable cochain complexes. Note that this implies that the maps $V/F^iV \to W/F^iW$ are weak equivalences of differentiable cochain complexes.*

A map $V \to W$ of differentiable pro-cochain complexes is a fibration *(respectively, a* cofibration*) if the map $V/F^iV \to W/F^iW$ are fibrations (cofibrations) for all i.*

As before, we need to define the completed direct sum and multilinear maps of differentiable cochain complexes.

B.6.0.8 Definition. *If $\{V_\alpha \mid \alpha \in A\}$ is a collection of complete filtered differentiable cochain complexes, indexed by some set A, then the* completed direct sum $\bigoplus_{\alpha \in A} V_\alpha$ *is defined to be the inverse limit*

$$\bigoplus\nolimits_{\alpha \in A} V_\alpha = \varprojlim_{i \in \mathbb{Z}_{\geq 0}} \bigoplus\nolimits_{\alpha \in A} \left(V_\alpha / F^i V_\alpha \right),$$

where on the right hand side we use the ordinary direct sum of differentiable spaces.

We can define the stalks of a differentiable pro-cochain complex $\mathrm{Stalk}_n(V)$ as the colimit

$$\mathrm{Stalk}_n(V) = \operatorname*{colim}_{0 \in U \subset \mathbb{R}^n} V(U),$$

where the colimit of the pro-cochain complexes $V(U)$ is completed as above. Thus, $\mathrm{Stalk}_n(V)$ is a pro-cochain complex, and

$$\mathrm{Stalk}_n(V)/F^i \,\mathrm{Stalk}_n(V) = \mathrm{Stalk}_n(V/F^iV).$$

B.6.0.9 Lemma. *A map $V \to W$ of differentiable pro-cochain complexes is an equivalence if and only if the maps $\mathrm{Stalk}_n(V) \to \mathrm{Stalk}_n(W)$ are weak equivalences of pro-cochain complexes.*

PROOF. Immediate. □

B.6.0.10 Lemma. *Let V_*, W_* be sequential directed systems of differentiable pro-cochain complexes, and let $V_* \to W_*$ be a map of directed systems.*

Suppose that the maps $V_i \to V_j$ *and* $W_i \to W_j$ *are all cofibrations and suppose that the maps* $V_i \to W_i$ *are all equivalences.*

Then the map

$$\operatorname{colim} V_i \to \operatorname{colim} W_j$$

is an equivalence.

PROOF. The proof is almost identical to the proof of lemma B.4.1.1. ☐

Similarly, we can have spectral sequences for inverse systems, but only under some more restrictive hypotheses.

B.6.0.11 Lemma. *Let* V_*, W_* *be sequential inverse systems of differentiable pro-cochain complexes, and let* $V_* \to W_*$ *be a map of inverse systems. Let* $V = \lim V_*$ *and* $W = \lim W_*$.

Suppose that

1. *The maps* $f_i : V_i \to V_{i-1}$, $g_i : W_i \to W_{i-1}$ *are fibrations of differentiable cochain complexes.*
2. *For each* k*, the inverse systems* $V_*/F^k V_*$ *and* $W_*/F^k W_*$ *are eventually constant, as in lemma* B.4.1.3.
3. *The maps* $\operatorname{Ker} f_i \to \operatorname{Ker} g_i$ *are quasi-isomorphisms of differentiable cochain complexes.*

Then, the map $V \to W$ *is a quasi-isomorphism of differentiable pro-cochain complexes.*

PROOF. This follows immediately from lemma B.4.1.3. ☐

B.6.1. Differentiable pro-cochain complexes form a multicategory, just like differentiable cochain complexes.

B.6.1.1 Definition. *Let* $V_1, \dots, V_k, W$ *be differentiable pro-cochain complexes. In the multicategory of differentiable pro-cochain complexes, an element of* $\operatorname{Hom}(V_1, \dots, V_k; W)$ *is a smooth multilinear cochain map*

$$\Phi : V_1 \times \cdots \times V_k \to W = \lim W / F^i W$$

which preserves filtrations: if $v_i \in F^{r_i}(V_i)$*, then*

$$\Phi(v_1, \dots, v_k) \in F^{r_1 + \cdots + r_k} W.$$

Factorization algebras valued in differentiable pro-cochain complexes are defined as before.

B.7. Differentiable cochain complexes over a differentiable dg ring

The category of differentiable cochain complexes is a differential graded multi-category. Thus, we can talk about commutative differentiable dg algebras R. This is just a commutative dg algebra R, with the structure of a differentiable vector space, such that all the structure maps are smooth. Similarly, a commutative differentiable pro-algebra is a commutative dg algebra in the multi-category of differentiable pro-cochain complexes.

In either context, we can define an R-module M to be a differentiable (pro-)cochain complex equipped with an action of the commutative differentiable (pro-)algebra R, in the obvious way. We say a map $M \to M'$ is a weak equivalence if it is a weak equivalence (as defined above) in the category of differentiable (pro-)cochain complex.

In either context, we say a sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if it is exact in the category of differentiable (pro-)cochain complexes. A map $M_1 \to M_2$ is a cofibration if it can be extended to an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$; it is a fibration if it can be extended to an exact sequence $0 \to M_3 \to M_1 \to M_2 \to 0$.

The category of modules over a differentiable (pro-)dg algebra R is, as above, multi-category. In either case, the multi-maps

$$\mathrm{Hom}_R(M_1, \dots, M_n; N)$$

are the multi-maps in the category of differentiable (pro-)cochain complexes whose underlying multilinear map $M_1 \times \cdots \times M_n \to N$ are R-multilinear.

B.8. Classes of functions on the space of sections of a vector bundle

Let M be a manifold and E a graded vector bundle on M. Let $U \subset M$ be an open subset. In this section we will introduce some notation for various classes of functionals on sections $\mathscr{E}(U)$ of E on U. These spaces of functionals will all be graded differentiable pro-vector spaces.

B.8.1. We are interested in symmetric algebras on vector spaces of the form $\overline{\mathscr{E}}^!_c(U)$, $\mathscr{E}^!_c(U)$, etc. These symmetric algebras can be defined in two ways: either using the completed projective tensor product of topological vector spaces, or in terms of sections of bundles on U^n. We will explain both points of view.

Thus, let us first define $(\mathscr{E}(U))^{\otimes n}$ to be the tensor power defined using the completed projective tensor product on the topological vector space $\mathscr{E}(U)$. Then a more concrete description of this space is as follows. Let $E^{\boxtimes n}$ denotes the vector bundle on M^n obtained as the external tensor product, so

$$(\mathscr{E}(U))^{\otimes n} = \Gamma(U^n, E^{\boxtimes n})$$

is the space of smooth sections of $E^{\boxtimes n}$ on U^n. Similarly, we can identify

$$\begin{aligned}(\mathscr{E}_c(U))^{\otimes n} &= \Gamma_c(U^n, E^{\boxtimes n})\\ (\overline{\mathscr{E}}_c(U))^{\otimes n} &= \overline{\Gamma}_c(U^n, E^{\boxtimes n})\\ (\overline{\mathscr{E}}(U))^{\otimes n} &= \overline{\Gamma}(U^n, E^{\boxtimes n})\end{aligned}$$

where $\overline{\Gamma}$ indicates the space of distributional sections and the subscript c indicates compactly supported distributional sections.

We have already seen how to equip the various kinds of spaces of sections of a vector bundle with the structure of a differentiable vector space. Since the spaces listed above are expressed as sections of various kinds of a vector bundle on U^n, they all have the structure of differentiable vector spaces.

Symmetric (or exterior) powers of the spaces $\mathscr{E}_c(U)$, $\overline{\mathscr{E}}_c(U)$, $\mathscr{E}(U)$, $\overline{\mathscr{E}}(U)$ are defined by taking coinvariants of the tensor powers defined above with respect to the action of the symmetric group. These symmetric powers inherit the structure of differentiable vector space.

Thus, we can define, for example, the completed symmetric algebra

$$\begin{aligned}\widehat{\mathrm{Sym}}^* \mathscr{E}_c^!(U) &= \prod_n \mathrm{Sym}^n \mathscr{E}_c^!(U)\\ \widehat{\mathrm{Sym}}^* \overline{\mathscr{E}}_c^!(U) &= \prod_n \mathrm{Sym}^n \overline{\mathscr{E}}_c^!(U)\end{aligned}$$

Note that since $\overline{\mathscr{E}}_c^!(U)$ is dual to $\mathscr{E}(U)$, we can view $\widehat{\mathrm{Sym}}\, \overline{\mathscr{E}}_c^!(U)$ as the algebra of formal power series on $\mathscr{E}(U)$. Thus, we often write

$$\widehat{\mathrm{Sym}}\, \overline{\mathscr{E}}_c^!(U) = \mathscr{O}(\mathscr{E}(U)).$$

Similarly, $\widehat{\mathrm{Sym}}\, \mathscr{E}_c^!(U)$ is the algebra of formal power series on $\overline{\mathscr{E}}(U)$.

In a similar way, we can construct

$$\begin{aligned}\mathscr{O}(\overline{\mathscr{E}}(U)) &= \prod_n \mathrm{Sym}^n(\mathscr{E}_c^!(U))\\ \mathscr{O}(\overline{\mathscr{E}}_c(U)) &= \prod_n \mathrm{Sym}^n(\mathscr{E}^!(U)).\end{aligned}$$

These spaces of functionals are all products of the differentiable vector spaces of symmetric powers, and so they are themselves differentiable vector spaces. We will equip all of these spaces of functionals with the structure of a differentiable pro-vector space, induced by the filtration

$$F^i \mathscr{O}(\mathscr{E}(U)) = \prod_{n \geq i} \mathrm{Sym}^i \overline{\mathscr{E}}_c^!(U)$$

(and similarly for $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$).

The natural product $\mathscr{O}(\mathscr{E}(U))$ is compatible with the differentiable structure, making $\mathscr{O}(\mathscr{E}(U))$ into a commutative algebra in the multicategory of differentiable graded pro-vector spaces. The same holds for the spaces of functionals $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

B.8.2. One-forms. Recall that if V is a vector space, we can define the space of one-forms on V (treated as formal scheme) as

$$\Omega^1(V) = \mathscr{O}(V) \otimes V^\vee.$$

Similarly, we can define

$$\Omega^1(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}^!_c(U),$$

where $\otimes$ denotes the completed projective tensor product.

In concrete terms,

$$\Omega^1(\mathscr{E}(U)) = \prod_n \operatorname{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!_c(U)$$

and we can identify the space

$$\operatorname{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!_c(U)) \subset \overline{\mathscr{E}}^!_c(U)^{\otimes n+1} = \overline{\Gamma}_c(U^{n+1}, (E^!)^{\boxtimes n+1})$$

as the space of compactly supported distributional sections of $(E^!)^{\boxtimes n+1}$ that are symmetric in the first n variables.

In this way, $\Omega^1(\mathscr{E}(U))$ becomes a differentiable pro-cochain complex, where the filtration is defined by

$$F^i\Omega^1(\mathscr{E}(U)) = \prod_{n \geq i-1} \operatorname{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!_c(U).$$

Further, $\Omega^1(\mathscr{E}(U))$ is a module for the commutative algebra $\mathscr{O}(\mathscr{E}(U))$, where the module structure is defined in the multicategory of differentiable pro-vector spaces.

If V is a finite-dimensional vector space, the exterior derivative map

$$\mathrm{d} : \mathscr{O}(V) \to \mathscr{O}(V) \otimes V^\vee$$

is, in components, just the composition

$$\operatorname{Sym}^{n+1} V^\vee \to (V^\vee)^{\otimes n+1} \to \operatorname{Sym}^n(V^\vee) \otimes V^\vee$$

where the maps are the inclusion followed by the natural projection (up to an overall combinatorial constant).

We can, in a similar way, define the exterior derivative

$$\mathrm{d} : \mathscr{O}(\mathscr{E}(U)) \to \Omega^1(\mathscr{E}(U))$$

by saying that on components it is given by the same formula as in the finite-dimensional case.

B.8.3. Other classes of sections of a vector bundle. Before we introduce our next class of functionals — those with proper support — we need to introduce some further notation concerning classes of sections of a vector bundle.

Let M be a manifold, and let $f : M \to N$ be a fibration. Let E be a vector bundle on M. We say a section $s \in \Gamma(M, E)$ has *relative compact support* if the map

$$f : \operatorname{Supp}(s) \to N$$

is proper. We let $\Gamma_{c/f}(M, E)$ denote the space of sections with relative compact support. This is a differentiable vector space: if X is an auxiliary manifold, a smooth map $X \to \Gamma_{c/f}(M, E)$ is a section of the bundle $\pi_M^* E$ on $X \times M$ which has relative compact support relative to the map

$$M \times X \to N \times X.$$

(It is straightforward to write down a flat connection on $C^\infty(X, \Gamma_{c/f}(M, E))$, using arguments of the type described in section B.2.)

Next, we need to consider spaces of the form $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$, where M and N are manifolds and E, F are vector bundles on M and N respectively. Of course, we can give an abstract definition using the projective tensor product, but we want a more geometric interpretation.

There are several ways to identify this space geometrically. We will view $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ as a subspace

$$\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N) \subset \overline{\mathscr{E}}(M) \otimes \overline{\mathcal{F}}(N).$$

It consists of those elements D with the property that, if $\phi \in \mathscr{E}_c^!(M)$, then map

$$\begin{array}{rccc} D(\phi): & \mathcal{F}_c^!(N) & \to & \mathbb{R} \\ & \psi & \mapsto & D(\phi \otimes \psi) \end{array}$$

comes from an element of $\mathcal{F}(N)$.

Alternatively, $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ is the space of continuous linear maps from $\mathscr{E}_c^!(M)$ to $\mathcal{F}(N)$.

We can similarly define $\overline{\mathscr{E}}_c(M) \otimes \mathcal{F}(N)$ as the subspace of those elements of $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ that have compact support relative to the projection $M \times N \to N$.

These spaces form differentiable vector spaces in a natural way: a smooth map from an auxiliary manifold X to $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ is an element of $\overline{\mathscr{E}}(N) \otimes \mathcal{F}(N) \otimes C^\infty(X)$. Similarly, a smooth map to $\overline{\mathscr{E}}_c(M) \otimes \mathcal{F}(N)$ is an element of $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N) \otimes C^\infty(X)$ whose support is compact relative to the map $M \times N \times X \to N \times X$.

B.8.4. Functions with proper support. Recall that

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U).$$

We can thus define a subspace

$$\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}^!(U) \subset \Omega^1(\mathscr{E}_c(U)).$$

The Taylor components of elements of this subspace are in the space

$$\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!(U),$$

which in concrete terms is the S_n invariants of

$$\overline{\mathscr{E}}^!_c(U)^{\otimes n} \otimes \overline{\mathscr{E}}^!(U).$$

B.8.4.1 Definition. *A function* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has* proper support *if*

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}^!(U) \subset \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U)^\vee.$$

The reason for the terminology is as follows. Let $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ and let

$$\Phi_n \in \mathrm{Hom}(\mathscr{E}_c(U)^{\otimes n}, \mathbb{R})$$

be the nth term in the Taylor expansion of Φ.

Then, Φ has proper support if and only if, for all n, the composition with a projection map

$$\mathrm{Supp}(\Phi_n) \subset U^n \to U^{n-1}$$

is proper.

We will let

$$\mathscr{O}^P(\mathscr{E}_c(U)) \subset \mathscr{O}(\mathscr{E}_c(U))$$

be the subspace of functions with proper support. Note that functions with proper support are *not* a subalgebra.

Because $\mathscr{O}^P(\mathscr{E}_c(U))$ fits into a fiber square

$$\begin{tikzcd}
\mathscr{O}^P(\mathscr{E}_c(U)) \arrow[r] \arrow[d] & \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c(U)^\vee \arrow[d] \\
\mathscr{O}(\mathscr{E}_c(U)) \arrow[r] & \mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}_c(U)^\vee
\end{tikzcd}$$

it has a natural structure of a differentiable pro-vector space.

B.8.5. Functions with smooth first derivative.

B.8.5.1 Definition. *A function* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has* smooth first derivative *if* $\mathrm{d}\Phi$, *which is* a priori *an element of*

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U)$$

is an element of the subspace

$$\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U).$$

Note that we can identify, concretely, $\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U)$ with the space

$$\prod_n \operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \otimes \mathscr{E}^!(U)$$

and

$$\operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \otimes \mathscr{E}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\otimes n} \otimes \mathscr{E}^!(U).$$

(Spaces of the form $\mathscr{E}(U) \otimes \overline{\mathscr{E}}(U)$ were described concretely above.)

Thus $\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U)$ is a differentiable pro-vector space. It follows that the space of functionals with smooth first derivative is a differentiable pro-vector space, since it is defined by a fiber diagram of such objects.

An even more concrete description of the space $\mathscr{O}^{sm}(\mathscr{E}_c(U))$ of functionals with smooth first derivative is as follows.

B.8.5.2 Lemma. *A functional* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has smooth first derivative if each of its Taylor components*

$$D_n\Phi \in \operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\otimes n}$$

lies in the intersection of all the subspaces

$$\overline{\mathscr{E}}^!(U)^{\otimes k} \otimes \mathscr{E}^!(U) \otimes \overline{\mathscr{E}}^!(U)^{\otimes n-k-1}$$

for $0 \leq k \leq n-1$.

Proof. The proof is a simple calculation. □

Note that the space of functions with smooth first derivative is a subalgebra of $\mathscr{O}(\mathscr{E}_c(U))$. We will denote this subalgebra by $\mathscr{O}^{sm}(\mathscr{E}_c(U))$. Again, the space of functions with smooth first derivative is a differentiable pro-vector space, as it is defined as a fiber product.

We can also define the space of functions on $\mathscr{E}(U)$ with smooth first derivative, by requiring that the exterior derivative lies in

$$\mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U) \subset \Omega^1(\mathscr{E}(U)).$$

B.8.6. Functions with smooth first derivative and proper support. We are particularly interested in those functions which have both smooth first derivative and proper support. We will refer to this subspace as $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$. The differentiable structure on $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$ is, again, given by viewing it as defined by the fiber diagram

$$\begin{tikzcd}
\mathscr{O}^{P,sm}(\mathscr{E}_c(U)) \arrow[r] \arrow[d] & \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}^!(U) \arrow[d] \\
\mathscr{O}(\mathscr{E}_c(U)) \arrow[r] & \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U)
\end{tikzcd}$$

We have inclusions

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) \subset \mathscr{O}^{sm}(\mathscr{E}_c(U)),$$

where each inclusion has dense image.

B.9. Derivations

As before, let M be a manifold, E a graded vector bundle on M, and U an open subset of M. In this section we will define derivations of algebras of functions on $\mathscr{E}(U)$.

To start with, recall that, for V a finite dimensional vector space (which we treat as a formal scheme) and $\mathscr{O}(V) = \prod \operatorname{Sym}^n V^\vee$ the algebra of formal power series on V, we identify the space of continuous derivations of $\mathscr{O}(V)$ with $\mathscr{O}(V) \otimes V$. We view these derivations as the space of vector fields on V and use the notation $\operatorname{Vect}(V)$.

In a similar way, we can define the space of vector fields $\operatorname{Vect}(\mathscr{E}(U))$ of vector fields on $\mathscr{E}(U)$ as

$$\operatorname{Vect}(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}(U) = \prod_n \left(\operatorname{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \mathscr{E}(U) \right),$$

using the completed projective tensor product. We have already seen (section B.8) how to define the structure of diffeological pro-vector space on spaces of this nature.

In concrete terms, the Taylor expansion of an element of $X \in \operatorname{Vect}(\mathscr{E}(U))$ is given by a sequence of continuous symmetric multilinear maps

$$D_n X : \mathscr{E}(U) \times \cdots \times \mathscr{E}(U) \to \mathscr{E}(U).$$

More generally, if M is a smooth manifold and if $X : M \to \operatorname{Vect}(\mathscr{E}(U))$ is a smooth map, then the Taylor expansion of X is a sequence of continuous symmetric multilinear maps

$$\mathscr{E}(U) \times \cdots \times \mathscr{E}(U) \to \mathscr{E}(U) \otimes C^\infty(M) = \Gamma(U \times M, E\big|_U).$$

In this section we will show the following.

B.9.0.1 Proposition. Vect$(\mathscr{E}(U))$ *has a natural structure of Lie algebra in the multicategory of diffeological pro-vector spaces. Further,* $\mathscr{O}(\mathscr{E}(U))$ *has an action of the Lie algebra* Vect$(\mathscr{E}(U))$ *by derivations, where the structure map* Vect$(\mathscr{E}(U)) \times \mathscr{O}(\mathscr{E}(U)) \to \mathscr{O}(\mathscr{E}(U))$ *is smooth.*

PROOF. To start with, let's look at the case of a finite-dimensional vector space V, to get an explicit formula for the Lie bracket on Vect(V), and the action of Vect(V) on $\mathscr{O}(V)$. Then, we will see that these formulae make sense when $V = \mathscr{E}(U)$.

Let $X \in$ Vect(X), and let us consider the Taylor components $D_n X$, which are multilinear maps

$$V \times \cdots \times V \to V.$$

Our conventions are such that

$$D_n(X)(v_1, \ldots, v_n) = \left(\frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_n} X \right)(0) \in V$$

Here, we are differentiating vector fields on V using the trivialization of the tangent bundle to this formal scheme arising from the linear structure.

Thus, we can view $D_n X$ as in the endomorphism operad of the vector space V.

If $A : V^{\times n} \to V$ and $B : V^{\times m} \to V$, let us define

$$A \circ_i B(v_1, \ldots, v_{n+m-1}) = A(v_1, \ldots, v_{i-1}, B(v_i, \ldots, v_{i+m-1}), v_{i+m}, \ldots, v_{n+m-1}).$$

If A, B are symmetric (under S_n and S_m, respectively), then define

$$A \circ B = \sum_{i=1}^{n} A \circ_i B.$$

Then, if X, Y are vector fields, the Taylor components of $[X, Y]$ satisfy

$$D_n([X, Y]) = \sum_{k+l=n+1} c_{k,l} \left(D_k X \circ D_l Y - D_l Y \circ D_k X \right)$$

where $c_{k,l}$ are combinatorial constants which are irrelevant for our purposes.

Similarly, if $f \in \mathscr{O}(V)$, the Taylor components of f are multilinear maps

$$D_n f : V^{\times n} \to \mathbb{C}.$$

In a similar way, if X is a vector field, we have

$$D_n(Xf) = \sum_{k+l=n+1} c'_{k,l} D_k(X) \circ D_k(f).$$

Thus, we see that in order to define the Lie bracket on Vect$(\mathscr{E}(U))$, we need to give maps of diffeological vector spaces

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\otimes n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}, \mathscr{E}(U)) \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes (n+m-1)}, \mathscr{E}(U))$$

where here Hom indicates the space of continuous linear maps, treated as a diffeological vector space. Similarly, to define the action of $\mathrm{Vect}(\mathscr{E}(U))$ on $\mathscr{O}(\mathscr{E}(U))$, we need to define a composition map

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\otimes n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}) \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes n+m-1}).$$

We will treat the first case; the second is similar.

Now, if X is an auxiliary manifold, a smooth map

$$X \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}, \mathscr{E}(U))$$

is the same as a continuous multilinear map

$$\mathscr{E}(U)^{\times m} \to \mathscr{E}(U) \otimes C^\infty(X).$$

Here, "continuous" means for the product topology.

This is the same thing as a continuous $C^\infty(X)$-multilinear map

$$\Phi : (\mathscr{E}(U) \otimes C^\infty(X))^{\times m} \to \mathscr{E}(U) \otimes C^\infty(X).$$

If

$$\Psi : (\mathscr{E}(U) \otimes C^\infty(X))^{\times n} \to \mathscr{E}(U) \otimes C^\infty(X).$$

is another such map, then it is easy to define $\Phi \circ_i \Psi$ by the usual formula:

$$\Phi \circ_i \Psi(v_1, \dots, v_{n+m-1}) = \Phi(v_1, \dots v_{i-1}, \Psi_i(v_i, \dots, v_{m+i-1}), \dots, v_{n+m-1})$$

if $v_i \in \mathscr{E}(U) \otimes C^\infty(X)$. This map is $C^\infty(X)$ linear. □

Remark: It is not hard to show that $\mathrm{Vect}(\mathscr{E}(U))$, as defined above, is the space of all continuous derivations of the topological algebra $\mathscr{O}(\mathscr{E}(U))$; we will not need this fact.

B.10. The Atiyah-Bott lemma

In [AB67], Atiyah and Bott showed that for an elliptic complex $(\mathscr{E}, d)$ on a compact closed manifold M, with $\mathscr{E}$ the smooth sections of a $\mathbb{Z}$-graded vector bundle, there is a homotopy equivalence $(\mathscr{E}, d) \hookrightarrow (\overline{\mathscr{E}}, d)$ into the elliptic complex of distributional sections. The argument follows from the existence of parametrices for elliptic operators. This result was generalized by N.N. Tarkhanov to the non-compact case [Tar87].

Let M be a smooth manifold (which, in general, will not be compact).

B.10.0.2 Definition. *An* elliptic complex *on M is a graded vector bundle E, whose space of smooth sections we denote by $\mathscr{E}$, together with a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree* 1 *possessing the following property, known as ellipticity. Let $\pi^* E$ denote the pullback bundle along the projection map for the cotangent bundle $\pi : T^* M \to M$. The symbol $\sigma(Q)$ of Q is a cohomological degree* 1 *endomorphism of the vector bundle $\pi^* E$. We require that the complex of vector bundles $(\pi^* E, \sigma(Q))$ on $T^* M$ is exact away from the zero section.*

Let $\overline{\mathscr{E}}$ denote the complex of distributional sections of $\mathscr{E}$. We will endow both $\mathscr{E}$ and $\overline{\mathscr{E}}$ with their natural topologies.

B.10.0.3 Lemma (Tarkhanov, [Tar87]). *There is a continuous homotopy inverse to the natural inclusion*

$$(\mathscr{E}, Q) \hookrightarrow (\overline{\mathscr{E}}, Q).$$

This is Lemma 1.7 of [Tar87]. The continuous homotopy inverse

$$\Phi : \overline{\mathscr{E}} \to \mathscr{E}$$

is given by a kernel $K_\Phi \in \mathscr{E}^! \otimes \mathscr{E}$ with proper support. The homotopy $S : \overline{\mathscr{E}} \to \overline{\mathscr{E}}$ is a continuous linear map with

$$[\mathrm{d}, S] = \Phi - \mathrm{Id}\,.$$

The kernel K_S for S is a distribution, that is, an element of $\overline{\mathscr{E}}^! \otimes \overline{\mathscr{E}}$, with proper support.

Proof. We will reproduce the proof in [Tar87]. Choose a metric on E (Hermitian, if E is a complex vector bundle) and a volume form on M. Let Q^* be the adjoint to Q. We form the graded commutator $D = [Q, Q^*]$. This is an elliptic operator on each space $\mathscr{E}^i$, the cohomological degree i part of $\mathscr{E}$. Thus, by standard results in the theory of pseudo-differential operators, there is a parametrix P for D. The kernel K_P is an element of $\overline{\mathscr{E}}^! \otimes \overline{\mathscr{E}}$, and the corresponding operator $P : \mathscr{E}_c \to \overline{\mathscr{E}}$ is an inverse for D up to smoothing operators.

By multiplying K_P by a smooth function on $M \times M$ which is 1 in a neighborhood of the diagonal, we can assume that K_P has proper support. This means that P will extend to a map $\mathscr{E} \to \overline{\mathscr{E}}$ and will still be a parametrix: thus $P \circ D$ and $D \circ P$ both differ from the identity by smoothing operators.

The homotopy S is now defined by

$$S = Q^* P.$$

□

Note that the homotopy inverse $\Phi : \overline{\mathscr{E}} \to \mathscr{E}$ and the homotopy $S : \overline{\mathscr{E}} \to \overline{\mathscr{E}}$ we have constructed are maps of differentiable vector spaces (as well as being continuous).

APPENDIX C

Homological algebra with topological vector spaces

For the bulk of the book, we use the language of differentiable vector spaces as a way to keep track of the analytic structure on the vector spaces that form our factorization algebras. The advantage of working with differentiable vector spaces is that homological algebra with sheaves on a site is very well developed, so we don't need to develop any new techniques. The disadvantage is that we need to treat the category of differentiable vector spaces as a multi-category, and not as a symmetric monoidal category. Because differentiable vector spaces form a multi-category, it doesn't make sense to ask that a factorization algebra with values in this category takes disjoint unions to tensor products. Thus, factorization algebras with values in a multicategory are not as local as one would like: for instance, they do not form a sheaf of categories.

If we treat the observables of our field theory as topological vector spaces, however, then it is possible to restore the axiom that the factorization algebra assigns a tensor product to a disjoint union. For instance, we will see that for every field theory on a manifold M, the quantum observables $\mathrm{Obs}^q(U)$ on any open subset have the structure of a topological vector space over $\mathbb{C}[[\hbar]]$, and that for a suitable completed tensor product $\widehat{\otimes}$ we have a quasi-isomorphism

$$\mathrm{Obs}^q(U)\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathrm{Obs}^q(V)\cong\mathrm{Obs}^q(U\amalg V).$$

(We will explain precisely which tensor product we will use shortly).

Other aspects of homological algebra, however, are much more difficult with topological vector spaces than with differentiable vector spaces. For instance, it is not obvious what one should mean by a quasi-isomorphism of topological vector spaces. If we use the weakest notion – just a quasi-isomorphism when we forget the topology – then the completed tensor product will almost never respect quasi-isomorphisms. With stronger notions – for example, asking for a continuous homotopy equivalence – it is much harder to check that a map is a quasi-isomorphism.

The main result of this appendix is that there is a symmetric monoidal dg category which has the best of both worlds: it is a full subcategory of both differentiable cochain complexes and of topological cochain complexes, quasi-isomorphisms are the same as those in differentiable cochain complexes, and tensor product and Hom's in this category respect quasi-isomorphisms. Further, complexes of observables of a quantum field theory live in this category.

More precisely, we have the following result.

Theorem. *There exists a full subcategory* GeoVS^{ast} *of the category* DVS^* *of differentiable vector spaces, whose objects are called* geometric *cochain complexes. The category* GeoVS^{ast} *has the following properties.*

1. GeoVS^{ast} *is also a full subcategory of the category of locally-convex topological vector spaces.*
2. GeoVS^{ast} *has the structure of a symmetric monoidal category, such that the functor* $\mathrm{GeoVS}^{ast} \to \mathrm{DVS}^*$ *is a full embedding of multi-categories.*
3. *A map in* GeoVS^{ast} *is a quasi-isomorphism (in* DVS^**) if and only if it is a homotopy equivalence. In particular, the symmetric monoidal structure on* GeoVS^{ast} *preserves quasi-isomorphisms, as does the functor*
$$\mathrm{Hom} : \left(\mathrm{GeoVS}^{ast}\right)^{op} \times \mathrm{GeoVS}^{ast} \to \mathrm{DVS}^* .$$
4. GeoVS^{ast}*, viewed as a full subcategory of* DVS^**, is closed under the formation of all cones and summands. This implies that the category* GeoVS^{ast} *is a pre-triangulated dg category, and so a stable* ∞*-category, in the terminology of* [].
5. *If* E *is an elliptic complex on a manifold* M*, then the complexes* $\mathscr{E}_c$ *and* $\overline{\mathscr{E}}_c$ *are objects of* GeoVS^{ast}.

This theorem asserts the existence of a nice category of geometric cochain complexes. The reason this category is useful for us is that complexes of observables of a quantum field theory, which a priori are objects of the category ProDVS^* of pro-differentiable cochain complexes, actually are pro-objects in GeoVS^{ast}.

Let $\mathrm{Pro\,GeoVS}^{ast}$ denote the full subcategory of the category of ProDVS^* consisting of inverse sequences $\cdots \to V_1 \to V_0$ where the objects V_i are in GeoVS^{ast} and where the maps $V_i \to V_{i-1}$ are smoothly split (that is, they are split degreewise in the category DVS but the splitting might not be a cochain map). For formal reasons, the category $\mathrm{Pro\,GeoVS}^{ast}$ has the same properties we discussed above for GeoVS^{ast}. Namely,

1. $\mathrm{Pro\,GeoVS}^{ast}$ is a symmetric-monoidal dg category,
2. the functor $\mathrm{Pro\,GeoVS}^{ast} \to \mathrm{DVS}^*$ is a full embedding of multicategories,
3. a map in $\mathrm{Pro\,GeoVS}^{ast}$ is a quasi-isomorphism if and only if it is a homotopy equivalence,
4. tensor product and Hom respect quasi-isomorphisms,
5. $\mathrm{Pro\,GeoVS}^{ast}$ is a stable infinity-category.

Further, the category of quasi-free $\mathbb{C}[[\hbar]]$-modules in $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}$ inherits all these nice properties. Here, the tensor product is taken over the ring $\mathbb{C}[[\hbar]]$. We let $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}_{\hbar}$ denote this category (and, similarly, we let $\mathrm{ProDVS}^*_{\hbar}$ denote the category of quasi-free $\mathbb{C}[[\hbar]]$-modules in ProDVS^*.

The tensor product on GeoVS^{ast} and on $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}$ will be denoted by $\widehat{\otimes}_{\beta}$ (for reasons that will be clear later).

Theorem. *For any quantum field theory on a manifold M, the factorization algebra Obs^q of observables, which* a priori *is a factorization algebra in the multi-category $\mathrm{ProDVS}^*_{\hbar}$, is actually a factorization algebra in the symmetric monoidal category $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}_{\hbar}$.*

More precisely, for every open subset $U \subset M$, the pro-diffeological complex $\mathrm{Obs}^q(U)$ is an object in the full subcategory $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}_{\hbar} \subset \mathrm{ProDVS}^*_{\hbar}$. Further, because $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}_{\hbar}$ is a full sub multi-category of ProDVS^*, the $\mathbb{C}[[\hbar]]$-multilinear maps

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_n) \to \mathrm{Obs}^q(V)$$

(for disjoint opens U_i in V) extend to maps

$$\mathrm{Obs}^q(U_1)\widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} \dots \widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} \mathrm{Obs}^q(U_n) \to \mathrm{Obs}^q(V).$$

The map

$$\mathrm{Obs}^q(U_1)\widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} \dots \widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} \mathrm{Obs}^q(U_n) \to \mathrm{Obs}^q(U_1 \amalg \cdots \amalg U_n)$$

is a quasi-isomorphism in $\mathrm{Pro}\,\mathrm{GeoVS}^{ast}_{\hbar}$ (and therefore a homotopy equivalence).

There is a simplicial set of quantizations of a given classical field theory, where different quantizations connected by a homotopy should be regarded as equivalent. This simplicial set allows us to track the dependence of the field theory on various auxiliary choices, like that of a gauge-fixing condition.

If we have an n-simplex in this simplicial set, the factorization algebra of observables becomes a factorization algebra over $\Omega^*(\triangle^n)$. We will see that this family of factorization algebras also lives in the world of geometric pro-cochain complexes.

Theorem. *Suppose we have an n-simplex in the simplicial set of quantizations of a fixed classical field theory. Then, the factorization algebra $\mathrm{Obs}^q_{\triangle^n}$ of observables of this family of theories is a factorization algebra in the category $\mathrm{ProGeoVS}^*_{\hbar}$ over the dg ring $\Omega^*(\triangle^n)$ of cochains on the n-simplex.*

Further, $[n] \to [m]$ is a face or degeneracy map, then the corresponding restriction map of factorization algebras

$$\mathrm{Obs}^q_{\triangle^n} \to \mathrm{Obs}^q_{\triangle^m}$$

is a quasi-isomorphism in $\mathrm{ProGeoVS}^*_{\hbar}$, and therefore a homotopy equivalence.

Because Obs^q is a factorization algebra valued in $\mathrm{ProDVS}^*_\hbar$, we know that there is a quasi-isomorphism in $\mathrm{ProDVS}^*_\hbar$ between $\mathrm{Obs}^q(U)$ and the Čech complex constructed from any Weiss cover of U.

If $\mathfrak{U}$ is a Weiss cover of an open subset U, the Čech complex is denoted $\check{C}(\mathfrak{U}, \mathrm{Obs}^q)$. This Čech complex arises from a simplicial Čech complex $\check{C}_\triangle(\mathfrak{U}, \mathrm{Obs}^q)$. We defined the Čech complex just by taking the usual realization of the simplicial Čech complex, and observing that this makes sense in the category $\mathrm{ProDVS}^*_\hbar$. However, in homotopy theory, this is not really sufficient: we would like the Čech complex to be the homotopy colimit of the simplicial Čech complex.

It turns out that this is true in the dg category $\mathrm{ProGeoVS}^*$. More precisely, we have the following theorem.

Theorem. *Fix a quantum field theory on a manifold M. For any Weiss cover $\mathfrak{U}$ of an open subset U of M, the Čech complex $\check{C}(\mathfrak{U}, \mathrm{Obs}^q)$ is an object of* $\mathrm{ProGeoVS}^*_\hbar$.

It follows that the quasi-isomorphism in $\mathrm{ProDVS}^*_\hbar$

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q) \to \mathrm{Obs}^q(U)$$

is a homotopy equivalence.

Further, the Čech complex is the homotopy colimit in the dg category $\mathrm{ProGeoVS}^*$ *of the simplicial Čech object $\check{C}_\triangle(\mathfrak{U}, \mathrm{Obs}^q)$.*

In general, a homotopy colimit in a dg category (or $(\infty, 1)$-category) has a homotopy universal property similar to the universal property of a colimit in an ordinary category. We will show that the Čech complex has this homotopy universal property.

C.1. Bornological vector spaces

Our construction of the category GeoVS^* of geometric cochain complexes is a little intricate, and requires a detour into the theory of bornological and convenient vector spaces. Our main reference for this theory is [KM97].

C.1.0.4 Definition. *If E is a locally-convex topological vector space, the bornologification of E is the finest locally convex topology on E which has the same bounded sets as E. We say that E is bornological if it is the same as its bornologification.*

Remark: All locally-convex topological vector spaces will be Hausdorff, but not necessarily complete.

We let BVS denote the category of complete bornological vector spaces, which is a full subcategory of the category LCTVS of complete locally-convex topological vector spaces. The bornologification functor LCTVS $\to$ BVS is a right adjoint to the inclusion functor BVS $\hookrightarrow$ LCTVS. An alternative definition of BVS is that it is equivalent to the category of locally convex topological vector spaces with *bounded* (rather than continuous) linear maps. Bounded is, of course, a weaker condition.

We let DVS be the category of differentiable vector spaces. There is a notion of smooth map from a manifold M to a locally-convex topological vector space (see [KM97], Chapter 1). We can also differentiate smooth maps. In this way, we get a functor LCTVS $\to$ DVS. This functor factors through the bornologification functor LCTVS $\to$ BVS.

C.1.0.5 Proposition. *When restricted to* BVS*, this functor embeds* BVS *as a full subcategory of* DVS*.*

It follows that BVS is equivalent to the essential image of LCTVS in TVS.

This proposition is Corollary 4.6 in [KM97]. The point is that smooth maps between bornological vector spaces are the same as continuous maps, and that any topological vector space and its bornologification have the same smooth maps from any manifold.

The multicategory structure on DVS restricts to one on BVS.

C.1.0.6 Lemma. *The multicategory structure on* BVS *arises from a symmetric monoidal structure.*

The tensor product in this symmetric monoidal structure preserves all colimits.

PROOF. We need to show that, given any objects $E_1, \dots, E_n$ of BVS, there is an object $E_1 \otimes_\beta \dots \otimes_\beta E_n$ such that

$$\operatorname{Hom}(E_1 \times \cdots \times E_n, G) = \operatorname{Hom}(E_1 \otimes_\beta \dots \otimes_\beta E_n, G)$$

for all objects G of BVS. On the left hand side of this equation we have the space of smooth multilinear maps. According to Lemma 5.5 of [KM97], a map is smooth if and only if it is bounded, and in section 5.7 of [KM97] it is shown that the tensor product $E_1 \otimes_\beta \dots \otimes_\beta E_n$ exists. It is called the bornological tensor product.

The fact that the tensor product preserves colimits is the theorem in section 5.7 of [KM97]. □

Thus, we have constructed a symmetric monoidal category BVS with a full embedding BVS $\hookrightarrow$ DVS of multicategories.

The functor BVS $\to$ LCTVS preserves all colimits, as it is a left adjoint.

The functor BVS → DVS does not preserve all colimits, but it does preserve a class of colimits.

C.1.0.7 Proposition. *The functor* BVS → DVS *preserves countable coproducts and sequential colimits of closed embeddings.*

PROOF. Since the case of countable coproducts is a special case of that of sequential colimits of closed embeddings, we will prove the latter. Suppose we have a sequence $V_1 \to V_2 \dots$ of closed embeddings. We let $V = \operatorname{colim} V_i$.

We need to show that for all manifolds M, $C^\infty(M, V)$ is the colimit, in the category of sheaves on M, of the sheaf of smooth maps from M to V_i. By definition, a section of the colimit of these sheaves is a map from M to V which locally on M factors through a smooth map to some V_i.

It is clear that this colimit is a subspace of $C^\infty(M, V)$. Since V_i is a closed subspace of V, lemma 3.8 of [KM97] tells us that a map to V_i is smooth if and only if the composd map to V is smooth. We thus need to show that every smooth map from M to V locally factors through some V_i.

By [KM97], 52.8, a subset of V is bounded if and only if it is a bounded subset of some V_i. It thus suffices to show that a smooth map to V locally lies in a bounded subset. In fact this is true for continuous maps to any locally convex topological vector space. Indeed, given any point $p \in M$, choose a neighbourhood U of p whose closure $\overline{U}$ in M is compact. If $f : M \to V$ is continuous, then $f(\overline{U})$ is also compact and so bounded.

□

Note that the functor LCTVS → BVS preserves limits, because it is a right adjoint. It follows that BVS admits all limits (since LCTVS does). The functor BVS → LCTVS does not, however, preserve all limits.

C.1.0.8 Proposition. *The functor* BVS → DVS *preserves all limits.*

PROOF. Lemma 3.8 of [KM97] show that the functor LCTVS → DVS preserves limits. We need to show that BVS is closed inside DVS under formation of limits. Since BVS is the essential image of LCTVS, this is immediate. □

C.1.0.9 Corollary. *The functor* BVS → DVS *has a left adjoint, as does the functor* LCTVS → DVS*.*

PROOF. It suffices to construct the adjoint functor DVS → BVS, as the one to LCTVS is obtained by composing with the full embedding BVS ↪ LCTVS. This follows from the adjoint functor theorem, since this functor preserves limits and since BVS admits all limits. One can check easily that the set-theoretic criteria of the adjoint functor theorem holds. ☐

We won't really use this left adjoint much, but it's useful to know that it exists. It is immediate that the composition

$$\mathrm{BVS} \to \mathrm{DVS} \to \mathrm{BVS}$$

is the identity, because BVS is a full subcategory.

C.1.0.10 Corollary. BVS *admits all colimits.*

PROOF. Indeed, the category DVS admits all colimits and the functor DVS → BVS is colimit preserving.

Alternatively, see page 53 of [KM97]. ☐

C.1.1. Completeness. We are interested in topological vector spaces which have some notion of completeness. A good definition of completeness for our purposes was developed in [KM97].

C.1.1.1 Definition. *An object V in* BVS *or* LCTVS *is* c^∞-complete *if every smooth map $f : \mathbb{R} \to V$ admits an antiderivative.*

A c^∞-complete bornological vector space will be called a convenient vector space. *The category of convenient vector spaces (which is a full subcategory of that of bornological vector spaces) will be called* ConVS. *This is equivalent to the category of c^∞-complete topological vector spaces with bounded linear maps.*

Equivalent formulations of this definition are presented in [KM97], theorem 2.14.

C.1.1.2 Proposition. *The full subcategory* ConVS $\subset$ DVS *is closed under formation of all limits and under sequential colimits of sequences of closed embeddings.*

PROOF. This is theorem 2.15 of [KM97]. ☐

In particular, by the adjoint functor theorem, there are left adjoints BVS → ConVS and DVS → Convs. The functor BVS → ConVS sends a bornological vector space to its c^∞ completion, see theorem 4.29 of [KM97].

C.1.1.3 Lemma. *The multi-category structure on* ConVS *is represented by the c^∞ completion of the bornological tensor product. We will denote this completed tensor product by $V \widehat{\otimes}_\beta W$.*

PROOF. The bornological tensor product $V \otimes_\beta W$ has the universal property that a map from it is the same as a smooth bilinear map from $V \times W$. The completion $\overline{F}$ of a bornological vector space F has the universal property that a map from F to a convenient vector space is the same as a map from $\overline{F}$. It follows that, for any convenient vector space U, smooth bilinear maps from $V \times W$ to U are the same as maps from the completed bornological tensor product $V \widehat{\otimes}_\beta W$ to U. ☐

C.1.1.4 Lemma. *The category* ConVS *admits all colimits, and the tensor product* $\widehat{\otimes}_\beta$ *commutes with the tensor product.*

PROOF. The functor DVS $\to$ ConVS is a left adjoint and hence colimit preserving. The category DVS admits all colimits. It follows that ConVS admits all colimits and that these colimits can be computed by first computing them in DVS and then applying the left adjoint functor DVS $\to$ ConVS.

The fact that the tensor product commutes with colimits follows from the corresponding fact for BVS and the fact that the completion functor BVS $\to$ ConVS is symmetric monoidal and colimit preserving.

☐

The following fact will be useful sometimes.

C.1.1.5 Lemma. *The antiderivative map*

$$\begin{aligned} \int : C^\infty(\mathbb{R}, F) &\to C^\infty(\mathbb{R}, F) \\ f &\mapsto \widetilde{f}(t) = \int_0^t f(x)\mathrm{d}x. \end{aligned}$$

is smooth.

(Note that the antiderivative of a smooth curve in F is unique if we require that it vanishes at the origin in $\mathbb{R}$*.)*

PROOF. To check that the antiderivative map is smooth, it suffices to show that it takes smooth curves in $C^\infty(\mathbb{R}, F)$ to smooth curves. Thus, let $A : \mathbb{R}_x \times \mathbb{R}_y \to F$ be a smooth map, so that A gives a smooth map $\mathbb{R}_x \to C^\infty(\mathbb{R}_y, F)$. We need to show that

$$\int_y A : \mathbb{R}_x \times \mathbb{R}_x \to F$$

is also smooth.

Lemma 3.7 of [KM97] shows that $C^\infty(\mathbb{R}, F)$ is convenient whenever F is convenient. Thus, the curve $\mathbb{R}_y \to C^\infty(\mathbb{R}_x, F)$ arising from A has an anti-derivative which is also smooth. By uniqueness of antiderivatives, the result follows. □

C.1.1.6 Theorem. *The category* ConVS *has internal* Hom*'s. That is, there is a convenient vector space structure on* $\operatorname{Hom}(E, F)$ *for any two convenient vector spaces E, F, characterized by the fact that a smooth map from a manifold M to* $\operatorname{Hom}(E, F)$ *is a continuous map $E \to C^\infty(M, F)$, and the composition bilinear maps are smooth.*

Further, there is a Hom-tensor adjunction:

$$\operatorname{Hom}(E \widehat{\otimes}_\beta F, G) = \operatorname{Hom}(E, \operatorname{Hom}(F, G)).$$

PROOF. This is essentially the Theorem in section 5.7 of chapter 1 of [KM97]. More precisely, this theorem shows that the space of bounded linear maps between bornological vector spaces has a bornological topology such that there is a Hom-tensor adjunction

$$\operatorname{Hom}(E, \operatorname{Hom}(F, G)) = \operatorname{Hom}(E \otimes_\beta F, G).$$

It remains to check that a smooth map from a manifold M to $\operatorname{Hom}(E, F)$ is the same thing as a bounded linear map $E \to C^\infty(M, F)$.

In [KM97], section 3.1.1, a topology is given to the space of all smooth (possibly non-linear) maps $E \to F$ between bornological vector spaces. Call this space $C^\infty(E, F)$. Theorem 3.12 shows that the category of bornological vector spaces, with smooth (possibly non-linear) maps is Cartesian closed. This implies, that for any manifold M, a map $M \to C^\infty(E, F)$ is smooth if and only if the map $E \times M \to F$ is smooth.

To complete the proof that bornological vector spaces have a Hom-tensor adjunction, we need to show that a map from a manifold to $\operatorname{Hom}(E, F)$ is smooth if and only if it is smooth as a map to $C^\infty(E, F)$, so that a map to $\operatorname{Hom}(E, F)$ is smooth if and only if it arises from a map $E \to C^\infty(M, F)$. Note that this will also show that the composition maps in BVS are smooth multi-linear maps.

In general, a map from a manifold to a closed subspace of a topological vector space is smooth if and only if it is smooth to the ambient vector space. Thus, it suffices to show that $\operatorname{Hom}(E, F)$ (with its topology for which Hom-tensor adjunction holds) is bornologically isomorphic to a closed subspace of $C^\infty(E, F)$.

Note that the condition that a smooth map is linear is obviously a closed condition, so that there is a topology on the space of smooth maps so that it is a closed subspace of $C^\infty(E, F)$. Let us call the space of smooth linear maps, with this topology, $\operatorname{Hom}'(E, F)$.

We need to show that the bornologifications of $\operatorname{Hom}'(E, F)$ and of $\operatorname{Hom}(E, F)$ agree. To show this, it suffices to show that they have the same bounded sets. Because $\operatorname{Hom}'(E, F)$ is a closed subspace of $C^\infty(E, F)$, a subset of $\operatorname{Hom}'(E, F)$ is bounded if and only if it is bounded as a subspace of $C^\infty(E, F)$. Proposition 5.6 of [KM97] shows that a subset of $\mathrm{Hom}(E, F)$ is bounded if and only if it is bounded when viewed as a subset of $C^\infty(E, F)$.

So far, we have shown that the category BVS has internal Hom's and a Hom-tensor adjunction. Next, we need to show that the full subcategory ConVS also has internal Hom's. That is, if E, F are convenient vector spaces, we need to show that $\mathrm{Hom}(E, F)$ is also convenient.

It suffices to show that $C^\infty(E, F)$ is convenient, since a closed subspace of a convenient space is convenient. Now, according to the lemma in section 3.11 of [KM97], the space $C^\infty(E, F)$ is a limit of the spaces $C^\infty(\mathbb{R}, F)$ over smooth maps $\mathbb{R} \to E$. Since, by lemma 3.7, $C^\infty(\mathbb{R}, F)$ is convenient whenever F is, and by theorem 2.15, convenient vector spaces are closed under the formation of limits, the result follows.

Alternatively, to show that $\mathrm{Hom}(E, F)$ is convenient we need to show that any smooth map $f : R \to \mathrm{Hom}(E, F)$ admits an antiderivative.

To see this, note that such a smooth map is the same as a smooth map $E \to C^\infty(\mathbb{R}, F)$.Since the antiderivative map from $C^\infty(\mathbb{R}, F)$ to itself is smooth, the result follows.

So, we have shown that ConVS has internal Hom's. We finally need to show that ConVS has a Hom-tensor adjuntion. But this follows immediately from the fact that the functor ConvS $\to$ BVS is a full embedding of multicategories, and the fact that BVS has a Hom-tensor adjunction. □

Let us summarize the results we have so far.

(1) The categories LCTVS, BVS, ConVS, and DVS admit all (small) limits and colimits.
(2) ConVS and BVS are symmetric monoidal categories, whereas we treat DVS just as a multicategory. The tensor product on both ConVS and BVS are colimit-preserving.
(3) The categories ConVS, BVS, DVS all admit internal Hom's, and the functors ConVS $\to$ BVS $\to$ DVS are all functors of categories enriched in the multicategory DVS.
(4) Both ConVS and BVS admit a Hom-tensor adjunction.
(5) The colimit-preserving functor
$$\mathrm{BVS} \to \mathrm{ConVS}$$
is symmetric monoidal.
(6) The functors
$$\mathrm{ConVS} \to \mathrm{BVS} \to \mathrm{DVS}$$
are all right-adjoints (so limit preserving), and are full embeddings of multicategories.

(7) The functors

$$\mathrm{DVS} \to \mathrm{BVS} \to \mathrm{ConVS}$$

are left adjoints, and so colimit-preserving.

(8) The functors

$$\mathrm{LCTVS} \to \mathrm{BVS} \to \mathrm{DVS}$$

are all limit preserving, so that their adjoints are colimit preserving.

(9) The functor

$$\mathrm{BVS} \hookrightarrow \mathrm{LCTVS}$$

is a full embedding of categories and commutes with colimits. If we equip LCTVS with the multicategory structure coming from continuous multilinear maps, this functor is *not* a full embedding of multicategories.

C.2. Examples of bornological vector spaces

Any topological vector space can be regarded as a bornological vector space by applying the bornologification functor. In what follows we will give all topological vector spaces their bornological topology. In general this might differ from the usual topology, but it will have the same spaces of smooth maps. We will explain some examples where the bornological and the usual topologies coincide.

For any smooth connected manifold M, $C^\infty(M)$ is a Fréchet bornological vector space. The bornological structure is determined either by taking the bornologification of the usual topology, or by declaring that a smooth map $N \to C^\infty(M)$ is an element of $C^\infty(N \times M)$.

For any bornological vector space V, $C^\infty(M, V)$ is again a bornological vector space, where the bornology is determined either from the natural topology, or by saying that a smooth map $N \to C^\infty(M, V)$ is an element of $C^\infty(N \times M, V)$. (The fact that these define the same bornology follows from theorem 3.12 of [KM97], which that the category of bornological vector spaces and smooth (possibly non-linear) maps is cartesian closed).

As we have seen, the category of bornological vector spaces, viewed as a subcategory of DVS, is closed under sequential colimits of closed embeddings and under limits. From this we can construct bornological vector spaces modelling compactly-supported maps to a bornological vector space.

C.2.0.7 Definition. *Let V be a bornological vector space. If $K \subset M$ is a compact set, let $C^\infty_K(M, V)$ denote the space of smooth maps from $M \to V$ with support in K. This is the kernel of the map*

$$C^\infty(M, V) \to C^\infty(M \setminus K, V)$$

and hence is a bornological vector space.

We define

$$C_c^\infty(M, V) = \operatorname*{colim}_i C_{K_i}^\infty(M, V)$$

where the K_i are an increasing sequence of compact sets whose union is M.

The maps in this colimit are closed embeddings, so that we get the same answer if the colimis computed in DVS *or* BVS*. It follows that, for all other manifolds N,*

$$C^\infty(N, C_c^\infty(M, V)) = C_P^\infty(M \times N, V)$$

where the P means we consider maps whose support is a subset which maps properly to M.

If V is c^∞-complete then so is $C^\infty(M, V)$ and $C_c^\infty(M, V)$.

Let us explain some further examples. Let E be a vector bundle on a manifold M, and, as before, let $\mathscr{E}, \mathscr{E}_c, \overline{\mathscr{E}}, \overline{\mathscr{E}}_c$ refer to sections of E which are smooth, compactly supported, distributional, or compactly-supported and distributional. All of these spaces have natural topologies and so can be viewed as bornological vector spaces, and they are all c^∞-complete.

We explained in section **??** how to view these vector spaces as being equipped with a diffeological structure. It is easy to check that the diffeological structure discussed there is the same as the one that arises from the topology. Indeed, the Serre-Swan theorem tells us that any vector bundle is a direct summand of a trivial vector bundle, so we can reduce to the case that E is trivial. Then results of Grothendieck summarized in [Gro52] allow one to describe smooth maps to these various vector spaces using the theory of nuclear vector spaces; in this way we arrive at the description given earlier.

The following lemma will be useful later.

C.2.0.8 Lemma. *For any vector bundle E on a manifold M, the standard (nuclear) topologies on $\overline{\mathscr{E}}_c$, $\mathscr{E}$ and $\mathscr{E}_c$ are bornological. It follows that these spaces are convenient (because completeness in the locally-convex sense is stronger than c^∞-completeness).*

PROOF. Because every vector bundle is a summand of a trivial one, it suffices to prove the statement for the trivial vector bundle. According to [KM97], 52.29, the strong dual of a Fréchet Montel space is bornological. The space $C^\infty(M)$ of smooth functions on a manifold is Fréchet Montel, because every nuclear space is Schwartz (see pages 579-581 of [KM97]) and every Fréchet Schwarz space is Montel. Thus the strong dual of $C^\infty(M)$ is bornological, as desired.

Next, we will see that any Fréchet space is bornological. This follows immediately from proposition 14.8 of [Trè67] (see also the corollary on the following page). It follows that $C^\infty(M)$ is bornological, and that the same holds for $C_K^\infty(M)$ for any compact subset $K \subset M$.

Since bornological spaces are closed under formation of colimits, the same holds for $C_c^\infty(M)$.

$\square$

C.2.0.9 Lemma. *The space of smooth functions on an n-simplex is a complete nuclear Fréchet space and hence bornological (and convenient).*

PROOF. The n-simplex $\triangle^n$ is a closed subspace of $\mathbb{R}^n$. We can identify $C^\infty(\triangle^n)$ wit the quotient of $C^\infty(\mathbb{R}^n)$ by the closure of the ideal consisting of those functions which vanish on $\triangle^n$. (By a theorem of Whitney, this closure consists of those functions f whose ∞-jet vanishes at each point in $\triangle^n$). The quotient of a complete nuclear Fréchet space by a closed subspace is again a nuclear Fréchet space. $\square$

It follows immediately that $C^\infty(\triangle^n, \mathscr{E})$ and $C^\infty(\triangle^n, \mathscr{E}_c)$ (for $\mathscr{E}$, $\mathscr{E}_c$ as before) are bornological when equipped with their nuclear topology.

C.3. Convenient cochain complexes

Let us define the category ConVS* of convenient cochain complexes to be the full subcategory of the category DVS* consisting of those complexes of differentiable vector spaces which are degreewise convenient. The category ConVS* is a dg symmetric monoidal category, and the embedding ConVS$^* \hookrightarrow$ DVS* is a full embedding of multicategories.

C.3.0.10 Definition. *We say that a map of convenient cochain complexes is a quasi-isomorphism (respectively, fibration or cofibration) if it is when viewed as a map of differentiable cochain complexes.*

The dg symmetric monoidal category ConVS* has a notion of quasi-isomorphism, fibration, and cofibration. The general yoga of homological algebra tells us that we should only consider tensor products with a *flat* object. (An object is defined to be flat if tensor product with it preserves quasi-isomorphisms). Tensor products between non-flat objects should be replaced by derived tensor products, which are computed by replacing one of the objects by a flat one. We now see that there is a potential problem: it is not obvious which (if any!) objects of ConVS* are flat, so that we don't know how to define (or compute) the derived tensor product.

A similar problem occurs when we want to consider the Hom-complexes between two objects of ConVS*. The general philosophy tells us that $\mathrm{Hom}(V, W)$ is only a good object when V is projective, that is, when $\mathrm{Hom}(V, -)$ takes quasi-isomorphisms in ConVS* to quasi-isomorphisms of cochain complexes. If V is not projective, then we should consider the derived Hom-complex, which is defined by finding a projective replacement of V. Again, it is not obvious which, if any, objects of ConVS are projective.

We solve this problem by restricting to some very small subcategories which contain all the objects of interest.

C.3.0.11 Definition. *We define the category of* geometric objects *of* ConVS^* *to be the smallest full subcategory of* ConVS^* *which contains objects of the form* $C_c^\infty(M, E)$ *where* E *is an elliptic complex on a manifold* M*; and which is closed under the following operations.*

1. *Formation of cones.*
2. *Homotopy equivalences.*
3. *Formation of direct summands.*

By "cone" we just mean the usual formula for a cone of a map of cochain complexes, where the resulting object is viewed as an element of ConvS^**.*

We let GeoVS^* *denote the category of geometric objects in* ConVS^**.*

C.3.0.12 Theorem. *The category of geometric objects in* ConVS^* *is closed under the completed bornological tensor product. Thus, the multicategory structure on* GeoVS^* *inherited from that on* DVS^* *is representable by a symmetric monoidal structure.*

Further, if V, W *are geometric and* $W \to W'$ *is a quasi-isomorphism, then*

$$V \widehat{\otimes}_\beta W \to V \widehat{\otimes}_\beta W'$$

is also a quasi-isomorphism. (That is, in the category GeoVS^**, every object is flat).*

If V, W, W' *are as before, then the map*

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W')$$

is a quasi-isomorphism. That is, in GeoVS^**, every object is projective.*

This result is the main technical result of this appendix. The proof will be presented over the next several sections.

C.4. Elliptic objects are closed under tensor product

Before we embark on the proof of the statement that GeoVS^* is closed under tensor products, we need a definition.

C.4.0.13 Definition. *We say an geometric object* V *has complexity* 0 *if it is of the form* $C_c^\infty(M, E)$ *for some elliptic complex on a manifold* M*. We say, inductively, that an object has complexity* $\leq n$ *if it can be formed from objects of complexity* $\leq n-1$ *by applying the operations appearing in* *the definition of geometric objects: namely, cones, homotopy equivalences, and formation of direct summands. We say that an object has complexity n if it has complexity $\leq n$ but not $\leq n-1$.*

Basically all of our proofs proceed by induction on the complexity.

Let us now prove the first statement.

C.4.0.14 Lemma. *Let V, W be geometric objects. Then so is $V\widehat{\otimes}_\beta$.*

PROOF. We will do this by induction on the sum of the complexities of V and W. Similar arguments will be used for all the proofs of properties of geometric objects; we will give full details for this argument and be a little more sketchy later.

The initial case of the induction is when V and W both have complexity zero. In this case, $V = C_c^\infty(M, E)$ and $W = C_c^\infty(N, F)$ for elliptic complexes E, F on manifolds M, N. Thus, V and W, with their bornological topology, are sequential colimits of the nuclear Fréchet complexes $C_K^\infty(M, E)$ and $C_L^\infty(N, F)$ as K and L range over an exhausting family of compact subsets of M and N.

Because the completed bornological and completed projective tensor products coincide for Fréchet spaces, and the completed bornological tensor product commutes with colimits, we have

$$V\widehat{\otimes}_\beta = \operatorname*{colim}_{K,L} C^\infty_{K\times L}(M \times N, E \boxtimes F).$$

This is the same as $C_c^\infty(M \times N, E \boxtimes F)$, which is again a compactly supported sections of an elliptic complex.

Now suppose that V has complexity n and W has complexity m. We can assume (by switching V and W if necessary) that $n > 0$. Thus, V is obtained from objects of complexity $\leq n-1$ by operations of homotopy equivalence, cones, and sequential colimits of smoothly-split maps. By induction, we will assume that the tensor product of W with any object of complexity $\leq n-1$ is geometric. Thus, we need to show the following three statements.

(1) If A, B, W are geometric, and A, B have complexity $\leq n-1$, and if $f : A \to B$ is a map, then $\operatorname{Cone}(f)\widehat{\otimes}_\beta W$ is geometric.
(2) If A is geometric of complexity $\leq n-1$, and $A \to B$ is a homotopy equivalence, then $B\widehat{\otimes}_\beta W$ is geometric.
(3) If A has complexity $n-1$, and if A splits as a direct sum $A = B \oplus B'$, then $B\widehat{\otimes}_\beta W$ is geometric.

For the first statement, note that $\operatorname{Cone}(f)\widehat{\otimes}_\beta W$ is the same as $\operatorname{Cone}(f\widehat{\otimes}_\beta \operatorname{Id}_W)$, where $f\widehat{\otimes}_\beta \operatorname{Id}_W$ is a map $A\widehat{\otimes}_\beta W \to B\widehat{\otimes}_\beta W$. The result follows by the induction assumption that $A\widehat{\otimes}_\beta W$ and $B\widehat{\otimes}_\beta W$ are geometric. For the second, note that $A\widehat{\otimes}_\beta W \to B\widehat{\otimes}_\beta W$ is a homotopy equivalence; and for the third, note that $B\widehat{\otimes}_\beta W$ is a direct summand of $A\widehat{\otimes}_\beta W$.

Thus, we have proved by induction that GeoVS* is closed under tensor product. ☐

C.4.1. Elliptic objects are flat.

In this subsection, we will prove the following.

C.4.1.1 Proposition. *If V, W, W' are geometric objects, and if $W \to W'$ is a quasi-isomorphism, then so is*

$$V\widehat{\otimes}_\beta W \to V\widehat{\otimes}_\beta W'.$$

Before we prove this result, we need some subsidiary lemmas.

C.4.1.2 Lemma. *For all geometric objects W and manifolds M, the map*

$$C_c^\infty(M)\widehat{\otimes}_\beta W \to C_c^\infty(M, W)$$

is a quasi-isomorphism.

PROOF. We do this by induction on the complexity of W. In the case that W has complexity zero, this map is actually an isomorphism, as we have seen above **??**. Assume that the statement is true for all objects of complexity $\leq n-1$. We need to prove it for objects of complexity n.

Note that the functors $C_c^\infty(M, -)$ and $C_c^\infty(M)\widehat{\otimes}_\beta -$ both take cones to cones, homotopy equivalences to homotopy equivalences, and summands to summands. The induction step follows immediately from these facts. We will spell out the case of summands.

Suppose that A is an geometric object of complexity $n-1$ and that A can be written as a direct sum $A = B \oplus B'$. Then, both B and B' are geometric objects of complexity n.

The map

$$C_c^\infty(M)\widehat{\otimes}_\beta A \to C_c^\infty(M, A)$$

is a direct sum of the corresponding maps from B and B'. A direct sum of two maps is a quasi-isomorphism if and only if the two factors are quasi-isomorphisms. ☐

C.4.1.3 Lemma. *The functor*

$$C_c^\infty(M, -) : \mathrm{DVS}^* \to \mathrm{DVS}^*$$

preserves quasi-isomorphisms.

PROOF. By replacing a quasi-isomorphism by its cone, it suffices to show that if W is quasi-isomorphic to 0 then so is $C_c^\infty(M, W)$.

Saying that W is quasi-isomorphic to 0 is equivalent to saying that the stalks of the sheaves $C^\infty(N, W)$ on all manifolds N have no cohomology. In particular, the sheaf W_M on M of smooth maps to W on M has stalks with no cohomology.

Note also that W_M is a sheaf of modules for the sheaf C^∞_M of smooth functions on M. Thus, it admits partitions of unity. A partition of unity argument then implies that $C^\infty_c(M, W)$ has no cohomology.

More generally, for any manifold N, the space $C^\infty_P(N \times M, W)$ of smooth maps to W whose support maps properly to N has no cohomology (again by a partition of unity argument). Since this is the space of smooth maps to $C^\infty_c(M, W)$, it follows that $C^\infty_c(M, W)$ is quasi-isomorphic to 0 not just as a cochain complex but as an object of DVS*. ☐

C.4.1.4 Corollary. *If $W \to W'$ is a quasi-isomorphism of geometric objects, then the map $C^\infty_c(M)\widehat{\otimes}_\beta W \to C^\infty_c(M)\widehat{\otimes}_\beta W'$ is a quasi-isomorphism in* ConVS*.

Finally, we will show the following.

C.4.1.5 Proposition. *If $W \to W'$ is a quasi-isomorphism in* GeoVS*, *then for all geometric objects V, $V\widehat{\otimes}_\beta W \to V\widehat{\otimes}_\beta W'$ is a quasi-isomorphism.*

PROOF. We do this by induction on the complexity of V. Because the functor $-\widehat{\otimes}_\beta W$ takes homotopy equivalences, cones to cones, and summands to summands, the inductive argument we have used so far reduces us to the case that V is of complexity zero. For this case, we need to show that the map

$$C^\infty_c(M, E)\widehat{\otimes}_\beta W \to C^\infty_c(M, E) \otimes W'$$

is a quasi-isomorphism. If suffices to show this when E is just a single vector bundle, and not an elliptic complex. Using the fact that every vector bundle is a summand of a trivial bundle, it suffices to show this when E is just the trivial vector bundle. But this is the Corollary above. ☐

C.4.2. Ind-geometric objects are projective.

C.4.2.1 Definition. *An object of* ConVS* *is an ind-geometric object if it is a sequential colimit of a sequence of geometric objects $A_0 \to A_1 \dots$ where the connecting maps $A_i \to A_{i+1}$ are smoothly split.*

Smoothly split means split in the category of graded differentiable vector spaces, i.e. the splitting might not be a cochain map.

C.4.2.2 Lemma. *The category of ind-geometric objects is closed under the completed bornological tensor product. Further, if V, W, W' are ind-geometric objects and if $W \to W'$ is a quasi-isomorphism, so is $V \widehat{\otimes}_\beta W \to V \widehat{\otimes}_\beta W'$.*

PROOF. This follows immediately from the corresponding facts for geometric objects and the fact that the tensor product $\widehat{\otimes}_\beta$ commuts with colimits. □

The main result of this subsection is the following.

C.4.2.3 Proposition. *Let V be an ind-geometric object, and let $W \to W'$ be a quasi-isomorphism between two geometric objects. Then, the map*

$$\operatorname{Hom}(V, W) \to \operatorname{Hom}(V, W')$$

is a quasi-isomorphism of complexes of objects of ConVS.

We have seen that ConVS has internal Hom's.

The proof relies on the following technical lemma.

C.4.2.4 Lemma. *Let $\mathcal{D}_c(M)$ denote the space of compactly supported distributions on a manifold M. For all complete locally convex spaces F, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathrm{ConVS}}(\mathcal{D}_c(M), F_{born}) = C^\infty(M, F).$$

PROOF. Theorem 50.4 in [Trè67] states that for all complete nuclear spaces E and all complete locally-convex spaces F,

$$E \widehat{\otimes}_\pi F = \operatorname{Hom}_{cts}(E^\vee, F)$$

where $\widehat{\otimes}_\pi$ is the completed projective tensor product and $E^\vee$ is the strong dual of E. Here Hom_{cts} denotes the space of continous linear maps, and we view this just as an isomorphism of vector spaces with no topoloy.

In particular, we have

$$C^\infty(M) \widehat{\otimes}_\pi F = \operatorname{Hom}_{cts}(\mathcal{D}_c(M), F).$$

A result of Grothendieck [Gro52] tells us that, for all complete locally convex spaces F,

$$C^\infty(M) \widehat{\otimes}_\pi F = C^\infty(M, F).$$

Now, lemma **??** tells us that $\mathcal{D}_c(M)$, equipped with its nuclear topology, is bornological. It follows that, for all complete locally convex spaces F, we have

$$\operatorname{Hom}_{\mathrm{BVS}}(\mathcal{D}_c(M), F_{born}) = C^\infty(M, F).$$

□

Our ultimate goal is to show that every object in the category of geometric objects is projective, i.e. that Hom's between geometric objects preserve quasi-isomorphisms. It will be useful, however, to have a stronger result: that the Hom-complex from an geometric object preserves quasi-isomorphisms between objects in a larger category. Let us introduce this (rather technical) larger category.

C.4.2.5 Definition. *We say a convenient cochain complex is* LC-complete *if it is the bornologification of a complex which is complete in the locally convex sense.*

We say a convenient cochain complex is weakly LC-complete *if it is in the smallest subcategory which contains all LC-complete objects and is closed under formation of cones, homotopy equivalences, and direct summands.*

Note that all geometric objects are weakly LC-complete.

C.4.2.6 Lemma. *Let W be an object of* ConVS^* *which is weakly LC-complete.*

Then there is a natural quasi-isomorphism in DVS^*

$$\mathrm{Hom}_{\mathrm{ConVS}}(\mathcal{D}_c(M), W) \to C^\infty(M, W).$$

PROOF. Note that the map $\delta : M \to \mathcal{D}_c(M)$ sending a point p to δ_p is smooth. Indeed, a smooth map from N to $\mathcal{D}_c(M)$ is the same as a continuous linear map $C^\infty(M) \to C^\infty(N)$. If $f : N \to M$ is smooth, then the map

$$\delta \circ f : N \to \mathcal{D}_c(M)$$

arises from the map

$$f^* : C^\infty(M) \to C^\infty(N)$$

and so is obviously smooth.

The natural map

$$\mathrm{Hom}(\mathcal{D}_c(M), W) \to C^\infty(M, W)$$

is obtained by composition with $\delta : M \to \mathcal{D}_c(M)$.

We need to show that this map is a quasi-isomorphism if W is a finite geometric object. Since we want to show that it's a quasi-isomorphism in DVS^*, we need to show that for all manifolds N, the map

$$\mathrm{Hom}(\mathcal{D}_c(M), C^\infty(N, W)) \to C^\infty(M \times N, W)$$

is a quasi-isomorphism.

Both functors in this equation commute with cone and formation of summands, and take homotopy equivalences to homotopy equivalences. By induction on the complexity of W, we reduce to the case when W is LC-complete, in which case the result follows immediately.

□

Now we arrive at the proof of the main proposition of this section.

C.4.2.7 Proposition. *Let V, W, W' be convenient cochain complexes where V is ind-geometric and W, W' are weakly LC-complete. Let $W \to W'$ be a quasi-isomorphism.*

(In particular, both W and W' could be geometric).

Then, the map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W')$$

is a quasi-isomorphism of objects of DVS^*.

PROOF. The functor $\mathrm{Hom}(-, W)$ commutes with cones and formation of summands. It takes sequential colimits of smoothly split maps to sequential limits of surjective maps of cochain complexes. It also takes homotopy equivalenes to homotopy equivalences. Thus, the usual inductive argument reduces us to the case when $V = C_c^\infty(M, E)$ for E an elliptic complex on M. But this is homotopy equivalent to $\overline{\mathscr{E}}_c$, the space of compactly supported distributional sections of E, by the Atiyah-Bott lemma. A further simple induction allows us to replace V by $\mathcal{D}_c(M)$, the space of compactly supported distributions on M. We have seen that for all weakly LC-complete complexes W, we have a quasi-isomorphism

$$\mathrm{Hom}(\mathcal{D}_c(M), W) \cong C^\infty(M, W).$$

Since the functor $C^\infty(M, -)$ preserves quasi-isomorphisms, the result follows. □

C.4.2.8 Corollary. *Let W, W' be both weakly LC-complete and ind-geometric. (For example, they could both be geometric).*

Then, a map $f : W \to W'$ is a quasi-isomorphism if and only if it's a homotopy equivalence.

PROOF. Suppose the map is a quasi-isomorphism. Then, the map

$$\mathrm{Hom}(W', W) \xrightarrow{f \circ -} \mathrm{Hom}(W', W')$$

is a quasi-isomorphism as well, by proposition C.4.2.7. A right homotopy inverse g is provided by an element in $\mathrm{Hom}(W', W)$ such that $f \circ g$ is cohomologous to the identity of W'. In a similar way we can consturct a right homotopy inverse $h : W \to W'$ to g. A standard argument shows that h and f are homotopic. □

C.5. Convenient pro-cochain complexes

Recall from section **??** that our factorization algebras take values in the category ProDVS^* of differentiable pro-cochain complexes. Recall that a differentiable pro-cochain complex is a differentiable cochain complex V^* equipped with a complete decreasing filtration, so that $V = \varprojlim V/F^iV$. We also require that the maps $V/F^iV \to V/F^jV$ for $i > j$ are fibrations. Maps between differentiable pro-cochain complexes must preserve the filtrations.

C.5.0.9 Definition. *We say that a differentiable pro-cochain complex is a convenient pro-cochain complex if*

(1) The maps $V/F^iV \to V/F^jV$ are smoothly split.
(2) Each of the quotients V/F^iV are convenient cochain complexes.

We say a convenient pro-cochain complex is geometric *(or ind-geometric, or weakly LC-complete) if each* $\mathrm{Gr}^i\, V$ *is geometric (or ind-geometric, or weakly LC-complete).*

We let ProConVS* *and* ProGeoVS* *denote the category of convenient and geometric pro-cochain complexes. Maps in these categories must preserve filtrations, and a map of convenient or geometric pro-cochain complexes is a quasi-isomorphism if and only if it is when viewed as a map of differentiable pro-cochain complexes.*

Because we assume that the maps are smoothly split, we can write any convenient procochain complex

$$V = \prod_{i \geq 0} V_i$$

where each of the V_i are convenient, and the differential maps V_i to the product $\prod_{j \geq i} V_j$.

C.5.0.10 Definition. *If V, W are convenient pro-cochain complexes, we define the tensor product*

$$V \widehat{\otimes}_\beta W = \varprojlim \left(V/F^iV \right) \widehat{\otimes}_\beta \left(W/F^jW \right).$$

This tensor product has a natural filtration, where we define $F^r(V \widehat{\otimes}_\beta W)$ to be the natural map

$$V \widehat{\otimes}_\beta W \to \prod_{i+j=r} (V/F^iV) \widehat{\otimes}_\beta (W/F^jW).$$

This tensor product represents the multicategory structure on ProConVS* *which is inherited from that on* ProDVS*.

Let $\mathbb{Z}_{>0}$ denote the category whose objects are the positive integers and where there's a map $n \to m$ whenever $n > m$. There's a full embedding

$$\mathrm{ProConVS} \subset \mathrm{Fun}(\mathbb{Z}_{>0}, \mathrm{ConVS})$$

by sending V to the sequence V/F^iV. The difference between ProConVS and $\mathrm{Fun}(\mathbb{Z}_{>0}, \mathrm{ConVS})$ is that in the latter we do not require that the maps are smoothly split.

C.5.0.11 Lemma. *The tensor product on* ProConVS *commutes with all colimits which are also colimits in* $\mathrm{Fun}(\mathbb{Z}_{>0}, \mathrm{ConVS})$.

PROOF. This is immediate from the fact that the tensor product on ConVS commutes with colimits. ☐

In particular, the tensor product on ProConVS commutes with coproducts.

C.5.0.12 Definition. *We lift the Hom-complexes in* ProConVS* *to objects of* ProDVS* *as follows. We let* $F^r \operatorname{Hom}(V, W)$ *be the space of those maps* $V \to W$ *which, for all* i, *map* $F^i V \to F^{i+r} W$. *We make* $\operatorname{Hom}(V, W)$ *into a sheaf on the site of smooth manifolds by declaring that*

$$C^\infty(M, \operatorname{Hom}(V, W)) = \operatorname{Hom}(V, C^\infty(M, W)).$$

Then, one can check that

1. *The filtraton on* $\operatorname{Hom}(V, W)$ *is smoothly split.*
2. $\operatorname{Hom}(V, W) = \varprojlim \operatorname{Hom}(V, W) / F^r \operatorname{Hom}(V, W)$ *in the category* DVS*.
3. $\operatorname{Gr}^r \operatorname{Hom}(V, W) = \prod_i \operatorname{Hom}(\operatorname{Gr}^i V, \operatorname{Gr}^{i+r} W)$, *again in the category* DVS*.

All results stated earlier concerning the flatness and projectivity of geometric objects carry over to geometric pro-cochain complexes.

C.5.0.13 Proposition.

1. *If* V, W *are geometric pro-cochain complexes then so is* $V \widehat{\otimes}_\beta W$.
2. *If* V, W, W' *are geometric pro-cochain complexes and if* $W \to W'$ *is a quasi-isomorphism, then so is* $V \widehat{\otimes}_\beta W \to V \widehat{\otimes}_\beta W'$.
3. *Under the same hypothesis,* $\operatorname{Hom}(V, W) \to \operatorname{Hom}(V, W')$ *is a quasi-isomorphism of objects of* ProDVS*. *Therefore, a quasi-isomorphism of geometric pro-cochain complexes is a homotopy equivalence.*
4. *More generally, if* V *is an ind-geometric pro-cochain complex (meaning each* $\operatorname{Gr}^i V$*), and if* W, W' *are weakly LC-complete convenient pro-cochain complexes (meaning each* $\operatorname{Gr}^i W$ *and* $\operatorname{Gr}^i W'$ *are weakly LC-complete), then the map*

$$\operatorname{Hom}(V, W) \to \operatorname{Hom}(V, W')$$

is a quasi-isomorphism in ProDVS*.

PROOF. All statements follow essentially immediately by applying spectral sequence arguments together with the corresponding results in the category of ordinary, not pro, convenient cochain complexes. ☐

C.6. Pro-cochain complexes over $\mathbb{C}[[\hbar]]$

Let us view $\mathbb{C}[[\hbar]]$ (or $\mathbb{R}[[\hbar]]$ if we work over $\mathbb{R}$) as a pro-cochain complex by saying that $\hbar$ has degree 2, and that the filtration is induced from this grading.

We can then talk about geometric or convenient pro-cochain complexes which are $\mathbb{C}[[\hbar]]$-modules, meaning that they are $\mathbb{C}[[\hbar]]$-modules in the symmetric monoidal category of convenient pro-cochain complexes. Any $\mathbb{C}[[\hbar]]$-linear convenient pro-cochain complex$V$ has a canonical $\hbar$-adic filtration defined by saying that $F^i_\hbar V = \hbar^i V$. This filtration is automatically complete, because, if $F^n V$ denotes the filtration defining the pro-cochain complex structure, we have

$$F^i_\hbar V \subset F^{2i} V$$

so that modulo $F^n V$, the $\hbar$-adic filtration is finite. the image of multiplication by $\hbar^i$.

C.6.0.14 Definition. *A convenient pro-cochain complex V with an action of* $\mathbb{C}[[\hbar]]$ *is* quasi-free *if the $\hbar$-adic filtration is smoothly split, and if the maps*

$$\hbar : \operatorname{Gr}^i_\hbar V \to \operatorname{Gr}^{i+1}_\hbar V$$

given by multiplication by $\hbar$ on the associated graded of the $\hbar$-adic filtration, are isomorphisms.

Note that this implies that

$$\operatorname{Gr}_\hbar V = \operatorname{Gr}^0_\hbar V[[\hbar]]$$

so that the associated graded for the $\hbar$-adic filtration is actually free.

Since we assumed that the $\hbar$-adic filtration is smoothly split, there is an isomorphism of graded convenient pro-cochain complexes (without differentials)

$$V \cong \operatorname{Gr}^0_\hbar V[[\hbar]].$$

In what follows, all $\mathbb{C}[[\hbar]]$-linear convenient pro-cochain complexes will be quasi-free although we may not always specify this.

C.6.0.15 Definition. *Let V, W be quasi-free $\mathbb{C}[[\hbar]]$-modules in the category* ProConVS*. *We define the tensor product*

$$V \widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} W$$

to be the coequalizer in the category ProConVS* *of the diagram*

$$V \widehat{\otimes}_\beta \mathbb{C}[[\hbar]] \widehat{\otimes}_\beta W \rightrightarrows V \widehat{\otimes}_\beta W.$$

It is not obvious that this colimit exists in ProConVS*. However, the fact that V and W are quasi-free means that it does. To see this, note that we may as well assume that V and W are free, meaning that $V = V^0[[\hbar]]$ and $W = W^0[[\hbar]]$. Note also that

$$V^0[[\hbar]] = \varprojlim V^0 \widehat{\otimes}_\beta \mathbb{C}[\hbar]/\hbar^n = V^0 \widehat{\otimes}_\beta \mathbb{C}[[\hbar]]$$

by the way we defined the tensor product in the category ProConVS* and by the fact that the filtration on $\mathbb{C}[[\hbar]]$ is defined by giving $\hbar$ weight 2.

Recall also that the bornological tensor product – and it's completed version – commute with colimits. The tensor product on ProConVS also commutes with those colimits which are colimits in the category $\mathrm{Fun}(\mathbb{Z}_{>0}, \mathrm{ConVS}$ of sequences $\dots V_i \to V_{i-1} \cdots \to V_1$ of objects of ConVS. Therefore, in order to show that the desired colimit exists, it suffices to show that the coequalizer of the diagram

$$\mathbb{C}[[\hbar]] \otimes \mathbb{C}[[\hbar]] \otimes \mathbb{C}[[\hbar]] \rightrightarrows \mathbb{C}[[\hbar]] \otimes \mathbb{C}[[\hbar]]$$

taken in the category $\mathrm{Fun}(\mathbb{Z}_{>0}, \mathrm{ConVS})$ actually lives in the category ProConVS. This is easy to see.

C.6.0.16 Lemma. *Let us give the category* $\mathrm{ProConVS}^*_{\mathbb{C}[[\hbar]]}$ *of quasi-free* $\mathbb{C}[[\hbar]]$*modules in* $\mathrm{ProConVS}^*$ *the structure of multicategory induced from the symmetric monoidal structure we have just discussed.*

Then, the functor

$$\mathrm{ProConVS}^*_{\mathbb{C}[[\hbar]]} \to \mathrm{ProDVS}^*_{\mathbb{C}[[\hbar]]}$$

is a full embedding of multicategories, where the right hand side is endowed with the structure of $\mathbb{C}[[\hbar]]$*-multilinear, smooth, filtration-preserving maps.*

PROOF. It suffices to show that, given quasi-free $\mathbb{C}[[\hbar]]$-modules $V_1, \dots, V_n$ in $\mathrm{ProConVS}^*$, the map

$$V_1 \times \cdots \times V_n \to V_1 \widehat{\otimes}_{\beta, \mathbb{C}[[\hbar]]} \dots \widehat{\otimes}_{\beta, \mathbb{C}[[\hbar]]} V_n$$

is universal among smooth, filtration-preserving, $\mathbb{C}[[\hbar]]$-multilinear maps. This is straightforward. ☐

C.7. Observables as geometric pro-cochain complexes

So far, we have proved most of the statements given in the introduction. It remains to show that, for any quantum field theory, observables form a factorization algebra in the category of geometric pro-cochain complex.

C.7.0.17 Theorem. *Suppose we have a quantum field theory on a manifold M. Let* Obs^q *denote the factorization algebra of observables. This is a factorization algebra valued in the multi-category* $\mathrm{ProDVS}^*_{\mathbb{C}[[\hbar]]}$ *of* $\mathbb{C}[[\hbar]]$*-linear pro diffeological cochain complexes.*

Then,

1. *Each* $\mathrm{Obs}^q(U)$ *for open subsets $U \subset M$ is an geometric pro-cochain complex, which is quasi-free as a* $\mathbb{C}[[\hbar]]$*-module.*
2. *The smooth, filtraton-preserving,* $\mathbb{C}[[\hbar]]$*-linear maps*

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V)$$

for disjoint open subsets $U, V \subset M$, induce a quasi-isomorphism

$$\mathrm{Obs}^q(U) \widehat{\otimes}_\beta \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

Thus, Obs^q *is a factorization algebra valued in the symmetric monoidal category of geometric pro-cochain complexes.*

We have seen in **??** that there is a simplicial set of quantizations of a given classical field theory. If we have an n-simplex in this simplicial set, we find a family of theories over $\Omega^*(\triangle^n)$, and we have seen in **??** that there is a corresponding family of factorization algebras over $\Omega^*(\triangle^n)$. We would like this factorization algebra to also live in the category of geometric pro-cochain complexes, so that everything we do is in this category.

C.7.0.18 Theorem. *Further, if we have an n-simplex in the space of quantizations of a given classical theory, then observables for this form a factorization algebra in geometric pro-cochain complexes over the base ring* $\Omega^*(\triangle^n)$.

In particular, if we have two different quantizations of a given classical theory, which are homotopic, and whose corresponding factorization algebras are denoted Obs^q_0 *and* Obs^q_1*, then there is a factorization algebra valued in geometric pro-cochain complexes* $\mathrm{Obs}^q_{[0,1]}$ *together with quasi-isomorphisms*

$$\mathrm{Obs}^q_0 \leftarrow \mathrm{Obs}^q_{[0,1]} \to \mathrm{Obs}^q_1 .$$

The nice thing about geometric pro-cochain complexes is that every quasi-isomorphism is a homotopy equivalence. This, together with standard techniques from homotopical algebra, implies that, after replacing Obs^q_0 by a free resolution $\widetilde{\mathrm{Obs}}^q_0$, there is a quasi-isomorphism from $\widetilde{\mathrm{Obs}}^q_0$ to Obs^q_1. At some stage, to make such a theorem more clean, we should introduce the infinity-category of factorization algebras (modelling it as a simplicially enriched category, for example) by taking as objects, quasi-free factorization algebras, with the natural simplicial enrichment

PROOF. The proof of these theorems will take the rest of this section. Note that the statements are entirely functional-analytic: we have already showed, in chapter **??**, that a field theory gives us a factorization algebra valued in differentiable pro-cochain complexes, and that an n-simplex in the simplicial set gives rise to a family of factorization algebras over $\Omega^*(\triangle^n)$. We need to verify that these factorization algebras actually live in the full subcategory of geometric pro-cochain complexes.

However, since this is precisely the purpose for which geometric pro-cochain complexes were designed, the result is not so difficult. Suppose we have a family of quantum field theories on a manifold M over $\Omega^*(\triangle^n)$. Let $\mathrm{Obs}^q_{\triangle^n}$ denote the $\Omega^*(\triangle^n)$-linear factorization algebra associated to this family. Let $\mathscr{E}$ denote the sheaf of fields on M.

Recall that there is an isomorphism

$$\mathrm{Obs}^{cl}(U) \cong \widehat{\mathrm{Sym}}_{\pi}^{*}(\overline{\mathscr{E}}_{c}^{!}(U))$$

On the right hand side, $\widehat{\mathrm{Sym}}_{\pi}^{*}$ refers to the completed tensor product *taken using the projective tensor product* $\widehat{\otimes}_{\pi}$. We emphasize this point because the tensor product on convenient cochain complexes is defined using the completed bornological tensor product, which is in general different. We will denote the completed projective tensor product by $\widehat{\otimes}_{\pi}$ and corresponding symmetric powers by Sym_{π}^{k}. A priori, the completed projective tensor product of two convenient vector spaces is a topological vector space which may not be bornological. So we will replace it by its bornologification, which will be convenient (as it is the bornologification of something which is complete in the locally convex sense).

Theorem **??** shows that, for any open subset $U \subset M$, we have an isomorphism of differentiable pro-graded vector spaces complexes

$$\mathrm{Obs}_{\triangle^n}^{q}(U) \cong \mathrm{Obs}_{\triangle^n}^{cl}(U)[[\hbar]] = \widehat{\mathrm{Sym}}_{\pi}^{*}(\overline{\mathscr{E}}_{c}^{!}(U))[[\hbar]]$$

If we choose a parametrix Φ on U, then we can make this into an isomorphism of differential pro-cochain complexes where we endow the right hand side with the differential

$$\mathrm{d}_{\mathscr{E}} + \mathrm{d}_{\triangle^n} + \hbar \triangle_{\Phi} + \{I[\Phi], -\}_{\Phi}$$

where $I[\Phi]$ is the effective action functional for the restriction of the family of field theories on M to U, $\mathrm{d}_{\mathscr{E}}$ is the linear differential on the complex of fields, and $\mathrm{d}_{\triangle^n}$ is the de Rham differential on $\Omega^*(\triangle^n)$.

We need to show that this is an geometric pro-cochain complex. The first thing to note is that the defining filtration on $\mathrm{Obs}^q(U)$ is smoothly split. The associated graded of this filtration is

$$\mathrm{Gr}\,\mathrm{Obs}^{q}(U) \cong \Omega^{*}(\triangle^{n}, \widehat{\mathrm{Sym}}_{\pi}^{*}(\overline{\mathscr{E}}_{c}^{!}(U))[[\hbar]]$$

with the differential $\mathrm{d}_{\mathscr{E}} + \mathrm{d}_{tr^n} + \hbar \triangle_{\Phi}$.

To show that this is an geometric pro-cochain complex, we need to show that each Gr^n is. The filtration on $\mathrm{Obs}^q(U)$ is defined in such a way that

$$\mathrm{Gr}^{n}\,\mathrm{Obs}^{q}(U) = \Omega^{*}\left(\triangle^{n}, \oplus_{i=0}^{[n/2]} \hbar^{i}\,\mathrm{Sym}_{\pi}^{n-2i}\,\overline{E}_{c}^{!}(U)\right).$$

The differential as before is $\mathrm{d}_{\mathscr{E}} + \mathrm{d}_{\triangle^n} + \hbar \triangle_{\Phi}$. Evidently, Gr^n is obtained from the complexes $\Omega^*(\triangle^n, \mathrm{Sym}_{\pi}^{k}\,\overline{\mathscr{E}}_{c}^{!}(U))$ with differential $\mathrm{d}_{\mathscr{E}} + \mathrm{d}_{\triangle^n}$ by the formation of a finite number of cones. Since the category of geometric cochain complexes is closed under the formation of cones, it suffices to show that $\Omega^*(\triangle^n, \mathrm{Sym}_{\pi}^{k}\,\overline{\mathscr{E}}_{c}^{!}(U))$ is geometric.

Now, geometric cochain complexes are also closed under formation of direct summands, and this complex is a summand of

$$\Omega^{*}(\triangle^{n}, \overline{\mathscr{E}}_{c}^{!}(U)^{\widehat{\otimes}_{\pi} n}).$$

Now, $\Omega^*(\triangle^n)$ is continuously homotopy equivalent to the base field $\mathbb{R}$ (or $\mathbb{C}$), so that this complex is homotopy equivalent to $\overline{\mathscr{E}}^!_c(U)^{\widehat{\otimes}_\pi n}$.

Since $\overline{\mathscr{E}}^!_c(U)$ is homotopy equivalent to $\mathscr{E}^!_c(U)$, this complex is homotopy equivalent $\mathscr{E}^!_c(U)^{\widehat{\otimes}_\pi n}$, which is, in turn, geometric.

Now we have proved most of the statements we need. It remains to verify that the factorization structure map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V)$$

induces a quasi-isomorphism

$$\mathrm{Obs}^q(U) \widehat{\otimes}_{\beta,\mathbb{C}[[\hbar]]} \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

Since the $\mathbb{C}[[\hbar]]$-modules $\mathrm{Obs}^q(U)$ are quasi-free, it suffices to prove the corresponding statement for classical observables. Thus, we need to show that the map

$$\widehat{\mathrm{Sym}}^*_\pi(\overline{\mathscr{E}}^!_c(U)) \widehat{\otimes}_\beta \widehat{\mathrm{Sym}}^*_\pi(\overline{\mathscr{E}}^!_c(V)) \to \widehat{\mathrm{Sym}}^*_\pi(\overline{\mathscr{E}}^!_c(U) \oplus \overline{\mathscr{E}}^!_c(V))$$

is a quasi-isomorphism, where both sides are endowed with the differential arising from the linear differential on $\overline{\mathscr{E}}^!_c$. Since $\overline{\mathscr{E}}^!_c(U)$ is continuously homotopy equivalent to $\mathscr{E}^!_c(U)$, it suffices to prove the same statement if $\overline{\mathscr{E}}^!_c$ is replaced everywhere by $\mathscr{E}^!_c$.

Now, the completed bornological and completed projective tensor products are bornologically isomorphic for spaces of compactly-supported smooth sections of a vector bundle on a manifold (this is lemma **??** need to write a proof of this fact our avoid using it...). It follows that we need to show that the map

$$\widehat{\mathrm{Sym}}^*_\beta(\mathscr{E}^!_c(U)) \widehat{\otimes}_\beta \widehat{\mathrm{Sym}}^*_\beta(\mathscr{E}^!_c(V)) \to \widehat{\mathrm{Sym}}^*_\beta(\mathscr{E}^!_c(U) \oplus \mathscr{E}^!_c(V))$$

is an isomorphism. The way we have completed the bornological tensor product when we deal with pro-objects means that it suffices to show, for any convenient vector spaces A and B, that

$$\mathrm{Sym}^n_\beta(A \oplus B) = \oplus_{i+j=n} \mathrm{Sym}^i_\beta(A) \widehat{\otimes}_\beta \mathrm{Sym}^j_\beta(A).$$

But this statement is true in any symmetric monoidal category where the tensor product commutes with finite colimits.

□

Next, we will show the following.

C.7.0.19 Proposition. *If we have a countable Weiss cover $\mathfrak{U}$ of an open subset U of X, and if we have a prefactorization algebra $\mathcal{F}$ on X associated to a quantum field theory, then the Čech complex $\check{C}(\mathfrak{U}, \mathcal{F})$ is an object of the category of geometric pro-cochain complexes over $\mathbb{C}[[\hbar]]$.*

PROOF. To prove this, it suffices (since $\mathcal{F}(U))$ is a geometric pro-cochain complex) to show that the natural map

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a homotopy equivalence. We already know from **??** that it is a quasi-isomorphism. Further, corollary C.4.2.8 tells us that any quasi-isomorphism between ind-geometric and weakly LC-complete convenient cochain complexes is a homotopy equivalence. The same result holds (with the same argument) for convenient pro-cochain complexes. Since $\mathcal{F}(U)$ is geometric, it is automatically weakly LC-complete. Further, since $\mathfrak{U}$ is countable, it is clear that $\check{C}(\mathfrak{U}, \mathcal{F})$ is ind-geometric. Finally, since the explicit model we write down for $\mathcal{F}(U)$ is LC-complete, and since $\check{C}(\mathfrak{U}, \mathcal{F})$ is (as a graded vector space) a countable direct sum of spaces $\mathcal{F}(V)$ for various subsets V of U, $\check{C}(\mathfrak{U}, \mathcal{F})$ is also LC-complete. The result follows. $\square$

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