CYCLIC HOMOLOGY ARISING FROM ADJUNCTIONS

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ABSTRACT. Given a monad and a comonad, one obtains a distributive law between them from lifts of one through an adjunction for the other. In particular, this yields for any bialgebroid the Yetter-Drinfel'd distributive law between the comonad given by a module coalgebra and the monad given by a comodule algebra. It is this self-dual setting that reproduces the cyclic homology of associative and of Hopf algebras in the monadic framework of Böhm and Ştefan. In fact, their approach generates two duplicial objects and morphisms between them which are mutual inverses if and only if the duplicial objects are cyclic. A 2-categorical perspective on the process of twisting coefficients is provided and the rôle of the two notions of bimonad studied in the literature is clarified.

CONTENTS

1. INTRODUCTION

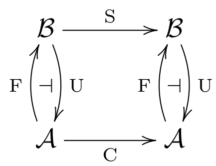
1.1. **Background and aim.** The Dold-Kan correspondence generalises chain complexes in abelian categories to general simplicial objects, and thus homological algebra to homotopical algebra. The classical homology theories defined by an augmented algebra (such as group, Lie algebra, Hochschild, de Rham and Poisson homology) become expressed as the homology of suitable comonads $\mathbb{T}$, defined via simplicial objects $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ obtained from the bar construction (see, *e.g.*, [Wei94]).

Connes' cyclic homology created a new paradigm of homology theories defined in terms of mixed complexes [Kas87, DK85]. The homotopical counterparts are cyclic [Con83] or more generally duplicial objects [DK85, DK87], and Böhm and Ştefan [BŞ08] showed how $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ becomes duplicial in the presence of a second comonad $\mathbb{S}$ compatible in a suitable sense with N, M and $\mathbb{T}$.

The aim of the present article is to study how the cyclic homology of associative algebras and of Hopf algebras in the original sense of Connes and Moscovici [CM98] fits into this monadic formalism, extending the construction from [KK11], and to clarify the rôle of different notions of bimonad in this generalisation.

1.2. **Distributive laws arising from adjunctions.** Inspired by [MW14, AC12] we begin by describing the relation of distributive laws between (co)monads and of lifts of one of them through an adjunction for the other. In particular, we have:

Theorem. Let $\mathrm{F} \dashv \mathrm{U}$ be an adjunction, $\mathbb{B} := (\mathrm{B}, \mu, \eta)$, $\mathrm{B} = \mathrm{UF}$, and $\mathbb{T} = (\mathrm{T}, \Delta, \varepsilon)$, $\mathrm{T} = \mathrm{FU}$, be the associated (co)monads, and $\mathbb{S} = (\mathrm{S}, \Delta^{\mathrm{S}}, \varepsilon^{\mathrm{S}})$ and $\mathbb{C} = (\mathrm{C}, \Delta^{\mathrm{C}}, \varepsilon^{\mathrm{C}})$ be comonads with a lax isomorphism $\Omega\colon \mathrm{CU} \to \mathrm{US}$,



If $\Lambda\colon \mathrm{FC} \to \mathrm{SF}$ corresponds under the adjunction to $\Omega\mathrm{F} \circ \mathrm{C}\eta\colon \mathrm{C} \to \mathrm{USF}$, where η is the unit of B, then the following are (mixed) distributive laws:

$$\theta\colon \mathrm{BC} = \mathrm{UFC} \xrightarrow{\mathrm{U}\Lambda} \mathrm{USF} \xrightarrow{\Omega^{-1}\mathrm{F}} \mathrm{CUF} = \mathrm{CB},$$

$$\chi\colon \mathrm{TS} = \mathrm{FUS} \xrightarrow{\mathrm{F}\Omega^{-1}} \mathrm{FCU} \xrightarrow{\Lambda\mathrm{U}} \mathrm{SFU} = \mathrm{ST}.$$

See Theorem 2.5 on p. 5 for a more detailed statement. For Eilenberg-Moore adjunctions ($\mathcal{B} = \mathcal{A}^{\mathbb{B}}$), such lifts $\mathbb{S}$ of a given comonad $\mathbb{C}$ correspond bijectively to mixed distributive laws between $\mathbb{B}$ and $\mathbb{C}$ (a dual statement holds for coKleisli adjunctions $\mathcal{A} = \mathcal{B}_{\mathbb{T}}$), *cf.* Section 2.4.

Sections 2–4 contain various technical results that we would like to add to the theory developed in [BŞ08], while the final two Sections 5 and 6 discuss examples.

First, we further develop the 2-categorical viewpoint of [BŞ12], interpreting the comparison functor from $\mathcal{B}$ to the Eilenberg-Moore category $\mathcal{A}^{\mathbb{B}}$ of $\mathbb{B}$ as a 1-cell in the 2-category of mixed distributive laws, and the passage from mixed distributive laws between

$\mathbb{B}, \mathbb{C}$ to distributive laws between $\mathbb{T}, \mathbb{S}$ in the case of an Eilenberg-Moore adjunction as the application of a 2-functor (Sections 2.5 and 2.6).

Secondly, Section 2.7 describes how different lifts S, V of a given functor C are related by a generalised Galois map $\Gamma^{\mathrm{S},\mathrm{V}}$ that will be used in subsequent sections.

1.3. **Coefficients.** In Section 3, we discuss left and right χ-coalgebras N respectively M that serve as coefficients of cyclic homology.

The structure of right χ-coalgebras is easily described in terms of $\mathbb{C}$-coalgebra structures on UM (Proposition 3.2). In the example from [KK11] associated to a Hopf algebroid H, these are simply right H-modules and left H-comodules, see Section 5.6 below.

The structure of left χ-coalgebras is more intricate. In the Hopf algebroid example, we present a construction from Yetter-Drinfel'd modules, but we do not have an analogue of Proposition 3.2 which characterises left χ-coalgebras in general. The Yetter-Drinfel'd condition is necessary for the well-definedness of the left χ-coalgebra structure, but not for that of the resulting duplicial object, see again Section 5.6.

The remainder of Section 3 explains the structure of entwined χ-coalgebras, which in the Hopf algebroid case are given by Hopf modules; these are homologically trivial (Proposition 4.5) and can be also interpreted as 1-cells to respectively from the trivial distributive law (Propositions 3.4 and 3.5). One reason for discussing them is to point out that general χ-coalgebras can not be reinterpreted as 1-cells.

1.4. **Duplicial objects.** Section 4 recalls the construction of duplicial objects. We emphasize the self-duality of the situation by defining in fact two duplicial objects $\mathrm{C}_{\mathbb{T}}(\mathrm{N},\mathrm{M})$ and $\mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N},\mathrm{M})$, arising from bar resolutions using $\mathbb{T}$ respectively $\mathbb{S}$. There is a canonical pair of morphisms of duplicial objects between these which are mutual inverses if and only if the two objects are cyclic (Proposition 4.4).

Furthermore, we describe in Section 4.6 the process of twisting a pair of coefficients M, N by what we called a factorisation in [KS14]. This is motivated by the example of the twisted cyclic homology of an associative algebra [KMT03] and constitutes our main application of the 2-categorical language.

1.5. **Hopf monads.** One of our motivations in this project is to understand how various notions of bimonads studied in the literature lead to examples of the above theory that generalise known ones arising from bialgebras and bialgebroids.

All give rise to distributive laws, but it seems to us that opmodule adjunctions over opmonoidal adjunctions as studied recently by Aguiar and Chase [AC12] are the underpinning of the cyclic homology theories from noncommutative geometry: such adjunctions are associated to opmonoidal adjunctions

$$\mathcal{E} \underset{\mathrm{E}}{\overset{\mathrm{H}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{H}\ ,$$

so here $\mathcal{H}$ and $\mathcal{E}$ are monoidal categories, E is a strong monoidal functor and H is an opmonoidal functor, see Section 5.1. In the key example, $\mathcal{H}$ is the category H-Mod of modules over a bialgebroid H and $\mathcal{E}$ is the category of bimodules over the base algebra A of H. In the special case of the cyclic homology of an associative algebra A, we have $\mathcal{H} = \mathcal{E}$ and $\mathrm{H} = \mathrm{E} = \mathsf{id}$, so this adjunction is irrelevant. Now the actual opmodule adjunctions defining cyclic homology are formed by an $\mathcal{H}$-module category $\mathcal{B}$ and an $\mathcal{E}$-module category $\mathcal{A}$. In the example, one can pick any H-module coalgebra C and any H-comodule algebra B, take $\mathcal{B}$ to be the category B-Mod of B-modules, $\mathcal{A}$ be the category A-Mod of A-modules, and the pair of comonads $\mathbb{S}, \mathbb{C}$ is given by $C \otimes_A -$. To obtain the cyclic homology of an associative algebra one takes $\mathcal{B}$ to be the category of A-bimodules (or rather right A^{e}-modules). Another very natural example is given by a quantum homogeneous space [MS99], where $A = k$ is commutative, H is a Hopf algebra, B is a left coideal subalgebra and $C := A/AB^+$ where B^+ is the kernel of the counit of H restricted to B. So here the distributive law arises from the fact that B admits a C-Galois extension to a Hopf algebra H; following, *e.g.,* [MM02] we call (B, C) a Doi-Koppinen datum.

Bimonads in the sense of Mesablishvili and Wisbauer also provide examples of the theory considered. There is no monoidal structure required on the categories involved, but instead we have $\mathrm{B} = \mathrm{C}$, see Section 6. At the end of the paper we give an example of such a bimonad which is not related to bialgebroids and noncommutative geometry, but indicates potential applications of cyclic homology in computer science.

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2. DISTRIBUTIVE LAWS

2.1. **Distributive laws.** We assume the reader is familiar with (co)monads and their (co)algebras (see, *e.g.,* [ML98]), but we briefly recall the notions of (co)lax morphisms and distributive laws, see, *e.g.,* [Lei04] for more background.

Definition 2.1. Let $\mathbb{B} = (\mathrm{B}, \mu^{\mathrm{B}}, \eta^{\mathrm{B}})$ and $\mathbb{A} = (\mathrm{A}, \mu^{\mathrm{A}}, \eta^{\mathrm{A}})$ be monads on categories $\mathcal{C}$ respectively $\mathcal{D}$, and let $\Sigma\colon \mathcal{C} \to \mathcal{D}$ be a functor. A natural transformation $\sigma\colon \mathrm{A}\Sigma \to \Sigma\mathrm{B}$ is called a *lax morphism of monads* if the two diagrams

$$\begin{array}{ccccc} \mathrm{AA}\Sigma & \xrightarrow{\mathrm{A}\sigma} & \mathrm{A}\Sigma\mathrm{B} & \xrightarrow{\sigma\mathrm{B}} & \Sigma\mathrm{BB} \\ \downarrow{\scriptstyle \mu^{\mathrm{A}}\Sigma} & & & & \downarrow{\scriptstyle \Sigma\mu^{\mathrm{B}}} \\ \mathrm{A}\Sigma & & \xrightarrow[\sigma]{} & & \Sigma\mathrm{B} \end{array} \qquad \begin{array}{ccc} \Sigma & \xrightarrow{\eta^{\mathrm{A}}\Sigma} & \mathrm{A}\Sigma \\ & \searrow{\scriptstyle \Sigma\eta^{\mathrm{B}}} & \downarrow{\scriptstyle \sigma} \\ & & \Sigma\mathrm{B} \end{array}$$

commute. We denote this by $\sigma\colon \mathbb{A}\Sigma \to \Sigma\mathbb{B}$.

Analogously, one defines *colax morphisms* $\sigma\colon \Sigma\mathbb{A} \to \mathbb{B}\Sigma$, where $\Sigma\colon \mathcal{D} \to \mathcal{C}$ and $\mathbb{A}, \mathbb{B}$ are as before, and (co)lax morphism of comonads.

Definition 2.2. A *distributive law* $\chi\colon \mathbb{AB} \to \mathbb{BA}$ between monads $\mathbb{A}, \mathbb{B}$ is a natural transformation $\chi\colon \mathrm{AB} \to \mathrm{BA}$ which is both a lax and a colax morphism of monads.

Analogously, one defines distributive laws between comonads and *mixed distributive law* [Bur73] between monads and comonads.

2.2. **The** 2**-categories** Dist **and** Mix**.** Since this will simplify the presentation of some results, we turn comonad and mixed distributive laws into the 0-cells of 2-categories Dist respectively Mix. This closely follows Street [Str72], see also [KS14]:

Definition 2.3. We denote by Dist the 2-category whose

(1) 0-cells are quadruples $(\mathcal{B}, \chi, \mathbb{T}, \mathbb{S})$ where $\chi\colon \mathbb{TS} \to \mathbb{ST}$ is a comonad distributive law on a category $\mathcal{B}$,
(2) 1-cells $(\mathcal{B}, \chi, \mathbb{T}, \mathbb{S}) \to (\mathcal{D}, \tau, \mathbb{G}, \mathbb{C})$ are triples (Σ, σ, γ), where $\Sigma\colon \mathcal{B} \to \mathcal{D}$ is a functor, $\sigma\colon \mathbb{G}\Sigma \to \Sigma\mathbb{T}$ is a lax morphism of comonads and $\gamma\colon \Sigma\mathbb{S} \to \mathbb{C}\Sigma$ is a colax morphism of comonads satisfying the Yang-Baxter equation, *i.e.,*

$$\begin{array}{ccccccc} & & \Sigma\mathrm{TS} & \xrightarrow{\Sigma\chi} & \Sigma\mathrm{ST} & & \\ & \nearrow^{\sigma\mathrm{S}} & & & & \searrow^{\gamma\mathrm{T}} & \\ \mathrm{G}\Sigma\mathrm{S} & & & & & & \mathrm{C}\Sigma\mathrm{T} \\ & \searrow_{\mathrm{G}\gamma} & & & & \nearrow_{\mathrm{C}\sigma} & \\ & & \mathrm{GC}\Sigma & \xrightarrow[\tau\Sigma]{} & \mathrm{CG}\Sigma & & \end{array}$$

commutes, and

(3) 2-cells $(\Sigma, \sigma, \gamma) \Rightarrow (\Sigma', \sigma', \gamma')$ are natural transformations $\alpha\colon \Sigma \to \Sigma'$ for which the diagrams

$$\begin{array}{ccc} \mathrm{G}\Sigma & \xrightarrow{\mathrm{G}\alpha} & \mathrm{G}\Sigma' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ \Sigma\mathrm{T} & \xrightarrow[\alpha\mathrm{T}]{} & \Sigma'\mathrm{T} \end{array} \qquad\qquad \begin{array}{ccc} \Sigma\mathrm{S} & \xrightarrow{\alpha\mathrm{S}} & \Sigma'\mathrm{S} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ \mathrm{C}\Sigma & \xrightarrow[\mathrm{C}\alpha]{} & \mathrm{C}\Sigma' \end{array}$$

commute.

In the sequel, we will denote 1-cells diagrammatically as:

$$\begin{array}{c} \mathbb{S} \overset{\chi}{\frown} \mathbb{T} \\ \mathcal{B} \\ \Big\downarrow{\scriptstyle(\Sigma,\sigma,\gamma)} \\ \mathcal{D} \\ \mathbb{C} \underset{\tau}{\smile} \mathbb{G} \end{array}$$

In a similar way, we define the 2-category Mix of mixed distributive laws.

2.3. **Distributive laws arising from adjunctions.** The topic of this paper is distributive laws that are compatible in a specific way with an adjunction for one of the involved comonads: let $\mathbb{B} = (\mathrm{B}, \mu, \eta)$ be a monad on a category $\mathcal{A}$. Suppose

$$\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B} \qquad (\mathrm{F} \dashv \mathrm{U})$$

is an adjunction for $\mathbb{B}$, that is, $\mathrm{B} = \mathrm{UF}$, and let $\mathbb{T} := (\mathrm{T}, \Delta, \varepsilon)$ with $\mathrm{T} := \mathrm{FU}$ be the induced comonad on $\mathcal{B}$.

Definition 2.4. If $\mathrm{S}\colon \mathcal{B} \to \mathcal{B}$ and $\mathrm{C}\colon \mathcal{A} \to \mathcal{A}$ are endofunctors for which the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathrm{S}} & \mathcal{B} \\ {\scriptstyle\mathrm{U}}\downarrow & & \downarrow{\scriptstyle\mathrm{U}} \\ \mathcal{A} & \xrightarrow[\mathrm{C}]{} & \mathcal{A} \end{array}$$

commutes up to a natural isomorphism $\Omega\colon \mathrm{CU} \to \mathrm{US}$, then we call C an *extension of* S and S a *lift of* C *through the adjunction*.

In general, any natural transformation $\Omega\colon \mathrm{CU} \to \mathrm{US}$ uniquely determines a *mate* $\Lambda\colon \mathrm{FC} \to \mathrm{SF}$ that corresponds to

$$\mathrm{C} \xrightarrow{\mathrm{C}\eta} \mathrm{CUF} \xrightarrow{\Omega\mathrm{F}} \mathrm{USF}$$

under the adjunction [Lei04]. The following theorem constructs a canonical pair of distributive laws from this mate of Ω:

Theorem 2.5. *Suppose that* S, C, *and* Ω *are as in Definition 2.4. Then:*

(1) *The natural transformation*

$$\theta\colon \mathrm{BC} = \mathrm{UFC} \xrightarrow{\mathrm{U}\Lambda} \mathrm{USF} \xrightarrow{\Omega^{-1}\mathrm{F}} \mathrm{CUF} = \mathrm{CB}$$

is a lax endomorphism of the monad $\mathbb{B}$.

(2) *The natural transformation*

$$\chi\colon \mathrm{TS} = \mathrm{FUS} \xrightarrow{\mathrm{F}\Omega^{-1}} \mathrm{FCU} \xrightarrow{\Lambda\mathrm{U}} \mathrm{SFU} = \mathrm{ST}$$

is a lax endomorphism of the comonad $\mathbb{T}$.

(3) *The lax morphism θ is unique such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{UFCU} & & \xrightarrow{\theta\mathrm{U}} & & \mathrm{CUFU} \\ {\scriptstyle \mathrm{UF}\Omega}\downarrow & & & & \downarrow{\scriptstyle \mathrm{CU}\varepsilon} \\ \mathrm{UFUS} & \xrightarrow[\mathrm{U}\varepsilon\mathrm{S}]{} & \mathrm{US} & \xrightarrow[\Omega^{-1}]{} & \mathrm{CU} \end{array}$$

(4) *The lax morphism χ is unique such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{US} & \xrightarrow{\Omega^{-1}} & \mathrm{CU} & \xrightarrow{C\eta\mathrm{U}} & \mathrm{CUFU} \\ {\scriptstyle \eta\mathrm{US}}\downarrow & & & & \downarrow{\scriptstyle \Omega\mathrm{FU}} \\ \mathrm{UFUS} & & \xrightarrow[\mathrm{U}\chi]{} & & \mathrm{USFU} \end{array}$$

(5) *If* C *is part of a comonad* $\mathbb{C} = (\mathrm{C}, \Delta^{\mathrm{C}}, \varepsilon^{\mathrm{C}})$ *and* S *is part of a comonad* $\mathbb{S} = (\mathrm{S}, \Delta^{\mathrm{S}}, \varepsilon^{\mathrm{S}})$ *and* Ω *is a lax morphism of comonads, then* θ *is a mixed distributive law and* χ *is a comonad distributive law.*

Proof. To prove (1), observe that the unit compatibility condition for θ is commutativity of the diagram

$$\begin{array}{ccc} \mathrm{UFC} & \xrightarrow{\mathrm{U}\Lambda} & \mathrm{USF} \\ {\scriptstyle \eta\mathrm{C}}\uparrow & & \downarrow{\scriptstyle \Omega^{-1}\mathrm{F}} \\ \mathrm{C} & \xrightarrow[\mathrm{C}\eta]{} & \mathrm{CUF} \end{array}$$

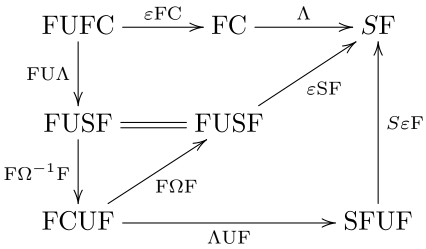
This diagram commutes if and only if the same diagram post-composed with $\Omega\mathrm{F}$ commutes, which is exactly the fact that $\Omega\mathrm{F} \circ \mathrm{C}\eta$ corresponds to Λ under the adjunction. The multiplication compatibility condition is given by commutativity of

$$\begin{array}{ccccc} \mathrm{BBC} & \xrightarrow{\mu\mathrm{C}} & \mathrm{BC} & \xrightarrow{\theta} & \mathrm{CB} \\ {\scriptstyle \mathrm{B}\theta}\downarrow & & & & \uparrow{\scriptstyle \mathrm{C}\mu} \\ \mathrm{BCB} & & \xrightarrow[\theta\mathrm{B}]{} & & \mathrm{CBB} \end{array}$$

which can be written as the outside of the diagram

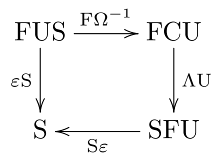
$$\begin{array}{ccccccc} \mathrm{UFUFC} & \xrightarrow{\mathrm{U}\varepsilon\mathrm{FC}} & \mathrm{UFC} & \xrightarrow{\mathrm{U}\Lambda} & \mathrm{USF} & \xrightarrow{\Omega^{-1}\mathrm{F}} & \mathrm{CUF} \\ {\scriptstyle \mathrm{UFU}\Lambda}\downarrow & & & & & & \\ \mathrm{UFUSF} & & & & \uparrow{\scriptstyle \mathrm{US}\varepsilon\mathrm{F}} & & \uparrow{\scriptstyle \mathrm{CU}\varepsilon\mathrm{F}} \\ {\scriptstyle \mathrm{UF}\Omega^{-1}\mathrm{F}}\downarrow & & & & & & \\ \mathrm{UFCUF} & & \xrightarrow[\mathrm{U}\Lambda\mathrm{UF}]{} & & \mathrm{USFUF} & \xrightarrow[\Omega^{-1}\mathrm{FUF}]{} & \mathrm{CUFUF} \end{array}$$

which will commute if both inner squares commute. The right-hand square commutes by naturality of Ω. The left-hand square is obtained by applying U to the outside of the diagram

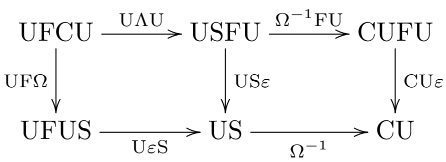


which commutes: the upper shape commutes by naturality of ε, the left-hand triangle clearly commutes, and the right-hand triangle commutes since both morphisms are mapped to Ω by the adjunction.

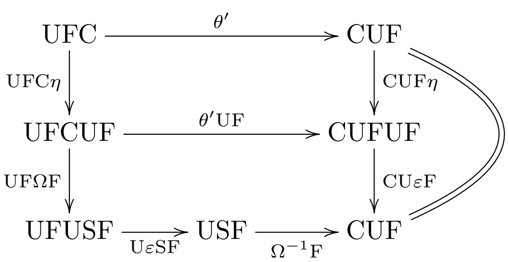
The proof for part (2) is similar to that of part (1). For part (3), observe that the counit condition for χ amounts to the commutativity of the diagram:



If we precompose this with $\mathrm{F}\Omega^{-1}$ and then apply U, we get the left-hand square of the diagram



The right-hand square commutes by naturality of Ω^{-1}, so the outer square commutes too, which is exactly the condition in part (3). Suppose that θ' is another lax morphism which makes the diagram commute. Consider the diagram:

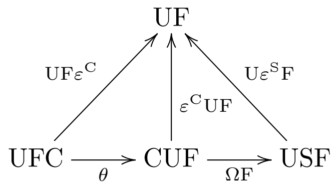


The rightmost shape commutes by one of the triangle identities for the adjunction, the bottom square commutes by hypothesis, and the upper square commutes by naturality of θ'. Therefore, the outer diagram commutes which says exactly that

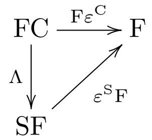
$$\theta' = \Omega^{-1}\mathrm{F} \circ \mathrm{U}(\varepsilon\mathrm{SF} \circ \mathrm{F}\Omega\mathrm{F} \circ \mathrm{FC}\eta) = \Omega^{-1}\mathrm{F} \circ \mathrm{U}\Lambda = \theta.$$

For part (4), the displayed diagram commutes for similar reasons to the diagram in part (3). Let χ' be another lax morphism such that the diagram commutes. Going round the diagram clockwise shows that χ and χ' are mapped to the same morphism under the adjunction, so $\chi = \chi'$.

For part (5), we will show that θ is a mixed distributive law, and remark that the proof that χ is a comonad distributive law is similar. Consider the following diagram:



The left hand triangle, which is the counit compatibility condition for θ, will commute if the right-hand and outer triangle commute. The right-hand triangle commutes because Ω is lax by hypothesis. The outer triangle is just U applied to the diagram



This commutes since the mate of a lax morphism is always colax [Lei04, p180]. By a similar argument, θ is compatible with the comultiplication. □

Definition 2.6. A comonad distributive law χ as in Theorem 2.5 is said to *arise from the adjunction* $\mathrm{F} \dashv \mathrm{U}$.

Example 2.7. A trivial example which will nevertheless play a rôle below is the case where $\mathrm{C} = \mathrm{B}$, $\mathrm{S} = \mathrm{T}$, and $\Omega = \mathsf{id}$. In this case, χ and θ are given by

$$\mathrm{TT} = \mathrm{FUFU} \xrightarrow{\varepsilon \mathrm{FU}} \mathrm{FU} \xrightarrow{\mathrm{F}\eta\mathrm{U}} \mathrm{FUFU} = \mathrm{TT},$$

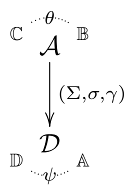
$$\mathrm{BB} = \mathrm{UFUF} \xrightarrow{\mathrm{U}\varepsilon\mathrm{F}} \mathrm{UF} \xrightarrow{\mathrm{UF}\eta} \mathrm{UFUF} = \mathrm{BB}.$$

2.4. **The Eilenberg-Moore and the coKleisli cases.** Functors do not necessarily lift respectively extend through an adjunction (for example, the functor on Set which assigns the empty set to each set does not lift to k-Mod), and if they do, they may not do so uniquely. Theorem 2.5 says only that once a lift respectively extension is chosen, there is a unique compatible pair of lax endomorphisms θ and χ.

One extremal situation in which specifying a lax endomorphism $\theta\colon \mathrm{C}\mathbb{B} \to \mathbb{B}\mathrm{C}$ uniquely determines a lift S of C is when $\mathcal{B}$ is the Eilenberg-Moore category $\mathcal{A}^{\mathbb{B}}$. In this case, S is defined on objects (X, α) by $\mathrm{S}(X, \alpha) = (\mathrm{C}X, \mathrm{C}\alpha \circ \theta X)$. Using Theorem 2.5 (with $\Omega = \mathsf{id}$), one recovers θ, see, *e.g.*, [App65, Joh75].

Dually, one can take $\mathcal{A}$ to be the coKleisli category $\mathcal{B}_{\mathbb{T}}$ in which case a lax endomorphism χ yields an extension C of a functor S. This means that every comonad distributive law and every mixed distributive law arises from an adjunction.

2.5. **The comparison functor is a 1-cell.** Let $\mathrm{F} \dashv \mathrm{U}$ be an adjunction and let $\mathbb{S}$ be the lift of a comonad $\mathbb{C}$ through the adjunction via Ω as in Section 2.3. Suppose we have a 1-cell



in the 2-category Mix. Let us denote with tildes the lifts of $\mathbb{A}, \mathbb{D}$, and ψ to the Eilenberg-Moore category $\mathcal{D}^{\mathbb{A}}$ outlined in Section 2.4. This gives rise to a 1-cell

$$\begin{array}{c} \mathbb{S} \overset{\chi}{\frown} \mathbb{T} \\ \mathcal{B} \\ \Big\downarrow {\scriptstyle (\tilde{\Sigma},\tilde{\sigma},\tilde{\gamma})} \\ \mathcal{D}^{\mathbb{A}} \\ \tilde{\mathbb{D}} \underset{\tilde{\psi}}{\smile} \tilde{\mathbb{A}} \end{array}$$

in Dist, where $\tilde{\Sigma}$ is defined on objects by

$$\tilde{\Sigma}X = \left(\Sigma\mathrm{U}X,\ \mathrm{A}\Sigma\mathrm{U}X \xrightarrow{\sigma\mathrm{U}X} \Sigma\mathrm{BU}X = \Sigma\mathrm{UFU}X \xrightarrow{\Sigma\mathrm{U}\varepsilon X} \Sigma\mathrm{U}X\right)$$

and on morphisms by $\tilde{\Sigma}f = \Sigma\mathrm{U}f$. The lax morphism $\tilde{\sigma}$ is defined by

$$\mathrm{A}\Sigma\mathrm{U}X \xrightarrow{\sigma\mathrm{U}X} \Sigma\mathrm{BU}X = \Sigma\mathrm{UT}X$$

and the colax morphism $\tilde{\gamma}$ is defined by

$$\Sigma\mathrm{US}X \xrightarrow{\Sigma\Omega^{-1}X} \Sigma\mathrm{CU}X \xrightarrow{\gamma\mathrm{U}X} \mathrm{D}\Sigma\mathrm{U}X$$

In the case that $\mathcal{A} = \mathcal{D}$, $\mathbb{B} = \mathbb{A}$, $\mathbb{C} = \mathbb{D}$, $\psi = \theta$ and $(\Sigma, \sigma, \gamma) = (\mathsf{id}, \mathsf{id}, \mathsf{id})$ is the trivial 1-cell, we get that $\tilde{\Sigma}$ is the *comparison functor* $\mathcal{B} \to \mathcal{A}^{\mathbb{B}} = \mathcal{D}^{\mathbb{A}}$.

2.6. **Interpretation as a 2-functor.** Consider the case that $\mathcal{B} = \mathcal{A}^{\mathbb{B}}$, $\mathbb{T} = \tilde{\mathbb{B}}$, $\mathbb{S} = \tilde{\mathbb{C}}$, and $\chi = \tilde{\theta}$. Since any 2-cell $\alpha\colon \Sigma \to \Sigma'$ lifts to a natural transformation $\tilde{\alpha}\colon \tilde{\Sigma} \to \tilde{\Sigma}'$, we can encode the above construction as the action of a 2-functor:

Proposition 2.8. *The assignment*

$$\begin{array}{c} \mathbb{C} \overset{\theta}{\frown} \mathbb{B} \\ \mathcal{A} \\ \Big\downarrow {\scriptstyle (\Sigma,\sigma,\gamma)} \\ \mathcal{D} \\ \mathbb{D} \underset{\psi}{\smile} \mathbb{A} \end{array} \longmapsto \begin{array}{c} \tilde{\mathbb{C}} \overset{\tilde{\theta}}{\frown} \tilde{\mathbb{B}} \\ \mathcal{A}^{\mathbb{B}} \\ \Big\downarrow {\scriptstyle (\tilde{\Sigma},\tilde{\sigma},\tilde{\gamma})} \\ \mathcal{D}^{\mathbb{A}} \\ \tilde{\mathbb{D}} \underset{\tilde{\psi}}{\smile} \tilde{\mathbb{A}} \end{array} \qquad \begin{array}{c} (\Sigma, \sigma, \gamma) \\ \Big\Downarrow {\scriptstyle \alpha} \\ (\Sigma', \sigma', \gamma') \end{array} \longmapsto \begin{array}{c} (\tilde{\Sigma}, \tilde{\sigma}, \tilde{\gamma}) \\ \Big\Downarrow {\scriptstyle \tilde{\alpha}} \\ (\tilde{\Sigma}', \tilde{\sigma}', \tilde{\gamma}') \end{array}$$

is a 2*-functor* $i\colon$ Mix $\to$ Dist.

Analogously, we obtain a 2-functor $j\colon$ Dist $\to$ Mix by taking extensions to coKleisli categories. It is those distributive laws in the image of the 2-functor i that are the main object of study in this paper.

2.7. **The Galois map.** Theorem 2.5 yields comonad distributive laws from lifts through an adjunction, and different lifts produce different distributive laws. Here we describe how these are related in terms of suitable generalisations of the Galois map from the theory of Hopf algebras.

Definition 2.9. If $\mathrm{S}, \mathrm{V}\colon \mathcal{B} \to \mathcal{B}$ are lifts of $\mathrm{C}\colon \mathcal{A} \to \mathcal{A}$ through $\mathrm{F} \dashv \mathrm{U}$ with isomorphisms $\Omega\colon \mathrm{CU} \to \mathrm{US}$ and $\Phi\colon \mathrm{CU} \to \mathrm{UV}$, we define a natural isomorphism

$$\Gamma^{\mathrm{S,V}}\colon \mathcal{B}(\mathrm{F}-, \mathrm{S}-) \to \mathcal{B}(\mathrm{F}-, \mathrm{V}-)$$

of functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to$ Set on components by the composition

$$\mathcal{B}(\mathrm{F}X, \mathrm{S}Y) \longrightarrow \mathcal{A}(X, \mathrm{US}Y) \longrightarrow \mathcal{A}(X, \mathrm{UV}Y) \longrightarrow \mathcal{B}(\mathrm{F}X, \mathrm{V}Y),$$

where the middle map is induced by $\Phi_Y \circ \Omega_Y^{-1}\colon \mathrm{US}Y \to \mathrm{UV}Y$ and the outer ones are induced by the adjunction $\mathrm{F} \dashv \mathrm{U}$. We call $\Gamma^{\mathrm{S,V}}$ the *Galois map* of the pair (S, V).

The following properties are easy consequences of the definition:

Proposition 2.10. *Let* S *and* V *be two lifts of an endofunctor* C *through an adjunction* $\mathrm{F} \dashv \mathrm{U}$*. Then:*

(1) *The inverse of* $\Gamma^{\mathrm{S},\mathrm{V}}$ *is given by* $\Gamma^{\mathrm{V},\mathrm{S}}$.
(2) *The Galois map* $\Gamma^{\mathrm{S},\mathrm{V}}$ *maps a morphism* $f\colon \mathrm{F}X \to \mathrm{S}Y$ *to*

$$\mathrm{F}X \xrightarrow{\mathrm{F}\eta X} \mathrm{FUF}X \xrightarrow{\mathrm{FU}f} \mathrm{FUS}Y \xrightarrow{\mathrm{F}(\Phi_Y \circ \Omega_Y^{-1})} \mathrm{FUV}Y \xrightarrow{\varepsilon \mathrm{V}Y} \mathrm{V}Y.$$

(3) *If* χ^{S} *and* χ^{V} *denote the lax morphisms determined by the two lifts, then*

$$\Gamma^{\mathrm{S},\mathrm{V}}(\chi^{\mathrm{S}}) = \chi^{\mathrm{V}}.$$

So, in the applications of Theorem 2.5, all distributive laws obtained from different lifts of a given comonad through an adjunction are obtained from each other by application of the appropriate Galois map.

The Galois map also relates different lifts of B itself: recall the trivial Example 2.7 of Theorem 2.5, where $\mathrm{C} = \mathrm{B}$ and $\mathrm{S} = \mathrm{T}$, and let V be any other lift of B through the adjunction. By taking X to be $\mathrm{U}Y$ for an object Y of $\mathcal{B}$, one obtains a Galois map $\Gamma^{\mathrm{T},\mathrm{V}}\colon \mathcal{B}(\mathrm{T}-,\mathrm{T}-) \to \mathcal{B}(\mathrm{T}-,\mathrm{V}-)$ that we can evaluate on $\mathrm{id}\colon \mathrm{T}Y \to \mathrm{T}Y$, which produces a natural transformation $\mathrm{T} \to \mathrm{V}$ that we denote by slight abuse of notation by $\Gamma^{\mathrm{T},\mathrm{V}}$ as well.

Adapting [MW10, Definition 1.3], we define:

Definition 2.11. We say that F is V-*Galois* if

$$\Gamma^{\mathrm{T},\mathrm{V}}\colon \mathrm{T} = \mathrm{FU} \xrightarrow{\mathrm{F}\eta\mathrm{U}} \mathrm{FUFU} = \mathrm{FUT} \xrightarrow{\mathrm{F}\Phi} \mathrm{FUV} \xrightarrow{\varepsilon\mathrm{V}} \mathrm{V}$$

is an isomorphism.

The following proposition provides the connection to Hopf algebra theory:

Proposition 2.12. *If* F *is* V*-Galois and* $\theta\colon \mathrm{BB} \to \mathrm{BB}$ *is the lax morphism arising from the lift* V *of* B*, then the natural transformation*

$$\beta\colon \mathrm{BB} \xrightarrow{\mathrm{B}\eta\mathrm{B}} \mathrm{BBB} \xrightarrow{\theta\mathrm{B}} \mathrm{BBB} \xrightarrow{\mathrm{B}\mu} \mathrm{BB}$$

is an isomorphism.

Proof. If F is V-Galois, then $\mathrm{U}\Gamma^{\mathrm{T},\mathrm{V}}\mathrm{F}$ is an isomorphism

$$\mathrm{UTF} = \mathrm{UFUF} \xrightarrow{\mathrm{UF}\eta\mathrm{UF}} \mathrm{UFUFUF} = \mathrm{UFUTF} \xrightarrow{\mathrm{UF}\Phi\mathrm{F}} \mathrm{UFUVF} \xrightarrow{\mathrm{U}\varepsilon\mathrm{VF}} \mathrm{UVF}.$$

Let now $\chi\colon \mathrm{TV} \to \mathrm{VT}$ be the lax morphism corresponding to θ as in Theorem 2.5. Inserting $\varepsilon\mathrm{V} = (\mathrm{V}\varepsilon) \circ \chi$ and $\mathrm{U}\chi \circ \mathrm{UF}\Phi = \Phi\mathrm{FU} \circ \theta\mathrm{U}$ and $\mathrm{B} = \mathrm{UF}$, the isomorphism becomes

$$\mathrm{UTF} = \mathrm{BB} \xrightarrow{\mathrm{B}\eta\mathrm{B}} \mathrm{BBB} \xrightarrow{\theta\mathrm{B}} \mathrm{BBB} = \mathrm{BUFUF} \xrightarrow{\Phi\mathrm{FUF}} \mathrm{UVFUF} \xrightarrow{\mathrm{UV}\varepsilon\mathrm{F}} \mathrm{UVF}$$

Finally, we have by construction $\mathrm{U}\varepsilon\mathrm{F} = \mu$, and using the naturality of Φ this gives $\mathrm{UV}\varepsilon\mathrm{F} \circ \Phi\mathrm{FUF} = \Phi\mathrm{F} \circ \mathrm{BU}\varepsilon\mathrm{F}$. Hence composing the above isomorphism with $\Phi^{-1}\mathrm{F}$ gives β. □

It is this associated map β that is used to distinguish Hopf algebras amongst bialgebras, see Section 6 below.

3. Coefficients

3.1. **Coalgebras over distributive laws.** Let $\mathbb{T} = (\mathrm{T}, \Delta^{\mathrm{T}}, \varepsilon^{\mathrm{T}})$ and $\mathbb{S} = (\mathrm{S}, \Delta^{\mathrm{S}}, \varepsilon^{\mathrm{S}})$ be comonads on a category $\mathcal{B}$, and let $\chi\colon \mathbb{TS} \to \mathbb{ST}$ be a distributive law. We now discuss χ-coalgebras, which serve as coefficients in the homological constructions in the next section.

Definition 3.1. A *right χ-coalgebra* is a triple $(\mathrm{M}, \mathcal{Y}, \rho)$, where $\mathrm{M}\colon \mathcal{Y} \to \mathcal{B}$ is a functor and $\rho\colon \mathrm{TM} \to \mathrm{SM}$ is a natural transformation such that the diagrams

$$
\begin{array}{ccccc}
\mathrm{TM} & \xrightarrow{\Delta^{\mathrm{T}}\mathrm{M}} & \mathrm{TTM} & \xrightarrow{\mathrm{T}\rho} & \mathrm{TSM} \\
\downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \chi\mathrm{M}} \\
\mathrm{SM} & \xrightarrow[\Delta^{\mathrm{S}}\mathrm{M}]{} & \mathrm{SSM} & \xleftarrow[\mathrm{S}\rho]{} & \mathrm{STM}
\end{array}
\qquad
\begin{array}{ccc}
 & & \mathrm{TM} \\
 & \swarrow{\scriptstyle \varepsilon^{\mathrm{T}}\mathrm{M}} & \downarrow{\scriptstyle \rho} \\
\mathrm{M} & \xleftarrow[\varepsilon^{\mathrm{S}}\mathrm{M}]{} & \mathrm{SM}
\end{array}
$$

commute. Dually, we define *left χ-coalgebras* $(\mathrm{N}, \mathcal{Z}, \lambda)$.

The following characterises right χ-coalgebras in the setting of Theorem 2.5.

Proposition 3.2. *In the situation of Theorem 2.5, let $\mathrm{M}\colon \mathcal{Y} \to \mathcal{B}$ be a functor.*

(1) *Right χ-coalgebra structures ρ on M correspond to $\mathbb{C}$-coalgebra structures ∇ on the functor $\mathrm{UM}\colon \mathcal{Y} \to \mathcal{A}$.*
(2) *Let S and V be two lifts of the functor C through the adjunction, and let χ^{S} and χ^{V} denote the comonad distributive laws determined by the lifts S and V respectively. Then the Galois map $\Gamma^{\mathrm{S},\mathrm{V}}$ maps right χ^{S}-coalgebra structures ρ^{S} on M bijectively to right χ^{V}-coalgebra structures ρ^{V} on M.*

Proof. For part (1), right χ-coalgebra structures $\rho\colon \mathrm{FUM} \to \mathrm{SM}$ are mapped under the adjunction to $\nabla\colon \mathrm{UM} \to \mathrm{USM} \cong \mathrm{CUM}$. Part (2) follows immediately since the Galois map is the composition of the adjunction isomorphisms and $\Phi \circ \Omega^{-1}$. $\square$

3.2. **Entwined χ-coalgebras.** In the remainder of this section, we discuss a class of coefficients that lead to contractible simplicial objects, see Proposition 4.5 below. In the Hopf algebroid setting, these are the Hopf (or entwined) modules as studied in [AC12, BM98]. First, we recall:

Definition 3.3. A $\mathbb{T}$*-coalgebra* is a triple $(\mathrm{M}, \mathcal{Y}, \nabla)$, where $\mathrm{M}\colon \mathcal{Y} \to \mathcal{B}$ is a functor and $\nabla\colon \mathrm{M} \to \mathrm{TM}$ is a natural transformation such that the diagrams

$$
\begin{array}{ccc}
\mathrm{M} & \xrightarrow{\nabla} & \mathrm{TM} \\
\downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \Delta^{\mathrm{T}}\mathrm{M}} \\
\mathrm{TM} & \xrightarrow[\mathrm{T}\nabla]{} & \mathrm{TTM}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{M} & \xrightarrow{\nabla} & \mathrm{TM} \\
 & \searrow\!\!\!\searrow & \downarrow{\scriptstyle \varepsilon^{\mathrm{T}}\mathrm{M}} \\
 & & \mathrm{M}
\end{array}
$$

commute.

Dually, one defines $\mathbb{T}$-opcoalgebras $(\mathrm{N}, \mathcal{Z}, \nabla)$ where $\nabla\colon \mathrm{N} \to \mathrm{NT}$, as well as algebras and opalgebras involving monads. Note that $\mathbb{T}$-coalgebras can be equivalently viewed as 1-cells from respectively to the trivial distributive law:

Proposition 3.4. *Given an $\mathbb{S}$-coalgebra $(\mathrm{M}, \mathcal{Y}, \nabla^{\mathrm{S}})$ and a $\mathbb{T}$-opcoalgebra $(\mathrm{N}, \mathcal{Z}, \nabla^{\mathrm{T}})$, there is a pair of 1-cells*

$$
\begin{array}{c}
\mathsf{id} \quad \overset{\mathsf{id}}{\mathcal{Y}} \quad \mathsf{id} \\
\Big\downarrow \big(\mathrm{M},\, \varepsilon^{\mathrm{T}}\mathrm{M},\, \nabla^{\mathrm{S}}\big) \\
\mathbb{S} \quad \underset{\chi}{\mathcal{B}} \quad \mathbb{T}
\end{array}
\qquad
\begin{array}{c}
\mathbb{S} \quad \overset{\chi}{\mathcal{B}} \quad \mathbb{T} \\
\Big\downarrow \big(\mathrm{N},\, \nabla^{\mathrm{T}},\, \mathrm{N}\varepsilon^{\mathrm{S}}\big) \\
\mathsf{id} \quad \underset{\mathsf{id}}{\mathcal{Z}} \quad \mathsf{id}
\end{array}
$$

and all 1-cells $\mathsf{id} \to \chi$ respectively $\chi \to \mathsf{id}$ are of this form.

Furthermore, these 1-cells can also be viewed as χ-coalgebras:

Proposition 3.5. *Let $\chi\colon \mathbb{TS} \to \mathbb{ST}$ be a comonad distributive law. Then:*

(1) *Any $\mathbb{S}$-coalgebra $(\mathrm{M}, \mathcal{Y}, \nabla^{\mathrm{S}})$ defines a right χ-coalgebra $(\mathrm{M}, \mathcal{Y}, \varepsilon^{\mathrm{T}}\nabla^{\mathrm{S}})$.*
(2) *Any $\mathbb{T}$-opcoalgebra $(\mathrm{N}, \mathcal{Z}, \nabla^{\mathrm{T}})$ defines a left χ-coalgebra $(\mathrm{N}, \mathcal{Z}, \nabla^{\mathrm{T}}\varepsilon^{\mathrm{S}})$.*

Definition 3.6. If a χ-coalgebra arises from an (op)coalgebra as in Proposition 3.5, then we call the χ-coalgebra *entwined*.

Note, however, that there is no obvious way to associate a 1-cell in Dist to an arbitrary right or left χ-coalgebra.

3.3. **Entwined algebras.** Finally, we describe how entwined χ-coalgebras are in some sense lifts of entwined algebras; throughout, $\theta\colon \mathbb{BC} \to \mathbb{CB}$ is a mixed distributive law between a monad $\mathbb{B}$ and a comonad $\mathbb{C}$ on a category $\mathcal{A}$.

Definition 3.7. Let $\mathrm{M}\colon \mathcal{Y} \to \mathcal{A}$ be a functor which has a $\mathbb{B}$-algebra structure $\beta\colon \mathrm{BM} \to \mathrm{M}$ and a $\mathbb{C}$-coalgebra structure $\nabla\colon \mathrm{M} \to \mathrm{CM}$. We say that the quadruple $(\mathrm{M}, \mathcal{Y}, \beta, \nabla)$ is an *entwined algebra with respect to* θ if the diagram

$$\begin{array}{ccccc} \mathrm{BM} & \xrightarrow{\beta} & \mathrm{M} & \xrightarrow{\nabla} & \mathrm{CM} \\ {\scriptstyle \mathrm{B}\nabla}\downarrow & & & & \uparrow{\scriptstyle \mathrm{C}\beta} \\ \mathrm{BCM} & & \xrightarrow[\theta\mathrm{M}]{} & & \mathrm{CBM} \end{array} \tag{3.1}$$

commutes.

Dually we define an entwined opalgebra structure on a functor $\mathrm{N}\colon \mathcal{A} \to \mathcal{Z}$ for a distributive law $\mathbb{CB} \to \mathbb{BC}$.

The following proposition explains the relation between entwined algebras and entwined right χ-coalgebras for distributive laws χ arising from an adjunction:

Proposition 3.8. *In the situation of Theorem 2.5, let* $\mathrm{M}\colon \mathcal{Y} \to \mathcal{B}$ *be a functor and let* $\nabla\colon \mathrm{M} \to \mathrm{SM}$ *be a natural transformation.*

(1) *If* ∇ *is an* $\mathbb{S}$*-coalgebra structure, then the structure morphisms*

$$\mathrm{BUM} = \mathrm{UFUM} \xrightarrow{\mathrm{U}\varepsilon\mathrm{M}} \mathrm{UM}, \qquad \mathrm{UM} \xrightarrow{\mathrm{U}\nabla} \mathrm{USM} \xrightarrow{\Omega^{-1}} \mathrm{CUM}$$

turn UM *into an entwined algebra with respect to* θ*.*

(2) *If* $\mathcal{B} = \mathcal{A}^{\mathbb{B}}$*, then the converse of (1) holds.*

Proof. For part (1), the morphism $\mathrm{BUM} \to \mathrm{UM}$ is the $\mathbb{B}$-algebra structure on M given by the comparison functor, and the morphism $\mathrm{UM} \to \mathrm{CUM}$ is the $\mathbb{C}$-coalgebra structure given by Proposition 3.2. The commutativity of (3.1) follows by applying the functor U to the Yang-Baxter condition for the 1-cell $\big(\mathrm{M},\ \varepsilon^{\mathrm{T}}\mathrm{M},\ \nabla^{\mathrm{S}}\big)$ of Proposition 3.4. For part (2), condition (3.1) means exactly that the $\mathbb{C}$-coalgebra structure defines a morphism in $\mathcal{A}^{\mathbb{B}}$, and hence lifts to an $\mathbb{S}$-coalgebra structure. ☐

Dually, entwined opalgebra structures on a $\mathbb{B}$-opalgebra $(\mathrm{N}, \mathcal{Z}, \omega)$ are related to left χ-coalgebras if the codomain $\mathcal{Z}$ of N is a category with coequalisers. First, we define a functor $\mathrm{N}_{\mathbb{B}}\colon \mathcal{A}^{\mathbb{B}} \to \mathcal{Z}$ that takes a $\mathbb{B}$-algebra morphism $f\colon (X, \alpha) \to (Y, \beta)$ to $\mathrm{N}_{\mathbb{B}}(f)$ defined using coequalisers:

$$\begin{array}{ccccc} \mathrm{NB}X & \overset{\omega_X}{\underset{\mathrm{N}\alpha}{\rightrightarrows}} & \mathrm{N}X & \xrightarrow{q_{(X,\alpha)}} & \mathrm{N}_{\mathbb{B}}(X,\alpha) \\ {\scriptstyle \mathrm{NB}f}\downarrow & & {\scriptstyle \mathrm{N}f}\downarrow & & \vdots\,{\scriptstyle \mathrm{N}_{\mathbb{B}}(f)} \\ \mathrm{NB}Y & \overset{\omega_Y}{\underset{\mathrm{N}\beta}{\rightrightarrows}} & \mathrm{N}Y & \xrightarrow[q_{(Y,\beta)}]{} & \mathrm{N}_{\mathbb{B}}(Y,\beta) \end{array}$$

Thus $\mathrm{N}_{\mathbb{B}}$ generalises the functor $- \otimes_B N$ defined by a left module N over a ring B on the category of right B-modules.

Suppose that θ is invertible, and that N admits the structure of an entwined θ^{-1}-opalgebra, with coalgebra structure $\nabla\colon \mathrm{N} \to \mathrm{CN}$. There are two commutative diagrams:

$$\begin{CD}
\mathrm{NB}X @>{\omega_X}>> \mathrm{N}X \\
@V{\nabla_{\mathrm{B}X}}VV @. \\
\mathrm{NCB}X @. @VV{\nabla_X}V \\
@V{\mathrm{N}\theta_X^{-1}}VV @. \\
\mathrm{NBC}X @>>{\omega_{\mathrm{C}X}}> \mathrm{NC}X
\end{CD}
\qquad
\begin{CD}
\mathrm{NB}X @>{\mathrm{N}\alpha}>> @. \mathrm{N}X \\
@V{\nabla_{\mathrm{B}X}}VV @. @. \\
\mathrm{NCB}X @. @. @VV{\nabla_X}V \\
@V{\mathrm{N}\theta_X^{-1}}VV @. @. \\
\mathrm{NBC}X @>>{\mathrm{N}\theta_X}> \mathrm{NCB}X @>>{\mathrm{NC}\alpha}> \mathrm{NC}X
\end{CD}$$

Hence, using coequalisers, ∇ extends to a natural transformation $\tilde{\nabla}\colon \mathrm{N}_{\mathbb{B}} \to \mathrm{N}_{\mathbb{B}}\tilde{\mathrm{C}}$, and in fact it gives $\mathrm{N}_{\mathbb{B}}$ the structure of a $\tilde{\mathbb{C}}$-opcoalgebra. Since $\tilde{\theta}^{-1}\colon \tilde{\mathbb{C}}\tilde{\mathbb{B}} \to \tilde{\mathbb{B}}\tilde{\mathbb{C}}$ is a comonad distributive law on $\mathcal{A}^{\mathbb{B}}$, Proposition 3.5 gives us the following:

Proposition 3.9. *The triple* $(\mathrm{N}_{\mathbb{B}}, \mathcal{Z}, \tilde{\nabla}\varepsilon)$ *is an entwined left* $\tilde{\theta}^{-1}$*-coalgebra.*

4. Duplicial objects

4.1. **The bar and opbar resolutions.** Let $\mathbb{T} = (\mathrm{T}, \Delta, \varepsilon)$ be a comonad on a category $\mathcal{B}$, and let $\mathrm{M}\colon \mathcal{Y} \to \mathcal{B}$ be a functor.

Definition 4.1. The *bar resolution of* M is the simplicial functor $\mathrm{B}(\mathbb{T}, \mathrm{M})\colon \mathcal{Y} \to \mathcal{B}$ defined by

$$\mathrm{B}(\mathbb{T}, \mathrm{M})_n = \mathrm{T}^{n+1}\mathrm{M}, \qquad d_i = \mathrm{T}^i\varepsilon\mathrm{T}^{n-i}\mathrm{M}, \qquad s_j = \mathrm{T}^j\Delta\mathrm{T}^{n-j}\mathrm{M},$$

where the face and degeneracy maps above are given in degree n. The *opbar resolution of* M, denoted $\mathrm{B}^{\mathrm{op}}(\mathbb{T}, \mathrm{M})$, is the simplicial functor obtained by taking the opsimplicial simplicial functor of $\mathrm{B}(\mathbb{T}, \mathrm{M})$. Explicitly:

$$\mathrm{B}^{\mathrm{op}}(\mathbb{T}, \mathrm{M})_n = \mathrm{T}^{n+1}\mathrm{M}, \qquad d_i = \mathrm{T}^{n-i}\varepsilon\mathrm{T}^i\mathrm{M}, \qquad s_j = \mathrm{T}^{n-j}\Delta\mathrm{T}^j\mathrm{M}.$$

Given any functor $\mathrm{N}\colon \mathcal{B} \to \mathcal{Z}$, we compose it with the above simplicial functors to obtain new simplicial functors that we denote by

$$\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M}) := \mathrm{NB}(\mathbb{T}, M), \qquad \mathrm{C}^{\mathrm{op}}_{\mathbb{T}}(\mathrm{N}, \mathrm{M}) := \mathrm{NB}^{\mathrm{op}}(\mathbb{T}, \mathrm{M}).$$

4.2. **Duplicial objects.** Duplicial objects were defined by Dwyer and Kan [DK85] as a mild generalisation of Connes' cyclic objects [Con83]:

Definition 4.2. A *duplicial object* is a simplicial object (C, d_i, s_j) together with additional morphisms $t\colon C_n \to C_n$ satisfying

$$d_i t = \begin{cases} t d_{i-1}, & 1 \leqslant i \leqslant n, \\ d_n, & i = 0, \end{cases} \qquad s_j t = \begin{cases} t s_{j-1}, & 1 \leqslant j \leqslant n, \\ t^2 s_n, & j = 0. \end{cases}$$

A duplicial object is *cyclic* if $T := t^{n+1} = \mathsf{id}$.

Equivalently, a duplicial object is a simplicial object which has in each degree an *extra degeneracy* $s_{-1}\colon C_n \to C_{n+1}$. This corresponds to t via

$$s_{-1} := t s_n, \quad t = d_{n+1} s_{-1}.$$

This turns a duplicial object also into a cosimplicial object, and hence a duplicial object C in an additive category carries a boundary and a coboundary map

$$b := \sum_{i=0}^{n} (-1)^i d_i, \quad s := \sum_{j=-1}^{n} (-1)^j s_j.$$

Dwyer and Kan called such chain and cochain complexes *duchain complexes* and showed that the normalised chain complex functor yields an equivalence between duplicial objects and duchain complexes in an abelian category, thus extending the classical Dold-Kan correspondence between simplicial objects and chain complexes.

If $f_n \in \mathbb{Z}[x]$ is given by $1 - xf_n(x) = (1-x)^{n+1}$ and $B := sf_n(bs)$, then one has

$$B^2 = 0, \quad bB + Bb = \mathsf{id} - T,$$

and in this way cyclic objects give rise to mixed complexes (C, b, B) in the sense of [Kas87] that can be used to define *cyclic homology*.

4.3. **The Böhm-Ştefan construction.** Let $(\mathcal{B}, \chi, \mathbb{T}, \mathbb{S})$ be a 0-cell in Dist, and let $(\mathrm{M}, \mathcal{Y}, \rho)$ and $(\mathrm{N}, \mathcal{Z}, \lambda)$ be right and left χ-coalgebras respectively. By abuse of notation, we let χ^n denote both natural transformations $\mathrm{T}^n\mathrm{S} \to \mathrm{S}\mathrm{T}^n$ and $\mathrm{T}\mathrm{S}^n \to \mathrm{S}^n\mathrm{T}$ obtained by repeated application of χ (up to horizontal composition of identities), where $\chi^0 = \mathsf{id}$. We furthermore define natural transformations

$$t_n^{\mathbb{T}} \colon \mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})_n \to \mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})_n, \quad t_n^{\mathbb{S}} \colon \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})_n \to \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})_n$$

by the diagrams

$$\begin{array}{ccc} \mathrm{NT}^n\mathrm{SM} & \xrightarrow{\mathrm{N}\chi^n\mathrm{M}} & \mathrm{NST}^n\mathrm{M} \\ {\scriptstyle \mathrm{NT}^n\rho}\uparrow & & \downarrow{\scriptstyle \lambda\mathrm{T}^n\mathrm{M}} \\ \mathrm{NT}^{n+1}\mathrm{M} & \xrightarrow[t_n^{\mathbb{T}}]{} & \mathrm{NT}^{n+1}\mathrm{M} \end{array} \qquad \begin{array}{ccc} \mathrm{NTS}^n\mathrm{M} & \xrightarrow{\mathrm{N}\chi^n\mathrm{M}} & \mathrm{NS}^n\mathrm{TM} \\ {\scriptstyle \lambda\mathrm{S}^n\mathrm{M}}\uparrow & & \downarrow{\scriptstyle \mathrm{NS}^n\rho} \\ \mathrm{NS}^{n+1}\mathrm{M} & \xrightarrow[t_n^{\mathbb{S}}]{} & \mathrm{NS}^{n+1}\mathrm{M} \end{array}$$

Theorem 4.3. *The simplicial functors* $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ *and* $\mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})$ *become duplicial functors with duplicial operators given by* $t^{\mathbb{T}}$ *respectively* $t^{\mathbb{S}}$.

Proof. The first operator being duplicial is exactly the case considered in [BŞ08], and the second follows from a slight modification of their proof. □

4.4. **Cyclicity.** For each $n \geqslant 0$, we define a morphism $R_n \colon \mathrm{NT}^{n+1}\mathrm{M} \to \mathrm{NS}^{n+1}\mathrm{M}$ in the following way. For each $0 \leqslant i \leqslant n$, let $r_{i,n}$ denote the morphism

$$\mathrm{NS}^i\mathrm{T}^{n+1-i}\mathrm{M} \xrightarrow{\mathrm{NS}^i\mathrm{T}^{n-i}\rho} \mathrm{NS}^i\mathrm{T}^{n-i}\mathrm{SM} \xrightarrow{N\mathrm{S}^i\chi^{n-i}\mathrm{M}} \mathrm{NS}^{i+1}\mathrm{T}^{n-i}\mathrm{M}.$$

Then set

$$R_n := r_{n,n} \circ \cdots \circ r_{0,n}.$$

Similarly, we can define a morphism $L_n \colon \mathrm{NS}^{n+1}\mathrm{M} \to \mathrm{NT}^{n+1}\mathrm{M}$ whose definition involves the left χ-coalgebra structure λ on N.

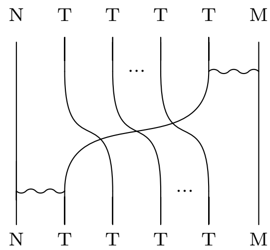
Proposition 4.4. *The above construction defines two morphisms*

$$\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M}) \xrightarrow{R} \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M}), \qquad \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M}) \xrightarrow{L} \mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$$

of duplicial functors. Furthermore, $L \circ R = \mathsf{id}$ *if and only if* $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ *is cyclic, and* $R \circ L = \mathsf{id}$ *if and only if* $\mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})$ *is cyclic.*

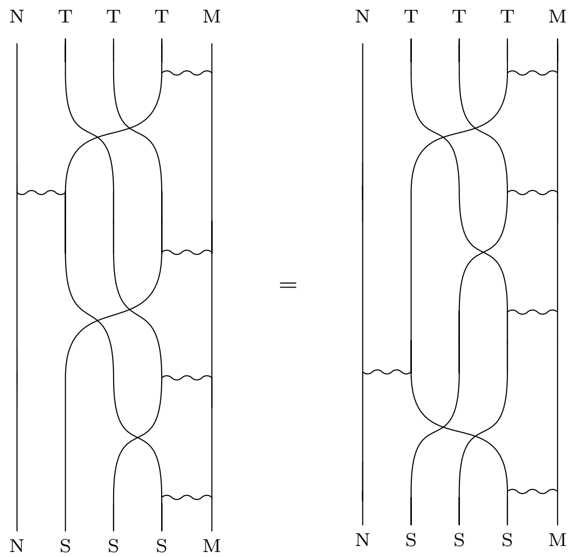
Proof. This is verified by straightforward computation. However, it is convenient to use a diagrammatic calculus as, *e.g.*, in [BŞ08], in which natural transformations $\mathrm{NVM} \to \mathrm{NWM}$ are visualised as string diagrams, where V and W are words in S, T. For example

$t^{\mathbb{T}}$ will be represented by the diagram

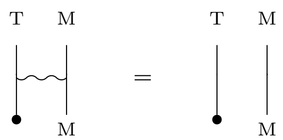


Crossing of strings represents the distributive law χ and the bosonic propagators represent the χ-coalgebra structures $\lambda\colon \mathrm{NS} \to \mathrm{NT}$ respectively $\rho\colon \mathrm{TM} \to \mathrm{SM}$.

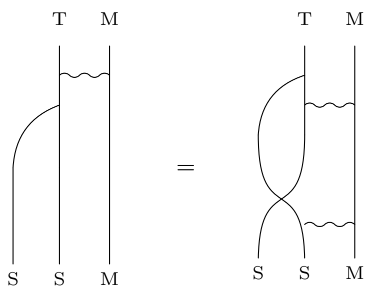
As a demonstration, the relation $Rt^{\mathbb{T}} = t^{\mathbb{S}}R$ for $n = 2$ becomes



which reflects the naturality of λ, ρ, and χ. Analogously, the identities $Rd_i = d_iR$ and $Rs_j = s_jR$ follow from the commutative diagrams in Definition 3.1, which are represented diagrammatically by



respectively



Similarly, L is a morphism of duplicial objects, and one has $(L \circ R)_n = (t_n^{\mathbb{T}})^{n+1}$ and $(R \circ L)_n = (t_n^{\mathbb{C}})^{n+1}$. □

4.5. **The case of entwined coalgebras.** As we had announced above, entwined coalgebras lead to trivial simplicial objects:

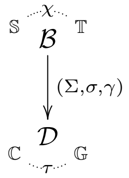
Proposition 4.5. *Let $\chi\colon \mathbb{TS} \to \mathbb{ST}$ be a comonad distributive law on a category $\mathcal{B}$, and let $(\mathrm{M}, \mathcal{Y}, \rho)$ and $(\mathrm{N}, \mathcal{Z}, \lambda)$ be left and right χ-coalgebras respectively. Suppose also that $\mathcal{Z}$ is an abelian category. If either of $(\mathrm{N}, \mathcal{Z}, \lambda)$, $(\mathrm{M}, \mathcal{Y}, \rho)$ is entwined, then the chain complexes associated to both $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ and $\mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})$ are contractible.*

Proof. If $(\mathrm{N}, \mathcal{Z}, \lambda)$ is entwined, there is a $\mathbb{T}$-opcoalgebra structure $\nabla\colon \mathrm{N} \to \mathrm{NT}$ on N. The morphisms $\nabla\mathrm{T}^n\mathrm{M}\colon \mathrm{NT}^{n+1}\mathrm{M} \to \mathrm{NT}^{n+2}\mathrm{M}$ provide a contracting homotopy for the complex associated to $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$, and the morphisms

$$\mathrm{NS}^{n+1}\mathrm{M} \xrightarrow{\nabla\mathrm{S}^{n+1}\mathrm{M}} \mathrm{NTS}^{n+1}\mathrm{M} \xrightarrow{\mathrm{N}\chi^{n+1}\mathrm{M}} \mathrm{NS}^{n+1}\mathrm{TM} \xrightarrow{\mathrm{NS}^{n+1}\rho} \mathrm{NS}^{n+2}\mathrm{M}$$

provide a contracting homotopy for the complex associated to $\mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}, \mathrm{M})$. The other case is similar. □

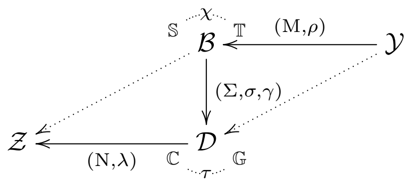
4.6. **Twisting by 1-cells.** In this section, we show how factorisations of distributive laws as considered in [KS14] give rise to morphisms between duplicial functors of the form considered above. To this end, fix a 1-cell in the 2-category Dist:



Lemma 4.6. *Let $(\mathrm{M}, \mathcal{Y}, \rho)$ be a right χ-coalgebra. Then $(\Sigma\mathrm{M}, \mathcal{Y}, \gamma\mathrm{M} \circ \Sigma\rho \circ \sigma\mathrm{M})$ is a right τ-coalgebra.*

Proof. This is proved for the case that $\chi = \tau$ in [KS14], but the same proof applies to this slightly more general situation. □

Dually, left τ-coalgebras $(\mathrm{N}, \mathcal{Z}, \rho)$ define left χ-coalgebras $(\mathrm{N}\Sigma, \mathcal{Z}, \mathrm{N}\sigma \circ \lambda\Sigma \circ \mathrm{N}\gamma)$. The following diagram illustrates the situation:



The dotted arrows represent the induced χ-coalgebras from Lemma 4.6.

Hence Theorem 4.3 and Lemma 4.6 yield duplicial structures on the simplicial functors

$$\mathrm{C}_{\mathbb{T}}(\mathrm{N}\Sigma, \mathrm{M}), \quad \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}\Sigma, \mathrm{M}), \quad \mathrm{C}_{\mathbb{G}}(\mathrm{N}, \Sigma\mathrm{M}), \quad \mathrm{C}^{\mathrm{op}}_{\mathbb{C}}(\mathrm{N}, \Sigma\mathrm{M}),$$

and from Proposition 4.4 we obtain morphisms

$$\mathrm{C}_{\mathbb{T}}(\mathrm{N}\Sigma, \mathrm{M}) \xrightarrow{R^\chi} \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}\Sigma, \mathrm{M}), \qquad \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}\Sigma, \mathrm{M}) \xrightarrow{L^\chi} \mathrm{C}_{\mathbb{T}}(\mathrm{N}\Sigma, \mathrm{M}),$$

$$\mathrm{C}_{\mathbb{G}}(\mathrm{N}, \Sigma\mathrm{M}) \xrightarrow{R^\tau} \mathrm{C}^{\mathrm{op}}_{\mathbb{C}}(\mathrm{N}, \Sigma\mathrm{M}), \qquad \mathrm{C}^{\mathrm{op}}_{\mathbb{C}}(\mathrm{N}, \Sigma\mathrm{M}) \xrightarrow{L^\tau} \mathrm{C}_{\mathbb{G}}(\mathrm{N}, \Sigma\mathrm{M})$$

of duplicial objects which determine the cyclicity of each functor.

Additionally, repeated application of $\sigma\colon \mathrm{G}\Sigma \to \Sigma\mathrm{T}$ and $\gamma\colon \Sigma\mathrm{S} \to \mathrm{C}\Sigma$ yields two duplicial morphisms

$$\mathrm{C}_{\mathbb{G}}(\mathrm{N}, \Sigma\mathrm{M}) \longrightarrow \mathrm{C}_{\mathbb{T}}(\mathrm{N}\Sigma, \mathrm{M}), \qquad \mathrm{C}^{\mathrm{op}}_{\mathbb{S}}(\mathrm{N}\Sigma, \mathrm{M}) \longrightarrow \mathrm{C}^{\mathrm{op}}_{\mathbb{C}}(\mathrm{N}, \Sigma\mathrm{M}).$$

Note that for arbitrary functors M and N these are simplicial morphisms which become duplicial morphisms if M and N have coalgebra structures.

5. Hopf monads and Hopf algebroids

5.1. **Opmodule adjunctions.** One example of Theorem 2.5 is provided by an opmonoidal adjunction between monoidal categories:

Definition 5.1. An adjunction

$$(\mathcal{E}, \otimes_{\mathcal{E}}, \mathbf{1}_{\mathcal{E}}) \underset{\mathrm{E}}{\overset{\mathrm{H}}{\rightleftarrows}} (\mathcal{H}, \otimes_{\mathcal{H}}, \mathbf{1}_{\mathcal{H}})$$

between monoidal categories is *opmonoidal* if both H and E are opmonoidal functors.

Some authors call these *comonoidal adjunctions* or *bimonads*. Thus by definition, there are natural transformations

$$\Xi\colon \mathrm{H}(X \otimes_{\mathcal{E}} Y) \to \mathrm{H}X \otimes_{\mathcal{H}} \mathrm{H}Y, \quad \Psi\colon \mathrm{E}(K \otimes_{\mathcal{H}} L) \to \mathrm{E}K \otimes_{\mathcal{E}} \mathrm{E}L,$$

and Ψ is in fact an isomorphism, see [AC12, BLV11, McC02, MW14, Moe02] for more information. It follows that

$$\mathrm{H}(\mathbf{1}_{\mathcal{E}}) \otimes_{\mathcal{H}} - \qquad \mathrm{EH}(\mathbf{1}_{\mathcal{E}}) \otimes_{\mathcal{E}} -$$

form a compatible pair of comonads as in Theorem 2.5 whose comonad structures are induced by the natural coalgebra (comonoid) structures on $\mathbf{1}_{\mathcal{E}}$.

However, the examples we are more interested in arise from *opmodule adjunctions*

$$(\mathcal{A}, \otimes_{\mathcal{A}}) \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} (\mathcal{B}, \otimes_{\mathcal{B}})$$

over $\mathcal{E} \rightleftarrows \mathcal{H}$, *cf.* [AC12, Definition 4.1.1]. Here $\mathcal{B}$ is an $\mathcal{H}$-module category with action $\otimes_{\mathcal{B}}\colon \mathcal{H} \times \mathcal{B} \to \mathcal{B}$, whereas $\mathcal{A}$ is an $\mathcal{E}$-module category with action $\otimes_{\mathcal{A}}\colon \mathcal{E} \times \mathcal{A} \to \mathcal{A}$, and there are natural transformations

$$\Theta\colon \mathrm{F}(Y \otimes_{\mathcal{A}} Z) \to \mathrm{H}Y \otimes_{\mathcal{B}} \mathrm{F}Z, \quad \Omega\colon \mathrm{U}(L \otimes_{\mathcal{B}} M) \to \mathrm{E}L \otimes_{\mathcal{A}} \mathrm{U}M$$

with Ω being an isomorphism (see [AC12, Proposition 4.1.2]).

Now any coalgebra C in $\mathcal{H}$ defines a compatible pair of comonads

$$\mathrm{S} = C \otimes_{\mathcal{B}} -, \quad \mathrm{C} = \mathrm{E}C \otimes_{\mathcal{A}} -$$

on $\mathcal{B}$ respectively $\mathcal{A}$. It is such an instance of Theorem 2.5 that provides the monadic generalisation of the setting from [KK11], see Section 5.6.

5.2. **Bialgebroids and Hopf algebroids.** Opmonoidal adjunctions can be seen as categorical generalisations of bialgebras and more generally (left) bialgebroids. We briefly recall the definitions but refer to [Böh09, KK11] for further details and references.

Definition 5.2. If E is a k-algebra, then an *E-ring* is a k-algebra map $\eta : E \to H$.

In particular, when $E = A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$ is the *enveloping algebra* of a k-algebra A, then H carries two A-bimodule structures given by

$$a \triangleright h \triangleleft b := \eta(a \otimes_k b)h, \quad a \blacktriangleright h \blacktriangleleft b := h\eta(b \otimes_k a).$$

Definition 5.3. A *bialgebroid* is an A^{e}-ring $\eta : A^{\mathrm{e}} \to H$ for which ${}_{\triangleright}H_{\triangleleft}$ is a coalgebra in $(A^{\mathrm{e}}\text{-}\mathsf{Mod}, \otimes_A, A)$ whose coproduct $\Delta\colon H \to H_{\triangleleft} \otimes_A {}_{\triangleright}H$ satisfies

$$a \blacktriangleright \Delta(h) = \Delta(h) \blacktriangleleft a, \quad \Delta(gh) = \Delta(g)\Delta(h),$$

and whose counit $\varepsilon\colon H \to A$ defines a unital H-action on A given by $h(a) := \varepsilon(a \blacktriangleright h)$.

Finally, by a Hopf algebroid we mean *left* rather than *full* Hopf algebroid, so there is in general no antipode [KR13]:

Definition 5.4 ([Sch00])**.** A *Hopf algebroid* is a bialgebroid with bijective *Galois map*

$$\beta\colon {}_\blacktriangleright H \otimes_{A^{\mathrm{op}}} H_\triangleleft \to H_\triangleleft \otimes_A {}_\triangleright H, \quad g \otimes_{A^{\mathrm{op}}} h \mapsto \Delta(g)h.$$

As usual, we abbreviate

$$\Delta(h) =: h_{(1)} \otimes_A h_{(2)}, \qquad \beta^{-1}(h \otimes_A 1) =: h_+ \otimes_{A^{\mathrm{op}}} h_-. \tag{5.1}$$

5.3. **The opmonoidal adjunction.** Every E-ring H defines a forgetful functor

$$\mathrm{E}\colon H\text{-}\mathsf{Mod} \to E\text{-}\mathsf{Mod}$$

with left adjoint $\mathrm{H} = H \otimes_E -$. In the sequel, we abbreviate $\mathcal{H} := H\text{-}\mathsf{Mod}$ and $\mathcal{E} := E\text{-}\mathsf{Mod}$. If H is a bialgebroid, then $\mathcal{H}$ is monoidal with tensor product $K \otimes_{\mathcal{H}} L$ of two left H-modules K and L given by the tensor product $K \otimes_A L$ of the underlying A-bimodules whose H-module structure is given by

$$h(k \otimes_{\mathcal{H}} l) := h_{(1)}(k) \otimes_A h_{(2)}(l).$$

So by definition, we have $\mathrm{E}(K \otimes_{\mathcal{H}} L) = \mathrm{E}K \otimes_A \mathrm{E}L$. The opmonoidal structure Ξ on H is defined by the map [BLV11, AC12]

$$\mathrm{H}(X \otimes_A Y) = H \otimes_{A^{\mathrm{e}}} (X \otimes_A Y) \to \mathrm{H}X \otimes_{\mathcal{H}} \mathrm{H}Y = (H \otimes_{A^{\mathrm{e}}} X) \otimes_A (H \otimes_{A^{\mathrm{e}}} Y),$$
$$h \otimes_{A^{\mathrm{e}}} (x \otimes_A y) \mapsto (h_{(1)} \otimes_{A^{\mathrm{e}}} x) \otimes_A (h_{(2)} \otimes_{A^{\mathrm{e}}} y).$$

Schauenburg proved that this establishes a bijective correspondence between bialgebroid structures on H and monoidal structures on H-Mod [Sch98, Theorem 5.1]:

Theorem 5.5. *The following data are equivalent for an A^{e}-ring $\eta\colon A^{\mathrm{e}} \to H$:*

1. *A bialgebroid structure on H.*
2. *A monoidal structure $(\otimes, \mathbf{1})$ on H-*Mod *such that the adjunction*

$$(A^{\mathrm{e}}\text{-}\mathsf{Mod}, \otimes_A, A) \rightleftarrows (H\text{-}\mathsf{Mod}, \otimes, \mathbf{1})$$

induced by η is opmonoidal.

Consequently, we obtain an opmonoidal monad

$$\mathrm{EH} = {}_\blacktriangleright H_\blacktriangleleft \otimes_{A^{\mathrm{e}}} -$$

on $\mathcal{E} = A^{\mathrm{e}}$-Mod. This takes the unit object A to the cocentre $H \otimes_{A^{\mathrm{e}}} A$ of the A-bimodule ${}_\blacktriangleright H_\blacktriangleleft$, and the comonad $\mathrm{H}(\mathbf{1}_{\mathcal{E}}) \otimes_{\mathcal{E}} -$ is given by

$$(H \otimes_{A^{\mathrm{e}}} A) \otimes_A -,$$

where the A-bimodule structure on the cocentre is given by the actions $\triangleright, \triangleleft$ on H.

The lift to $\mathcal{H} = H$-Mod takes a left H-module L to $(H \otimes_{A^{\mathrm{e}}} A) \otimes_A L$ with action

$$g((h \otimes_{A^{\mathrm{e}}} 1) \otimes_A l) = (g_{(1)}h \otimes_{A^{\mathrm{e}}} 1) \otimes_A g_{(2)}l,$$

and the distributive law resulting from Theorem 2.5 is given by

$$\chi\colon g \otimes_{A^{\mathrm{e}}} ((h \otimes_{A^{\mathrm{e}}} 1) \otimes_A l) \mapsto (g_{(1)}h \otimes_{A^{\mathrm{e}}} 1) \otimes_A (g_{(2)} \otimes_{A^{\mathrm{e}}} l).$$

That is, it is the map induced by the *Yetter-Drinfel'd braiding*

$$H_\blacktriangleleft \otimes_A {}_\triangleright H \to H_\triangleleft \otimes_A {}_\triangleright H, \quad g \otimes_A h \mapsto g_{(1)}h \otimes_A g_{(2)}.$$

For $A = k$, that is, when H is a Hopf algebra, and also trivially when $H = A^{\mathrm{e}}$, the monad and the comonad on A^{e}-Mod coincide and are also a bimonad in the sense of Mesablishvili and Wisbauer, *cf.* Section 6. An example where the two are different is the Weyl algebra, or more generally, the universal enveloping algebra of a Lie-Rinehart algebra [Hue98]. In these examples, A is commutative but not central in H in general, so ${}_\blacktriangleright H_\blacktriangleleft \otimes_{A^{\mathrm{e}}} -$ is different from $H_\triangleleft \otimes_A -$.

5.4. **Doi-Koppinen data.** The instance of Theorem 2.5 that we are most interested in is an opmodule adjunction associated to the following structure:

Definition 5.6. A *Doi-Koppinen datum* is a triple (H, C, B) of an H-module coalgebra C and an H-comodule algebra B over a bialgebroid H.

This means that C is a coalgebra in the monoidal category H-Mod. Dually, the category H-Comod of left H-comodules is also monoidal, and this defines the notion of a comodule algebra. Explicitly, B is an A-ring $\eta_B \colon A \to B$ together with a coassociative coaction

$$\delta \colon B \to H_{\triangleleft} \otimes_A B, \quad b \mapsto b_{(-1)} \otimes_A b_{(0)},$$

which is counital and an algebra map,

$$\eta_B(\varepsilon(b_{(-1)}))b_{(0)} = b, \quad (bd)_{(-1)} \otimes (bd)_{(0)} = b_{(-1)}d_{(-1)} \otimes b_{(0)}d_{(0)}.$$

Similarly, as in the definition of a bialgebroid itself, for this condition to be well-defined one must also require

$$b_{(-1)} \otimes_A b_{(0)}\eta_B(a) = a \blacktriangleright b_{(-1)} \otimes_A b_{(0)}.$$

The key example that reproduces [KK11] is the following:

5.5. **The opmodule adjunction.** For any Doi-Koppinen datum (H, C, B), the H-coaction δ on B turns the Eilenberg-Moore adjunction $A\text{-Mod} \rightleftarrows B\text{-Mod}$ for the monad $\mathrm{B} := B \otimes_A -$ into an opmodule adjunction for the opmonoidal adjunction $\mathcal{E} \rightleftarrows \mathcal{H}$ defined in Section 5.3. The $\mathcal{H}$-module category structure of B-Mod is given by the left B-action

$$b(l \otimes_A m) := b_{(-1)}l \otimes_A b_{(0)}m,$$

where $b \in B$, $l \in L$ (an H-module), and $m \in M$ (a B-module).

Hence, as explained in Section 5.1, C defines a compatible pair of comonads $C \otimes_A -$ on B-Mod and A-Mod. The distributive law resulting from Theorem 2.5 generalises the Yetter-Drinfel'd braiding, as it is given for a B-module M by

$$\begin{aligned} \chi \colon B \otimes_A (C \otimes_A M) &\to C \otimes_A (B \otimes_A M), \\ b \otimes_A (c \otimes_A m) &\mapsto b_{(-1)}c \otimes_A (b_{(0)} \otimes_A m). \end{aligned}$$

5.6. **The main example.** If H is a bialgebroid, then $C := H$ is a module coalgebra with left action given by multiplication and coalgebra structure given by that of H. If H is a Hopf algebroid, then $B := H^{\mathrm{op}}$ is a comodule algebra with unit map $\eta_B(a) := \eta(1 \otimes_k a)$ and coaction

$$\delta \colon H^{\mathrm{op}} \to H_{\triangleleft} \otimes_A {}_{\blacktriangleright}H^{\mathrm{op}}, \quad b \mapsto b_- \otimes_A b_+.$$

In the sequel we write B as $- \otimes_{A^{\mathrm{op}}} H$ rather than $H^{\mathrm{op}} \otimes_A -$ to work with H only. Then the distributive law becomes

$$\begin{aligned} \chi \colon (H \otimes_A M) \otimes_{A^{\mathrm{op}}} H &\to H \otimes_A (M \otimes_{A^{\mathrm{op}}} H), \\ (c \otimes_A m) \otimes_{A^{\mathrm{op}}} b &\mapsto b_-c \otimes_A (m \otimes_{A^{\mathrm{op}}} b_+), \end{aligned}$$

for $b, c \in H$.

Proposition 3.2 completely characterises the right χ-coalgebras: in this example, they are given by right H-modules and left H-comodules M with right χ-coalgebra structure

$$\rho : m \otimes_{A^{\mathrm{op}}} h \mapsto h_-m_{(-1)} \otimes_A m_{(0)}h_+.$$

Recall furthermore that there is no analogue of Proposition 3.2 for left χ-coalgebras. However, the specific example of a Hopf algebroid might provide some indication towards such a result. Indeed, here one can carry out an analogous construction of left χ-coalgebras associated to (left-left) Yetter-Drinfel'd modules:

Definition 5.7. A *Yetter-Drinfel'd module* over H is a left H-comodule and left H-module N such that for all $h \in H, n \in N$, one has

$$(hn)_{(-1)} \otimes_A (hn)_{(0)} = h_{+(1)} n_{(-1)} h_- \otimes_A h_{+(2)} n_{(0)}.$$

Each such Yetter-Drinfel'd module defines a left χ-coalgebra

$$\mathrm{N} := - \otimes_H N \colon H^{\mathrm{op}}\text{-}\mathsf{Mod} \to k\text{-}\mathsf{Mod}$$

whose χ-coalgebra structure is given by

$$\lambda : (h \otimes_A x) \otimes_H n \mapsto (xn_{(-1)+} h_+ \otimes_{A^{\mathrm{op}}} h_- n_{(-1)-}) \otimes_H n_{(0)}.$$

The resulting duplicial object $\mathrm{C}_{\mathbb{T}}(\mathrm{N}, \mathrm{M})$ is the one studied in [KK11, Kow13].

Identifying $(- \otimes_{A^{\mathrm{op}}} H) \otimes_H N \cong - \otimes_{A^{\mathrm{op}}} N$, the χ-coalgebra structure becomes

$$\lambda : (h \otimes_A x) \otimes_H n \mapsto xn_{(-1)+} h_+ \otimes_{A^{\mathrm{op}}} h_- n_{(-1)-} n_{(0)}.$$

Using this identification, we give explicit expressions of the operators L_n and R_n as well as $t_n^{\mathbb{T}}$ that appeared in Sections 4.3 and 4.4: first of all, observe that the right H-module structure on $\mathrm{S}M := H_\triangleleft \otimes_A M$ is given by

$$(h \otimes_A m) g := g_- h \otimes_A m g_+,$$

whereas the right H-module structure on $\mathrm{T}M := M \otimes_{A^{\mathrm{op}}} H_\triangleleft$ is given by

$$(m \otimes_{A^{\mathrm{op}}} h) g := m \otimes_{A^{\mathrm{op}}} hg.$$

The cyclic operator from Section 4.3 then results as

$$\begin{aligned} &t_n^{\mathbb{T}}(m \otimes_{A^{\mathrm{op}}} h^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n \otimes_{A^{\mathrm{op}}} n) \\ &= m_{(0)} h^1_+ \otimes_{A^{\mathrm{op}}} h^2_+ \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n_+ \\ &\quad \otimes_{A^{\mathrm{op}}} (n_{(-1)} h^n_- \cdots h^1_- m_{(-1)})_+ \otimes_{A^{\mathrm{op}}} (n_{(-1)} h^n_- \cdots h^1_- m_{(-1)})_- n_{(0)}, \end{aligned}$$

and for the operators L and R from Section 4.4 one obtains with the help of the properties [Sch00, Prop. 3.7] of the translation map (5.1):

$$\begin{aligned} L_n : (h^1 \otimes_A \cdots \otimes_A h^{n+1} \otimes_A m) \otimes_H n \mapsto \\ (m n_{(-1)+} h^1_+ \otimes_{A^{\mathrm{op}}} h^1_- h^2_+ \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^{n+1}_- n_{(-1)-}) \otimes_H n_{(0)}, \end{aligned}$$

along with

$$\begin{aligned} R_n : (m \otimes_{A^{\mathrm{op}}} h^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n \otimes_{A^{\mathrm{op}}} 1) \otimes_H n \mapsto \\ (m_{(-n-1)} \otimes_A m_{(-n)} h^1_{(1)} \otimes_A m_{(-n+1)} h^1_{(2)} h^2_{(1)} \otimes_A \cdots \\ \otimes_A m_{(-1)} h^1_{(n)} h^2_{(n-1)} \cdots h^n_{(1)} \otimes_A m_{(0)}) \otimes_H h^1_{(n+1)} h^2_{(n)} \cdots h^n_{(2)} n. \end{aligned}$$

Compare these maps with those obtained in [KK11, Lemma 4.10]. Hence, one has:

$$\begin{aligned} &(L_n \circ R_n)\big((m \otimes_{A^{\mathrm{op}}} h^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n \otimes_{A^{\mathrm{op}}} 1) \otimes_H n\big) = \\ &\quad m_{(0)} (h^1_{(n+1)} h^2_{(n)} \cdots h^n_{(2)} n)_{(-1)+} m_{(-n-1)+} \otimes_{A^{\mathrm{op}}} m_{(-n-1)-} m_{(-n)+} h^1_{(1)+} \\ &\qquad \otimes_{A^{\mathrm{op}}} h^1_{(1)-} m_{(-n)-} m_{(-n+1)+} h^1_{(2)+} h^2_{(1)+} \otimes_{A^{\mathrm{op}}} \cdots \\ &\qquad \otimes_{A^{\mathrm{op}}} h^n_{(1)-} \cdots h^1_{(n)-} m_{(-1)-} (h^1_{(n+1)} \cdots h^n_{(2)} n)_{(-1)-} (h^1_{(n+1)} \cdots h^n_{(2)} n)_{(0)} \\ &\quad = m_{(0)} \big((h^1_{(2)} \cdots h^n_{(2)} n)_{(-1)} m_{(-1)}\big)_+ \otimes_{A^{\mathrm{op}}} h^1_{(1)+} \otimes_{A^{\mathrm{op}}} \cdots \\ &\qquad \otimes_{A^{\mathrm{op}}} h^n_{(1)+} \otimes_{A^{\mathrm{op}}} h^n_{(1)-} \cdots h^1_{(1)-} \big((h^1_{(2)} \cdots h^n_{(2)} n)_{(-1)} m_{(-1)}\big)_- (h^1_{(2)} \cdots h^n_{(2)} n)_{(0)}. \end{aligned}$$

Finally, if $M \otimes_{A^{\mathrm{op}}} N$ is a stable anti Yetter-Drinfel'd module [BŞ08], that is, if

$$m_{(0)} (n_{(-1)} m_{(-1)})_+ \otimes_{A^{\mathrm{op}}} (n_{(-1)} m_{(-1)})_- n_{(0)} = m \otimes_{A^{\mathrm{op}}} n$$

holds for all $n \in N$, $m \in M$, we conclude by

$$\begin{aligned}(L_n \circ R_n)(m \otimes_{A^{\mathrm{op}}} h^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n \otimes_{A^{\mathrm{op}}} n) \\ &= m \otimes_{A^{\mathrm{op}}} h^1_{(1)+} \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n_{(1)+} \otimes_{A^{\mathrm{op}}} h^n_{(1)-} \cdots h^1_{(1)-} h^1_{(2)} \cdots h^n_{(2)} n \\ &= m \otimes_{A^{\mathrm{op}}} h^1 \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} h^n \otimes_{A^{\mathrm{op}}} n.\end{aligned}$$

Observe that in [Kow13] this cyclicity condition was obtained for a different complex which, however, computes the same homology.

5.7. **The antipode as a 1-cell.** If $A = k$, then the four actions $\triangleright, \triangleleft, \blacktriangleright, \blacktriangleleft$ coincide and H is a Hopf algebra with antipode $S\colon H \to H$ given by $S(h) = \varepsilon(h_+)h_-$. The aim of this brief section is to remark that this defines a 1-cell that connects the two instances of Theorem 2.5 provided by the opmonoidal adjunction and the opmodule adjunction considered above.

Indeed, in this case we have $A^{\mathrm{e}}\text{-Mod} \cong A\text{-Mod} = k\text{-Mod}$, but $H^{\mathrm{op}}\text{-Mod} \neq H\text{-Mod}$ unless H is commutative. However, S defines a lax morphism $\sigma\colon -\otimes_k H\,\mathsf{id} \to H \otimes_k - \mathsf{id}$, given in components by

$$\sigma_X \colon X \otimes_k H \to H \otimes_k X, \quad x \otimes_k h \mapsto S(h) \otimes_k x.$$

The fact that this is a lax morphism is equivalent to the fact that S is an algebra anti-homomorphism. Also, the lifted comonads agree and are given by $H \otimes_k -$ with comonad structure given by the coalgebra structure of H; clearly, $\gamma = \mathsf{id}\colon \mathsf{id}H \otimes_k - \to H \otimes_k -\mathsf{id}$ is a colax morphism. Furthermore, the Yang-Baxter condition is satisfied, so we have that $(\mathsf{id}, \sigma, \gamma)$ is a 1-cell in the 2-category of mixed distributive laws. If we apply the 2-functor i to this, we get a 1-cell $(\Sigma, \tilde{\sigma}, \tilde{\gamma})$ between a comonad distributive law on the category of left H-modules and one on the category of right H-modules. The identity lifts to the functor $\Sigma\colon H\text{-Mod} \to \text{Mod-}H$ which sends a left H-module X to the right H-module with right action given by

$$x \lhd h := S(h)x.$$

6. Hopf monads à la Mesablishvili-Wisbauer

6.1. **Bimonads.** A *bimonad* in the sense of [MW11] is a sextuple $(\mathrm{A}, \mu, \eta, \Delta^{\mathrm{A}}, \varepsilon^{\mathrm{A}}, \theta)$, where $\mathrm{A}\colon \mathcal{C} \to \mathcal{C}$ is a functor, (A, μ, η) is a monad, $(\mathrm{A}, \Delta^{\mathrm{A}}, \varepsilon^{\mathrm{A}})$ is a comonad and $\theta\colon \mathrm{AA} \to \mathrm{AA}$ is a mixed distributive law satisfying a list of compatibility conditions.

In particular, μ and Δ^A are required to be compatible in the sense that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{AA} & \xrightarrow{\mu} & \mathrm{A} & \xrightarrow{\Delta^{\mathrm{A}}} & \mathrm{AA} \\ {\scriptstyle \mathrm{A}\Delta^{\mathrm{A}}}\downarrow & & & & \uparrow{\scriptstyle \mathrm{A}\mu} \\ \mathrm{AAA} & & \xrightarrow[\theta\mathrm{A}]{} & & \mathrm{AAA} \end{array} \tag{6.1}$$

The other defining conditions rule the compatibility between the unit and the counit with each other and with μ respectively Δ^{A}, see [MW11] for the details.

It follows immediately that we also obtain an instance of Theorem 2.5 in this situation: if we take $\mathcal{A} = \mathcal{C}^{\mathbb{B}}$ to be the Eilenberg-Moore category of the monad $\mathbb{B} = (\mathrm{A}, \mu, \eta)$ as in Section 2.4, then the mixed distributive law θ defines a lift $\mathbb{V} = (\mathrm{V}, \Delta^{\mathrm{V}}, \varepsilon^{\mathrm{V}})$ of the comonad $\mathbb{C} = (\mathrm{A}, \Delta^{\mathrm{A}}, \varepsilon^{\mathrm{A}})$ to $\mathcal{A}$.

Note that in general, neither $\mathcal{A}$ nor $\mathcal{C}$ need to be monoidal, so B is in general not an opmonoidal monad. Conversely, recall that for the examples of Theorem 2.5 obtained from opmonoidal monads, B need not equal C.

6.2. **Examples from bialgebras.** In the main example of bimonads in the above sense, we in fact do have $\mathrm{B} = \mathrm{C}$ and we are in the situation of Section 5.3 for a bialgebra H over $A = k$. The commutativity of (6.1) amounts to the fact that the coproduct is an algebra map.

This setting provides an instance of Proposition 2.10 since there are two lifts of $\mathrm{B} = \mathrm{C}$ from $\mathcal{A} = k\text{-}\mathsf{Mod}$ to $\mathcal{B} = H\text{-}\mathsf{Mod}$: the canonical lift $\mathrm{S} = \mathrm{T} = \mathrm{FU}$ which takes a left H-module L to the H-module $H \otimes_k L$ with H-module structure given by multiplication in the first tensor component, and the lift V which takes L to $H \otimes_k L$ with H-action given by the codiagonal action $g(h \otimes_k y) = g_{(1)}h \otimes_k g_{(2)}y$, that is, the one defining the monoidal structure on $\mathcal{B}$. Now the Galois map from Proposition 2.12 is the Galois map

$$H \otimes_k L \to H \otimes_k L, \quad g \otimes_k y \mapsto g_{(1)} \otimes_k g_{(2)}y$$

used to define left Hopf algebroids (when taking tensor products over $A \neq k$ resp. A^{op}), which for $A = k$ are simply Hopf algebras, and more generally Hopf monads in the sense of [LMW15, Theorem 5.8(c)].

6.3. **An example not from bialgebras.** Another example of a bimonad is the *nonempty list monad* $\mathbb{L}^+$ on Set, which assigns to a set X the set L^+X of all nonempty lists of elements in X, denoted $[x_1, \ldots, x_n]$. The monad multiplication is given by concatenation of lists and the unit maps x to $[x]$. The comonad comultiplication is given by $\Delta[x_1, \ldots, x_n] = [[x_1, \ldots, x_n], \ldots, [x_n]]$, the counit is $\varepsilon[x_1, \ldots, x_n] = x_1$, and the mixed distributive law

$$\theta \colon \mathbb{L}^+\mathbb{L}^+ \to \mathbb{L}^+\mathbb{L}^+$$

is defined as follows: given a list

$$[[x_{1,1}, \ldots, x_{1,n_1}], \ldots, [x_{m,1}, \ldots, x_{m,n_m}]]$$

in L^+X, its image under θX is the list with

$$\sum_{i=1}^{m} n_i(m - i + 1)$$

terms, given by the lexicographic order, that is

$$\Big[[x_{1,1}, x_{2,1}, x_{3,1} \ldots, x_{m,1}], \ldots, [x_{1,n_1}, x_{2,1}, x_{3,1}, \ldots, x_{m,1}], \\ [x_{2,1}, x_{3,1} \ldots, x_{m,1}], \ldots, [x_{2,n_2}, x_{3,1}, \ldots x_{m,1}], \\ \ldots, \\ [x_{m,1}], [x_{m,2}], \ldots, [x_{m,n_m}]\Big].$$

One verifies straightforwardly:

Proposition 6.1. *$\mathbb{L}^+$ becomes a bimonad on* Set *whose Eilenberg-Moore category is* $\mathsf{Set}^{\mathbb{L}^+} \cong \mathsf{SemiGp}$*, the category of (nonunital) semigroups.*

The second lift V of the comonad $\mathbb{L}^+$ that one obtains from the bimonad structure on SemiGp is as follows. Given a semigroup X, we have $\mathrm{V}X = \mathrm{L}^+X$ as sets, but the binary operation is given by

$$\mathrm{V}X \times \mathrm{V}X \to \mathrm{V}X$$
$$[x_1, \ldots, x_m][y_1, \ldots, y_n] := [x_1y_1, \ldots, x_my_1, y_1, \ldots, y_n].$$

Following Proposition 3.2, given a semigroup X, the unit turns the underlying set of X into an $\mathbb{L}^+$-coalgebra and hence we get a right χ-coalgebra structure on X. Explicitly, $\rho_X \colon \mathrm{T}X \to \mathrm{V}X$ is given by

$$\rho[x_1, \ldots, x_n] = [x_1 \cdots x_n, x_2 \cdots x_n, \ldots, x_n].$$

The image of ρ is known as the *left machine expansion* of X [BR84].

Proposition 6.2. *The only θ-entwined algebra is the trivial semigroup $\varnothing$.*

Proof. An $\mathbb{L}^+$-coalgebra structure $\beta\colon T \to \mathbb{L}^+T$ is equivalent to T being a forest of at most countable height (rooted) trees, where each level may have arbitrary cardinality. The structure map β sends x to the finite list of predecessors of x. A θ-entwined algebra is therefore such a forest, which also has the structure of a semigroup such that for all $x, y \in T$ with $\beta(y) = [y, y_1, \ldots, y_n]$ we have

$$\beta(xy) = [xy, xy_1, \ldots, xy_n, y, y_1, \ldots, y_n].$$

Let T be a θ-entwined algebra. If T is non-empty, then there must be a root. We can multiply this root with itself to generate branches of arbitrary height. Suppose that we have a branch of height two; that is to say, an element $y \in T$ with $\beta(y) = [y, x]$ (so, in particular, $x \neq y$). Then $\beta(xy) = [xy, y]$, but $\beta(xx) = [xx, xy, x, y]$. This is impossible since x and y cannot both be the predecessor of xy. □

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