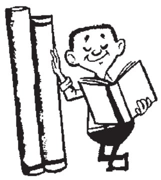
Factorial, Gamma and Beta Functions



Reading **Problems**

Outline

Background

Louis Franois Antoine Arbogast (1759 - 1803) a French mathematician, is generally credited with being the first to introduce the concept of the *factorial* as a product of a fixed number of terms in arithmetic progression. In an effort to generalize the factorial function to non-integer values, the *Gamma function* was later presented in its traditional integral form by Swiss mathematician Leonhard Euler (1707-1783). In fact, the integral form of the Gamma function is referred to as the second Eulerian integral. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Cristoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822 - 1901), as well as many others.[1] The first reported use of the gamma symbol for this function was by Legendre in 1839.[2]

The first Eulerian integral was introduced by Euler and is typically referred to by its more common name, the *Beta function*. The use of the Beta symbol for this function was first used in 1839 by Jacques P.M. Binet (1786 - 1856).

At the same time as Legendre and Gauss, Cristian Kramp (1760 - 1826) worked on the generalized factorial function as it applied to non-integers. His work on factorials was independent to that of Stirling, although Sterling often receives credit for this effort. He did achieve one “first” in that he was the first to use the notation n! although he seems not to be remembered today for this widely used mathematical notation[3].

A complete historical perspective of the Gamma function is given in the work of Godefroy[4] as well as other associated authors given in the references at the end of this chapter.

[1] http://numbers.computation.free.fr/Constants/Miscellaneous/gammaFunction.html
[2] Cajori, Vol.2, p. 271
[3] Elements d'arithmtique universelle , 1808
[4] M. Godefroy, *La fonction Gamma; Theorie, Histoire, Bibliographie*, Gauthier-Villars, Paris (1901)

Definitions

1. Factorial

$$n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 \qquad \text{for all integers, } n > 0$$

2. Gamma

also known as: generalized factorial, Euler's second integral

The factorial function can be extended to include all real valued arguments excluding the negative integers as follows:

$$z! = \int_0^\infty e^{-t}\, t^z\ dt \qquad z \neq -1, -2, -3, \ldots$$

or as the Gamma function:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\ dt = (z-1)! \qquad z \neq -1, -2, -3, \ldots$$

3. Digamma

also known as: psi function, logarithmic derivative of the gamma function

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad z \neq -1, -2, -3, \ldots$$

4. Incomplete Gamma

The gamma function can be written in terms of two components as follows:

$$\Gamma(z) = \gamma(z, x) + \Gamma(z, x)$$

where the incomplete gamma function, $\gamma(z, x)$, is given as

$$\gamma(z,x) = \int_0^x e^{-t}\, t^{z-1}\, dt \qquad\qquad x > 0$$

and its compliment, $\Gamma(z,x)$, as

$$\Gamma(z,x) = \int_x^\infty e^{-t}\, t^{z-1}\, dt \qquad\qquad x > 0$$

5. Beta

also known as: Euler's first integral

$$\begin{aligned} \mathrm{B}(y,z) &= \int_0^1 t^{y-1}\,(1-t)^{z-1}\, dt \\ &= \frac{\Gamma(y)\;\cdot\;\Gamma(z)}{\Gamma(y+z)} \end{aligned}$$

6. Incomplete Beta

$$\mathrm{B}_x(y,z) = \int_0^x t^{y-1}\,(1-t)^{z-1}\, dt \qquad\qquad 0 \le x \le 1$$

and the regularized (normalized) form of the incomplete Beta function

$$\mathrm{I}_x(y,z) = \frac{\mathrm{B}_x(y,z)}{\mathrm{B}(y,z)}$$

Theory

Factorial Function

The classical case of the integer form of the factorial function, $\boldsymbol{n!}$, consists of the product of $\boldsymbol{n}$ and all integers less than $\boldsymbol{n}$, down to 1, as follows

$$n! = \begin{cases} n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 & n = 1,2,3,\ldots \\ 1 & n = 0 \end{cases} \tag{1.1}$$

where by definition, $\mathbf{0! = 1}$.

The integer form of the factorial function can be considered as a special case of two widely used functions for computing factorials of non-integer arguments, namely the **Pochhammer's polynomial**, given as

$$(z)_n = \begin{cases} z(z+1)(z+2)\ldots(z+n-1) = \dfrac{\Gamma(z+n)}{\Gamma(z)} & n > 0 \\ \qquad = \dfrac{(z+n-1)!}{(z-1)!} & \\ 1 = 0! & n = 0 \end{cases} \tag{1.2}$$

and the **gamma function** (Euler's integral of the second kind).

$$\Gamma(z) = (z-1)! \tag{1.3}$$

While it is relatively easy to compute the factorial function for small integers, it is easy to see how manually computing the factorial of larger numbers can be very tedious. Fortunately given the recursive nature of the factorial function, it is very well suited to a computer and can be easily programmed into a function or subroutine. The two most common methods used to compute the integer form of the factorial are

direct computation: use iteration to produce the product of all of the counting numbers between $\boldsymbol{n}$ and $\mathbf{1}$, as in Eq. 1.1

recursive computation: define a function in terms of itself, where values of the factorial are stored and simply multiplied by the next integer value in the sequence

Another form of the factorial function is the double factorial, defined as

$$n!! = \begin{cases} n(n-2)\ldots 5\cdot 3\cdot 1 & n > 0 \quad \text{odd} \\ n(n-2)\ldots 6\cdot 4\cdot 2 & n > 0 \quad \text{even} \\ 1 & n = -1, 0 \end{cases} \tag{1.4}$$

The first few values of the double factorial are given as

$$\begin{array}{rclcrcl} 0!! &=& 1 & \qquad & 5!! &=& 15 \\ 1!! &=& 1 & & 6!! &=& 48 \\ 2!! &=& 2 & & 7!! &=& 105 \\ 3!! &=& 3 & & 8!! &=& 384 \\ 4!! &=& 8 & & 9!! &=& 945 \end{array}$$

While there are several identities linking the factorial function to the double factorial, perhaps the most convenient is

$$n! = n!!(n-1)!! \tag{1.5}$$

Potential Applications

1. *Permutations and Combinations:* The combinatory function $C(n,k)$ (n choose k) allows a concise statement of the Binomial Theorem using symbolic notation and in turn allows one to determine the number of ways to choose k items from n items, regardless of order.

   The combinatory function provides the binomial coefficients and can be defined as

   $$C(n,k) = \frac{n!}{k!(n-k)!} \tag{1.6}$$

   It has uses in modeling of noise, the estimation of reliability in complex systems as well as many other engineering applications.

Gamma Function

The factorial function can be extended to include non-integer arguments through the use of Euler's second integral given as

$$\boldsymbol{z! = \int_0^\infty e^{-t}\, t^z\, dt} \tag{1.7}$$

Equation 1.7 is often referred to as the *generalized factorial function*.

Through a simple translation of the $\boldsymbol{z-}$ variable we can obtain the familiar *gamma function* as follows

$$\boldsymbol{\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt = (z-1)!} \tag{1.8}$$

The gamma function is one of the most widely used special functions encountered in advanced mathematics because it appears in almost every integral or series representation of other advanced mathematical functions.

Let's first establish a direct relationship between the gamma function given in Eq. 1.8 and the integer form of the factorial function given in Eq. 1.1. Given the gamma function $\boldsymbol{\Gamma(z+1) = z!}$ use integration by parts as follows:

$$\boldsymbol{\int u\, dv = uv - \int v\, du}$$

where from Eq. 1.7 we see

$$\boldsymbol{u = t^z \quad \Rightarrow \quad du = zt^{z-1}\, dt}$$

$$\boldsymbol{dv = e^{-t}\, dt \quad \Rightarrow \quad v = -e^{-t}}$$

which leads to

$$\boldsymbol{\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z\, dt = \Big[-e^{-t}\, t^z\Big]_0^\infty + z\int_0^\infty e^{-t}\, t^{z-1}\, dt}$$

Given the restriction of $z > 0$ for the integer form of the factorial function, it can be seen that the first term in the above expression goes to zero since, when

$$\begin{aligned} t = 0 \quad &\Rightarrow \quad t^n \to 0 \\ t = \infty \quad &\Rightarrow \quad e^{-t} \to 0 \end{aligned}$$

Therefore

$$\Gamma(z+1) = z \underbrace{\int_0^\infty e^{-t}\, t^{z-1}\, dt}_{\Gamma(z)} = z\, \Gamma(z), \qquad\qquad z > 0 \tag{1.9}$$

When $z = 1 \quad \Rightarrow \quad t^{z-1} = t^0 = 1$, and

$$\Gamma(1) = 0! = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1$$

and in turn

$$\begin{aligned} \Gamma(2) &= 1\, \Gamma(1) = 1 \cdot 1 = 1! \\ \Gamma(3) &= 2\, \Gamma(2) = 2 \cdot 1 = 2! \\ \Gamma(4) &= 3\, \Gamma(3) = 3 \cdot 2 = 3! \end{aligned}$$

In general we can write

$$\Gamma(n+1) = n! \qquad\qquad n = 1, 2, 3, \ldots \tag{1.10}$$

The gamma function constitutes an essential extension of the idea of a factorial, since the argument z is not restricted to positive integer values, but can vary continuously.

From Eq. 1.9, the gamma function can be written as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

From the above expression it is easy to see that when $z = 0$, the gamma function approaches ∞ or in other words $\Gamma(0)$ is undefined.

Given the recursive nature of the gamma function, it is readily apparent that the gamma function approaches a singularity at each negative integer.

However, for all other values of z, $\Gamma(z)$ is defined and the use of the recurrence relationship for factorials, i.e.

$$\Gamma(z+1) = z\,\Gamma(z)$$

effectively removes the restriction that x be positive, which the integral definition of the factorial requires. Therefore,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad z \neq 0, -1, -2, -3, \ldots \tag{1.11}$$

A plot of $\Gamma(z)$ is shown in Figure 1.1.

Several other definitions of the Γ-function are available that can be attributed to the pioneering mathematicians in this area

<u>Gauss</u>

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\ldots(z+n)}, \qquad z \neq 0, -1, -2, -3, \ldots \tag{1.12}$$

<u>Weierstrass</u>

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma \cdot z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \tag{1.13}$$

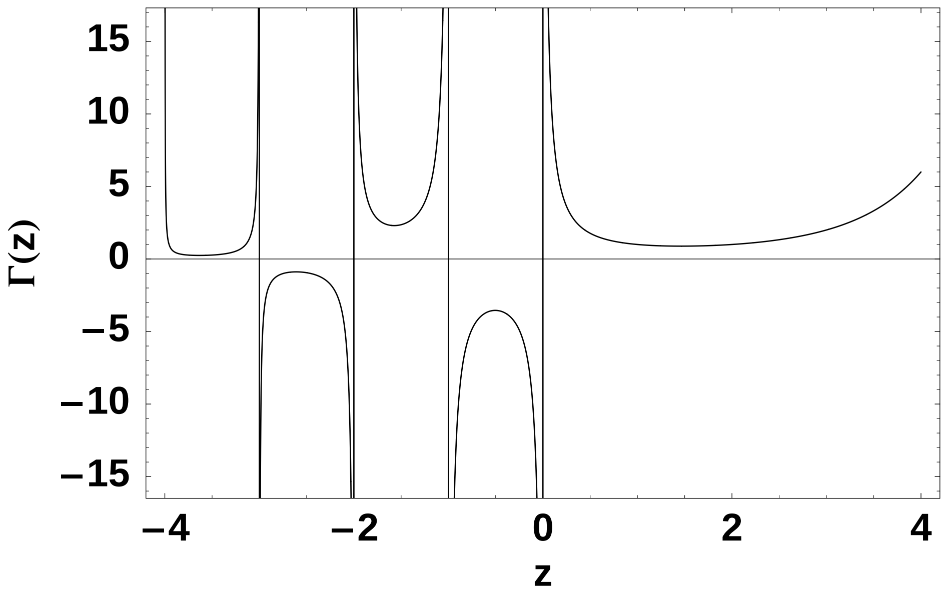


Figure 1.1: Plot of Gamma Function

where γ is the *Euler-Mascheroni constant*, defined by

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n) = 0.57721\ 56649\ 0\ldots \tag{1.14}$$

An excellent approximation of γ is given by the very simple formula

$$\gamma = \frac{1}{2}\left(\sqrt[3]{10} - 1\right) = 0.57721\ 73\ldots$$

Other forms of the gamma function are obtained through a simple change of variables, as follows

$$\Gamma(z) = 2\int_0^\infty y^{2z-1}\, e^{-y^2}\, dy \qquad \text{by letting } t = y^2 \tag{1.15}$$

$$\Gamma(z) = \int_0^1 \left(\ln\frac{1}{y}\right)^{z-1} dy \qquad \text{by letting } e^{-t} = y \tag{1.16}$$

Relations Satisfied by the Γ-Function

Recurrence Formula

$$\Gamma(z+1) = z\,\Gamma(z) \tag{1.17}$$

Duplication Formula

$$2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z) \tag{1.18}$$

Reflection Formula

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{1.19}$$

Some Special Values of the Gamma Function

Using Eq. 1.15 or Eq. 1.19 we have

$$\Gamma(1/2) = (-1/2)! = 2\,\underbrace{\int_0^\infty e^{-y^2}\,dy}_{I} = \sqrt{\pi} \tag{1.20}$$

where the solution to I is obtained from Schaum's Handbook of Mathematical Functions (Eq. 18.72).

Combining the results of Eq. 1.20 with the recurrence formula, we see

$$
\begin{aligned}
\Gamma(1/2) &= \sqrt{\pi} \\
\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2} \\
\Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4} \\
&\vdots \\
\Gamma\left(n+\frac{1}{2}\right) &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi} \qquad n = 1, 2, 3, \ldots
\end{aligned}
$$

For $z > 0$, $\Gamma(z)$ has a single minimum within the range $1 \leq z \leq 2$ at **1.46163 21450** where $\Gamma(z) = \mathbf{0.88560\ 31944}$. Some selected 10 decimal place values of $\Gamma(z)$ are found in Table 1.1.

Table 1.1: 10 Decimal Place Values of $\Gamma(z)$ for $1 \leq z \leq 2$

z	$\Gamma(z)$
1.0	1.00000 00000
1.1	0.95135 07699
1.2	0.91816 87424
1.3	0.89747 06963
1.4	0.88726 38175
1.5	0.88622 69255
1.6	0.89351 53493
1.7	0.90863 87329
1.8	0.93138 37710
1.9	0.96176 58319
2.0	1.00000 00000

For other values of z $(z \neq 0, -1, -2 \ldots)$, $\Gamma(z)$ can be computed by means of the recurrence formula.

Approximations

Asymptotic Representation of the Factorial and Gamma Functions

Asymptotic expansions of the factorial and gamma functions have been developed for $z >> 1$. The expansion for the factorial function is

$$z! = \Gamma(z+1) = \sqrt{2\pi z}\; z^z\; e^{-z}\; A(z) \tag{1.21}$$

where

$$A(z) = 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots \tag{1.22}$$

The expansion for the natural logarithm of the gamma function is

$$\begin{aligned} \ln\Gamma(z) \quad = \quad & \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} \\ & - \frac{1}{1680z^7} + \cdots \end{aligned} \tag{1.23}$$

The absolute value of the error is less than the absolute value of the first term neglected.

For large values of z, i.e. as $z \longrightarrow \infty$, both expansions lead to *Stirling's Formula*, given as

$$z! = \sqrt{2\pi}\; z^{z+1/2}\; e^{-z} \tag{1.24}$$

Even though the asymptotic expansions in Eqs. 1.21 and 1.23 were developed for very large values of z, they give remarkably accurate values of $z!$ and $\Gamma(z)$ for small values of z. Table 1.2 shows the relative error between the asymptotic expansion and known accurate values for arguments between $1 \leq z \leq 7$, where the relative error is defined as

$$\text{relative error} = \frac{\text{approximate value} - \text{accurate value}}{\text{accurate value}}$$

Table 1.2: Comparison of Approximate value of $z!$ by Eq. 1.21 and $\Gamma(z)$ by Eq. 1.23 with the Accurate values of Mathematica 5.0

z	$\frac{z!\ Eq.1.21}{z!\ \text{Mathematica}}$	error	$\frac{\Gamma(z)\ Eq.1.23}{\Gamma(z)\ \text{Mathematica}}$	error
1	0.99949 9469	-5.0×10^{-4}	0.99969 2549	-3.1×10^{-4}
2	0.99997 8981	-2.1×10^{-5}	0.99999 8900	-1.1×10^{-6}
3	0.99999 7005	-3.0×10^{-6}	0.99999 9965	-3.5×10^{-8}
4	0.99999 9267	-7.3×10^{-7}	0.99999 9997	-2.8×10^{-9}
5	0.99999 9756	-2.4×10^{-7}	0.99999 9999	-4.0×10^{-10}
6	0.99999 9901	-9.9×10^{-8}	0.99999 9999	-7.9×10^{-11}
7	0.99999 9954	-4.6×10^{-8}	0.99999 9999	-2.0×10^{-11}

The asymptotic expansion for $\Gamma(z)$ converges very quickly to give accurate values for relatively small values of z. The asymptotic expansion for $z!$ converges less quickly and does not yield 9 decimal place accuracy even when $z = 7$.

More accurate values of $\Gamma(z)$ for small z can be obtained by means of the recurrence formula. For example, if we want $\Gamma(1+z)$ where $0 \leq z \leq 1$, then by means of the recurrence formula we can write

$$\Gamma(1+z) = \frac{\Gamma(n+z)}{(1+z)(2+z)(3+z)\ldots(n-1+z)} \tag{1.25}$$

where n is an integer greater that 4. For $n = 5$ and $z = 0.3$, we have

$$\Gamma(1+0.3) = \frac{\Gamma(5.3)}{(1.3)(2.3)(3.3)(4.3)} = 0.89747\ 0699$$

This value can be compared with the 10 decimal place value given previously in Table 1.1. We observe that the absolute error is approximately 3×10^{-9}. Comparable accuracy can be obtained by means of the above equation with $n = 6$ and $0 \leq z \leq 1$.

Polynomial Approximation of $\Gamma(z+1)$ within $0 \le z \le 1$

Numerous polynomial approximations which are based upon the use of Chebyshev polynomials and the minimization of the maximum absolute error have been developed for varying degrees of accuracy. One such approximation developed for $0 \le z \le 1$ due to Hastings[8] is

$$\begin{aligned}\Gamma(z+1) &= z! \\ &= 1 + z(a_1 + z(a_2 + z(a_3 + z(a_4 + z(a_5 + \\ &\qquad z(a_6 + z(a_7 + a_8 z))))))) + \epsilon(z)\end{aligned} \tag{1.26}$$

where

$$|\epsilon(z)| \le 3 \times 10^{-7}$$

and the coefficients in the polynomial are given as

Table 1.3: Coefficients of Polynomial of Eq. 1.26

a_1	=	−0.57719 1652	a_5	=	−0.75670 4078
a_2	=	0.98820 5891	a_6	=	0.48219 9394
a_3	=	−0.89705 6937	a_7	=	−0.19352 7818
a_4	=	0.91820 6857	a_8	=	0.03586 8343

Series Expansion of $1/\Gamma(z)$ for $|z| \leq \infty$

The function $\mathbf{1/\Gamma(z)}$ is an entire function defined for all values of $\boldsymbol{z}$. It can be expressed as a series expansion according to the relationship

$$\frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} C_k z^k, \qquad |z| \leq \infty \tag{1.27}$$

where the coefficients C_k for $0 \leq k \leq 26$, accurate to $\mathbf{16}$ decimal places are tabulated in Abramowitz and Stegun[1]. For $\mathbf{10}$ decimal place accuracy one can write

$$\frac{1}{\Gamma(z)} = \sum_{k=1}^{19} C_k z^k \tag{1.28}$$

where the coefficients are listed below

Table 1.4: Coefficients of Expansion of $1/\Gamma(z)$ of Eq. 1.28

k	C_k	k	C_k
1	1.00000 00000	11	0.00012 80502
2	0.57721 56649	12	−0.00002 01348
3	−0.65587 80715	13	−0.00000 12504
4	−0.04200 26350	14	0.00000 11330
5	0.16653 86113	15	−0.00000 02056
6	−0.04219 77345	16	0.00000 00061
7	−0.00962 19715	17	0.00000 00050
8	0.00721 89432	18	−0.00000 00011
9	−0.00116 51675	19	0.00000 00001
10	−0.00021 52416		

Potential Applications

1. *Gamma Distribution:* The probability density function can be defined based on the Gamma function as follows:

$$\boldsymbol{f(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, x^{\alpha-1} e^{-x/\beta}}$$

This function is used to determine time based occurrences, such as:

- life length of an electronic component
- remaining life of a component
- waiting time between any two consecutive events
- waiting time to see the next event
- hypothesis tests
- confidence intervals

Digamma Function

The digamma function is the regularized (normalized) form of the logarithmic derivative of the gamma function and is sometimes referred to as the psi function.

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.29}$$

The digamma function is shown in Figure 1.2 for a range of arguments between $-4 \leq z \leq 4$.

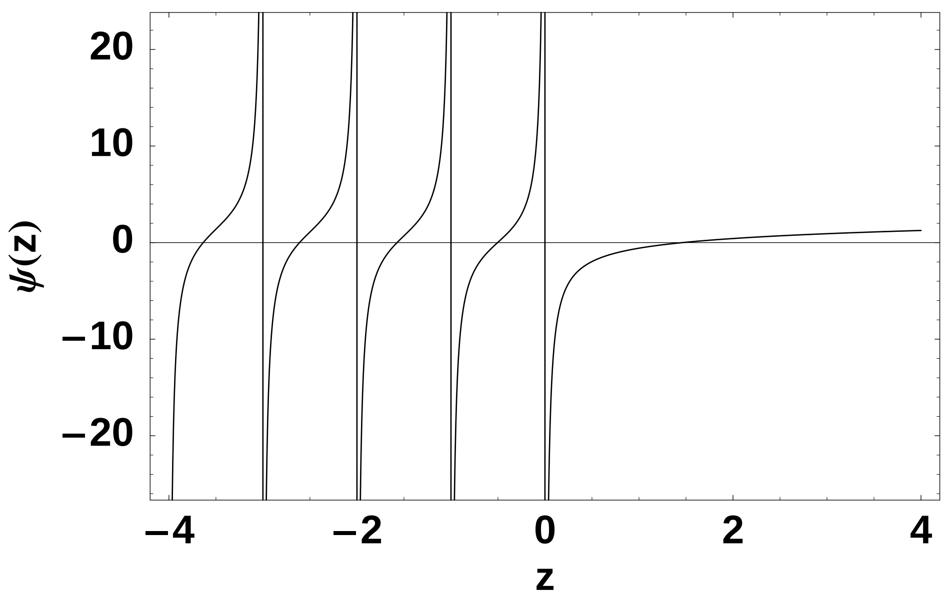


Figure 1.2: Plot of the Digamma Function

The ψ-function satisfies relationships which are obtained by taking the logarithmic derivative of the recurrence, reflection and duplication formulas of the Γ-function. Thus

$$\psi(z+1) = \frac{1}{z} + \psi(z) \tag{1.30}$$

$$\psi(1-z) - \psi(z) = \pi \cot(\pi z) \tag{1.31}$$

$$\psi(z) + \psi(z+1/2) + 2\ln 2 = 2\psi(2z) \tag{1.32}$$

These formulas may be used to obtain the following special values of the ψ-function:

$$\psi(1) = \Gamma'(1) = -\gamma \tag{1.33}$$

where γ is the Euler-Mascheroni constant defined in Eq. (1.14). Using Eq. (1.30)

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} \qquad n = 1, 2, 3, \ldots \tag{1.34}$$

Substitution of $z = 1/2$ into Eq. (1.32) gives

$$\psi(1/2) = -\gamma - 2\ln 2 = -1.96351\ 00260 \tag{1.35}$$

and with Eq. (1.30) we obtain

$$\psi(n+1/2) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}, \qquad n = 1, 2, 3, \ldots \tag{1.36}$$

Integral Representation of $\psi(z)$

The ψ-function has simple representations in the form of definite integrals involving the variable z as a parameter. Some of these are listed below.

$$\psi(z) = -\gamma + \int_0^1 (1-t)^{-1}(1-t^{z-1})\, dt, \qquad z > 0 \tag{1.37}$$

$$\psi(z) = -\gamma - \pi\ \cot(\pi\ z) + \int_0^1 (1-t)^{-1}(1-t^{-z})\, dt, \qquad z < 1 \tag{1.38}$$

$$\psi(z) = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right] dt, \qquad z > 0 \tag{1.39}$$

$$\begin{aligned} \psi(z) &= \int_0^\infty \left[e^{-t} - (1+t)^{-z}\right] \frac{dt}{t}, && z > 0 \\ &= -\gamma + \int_0^\infty \left[(1+t)^{-1} - (1+t)^{-z}\right] \frac{dt}{t}, && z > 0 \end{aligned} \tag{1.40}$$

$$\begin{aligned} \psi(z) &= \ln z + \int_0^\infty \left[\frac{1}{t} - \frac{1}{1-e^{-t}}\right] e^{-zt}\, dt, && z > 0 \\ &= \ln z - \frac{1}{2z} - \int_0^\infty \left[\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right] e^{-zt}\, dt, && z > 0 \end{aligned} \tag{1.41}$$

Series Representation of $\psi(z)$

The ψ-function can be represented by means of several series

$$\psi(z) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{z+k} - \frac{1}{1+k}\right) \qquad z \neq -1, -2, -3, \ldots \tag{1.42}$$

$$\psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(z+k)} \qquad z \neq -1, -2, -3, \ldots \tag{1.43}$$

$$\psi(z) = \ln z - \sum_{k=0}^{\infty}\left[\frac{1}{z+k} - \ln\left(1 + \frac{1}{z+k}\right)\right] \qquad z \neq -1, -2, -3, \ldots \tag{1.44}$$

Asymptotic Expansion of $\psi(z)$ for Large z

The asymptotic expansion of the ψ-function developed for large z is

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}} \qquad z \to \infty \tag{1.45}$$

where B_{2n} are the Bernoulli numbers

$$\begin{array}{lll} B_0 = 1 & \qquad & B_6 = 1/42 \\ B_2 = 1/6 & & B_8 = -1/30 \\ B_4 = -1/30 & & B_{10} = 5/66 \end{array} \tag{1.46}$$

The expansion can be expressed as

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots \qquad z \to \infty \tag{1.47}$$

The Incomplete Gamma Function $\gamma(z,x)$, $\Gamma(z,x)$

We can generalize the Euler definition of the gamma function by defining the incomplete gamma function $\gamma(z,x)$ and its compliment $\Gamma(z,x)$ by the following variable limit integrals

$$\gamma(z,x) = \int_0^x e^{-t}\, t^{z-1}\, dt \qquad z > 0 \tag{1.48}$$

and

$$\Gamma(z,x) = \int_x^\infty e^{-t}\, t^{z-1}\, dt \qquad z > 0 \tag{1.49}$$

so that

$$\gamma(z,x) + \Gamma(z,x) = \Gamma(z) \tag{1.50}$$

Figure 1.3 shows plots of $\gamma(z,x)$, $\Gamma(z,x)$ and $\Gamma(z)$ all regularized with respect to $\Gamma(z)$. We can clearly see that the addition of $\gamma(z,x)/\Gamma(z)$ and $\Gamma(z,x)/\Gamma(z)$ leads to a value of unity or $\Gamma(z)/\Gamma(z)$ for each value of z.

The choice of employing $\gamma(z,x)$ or $\Gamma(z,x)$ is simply a matter of analytical or computational convenience.

Some special values, integrals and series are listed below for convenience

Special Values of $\gamma(z,x)$ and $\Gamma(z,x)$ for "z" Integer (let $z = n$)

$$\gamma(1+n,x) = n! \left[1 - e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!}\right] \qquad n = 0,1,2,\ldots \tag{1.51}$$

$$\Gamma(1+n,x) = n!\, e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad n = 0,1,2,\ldots \tag{1.52}$$

$$\Gamma(-n,x) = \frac{(-1)^n}{n!} \left[\Gamma(0,x) - e^{-x} \sum_{k=0}^{n-1} (-1)^k \frac{k!}{x^{k+1}}\right] \qquad n = 1,2,3\ldots \tag{1.53}$$

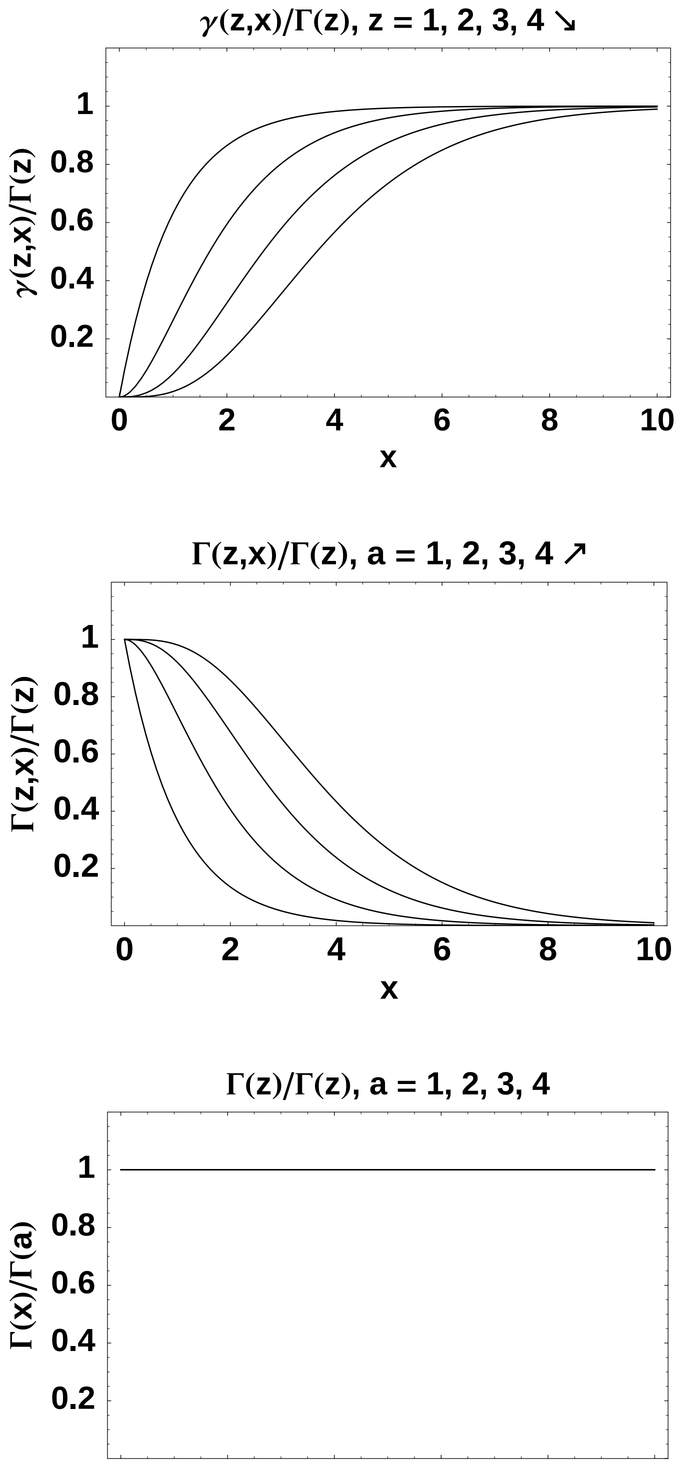


Figure 1.3: Plot of the Incomplete Gamma Function where

$$\frac{\gamma(z,x)}{\Gamma(z)} + \frac{\Gamma(z,x)}{\Gamma(z)} = \frac{\Gamma(z)}{\Gamma(z)}$$

Integral Representations of the Incomplete Gamma Functions

$$\gamma(z,x) = x^z \operatorname{cosec}(\pi z) \int_0^\pi e^{x\cos\theta} \cos(z\theta + x\sin\theta)\, d\theta \qquad x \neq 0, z > 0, z \neq 1, 2, \ldots \tag{1.54}$$

$$\Gamma(z,x) = \frac{e^{-x}x^z}{\Gamma(1-z)} \int_0^\infty \frac{e^{-t}\, t^{-z}}{x+t}\, dt \qquad z < 1,\ x > 0 \tag{1.55}$$

$$\Gamma(z,xy) = y^z e^{-xy} \int_0^\infty e^{-ty}\, (t+x)^{z-1}\, dt \qquad y > 0, x > 0, z > 1 \tag{1.56}$$

Series Representations of the Incomplete Gamma Functions

$$\gamma(z,x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{z+n}}{n!\, (z+n)} \tag{1.57}$$

$$\Gamma(z,x) = \Gamma(z) - \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{z+n}}{n!\, (z+n)} \tag{1.58}$$

$$\Gamma(z+x) = e^{-x}x^z \sum_{n=0}^{\infty} \frac{L_n^z(x)}{n+1} \qquad x > 0 \tag{1.59}$$

where $L_n^z(x)$ is the associated Laguerre polynomial.

Functional Representations of the Incomplete Gamma Functions

$$\gamma(z+1,x) = z\gamma(z,x) - x^z e^{-x} \tag{1.60}$$

$$\Gamma(z+1,x) = z\Gamma(z,x) + x^z e^{-x} \tag{1.61}$$

$$\frac{\Gamma(z+n,x)}{\Gamma(z+n)} = \frac{\Gamma(z,x)}{\Gamma(z)} + e^{-x} \sum_{k=0}^{n-1} \frac{x^{z+k}}{\Gamma(z+k+1)} \tag{1.62}$$

$$\frac{d\gamma(z,x)}{dx} = -\frac{d\Gamma(z,x)}{dx} = x^{z-1}\, e^{-x} \tag{1.63}$$

Asymptotic Expansion of $\Gamma(z,x)$ for Large x

$$\Gamma(z,x) = x^{z-1}\, e^{-x} \left[1 + \frac{(z-1)}{x} + \frac{(z-1)(z-2)}{x^2} + \cdots\right] \qquad x \to \infty \tag{1.64}$$

Continued Fraction Representation of $\Gamma(z,x)$

$$\Gamma(z,x) = \cfrac{e^{-x}\, x^z}{z + \cfrac{1-z}{1 + \cfrac{1}{x + \cfrac{2-z}{1 + \cfrac{2}{x + \cfrac{3-z}{1+\ldots}}}}}} \tag{1.65}$$

for $x > 0$ and $|z| < \infty$.

Relationships with Other Special Functions

$$\Gamma(0,x) = -\mathrm{Ei}(-x) \tag{1.66}$$

$$\Gamma(0,\ln 1/x) = -\ell i(x) \tag{1.67}$$

$$\Gamma(1/2,x^2) = \sqrt{\pi}(1-\mathrm{erf}(x)) = \sqrt{\pi}\,\mathrm{erfc}(x) \tag{1.68}$$

$$\gamma(1/2,x^2) = \sqrt{\pi}\,\mathrm{erf}(x) \tag{1.69}$$

$$\gamma(z,x) = z^{-1}\, x^z\, e^{-x}\, \mathrm{M}(1, 1+z, x) \tag{1.70}$$

$$\gamma(z,x) = z^{-1}\, x^z\, \mathrm{M}(z, 1+z, -x) \tag{1.71}$$

Beta Function $\mathrm{B}(a, b)$

Another definite integral which is related to the Γ-function is the Beta function $\mathbf{B}(a, b)$ which is defined as

$$\mathrm{B}(a, b) = \int_0^1 t^{a-1} \, (1-t)^{b-1} \, dt, \qquad a > 0, \quad b > 0 \tag{1.72}$$

The relationship between the $\mathbf{B}$-function and the Γ-function can be demonstrated easily. By means of the new variable

$$u = \frac{t}{(1-t)}$$

Therefore Eq. 1.72 becomes

$$\mathrm{B}(a, b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}} \, du \qquad a > 0, \quad b > 0 \tag{1.73}$$

Now it can be shown that

$$\int_0^\infty e^{-pt} \, t^{z-1} \, dt = \frac{\Gamma(z)}{p^z} \tag{1.74}$$

which is obtained from the definition of the Γ-function with the change of variable $s = pt$. Setting $p = 1 + u$ and $z = a + b$, we get

$$\frac{1}{(1+u)^{a+b}} = \frac{1}{\Gamma(a+b)} \int_0^\infty e^{-(1+u)t} \, t^{a+b-1} \, dt \tag{1.75}$$

and substituting this result into the Beta function in Eq. 1.73 gives

$$\begin{aligned} \mathrm{B}(a, b) &= \frac{1}{\Gamma(a+b)} \int_0^\infty e^{-t} \, t^{a+b-1} \, dt \int_0^\infty e^{-ut} \, u^{a-1} \, du \\ &= \frac{\Gamma(a)}{\Gamma(a+b)} \int_0^\infty e^{-t} \, t^{b-1} \, dt \\ &= \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} \end{aligned} \tag{1.76}$$

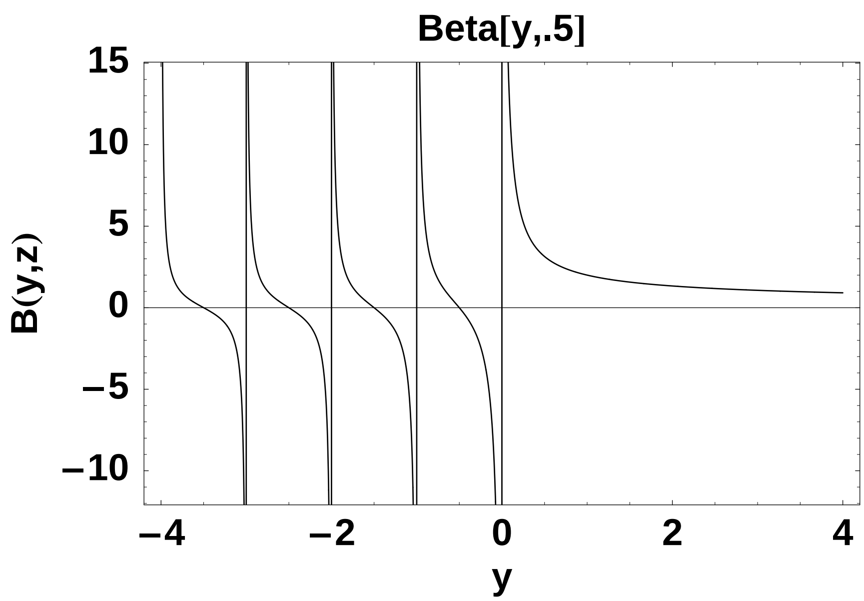


Figure 1.4: Plot of Beta Function

All the properties of the Beta function can be derived from the relationships linking the $\boldsymbol{\Gamma}$-function and the Beta function.

Other forms of the beta function are obtained by changes of variables. Thus

$$\mathrm{B}(a,b) = \int_0^\infty \frac{u^{a-1}\,du}{(1+u)^{a+b}} \qquad \text{by } t = \frac{u}{1-u} \tag{1.77}$$

$$\mathrm{B}(a,b) = 2\int_0^{\pi/2} \sin^{2a-1}\theta\ \cos^{2a-1}\theta\ d\theta \qquad \text{by } t = \sin^2\theta \tag{1.78}$$

Potential Applications

1. *Beta Distribution:* The Beta distribution is the integrand of the Beta function. It can be used to estimate the average time of completing selected tasks in time management problems.

Incomplete Beta Function $\mathrm{B}_x(a,b)$

Just as one can define an incomplete gamma function, so can one define the incomplete beta function by the variable limit integral

$$B_x(a,b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt \qquad\qquad 0 \le x \le 1 \tag{1.79}$$

with $a > 0$ and $b > 0$ if $x \neq 1$. One can also define

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} \tag{1.80}$$

Clearly when $x = 1$, $B_x(a,b)$ becomes the complete beta function and

$$I_1(a,b) = 1$$

The incomplete beta function and $I_x(a,b)$ satisfies the following relationships:

Symmetry

$$I_x(a,b) = 1 - I_{1-x}(b,a) \tag{1.81}$$

Recurrence Formulas

$$I_x(a,b) = xI_x(a-1,b) + (1-x)I_x(a,b-1) \tag{1.82}$$

$$(a+b-ax)I_x(a,b) = a(1-x)I_x(a+1,b-1) + bI_x(a,b+1) \tag{1.83}$$

$$(a+b)I_x(a,b) = aI_x(a+1,b) + bI_x(a,b+1) \tag{1.84}$$

Relation to the Hypergeometric Function

$$B_x(a,b) = a^{-1}x^a\,F(a,1-b;a+1;x) \tag{1.85}$$

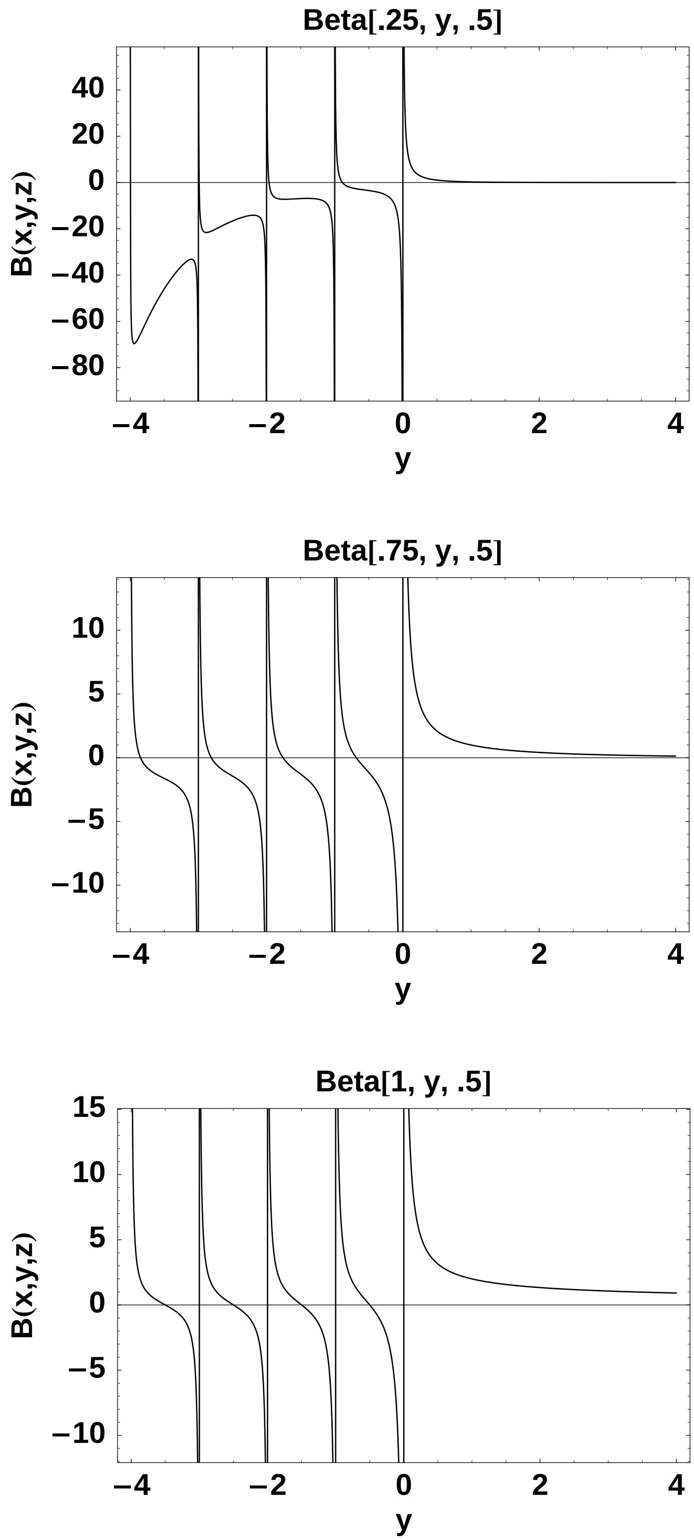


Figure 1.5: Plot of the Incomplete Beta Function

Assigned Problems

Problem Set for Gamma and Beta Functions

1. Use the definition of the gamma function with a suitable change of variable to prove that

   i) $$\int_0^\infty e^{-ax} x^n \, dx = \frac{1}{a^{n+1}} \, \Gamma(n+1) \qquad \text{with } n > -1, \; a > 0$$

   ii) $$\int_a^\infty \exp(2ax - x^2) \, dx = \frac{\sqrt{\pi}}{2} \, \exp(a^2)$$

2. Prove that

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{\sqrt{\pi}}{2} \, \frac{\Gamma([1+n]/2)}{\Gamma([2+n]/2)}$$

3. Show that

$$\Gamma\left(\frac{1}{2} + x\right) \; \Gamma\left(\frac{1}{2} - x\right) = \frac{\pi}{\cos \pi x}$$

   Plot your results over the range $-10 \leq x \leq 10$.

4. Evaluate $\Gamma\left(-\frac{1}{2}\right)$ and $\Gamma\left(-\frac{7}{2}\right)$.

5. Show that the area enclosed by the axes $x = 0, \; y = 0$ and the curve $x^4 + y^4 = 1$ is

$$\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{8\sqrt{\pi}}$$

   Use both the Dirichlet integral and a conventional integration procedure to substantiate this result.

6. Express each of the following integrals in terms of the gamma and beta functions and simplify when possible.

i) $\displaystyle\int_0^1 \left(\frac{1}{x} - 1\right)^{1/4} dx$

ii) $\displaystyle\int_a^b (b-x)^{m-1}(x-a)^{n-1}\, dx,$ with $b > a,\ m > 0,\ n > 0$

iii) $\displaystyle\int_0^\infty \frac{dt}{(1+t)\sqrt{t}}$

Note: Validate your results using various solution procedures where possible.

7. Compute to 5 decimal places

$$\frac{A}{4ab} = \frac{1}{2n} \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}$$

for $n = 0.2,\ 0.4,\ 0.8,\ 1.0,\ 2.0,\ 4.0,\ 8.0,\ 16.0,\ 32.0,\ 64.0,\ 100.0$

8. Sketch $x^3 + y^3 = 8$. Derive expressions of the integrals and evaluate them in terms of Beta functions for the following quantities:

a) the first quadrant area bounded by the curve and two axes
b) the centroid $(\overline{x}, \overline{y})$ of this area
c) the volume generated when the area is revolved about the $y-$axis
d) the moment of inertia of this volume about its axis

Note: Validate your results using various solution procedures where possible.

9. Starting with

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}\, dt}{\sqrt{t}}$$

and the transformation $y^2 = t$ or $x^2 = t$, show that

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty \exp\left[-(x^2+y^2)\right]\; dx\; dy$$

Further prove that the above double integral over the first quadrant when evaluated using polar coordinates (r, θ) yields

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

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