OPTIMIZED ROBUST CONTROL INVARIANT SETS FOR CONSTRAINED LINEAR DISCRETE-TIME SYSTEMS ⋆

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Abstract: In this paper we introduce the concept of optimized robust control invariance for a discrete-time, linear, time-invariant system subject to additive state disturbances. A novel characterization of a family of the robust control invariant sets is given. The existence of a constraint admissible member of this family can be checked by solving a *single linear programming problem.* The solution of the *same* linear programming problem yields the corresponding feedback controller. *Copyright© 2005 IFAC.*

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1. INTRODUCTION

The theory of set invariance plays a fundamental role in the control of constrained systems. The interested reader is referred to the important and comprehensive survey paper (Blanchini, 1999) for an introduction to set invariance and a number of relevant references. Two important issues, the computation of the minimal robust positively invariant (mRPI) set and the maximal robust positively invariant (MRPI), set are studied in detail in (Kolmanovsky and Gilbert, 1998).

From the control theory point of view, set invariance provides useful tools for the synthesis of reference governors (Gilbert and Kolmanovsky, 1999) and predictive controllers (Bemporad and Morari, 1999; Findeisen *et al.*, 2003; Mayne, 2001) with guaranteed invariance, stability and convergence properties. Since the mRPI set is the smallest invariant set for a system, it is also a suitable target set in robust time-optimal control (Bertsekas and Rhodes, 1971; Blanchini, 1992; Mayne and Schroeder, 1997) and plays an integral part in a novel robust predictive control method recently proposed in (Langson *et al.*, 2004).

It is the main purpose of this paper to provide a novel characterization of a family of the polytopic robust control invariant sets. Verifying existence

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of a constraint admissible member of this family as well as the computation of the corresponding feedback controller can be efficiently realized by solving a *single linear programming problem* (LP). This paper is organized as follows. Section 2 is concerned with the preliminaries. Section 3 addresses the robust control invariance issue. Section 4 provides an interesting comparison to existing methods. Finally, Section 5 indicates possible applications of the results and presents conclusions.

NOTATION: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$ and $\mathbb{N}_q \triangleq \{0, 1, \ldots, q\}$. Let $\mathbf{1}_t$ denote the vector $(1, 1, \ldots, 1)' \in \mathbb{R}^t$. Let $\text{abs}(A)$ denote the matrix whose elements are the absolute values of the corresponding components of the matrix A. Given two matrices A and B , $\text{vec}(A)$ denotes standard stack operator and $A \otimes B$ is the Kronecker product of matrices A and B. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is the closed and bounded polyhedron. Let $\mathbb{B}_p^n(r) \triangleq \{x \in \mathbb{R}^n \mid |x|_p \leq r\}$ be a p–norm ball in $\mathbb{R}^n$, where $r \geq 0$ and $|\cdot|_p$ denotes the vector p–norm. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^b$, we define $\bigoplus_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$. The support function of a set $\Pi \subset \mathbb{R}^n$, evaluated at $z \in \mathbb{R}^n$, is defined as $h(\Pi, z) \triangleq \sup_{\pi \in \Pi} z^T \pi$.

2. PRELIMINARY DEFINITIONS AND EXISTING RESULTS

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. The disturbance w is persistent, but contained in a convex and compact set $W \subset \mathbb{R}^n$ that contains the origin. We make the standing assumption that the couple (A, B) is controllable.

The system (2.1) is subject to the following set of hard state and control constraints:

$$(x, u) \in X \times U \tag{2.2}$$

where $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets respectively and both contain the origin as an interior point.

Most of the previous research considered the case $u = \mu(x) = Kx$ and the corresponding autonomous DLTI system:

$$x^+ = A_K x + w,\ A_K \triangleq (A + BK), \tag{2.3}$$

where $A_K \in \mathbb{R}^{n \times n}$ and all the eigenvalues of A_K are strictly inside the unit disk. Given any $K \in \mathbb{R}^{m \times n}$ let $X_K \triangleq \{x \mid x \in X,\ Kx \in U\} \subset \mathbb{R}^n$.

Definition 1. The set $\Omega \subset \mathbb{R}^n$ is a *robust positively invariant* (RPI) set for the system (2.3) and constraint set (X_K, W) if $\Omega \subseteq X_K$ and $A_K x + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$.

Definition 2. The *minimal robust positively invariant* (mRPI) set F_∞ for the system (2.3) and constraint set $(\mathbb{R}^n, W)$ is the RPI set for the system (2.3) and constraint set $(\mathbb{R}^n, W)$ that is contained in every closed, RPI set for the system (2.3) and constraint set $(\mathbb{R}^n, W)$.

The mRPI set F_∞ exists, is unique, compact and contains the origin (Kolmanovsky and Gilbert, 1998, Sect. IV). The mRPI set F_∞ is the limit of the set sequence $\{F_i\}$ defined by:

$$F_i \triangleq \bigoplus_{j=0}^{i-1} A_K^j W,\ i \in \mathbb{N}_+ \text{ and } F_0 \triangleq \{0\} \tag{2.4}$$

The mRPI set is then given by:

$$F_\infty = \text{closure}\left(\bigoplus_{i=0}^{\infty} A_K^i W\right) \tag{2.5}$$

It is impossible in general to obtain an explicit characterization of the mRPI set F_∞. In (Raković *et al.*, 2005) a method for computation of an ε ($\varepsilon > 0$) outer RPI approximation of the mRPI F_∞ is given:

Theorem 1. If $0 \in \text{interior}(W)$, then for all $\varepsilon > 0$, there exists $\zeta \in [0, 1)$ and a corresponding integer s such that the following set inclusions

$$A_K^s W \subseteq \zeta W \text{ and } \zeta(1-\zeta)^{-1} F_s \subseteq \mathbb{B}_p^n(\varepsilon) \tag{2.6}$$

are true. Furthermore, if (2.6) is satisfied, then the set $F_{(\zeta,s)}$ defined by:

$$F_{(\zeta,s)} \triangleq (1-\zeta)^{-1} F_s \tag{2.7}$$

where F_i is defined by (2.4), is an RPI set for the system (2.3) and constraint set $(\mathbb{R}^n, W)$ such that $F_\infty \subseteq F_{(\zeta,s)} \subseteq F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$.

This result can be extended to case when the origin is in the *relative interior* of W (Raković, 2005).

Definition 3. The set $\Omega \subset \mathbb{R}^n$ is a *robust control invariant* (RCI) set for the system (2.1) and constraint set (X, U, W) if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u \in U$ such that $Ax + Bu + w \in \Omega$ for all $w \in W$.

An RPI set for the system (2.3) and constraint set (X_K, W) exists if and only if $F_\infty \subseteq X_K$; this condition is not necessarily satisfied for an arbitrary selected stabilizing feedback controller K. In this note we provide a method for checking *existence of a RCI set for the system* (2.1) *and constraint set* (X, U, W) as well as *the computation of the corresponding control policy* via an optimization procedure.

3. ROBUST CONTROL INVARIANCE ISSUE

Let $M_i \in \mathbb{R}^{m\times n},\ i \in \mathbb{N}$ and for each $k \in \mathbb{N}$ let $\mathbf{M}_k \triangleq (M_0, M_1, \ldots, M_{k-2}, M_{k-1})$. An appropriate characterization of a family of RCI sets for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$ is given by the following sets for $k \geq n$:

$$R_k(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} D_i(\mathbf{M}_k)W \tag{3.1}$$

where the matrices $D_i(\mathbf{M}_k),\ i \in \mathbb{N}_k,\ k \geq n$ are defined by:

$$D_0(\mathbf{M}_k) = I,\ D_i(\mathbf{M}_k) \triangleq A^i + \sum_{j=0}^{i-1} A^{i-1-j}BM_j, i \geq 1 \tag{3.2}$$

providing that $\mathbf{M}_k$ satisfies:

$$D_k(\mathbf{M}_k) = \mathbf{0} \tag{3.3}$$

Since the couple (A, B) is assumed to be controllable, such a choice exists for all $k \geq n$. Let $\mathbb{M}_k$ denote the set of all matrices $\mathbf{M}_k$ satisfying condition (3.3):

$$\mathbb{M}_k \triangleq \{\mathbf{M}_k \mid D_k(\mathbf{M}_k) = \mathbf{0}\} \tag{3.4}$$

Theorem 2. (Raković, 2005) Given any $\mathbf{M}_k \in \mathbb{M}_k,\ k \geq n$ and the corresponding set $R_k(\mathbf{M}_k)$ there exists a control law $\mu : R_k(\mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\mu(x) \oplus W \subseteq R_k(\mathbf{M}_k),\ \forall x \in R_k(\mathbf{M}_k)$, i.e. the set $R_k(\mathbf{M}_k)$ is RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$.

The feedback control law $\mu : R_k(\mathbf{M}_k) \to \mathbb{R}^m$ in Theorem 2 is a selection from the set valued map:

$$\mathcal{U}(x) \triangleq \mathbf{M}_k\mathbf{W}(x) \tag{3.5}$$

where $\mathbf{M}_k \in \mathbb{M}_k$ and the set of *disturbance sequences* $\mathbf{W}(x)$ is defined for each $x \in R_k(\mathbf{M}_k)$ by:

$$\mathbf{W}(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in \mathbf{W}^k,\ D\mathbf{w} = x\}, \tag{3.6}$$

where $\mathbf{W}^k \triangleq W \times W \times \ldots \times W$ and $D = [D_{k-1}(\mathbf{M}_k)\ \ldots\ D_0(\mathbf{M}_k)]$. A $\mu(\cdot)$ satisfying Theorem 2 can be defined, for instance, as follows:

$$\mu(x) \triangleq \mathbf{M}_k\mathbf{w}^0(x) \tag{3.7a}$$

$$\mathbf{w}^0(x) \triangleq \arg\min_{\mathbf{w}}\{|\mathbf{w}|_2^2 \mid \mathbf{w} \in \mathbf{W}(x)\} \tag{3.7b}$$

The function $\mathbf{w}^0(\cdot)$ is piecewise affine, being the solution of a *parametric* quadratic programme; since the feedback control law $\mu : R_k(\mathbf{M}_k) \to \mathbb{R}^m$ is a linear map of a piecewise affine function it is piecewise affine.

Theorem 2 states that for any $k \geq n$ the RCI set $R_k(\mathbf{M}_k)$, *finitely determined by* k, is easily computed if W is a polytope. The set $R_k(\mathbf{M}_k)$ is parametrized by the matrix $\mathbf{M}_k$; this allows us to formulate an LP that yields the set $R_k(\mathbf{M}_k)$ while minimizing an appropriate norm of the set $R_k(\mathbf{M}_k)$.

3.1 Optimized Robust Control Invariance

We provide a full exposition for the case when:

$$W \triangleq \{Ed + f \mid |d|_\infty \leq \eta\} \tag{3.8}$$

where $d \in \mathbb{R}^t,\ E \in \mathbb{R}^{n\times t}$ and $f \in \mathbb{R}^n$. We are interested in the computation of a RCI set $R_k(\mathbf{M}_k)$ for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$ contained in a *'minimal'* p–norm ball, i.e. we wish to find $R_k^0 = R_k(\mathbf{M}_k^0)$ where:

$$(\mathbf{M}_k^0, \alpha^0) = \arg\min_{\mathbf{M}_k,\alpha}\{\alpha \mid R_k(\mathbf{M}_k) \subseteq \mathbb{B}_p(\alpha),\ \alpha > 0\} \tag{3.9}$$

We show that our problem can be posed as an LP if $p = 1, \infty$ by considering a more general problem:

$$\mathbb{P}_k :\ (\mathbf{M}_k^0, \alpha^0) = \arg\min_{\mathbf{M}_k,\alpha}\{\alpha \mid (\mathbf{M}_k, \alpha) \in \Omega\} \tag{3.10}$$

where

$$\Omega \triangleq \{(\mathbf{M}_k, \alpha) \mid \mathbf{M}_k \in \mathbb{M}_k,\ R_k(\mathbf{M}_k) \subseteq P(\alpha),\ \alpha > 0\}, \tag{3.11}$$

and $P(1)$ is a polytope that contains the origin in its interior so that $P(\alpha) \triangleq \{x \mid C_px \leq \alpha c_p\},\ \alpha > 0$ with $C_p \in \mathbb{R}^{q\times n}$ and $c_p \in \mathbb{R}^q$. Before proceeding we recall few preliminary and elementary results (Raković, 2005):

Proposition 1. Let Π be a non-empty set in $\mathbb{R}^n$ and $\Psi = \left\{\psi \in \mathbb{R}^n \mid f_i^T\psi \leq g_i,\ i \in \mathbb{N}_l\right\}$, where $f_i \in \mathbb{R}^n,\ g_i \in \mathbb{R}$. Then, $\Pi \subseteq \Psi$ if and only if $h(\Pi, f_i) \leq g_i$ for all $i \in \mathbb{N}_l$.

Proposition 2. Let each matrix $L_k \in \mathbb{R}^{n\times m}$ and each Φ_k be a non-empty, compact set in $\mathbb{R}^m$ for all $k \in \mathbb{N}_K$. If $\Pi = \bigoplus_{k=0}^K L_k\Phi_k$, then $h(\Pi, z) = \sum_{k=0}^K \max_{\phi\in\Phi_k}(z^TL_k)\phi$.

The fact that $\max_d\{a'd \mid |d|_\infty \leq \eta\} = \eta|a|_1$ (Horn and Johnson, 1985) allows one to establish the following result:

Proposition 3. Let matrices $A \in \mathbb{R}^{n\times n},\ C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{n\times p}$ and $M \in \mathbb{R}^{p\times n}$ and let $w \in W$

where $W = \{Ed + f \mid |d|_\infty \leq \eta\}$ and $E \in \mathbb{R}^{n\times t}$ and $f \in \mathbb{R}^n$. Then

$$\max_{w\in W} C(A+DM)w = \eta \text{abs}(C(A+DM)E)\mathbf{1}_t + C(A+DM)f \quad (3.12)$$

where the maximization is taken row-wise. Moreover, there exists a matrix $L \in \mathbb{R}^{q\times t}$ such that

$$-L \leq C(A+DM)E \leq L, \quad (3.13)$$

where the inequality is element-wise, and the solution to (3.12) satisfies

$$\max_{w\in W} C(A+DM)w = \eta L \mathbf{1}_q + C(A+DM)f \quad (3.14)$$

Proposition 1 implies that the set inclusion $R_k(\mathbf{M}_k) \subseteq P(\alpha)$ is true if and only if:

$$\max_{x\in R_k(\mathbf{M}_k)} C_p x \leq \alpha c_p, \quad (3.15)$$

where the maximization is taken row-wise. It follows from Propositions 2 and 3 that there exist a set of matrices $L_i \in \mathbb{R}^{q\times t},\ i \in \mathbb{N}_{k-1}$ such that:

$$\max_{x\in R_k(\mathbf{M}_k)} C_p x = \sum_{i=0}^{k-1}(\eta L_i \mathbf{1}_t + C_p D_i(\mathbf{M}_k) f) \quad (3.16)$$

where $\Lambda_k \triangleq \{L_0, L_1, \ldots, L_{k-1}\}$ and each L_i satisfies:

$$-L_i \leq C_p D_i(\mathbf{M}_k) E \leq L_i,\ i \in \mathbb{N}_{k-1} \quad (3.17)$$

Since each $D_i(\mathbf{M}_k)$ is affine in $\mathbf{M}_k$ it follows by the basic properties of the Kronecker product (in particular $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$) that the set inclusion $R_k(\mathbf{M}_k) \subseteq P(\alpha)$ can be expressed as a set of linear inequalities in $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$. The condition $\mathbf{M}_k \in \mathbb{M}_k$ is a set of linear equalities in $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$. Since the cost (of $\mathbb{P}_k$) is a linear function of $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$ we can state the following:

Proposition 4. The minimization problem $\mathbb{P}_k$ defined in (3.10) is a linear programming problem.

An LP formulation of the problem $\mathbb{P}_k$ is:

$$\mathbb{P}_k: \quad \min_\gamma \{\alpha \mid \gamma \in \Gamma\} \quad (3.18)$$

where $\gamma \triangleq (\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$ and:

$$\begin{aligned}\Gamma \triangleq \{\gamma \mid & \sum_{i=0}^{k-1}(\eta L_i \mathbf{1}_t + C_p D_i(\mathbf{M}_k) f) \leq \alpha c, \\ & -L_i \leq C_p D_i(\mathbf{M}_k) E \leq L_i,\ i \in \mathbb{N}_{k-1}, \\ & \mathbf{M}_k \in \mathbb{M}_k, \alpha > 0\}\end{aligned} \quad (3.19)$$

3.2 Optimized Robust Control Invariance Under Constraints

In this case it is possible to formulate an LP, whose feasibility establishes existence of a RCI set $R_k(\mathbf{M}_k)$ for the system (2.1) and constraint set (X, U, W). The control law $\mu(x)$ satisfies $\mu(x) \in U(\mathbf{M}_k)$ for all $x \in R_k(\mathbf{M}_k)$ where:

$$U(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} M_i W \quad (3.20)$$

The state and control constraints (2.2) are satisfied if:

$$R_k(\mathbf{M}_k) \subseteq \alpha X,\ U(\mathbf{M}_k) \subseteq \beta U \quad (3.21)$$

where $\alpha X \triangleq \{x \mid C_x x \leq \alpha c_x\}$, $\beta U \triangleq \{u \mid C_u u \leq \beta c_u\}$, (with $C_x \in \mathbb{R}^{q_x\times n}$, $c_x \in \mathbb{R}^{q_x}$, $C_u \in \mathbb{R}^{q_u\times n}$, $c_u \in \mathbb{R}^{q_u}$) and $(\alpha, \beta) \in [0,1]\times[0,1]$.

Let now:

$$\begin{aligned}\bar{\Omega} \triangleq \{(\mathbf{M}_k, \alpha, \beta, \delta) \mid & \mathbf{M}_k \in \mathbb{M}_k,\ R_k(\mathbf{M}_k) \subseteq \alpha X, \\ & U(\mathbf{M}_k) \subseteq \beta U, \\ & (\alpha, \beta) \in [0,1]\times[0,1], \\ & q_\alpha \alpha + q_\beta \beta \leq \delta\}\end{aligned} \quad (3.22)$$

where $R_k(\mathbf{M}_k)$ is given by (3.1) and $U(\mathbf{M}_k)$ by (3.20). Consider the following minimization problem:

$$\bar{\mathbb{P}}_k : (\mathbf{M}_k^0, \alpha^0, \beta^0, \delta^0) = \arg\min_{\mathbf{M}_k,\alpha,\beta,\delta}\{\delta \mid (\mathbf{M}_k, \alpha, \beta, \delta) \in \bar{\Omega}\} \quad (3.23)$$

Proposition 5. The minimization problem $\bar{\mathbb{P}}_k$ is a linear programming problem.

The problem $\bar{\mathbb{P}}_k$ is an LP:

$$\bar{\mathbb{P}}_k: \quad \min_\gamma \{\delta \mid \gamma \in \bar{\Gamma}\} \quad (3.24)$$

where $\gamma \triangleq (\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \text{vec}(\Theta_k), \alpha, \beta, \delta)$ and :

$$\begin{aligned}\bar{\Gamma} \triangleq \{\gamma \mid & \sum_{i=0}^{k-1}(\eta L_i \mathbf{1}_t + C_x D_i(\mathbf{M}_k) f) \leq \alpha c_x, \\ & -L_i \leq C_x D_i(\mathbf{M}_k) E \leq L_i,\ i \in \mathbb{N}_{k-1}, \\ & \sum_{i=0}^{k-1}(\eta T_i \mathbf{1}_t + C_u S_i \mathbf{M}_k f) \leq \beta c_u, \\ & -T_i \leq C_u S_i \mathbf{M}_k E \leq T_i,\ i \in \mathbb{N}_{k-1}, \\ & (\alpha, \beta) \in [0,1]\times[0,1], \\ & \mathbf{M}_k \in \mathbb{M}_k,\ q_\alpha \alpha + q_\beta \beta \leq \delta\}\end{aligned} \quad (3.25)$$

where $\Theta_k \triangleq \{T_0, T_1, \ldots T_{k-1}\}$ (each $T_i \in \mathbb{R}^{q_u\times t}$) and S_i is selection matrix of the form $S_i = [\mathbf{0}\ \mathbf{0}\ \ldots\ I\ \ldots\ \mathbf{0}\ \mathbf{0}]$. It is possible to specify a variety of objective functions by minor modification of the definition of the set $\bar{\Omega}$ (3.22) and still obtain a tractable convex optimization problem. However, an appropriate objective function is the minimization of $q_\alpha \alpha, q_\beta \beta$ subject to the existence of a RCI set $R_k(\mathbf{M}_k)$ for the system (2.1) and constraint set $(\alpha X, \beta U, W)$. *The weights q_α and q_β express a preference for relative contraction of the state and control constraint sets.*

The solution $\mathbf{M}_k^0$ to problem $\bar{\mathbb{P}}_k$ (which exists if $\bar{\Omega} \neq \emptyset$) yields a set $R_k^0 \triangleq R_k(\mathbf{M}_k^0)$ and feedback control law $\mu^0(x) = \mathbf{M}_k^0 \mathbf{w}^0(x)$ satisfying $R_k^0 \subseteq \alpha^0 X$ and $\mu^0(x) \in U(\mathbf{M}_k) \subseteq \beta^0 U$ for all $x \in R_k^0$. It follows from Theorem 2 and the discussion above that the set R_k^0, if it exists, is RPI for system $x^+ = Ax + B\mu^0(x) + w$ and constraint set (X_{μ^0}, W), where $X_{\mu^0} \triangleq \alpha^0 X \cap \{x \mid \mu^0(x) \in \beta^0 U\}$. There might exist more than one set $R_k(\mathbf{M}_k)$ that yields the optimal cost δ^0. The cost function can be modified. For instance, an appropriate choice is a positively weighted quadratic norm of the decision variable γ that yields a unique solution, since in this case problem becomes a quadratic programming problem of the form $\min_\gamma \{|\gamma|_Q^2 \mid \gamma \in \bar{\Gamma}\}$, where Q is positive definite and it represents the suitable weight. A relevant observation is:

Proposition 6. Suppose that the problem $\bar{\mathbb{P}}_k$ is feasible for some $k \in \mathbb{N}$ and the optimal value of δ_k is δ_k^0, then for every integer $s \geq k$ the problem $\bar{\mathbb{P}}_s$ is also feasible and the corresponding optimal value of δ_s satisfies $\delta_s^0 \leq \delta_k^0$.

If the origin is an interior point of W, the condition (3.3) can be replaced by the following condition:

$$\mathbf{M}_k \in \bar{\mathbb{M}}_k \triangleq \{\mathbf{M}_k \mid D_k(\mathbf{M}_k) W \subseteq \varphi W\} \quad (3.26)$$

for $\varphi \in [0, 1)$ and $k \geq n$. A family of the sets $R_{(\varphi,k)}(\mathbf{M}_k)$ defined by:

$$R_{(\varphi,k)}(\mathbf{M}_k) \triangleq (1 - \varphi)^{-1} R_k(\mathbf{M}_k) \quad (3.27)$$

for couples (φ, k) such that (3.26) is true, is a family of the polytopic RCI sets:

Theorem 3. Given any couple $(\varphi, \mathbf{M}_k) \in [0, 1) \times \bar{\mathbb{M}}_k$, $k \geq n$ and the corresponding set $R_{(\varphi,k)}(\mathbf{M}_k)$, there exists a control law $\mu : R_{(\varphi,k)}(\mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\mu(x) \oplus W \subseteq R_{(\varphi,k)}(\mathbf{M}_k)$, $\forall x \in R_{(\varphi,k)}(\mathbf{M}_k)$.

4. COMPARISON

A theoretical comparison of the proposed procedure with the previous results is given in (Raković, 2005). The advantages of our method lie in the facts that: (i) hard state and control constraints are incorporated directly into the optimization problem and, (ii) the feedback control law $\mu : R_k(\mathbf{M}_k) \to U$ is piecewise affine function of $x \in R_k(\mathbf{M}_k)$. These advantages are illustrated bellow by a numerical example:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \quad (4.1)$$

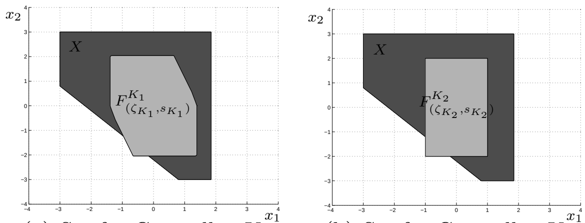
where $w \in W \triangleq \{w \in \mathbb{R}^2 \mid |w|_\infty \leq 1\}$. The hard state and control constraints are:

$$X = \{x \mid -3 \leq x^1 \leq 1.85,\ -3 \leq x^2 \leq 3,\ x^1 + x^2 \geq -2.2\},\ U = \{u \mid |u| \leq 2.4\} \quad (4.2)$$

where x^i is the i^{th} coordinate of a vector x. In the first attempt we obtain the closed loop dynamics by applying two linear stabilizing state feedback control laws:

$$K_1 = -[0.72\ 0.98],\ K_2 = -[1\ 1] \quad (4.3)$$

and compute the corresponding sets $F^K_{(\zeta_K, s_K)}$ by application of Algorithm 1 of (Raković *et al.*, 2005). The computed sets violate the state con-



(a) Set for Controller K_1 (b) Set for Controller K_2

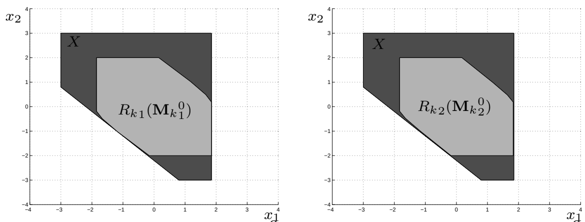
Fig. 1. Invariant Approximations of $F_\infty(K_i)$: Sets $F^{K_i}_{(\zeta_{K_i}, s_{K_i})}$, $i = 1, 2$

straints as illustrated in Figure 1. The corresponding control polytopes are:

$$U(K_1) = \{u \mid |u| \leq 2.4680\},\ U(K_2) = \{u \mid |u| \leq 3\}, \quad (4.4)$$

where $U(K) \triangleq K F^K_{(\zeta_K, s_K)}$ so that the control constraints are also violated.

By solving the optimization problem $\bar{\mathbb{P}}_k$ with the



(a) Set for Controller $\mathbf{M}_{k1}^0$ (b) Set for Controller $\mathbf{M}_{k2}^0$

Fig. 2. Invariant Sets $R_{ki}(\mathbf{M}_{ki}^0)$, $i = 1, 2$

following design parameters:

$$(k, q_\alpha, q_\beta)_1 = (5, 0, 1),\ (k, q_\alpha, q_\beta)_2 = (5, 1, 0) \quad (4.5)$$

we computed the following matrices $\mathbf{M}_{ki}^0$, $i = 1, 2$:

$$\mathbf{M}_{k1}^0 = \begin{bmatrix} -0.4875 & -1 \\ 0.2199 & 0 \\ 0.1154 & 0 \\ 0.0596 & 0 \\ 0.0926 & 0 \end{bmatrix},\ \mathbf{M}_{k2}^0 = \begin{bmatrix} -0.5038 & -1 \\ 0.2456 & 0 \\ 0.1132 & 0 \\ 0.0521 & 0 \\ 0.0930 & 0 \end{bmatrix} \quad (4.6)$$

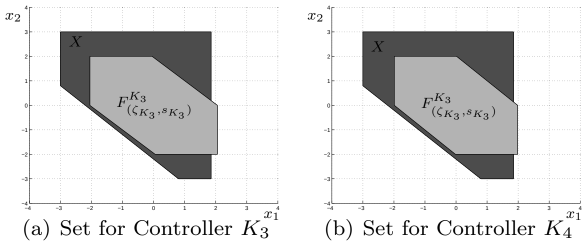
The sets constructed from the solution of the optimization problem $\bar{\mathbb{P}}_k$ satisfy state and control constraints as it can be seen from Figure 2 and from the fact that:

$$U(\mathbf{M}_{k_i}^{0}) = \{u \mid |u| \leq 1.975\},\ i = 1, 2, \quad (4.7)$$

To make our comparison as fair as possible, we consider also the following two linear state feedback control laws, constructed from the first row of the optimized matrices $\mathbf{M}_k$:

$$K_3 = -[0.4875\ 1],\ K_4 = -[0.5038\ 1] \quad (4.8)$$

The corresponding sets $F^{K}_{(\zeta_K, s_K)}$ are shown in Figure 3. The control constraints are satisfied, but the computed sets violate the state constraints.



(a) Set for Controller K_3 (b) Set for Controller K_4

Fig. 3. Invariant Approximations of $F_\infty(K_i)$: Sets $F^{K_i}_{(\zeta_{K_i}, s_{K_i})}$, $i = 3, 4$

5. APPLICATIONS AND CONCLUSIONS

The results of this paper can be used in the design of robust reference governors, predictive controllers and time-optimal controllers for constrained, linear discrete time systems subject to additive, but bounded disturbances.

The main contribution of this note is a novel characterization of a family of polytopic robust control invariant sets for which the corresponding control law is non-linear (piecewise affine) enabling better results to be obtained compared with existing methods where the control law is linear. Construction of a member of this family contained in the minimal p–norm ball or reference polytopic set can be obtained from the solution of an appropriately specified LP. The *optimized robust control invariance* algorithms were illustrated by an example, in which significant improvements over existing methods was illustrated.

The results can be extended to the case when disturbance belongs to an arbitrary polytope. Moreover, it is also possible to extend the results to the case when the system dynamics are parametrically uncertain.

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