ON THE CATEGORY OF BANACH SPACES

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Abstract. These notes are targeted mainly at final year undergraduate students and present Whitley's proof of Phillips' theorem that c_0 has no closed complement in ℓ^∞ which means that c_0 is not injective as an object of the category of Banach spaces. All the material is taken from Werner's *Funktionalanalysis* [5, Section IV.6].

1. Introduction

These notes are based on Section IV.6 of Werner's *Funktionalanalysis* [5] and present Whitley's neat and clever proof that c_0 has no closed complement in ℓ^∞ (a result originally due to Phillips). I include this in our lecture course on functional analysis at the end of the Banach space theory section as I find it a superb illustration of the subtlety of the topology of Banach spaces and of their homological properties, and in fact much more generally of a doable but tricky and unexpected proof of a harmless looking statement. Last but not least the proof fits perfectly into a one-hour lecture if one is well-prepared and determined (so I usually fail) and it allows me to review some linear algebra before getting into Hilbert spaces. As [5] is unfortunately not available in English and I do not know another textbook that treats the matter in this detail, I have typed up these notes for the students. Maybe one day I will expand this further, so any comments and suggestions are highly welcome.

2. On direct sums and idempotent morphisms

2.1. **Linear algebra.** We begin by recalling some linear algebra. In all of this section, we consider vector spaces over an arbitrary field $\mathbb{F}$.

Proposition 1. *If V is a subspace of a vector space U, then there exists a second subspace $W \subset U$ such that $U = V \oplus W$.*

Proof. Choose a basis $B_V \in \mathcal{P}(V)$ of V. As this is a linearly independent subset of U it can be completed to a basis B_U of the whole of U. Then define W as the span of the newly added basis vectors, $W := \mathrm{span}_{\mathbb{F}}(B_U \setminus B_V)$. This is easily verified to do the job. $\square$

Note that W is by no means unique. However, the possible choices of W can be easily characterised in terms of linear maps $\pi : U \to U$ which are *idempotent*, meaning that $\pi = \pi \circ \pi$:

Proposition 2. *Given $V \subset U$, there is a bijective correspondence between subspaces $W \subset U$ with $U = V \oplus W$ and idempotent linear maps $\pi : U \to U$ having kernel $\ker \pi = V$.*

Proof. Given π, put $W = \operatorname{im} \pi$. Then $V \cap W = 0$ because if $x \in \ker \pi$ is also in the image of π, $x = \pi(y)$, then $\pi \circ \pi = \pi$ implies

$$0 = \pi(x) = \pi(\pi(y)) = (\pi \circ \pi)(y) = \pi(y) = x. \tag{1}$$

Furthermore, every element $x \in U$ can be written as $y + z$ with $y := x - \pi(x)$ and $z := \pi(x)$. Clearly $z \in W = \operatorname{im} \pi$ but we also have

$$\pi(y) = \pi(x) - \pi(\pi(x)) = \pi(x) - \pi(x) = 0 \tag{2}$$

using once more that π is idempotent, so $y \in V = \ker \pi$ and we have shown that $U = V + W$. In combination, $U = V \oplus W$.

Conversely, $U = V \oplus W$ means that we can write every $x \in U$ uniquely as $y + z$ with $y \in V, z \in W$ and we can simply define the idempotent map π corresponding to this choice of W by $\pi(x) := z$. $\square$

In particular, $W = \operatorname{im} \pi$ is by the first isomorphism theorem isomorphic to the quotient vector space U/V, so the complement is at least unique up to a canonical isomorphism.

2.2. **Functional analysis.** The operation of a direct sum exists for many other types of mathematical objects. There is an abstract definition (see textbooks on *category theory*), but instead of spending time on this let us consider just one simple example from algebra that illustrates the matter before we begin with our main story:

Example 1. As for vector spaces, the direct sum $G \oplus H$ of abelian groups G, H is the product $G \times H$ of the underlying sets with componentwise group operation. However, here the analogue of Proposition 1 is not true: the additive subgroup $2\mathbb{Z} \subset \mathbb{Z}$ of even integers has no complement, that is, there is no subgroup $G \subset \mathbb{Z}$ such that $\mathbb{Z} = 2\mathbb{Z} \oplus G$.

Note that in Section 2.1 we were talking about *internal* direct sums, V, W were from start subspaces of a vector space U, while in the above example this gets somewhat mixed up with the notion of an *external* direct sum as we talk about taking any two abelian groups G, H and forming their direct sum which a posteriori can be shown to be the internal direct sum of two subgroups isomorphic to G and H. However, I think we can also skip the somewhat tedious and not very inspiring discussion of the precise relation between internal and external direct sums, once we are ready we will prove a result that clarifies that point.

What we want to talk about here is how subtle this topic is when dealing with *Banach spaces*, that is, normed vector spaces $(U, \|\cdot\|)$ over $\mathbb{F} = \mathbb{C}$ which are complete with respect to the metric $d(x, y) := \|x-y\|$. We assume that the reader is familiar with the most standard material on Banach spaces but will try to recall some of that on the way.

So, let us return to the beginning and carefully clarify what the various notions we used in Section 2.1 should mean for Banach spaces.

Definition 1. *A* Banach subspace *V of a Banach space U is a vector subspace which is closed (in the topology of U defined by the norm).*

Equivalently, V is a vector subspace which is a Banach space in its own right with respect to the norm of U restricted to V.

However, how about direct sums? Here is one definition:

Definition 2. *If U is a Banach space and $V, W \subset U$ are Banach subspaces, then we write $U = V \oplus W$ and say U is the* direct sum *of V and W if $U = V \oplus W$ as vector spaces.*

This definition is supposed to trigger some obvious questions: should the direct sum really just refer to the underlying vector space? Do we not want compatibility with the topological structure? Well, it turns out that we get this for free, but this is a nontrivial result and in order to discuss it we first should introduce another notion:

Definition 3. *A* morphism *$\varphi : V \to U$ of Banach spaces U, V is a linear map of the underlying vector spaces which is continuous with respect to the topologies defined by the norms.*

The first result on linear maps between Banach spaces that is proven in any course on functional analysis says that the morphisms $\varphi : V \to U$ are exactly what one traditionally calls *bounded linear operators*, that is, linear maps $\varphi : V \to U$ for which there is a constant $C > 0$ such that $\|\varphi(x)\| \leq C\|x\|$ for all $x \in V$. The *norm* $\|\varphi\|$ of φ is the infimum of all such C, so that for every morphism φ of Banach spaces we have

$$\|\varphi(x)\| \leq \|\varphi\|\|x\| \quad \forall x \in V. \tag{3}$$

At this moment one might ask why a morphism of a Banach space should not respect the structure complelety in the sense that it should be also an *isometry*, meaning $\|\varphi(x)\| = \|x\|$ for all $x \in V$. However, note that then $\varphi(x) = 0$ implies $x = 0$, so isometries are necessarily injective, and this is a bit more restrictive than what we want. Still, I think a lot of the mysteries in Banach space theory arise exactly from this fact that we study objects with a certain structure but allow for morphisms that do not really respect that structure completeley.

For example, let us contemplate the relation between Banach subspaces and monomorphisms (injective morphisms, that is, bounded linear operators φ with kernel $\ker\varphi = 0$) in the category of Banach spaces.

Proposition 3. *If $\varphi : V \to U$ is a monomorphism of Banach spaces, then $\operatorname{im}\varphi$ is a Banach subspace of U if and only if there exists $c > 0$ such that $\|\varphi(x)\| \geq c\|x\|$ for all $x \in V$.*

Proof. Like Conway [2, Exercise III.13.5] I leave this as an exercise. □

Example 2. The main example of a Banach space we will need is the vector space ℓ^∞ of all bounded sequences $x = \{x_n\}_{n\in\mathbb{N}}$ of complex numbers equipped with the norm

$$\|x\| := \sup_{n\in\mathbb{N}} |x_n|. \tag{4}$$

See any textbook on functional analysis for more information about this Banach space. Now consider the operator $\varphi : \ell^\infty \to \ell^\infty$ that rescales the n-term of a sequence $x = \{x_n\} \in \ell^\infty$ by $\frac{1}{n}$,

$$\varphi(x)_n := \frac{1}{n}x_n. \tag{5}$$

This is an injective bounded linear operator of norm $\|\varphi\| = 1$ violating the condition in the proposition, so $\operatorname{im}\varphi$ is not closed.

So one has to be very careful when assuming that the obvious generalisations of facts from linear algebra are true for Banach spaces. Sometimes they might be wrong, and sometimes they might be much deeper results than one thinks they are. Here is an example:

Theorem 1. *If $\varphi : V \to U$ is a morphism of Banach spaces which is bijective as a map of sets, then the inverse map $\varphi^{-1} : V \to U$ is a morphism of Banach spaces as well.*

This *inverse mapping theorem* is usually served as a dessert following the highly nontrivial *open mapping theorem* which says that bounded linear maps are open, that is, map open sets to open stes. See e.g. [2, Section III.4] for the full story and proofs. Note that in the category of topological spaces the analogous theoremis wrong, the inverse of a bijective continuous map might be not continuous, while for linear maps between vector spaces the statement is true but pretty banal. It is in functional analysis where linear algebra meets topology where a lot of things results turn out to be true but also to be surprisingly deep.

Now we are ready to come back to direct sums of Banach spaces. Recall that at the start of Section 2.2 I have confused you by talking about internal and external direct sums. Let me now explain what the fuss is about in the conrete setting of Banach spaces. On the one hand, there is Definition 2 which explains what we mean when saying that a Banach space is the direct sum of two Banach subspaces. On the other hand, suppose V, W are any two Banach spaces, not necessarily embedded into soem other Banach space U. Then we can take $V \times W$ as sets and turn this into a vector space in the usual way with componentwise operations. However, we can now also deifne a norm on that vector space by putting

$$\|(x, y)\| := \|x\| + \|y\|, \quad x \in V, y \in W, \tag{6}$$

where $\|x\|$, $\|y\|$ are of course the norms taken in V and W. An easy consideration proves that this indeed deifnes a norm on $V \times W$, and that the normed space $V \times W$ is a Banach space as V, W are Banach. This is called the *external* direct sum of V, W (and a direct sum in the proper categorical meaning of the word), but in fact we have:

Proposition 4. *If V, W are Banach subspaces of a Banach space U and $U = V \oplus W$, then $U \simeq V \times W$ as Banach spaces.*

Proof. In any category, an isomorphism is by deifnition a morphism $\varphi : U \to U'$ for which there is a morphism $\varphi^{-1} : U' \to U$ such that the two compositions of the morphisms are the identity morphisms on U respectively U'. In our concrete situation, every $x \in U$ can be uniquely written as $y + z$ with $y \in V, z \in W$, so we can deifne

$$\varphi : U \to V \times W \tag{7}$$

by mapping x to (y, z). This is quite easily seen to be a bijective bounded linear operator between Banach spaces of norm 1 as the triangle inequality in U says that

$$\|x\| = \|y + z\| \leq \|y\| + \|z\| = \|\varphi(x)\|. \tag{8}$$

However, be aware that for proving that $\varphi^{-1} : (y, z) \mapsto y + z$ is a bounded you will have to use some sledgehammer such as Theorem 1, you are bound to fail proving this in an elementary way! □

So there is no need to distinguish between internal and external direct sums, but again, while this is almost trivial in most situations in algebra it requires much work in the category of Banach spaces. By the way, one can also easily check now that the direct sum of Banach spaces is as a topological space the product of the two factors.

Anyway, the above is part of [5, Satz IV.6.3], and the second part settles the question about the analogue of Proposition 2:

Proposition 5. *If $V \subset U$ are Banach spaces, then there is a bijective correspondence between Banach subspaces $W \subset U$ with $U = V \oplus W$ and idempotent morphisms $\pi : U \to U$ having kernel* $\ker \pi = V$.

Proof. Proposition 2 provides us with the correspondence. What has to be investigated here is that the idempotent linear map $\pi : U \to U$ with kernel V and image W from Proposition 2 is a morphism of Banach spaces (i.e. is bounded) if both V, W are closed, and conversely that if π is an idempotent morphism that both $\ker \pi$ and $\operatorname{im} \pi$ are closed.

For the latter fact one observes first that $\ker \pi$ is the preimage of a closed set, namely $\{0\} \subset U$, so if π is continuous, its kernel is closed. A priori one feels that the image is more tricky to deal with (recall Proposition 3). however, if π is an idempotent morphism, then so is $\eta := \mathrm{id}_U - \pi$, and as $\operatorname{im} \pi = \ker \eta$ this is closed by the same argument applied to η rather than π.

The former claim follows from Proposition 4 respectively the inverse mapping theorem as used in its proof: we have seen there that

$$V \times W \to U, \quad (y, z) \mapsto y + z \tag{9}$$

is a continuous map, hence bounded. So there exists $C > 0$ such that

$$\|y\| + \|z\| \leq C(\|y + z\|) \quad \forall y \in V, z \in W. \tag{10}$$

In particular, $\|z\| \leq C(\|y + z\|)$, so the projection $\pi : y + z \mapsto z$ is bounded with norm $\|\pi\| \leq C$. □

The last remark made in Section 2.1 is also true but we would have to deifne the quotient of Banach spaces to discuss this so we skip that, see again [5, Satz IV.6.3].

2.3. **Phillips' theorem.** After all the discussion above we can now finally get to the quesiton we really want to reflect upon:

Definition 4. *A Banach subspace V of a Banach space U is* complemeneted *if $U = V \oplus W$ for some Banach subspace $W \subset U$.*

So, are all Banach subspaces complemented? That is, is the analogue of Proposition 1 true in the category of Banach spaces? Or, if you want to ask the same question in more abstract terms: are all Banach spaces *injective* objects in the category of Banach space?

The answer is no, not at all, and that deciding wheter a given $V \subset U$ is complemented is pretty obscure. And what we will do in the remaining second part of this note is demonstrating this by proving the following:

Theorem 2. *The Banach subspace $c_0 \subset \ell^\infty$ consisting of all sequences that converge to zero is not complemented.*

So whenever we take a vector space complemenet $W \subset \ell^\infty$ of c_0 using Proposition 1 it can not be closed, and the corresponding idempotent linear map π from Proposition 2 is necessarily unbounded.

This was proved by Phillips in 1940 [3] as an application of some quite detailed study of morphisms between ℓ^p-spaces (if you try to read the original article be aware that his notaiton is not quite ours, for example you will find on p. 356 the sentence "For notational convenience, we write $c_0 = l^\infty$" which is not so convenient for us). Anyway, the proof I present and that I learned from [5] is a beautiful direct argument due to Whitley [4]. Conway mentions the result and these two references on [2, p 94] and I suggest you have a look at the further fascinating remarks he makes there. One should maybe only add to this that in the same year 1966 when Whitley wrote up his proof, Conway himself wrote [1] which he is too modest to mention in [2].

Whitley's himself also admits that his proof is very related to proofs in the literature and that the referee (Conway?) has pointed this out to him. But let us not spend too much time on history and moral decline in mathematics and instead prove the thing.

3. The proof of Phillips' theorem

3.1. **Krause's lemma and its implication.** We begin by constructing some sets that will be needed in the proof. Whitley attributes at least this particular constuction to Arthur Krause:

Lemma 1. *There ecists an uncountable set $\{N_x\}$ of infinite sets N_x of natural numbers such that each two of them have finite intersection.*

Proof. Pick a numbering of $\mathbb{Q}$ (which we recall is countable)

$$\mathbb{Q} = \{q_1, q_2, q_3, \dots\} \tag{11}$$

and for every irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ a sequence of rational numbers $\{x_i\}$ converging to x. Then define

$$N_x := \{n \in \mathbb{N} \mid q_n = x_i \text{ for some } i\}. \tag{12}$$

There are uncountably many irrational numbers, and if N_x were finite, then the sequence $\{x_i\}$ that converges to x has only finitely many different terms and hence becomes stable which is impossible as $x_i \in \mathbb{Q}$ but $x \notin \mathbb{Q}$. So N_x is infinite. Finally, if $N_x \cap N_y$ is infinite, then there is a corresponding infinite subsequence common to both $\{x_i\}$ and $\{y_j\}$ and that must converge to both x and to y, hence $x = y$. □

So, what has this to do with our story? Well, we can now consider the characteristic function $k^x := \chi_{N_x}$ of N_x which is the sequence $\{k^x_n\} \in \ell^\infty$ whose n-th term is

$$k^x_n := \begin{cases} 1 & n \in N_x \\ 0 & n \notin N_x. \end{cases} \tag{13}$$

By construction of N_x the sequences k^x, k^y have for $x \neq y$ at most a finite number of 1's in common, and this allows us to say something about the norm of a linear combination of these sequences if we allow for a small perturbation that belongs to c_0:

Lemma 2. *For all $x_1, \ldots, x_r \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ there exists $d \in c_0$ such that*

$$\|\lambda_1 k^{x_1} + \ldots + \lambda_r k^{x_r} - d\| = \max\{|\lambda_1|, \ldots, |\lambda_r|\}. \tag{14}$$

Proof. This is in fact pretty banal but a bit awkward to write up formally: given $n \in \mathbb{N}$ define

$$S_n := \{j \in \{1, \ldots, r\} \mid n \in N_{x_j}\}. \tag{15}$$

Then the n-th term k_n in the sequence $k := \lambda_1 k^{x_1} + \ldots + \lambda_r k^{x_r} \in \ell^\infty$ is $\sum_{j \in S_n} \lambda_j$, as the set S_n are simply those indices j for which $k^{x_j}_n$ is not zero so that $\lambda_j k^{x_j}$ contributes a λ_j to k_n.

The main point is that only for finitely many $n \in \mathbb{N}$ the set S_n can have more than one element: $i, j \in S_n$ for $i \neq j$ means nothing but $n \in N_{x_i} \cap N_{x_j}$. However, recall that this intersection was finite, so

$$N := \bigcup_{1 \leq i < j \leq r} N_{x_i} \cap N_{x_j} \tag{16}$$

is a finite set. Hence the sequence d with n-th term

$$d_n := \begin{cases} \sum_{j \in S_n} \lambda_j & n \in N \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

has only finitely many nonzero terms, and in particular belongs to c_0. This sequence concides with k in all terms except those for which $|S_n| = 1$ where $k_n = \lambda_j$ for some $j \in \{1, \ldots, r\}$ while $d_n = 0$. So, $k - d$ is a sequence all of whose nonzero terms areone of the λ_j's. Now the claim should be clear. □

3.2. **Assume Phillips'theorem is wrong.** Now assume there is a Banach subspace $W \subset \ell^\infty$ with $\ell^\infty = c_0 \oplus W$ and let $\pi : \ell^\infty \to \ell^\infty$ be the idempotent morphism corresponding to this decomposition, that is, $\ker \pi = c_0, \operatorname{im} \pi = W$ and we know that π is bounded.

The contradiction will arise from the following fact:

Lemma 3. *If we define*

$$w^x := k^x - \pi(k^x) \in W, \tag{18}$$

then for every continuous linear functional $f \in (\ell^\infty)^*$ *the set*

$$I_f := \{x \in \mathbb{R} \setminus \mathbb{Q} \mid f(w^x) \neq 0\} \tag{19}$$

is countable.

I put this into a new subsection to help you keeping track of where we are.

3.3. **Proof of Lemma 3.** Indeed, for $N \in \mathbb{N}$

$$I_f^N := \{x \in \mathbb{R} \setminus \mathbb{Q} \mid |f(v^x)| > \frac{1}{N}\}$$

is finite: if we fix r pairwise different irrational numbers

$$x_1, \ldots, x_r \in I_f^N, \tag{20}$$

then Now η is continuous and $\eta(y) = 0$ ($c_0 = \ker \eta$), so if we fix λ_i with $|\lambda_i| = 1$ such that

$$f(\lambda_i v^{x_i}) = |f(v^{x_i})| \geq \frac{1}{N},$$

then we get

$$\begin{aligned} \frac{1}{N} r &\leq |\sum_{i=1}^r f(\lambda_i v^{x_i})| = |f(\sum_{i=1}^r \lambda_i v^{x_i})| \\ &\leq \|f\| \| \sum_{i=1}^r \lambda_i v^{x_i} \|_\infty \leq \|f\| \|\eta\| \| \sum_{i=1}^r \lambda_i e^{x_i} - y\|_\infty \\ &= \|f\| \|\eta\|. \end{aligned}$$

This means that for fixed N, the number r of x_i's in (20) is at most $N\|f\|\|\eta\|$, so I_f^N is indeed finite. And since I_f from (19) is the union $\bigcup_{N \geq N} I_f^N$, this set is indeed at most countable as claimed.

Final step of the proof: consider the sequence of functionals

$$f_i \in (\ell^\infty)^*, \quad \{x_n\} \mapsto x_i, \quad i \in \mathbb{N}.$$

There are countably many of these, and they separate the points in ℓ^∞, that is, a sequence $w \in \ell^\infty$ is zero if and only if $f_i(w) = 0$ for all i. Now, for each single f_i we know that the set I_f of $x \in \mathbb{R} \setminus \mathbb{Q}$ with $f_i(v^x) \neq 0$ is countable. Since there are countably many f_i's, the set of $x \in \mathbb{R} \setminus \mathbb{Q}$ for which $f_i(v^x) \neq 0$ for some $i \in \mathbb{N}$ is also at most countable, so the remaining v^y must be zero, a contradiction.

References

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