

# IMAGES OF MANIFOLDS WITH SEMI-AMPLE ANTI-CANONICAL DIVISOR

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ABSTRACT. We prove that if  $f: X \rightarrow Z$  is a smooth surjective morphism between projective manifolds and if  $-K_X$  is semi-ample, then  $-K_Z$  is also semi-ample. This was conjectured by Fujino and Gongyo. We list several counter-examples to show that this fails without the smoothness assumption on  $f$ .

We prove the above result by proving some results concerning the moduli divisor of the canonical bundle formula associated to a klt-trivial fibration  $(X, B) \rightarrow Z$ .

## 1. INTRODUCTION

All the varieties in this paper are defined over  $\mathbb{C}$  unless stated otherwise. Recall that a divisor  $D$  on a normal projective variety  $X$  is said to be *semi-ample* if the linear system  $|mD|$  is base point free for some  $m > 0$ , i.e.  $\mathcal{O}_X(mD)$  is generated by global sections.

**Main result.** The main result of this paper is the following:

**Theorem 1.1.** *Let  $f: X \rightarrow Z$  be a smooth surjective morphism between smooth projective varieties. If  $-K_X$  is semi-ample, then  $-K_Z$  is also semi-ample.*

This was conjectured by Fujino and Gongyo [FG11] (see also [FG12]). Although the statement might make the impression that the proof should be straightforward but in fact we have to rely on very deep consequences of the minimal model program in birational geometry and of Hodge theory. The proof is given in Section 5.

Fujino and Gongyo proved the conjecture when  $-K_X$  is big. They also proved some other special cases (see Remark 4.2 and Theorem 4.4 of [FG11]). The theorem does not hold without the smoothness assumption on  $f$  by a counter-example due to H. Sato (see Section 5 where we give additional counter-examples).

There are results similar to Theorem 1.1 in the literature. We mention some of them for completeness. Let  $f: X \rightarrow Z$  be a smooth surjective morphism between smooth projective varieties. Then, it is known that (see [FG11], §5 for more details):

- if  $-K_X$  is nef, then  $-K_Z$  is nef (Miyaoaka [M93])
- if  $-K_X$  is ample, then  $-K_Z$  is ample (Kollár-Miyaoka-Mori [KMM92])

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- if  $-K_X$  is nef and big, then  $-K_Z$  is nef and big (Fujino-Gongyo [FG11]).

To prove Theorem 1.1, we do not rely on these known results but we use some of the ideas of [FG11].

In general, varieties with nef anti-canonical divisor have interesting properties. Let  $X$  be a smooth projective variety with  $-K_X$  nef. Then, for any surjective morphism  $g: X \rightarrow Y$  with  $Y$  smooth projective, Zhang [Zh96] proved that either  $Y$  is uniruled or  $\kappa(Y) = 0$ . On the other hand, Lu, Tu, Zhang, and Zheng [LTZZ10] proved that the Albanese map  $X \rightarrow A$  is semi-stable (it is conjectured that this map should be smooth).

**The semi-ampleness conjecture of moduli divisors.** To prove Theorem 1.1, we need to prove some results concerning the moduli divisor associated to certain fibrations. A very hard conjecture claims that this moduli divisor is semi-ample. Unfortunately, formulation of the conjecture and of our results are too technical so we content ourselves with giving a rough picture (see Section 3 for more details).

Assume that  $f: X \rightarrow Z$  is a contraction of normal projective varieties, and that  $(X, B)$  is a Kawamata log terminal (klt) pair such that  $K_X + B \sim_{\mathbb{Q}} 0/Z$ , i.e.  $K_X + B \sim_{\mathbb{Q}} f^*N$  for some  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $N$ . Such a contraction is called a *klt-trivial fibration*. By a construction of Kawamata [Ka97],[Ka98] we have a decomposition

$$N \sim_{\mathbb{Q}} K_Z + B_Z + M_Z$$

where  $B_Z$  is defined using the singularities of  $(X, B)$  and of the fibres of  $f$  over the codimension one points of  $Z$ . The part  $B_Z$  is called the *discriminant part* and the part  $M_Z$  is called the *moduli part*. The moduli part is expected to reflect the variation of the fibres of  $f$  in their moduli space and this is deeply connected with the variation of the Hodge structure on these fibres. To fully understand these variations one needs to consider birational morphisms  $\sigma: Y \rightarrow Z$  where  $Y$  is smooth. We refer to  $Y$  simply as a resolution. One can define a decomposition  $K_Y + B_Y + M_Y$  in a similar way so that

$$K_Y + B_Y + M_Y \sim_{\mathbb{Q}} \sigma^*(K_Z + B_Z + M_Z)$$

The semi-ampleness conjecture then states:

**Conjecture 1.2** (cf. [Ka97], [Am99],[PSh09]). *If  $Y$  is a sufficiently high resolution, then  $M_Y$  is semi-ample.*

We do not prove the conjecture but we show that there is a suitable divisor  $G$  such that  $\alpha M_Y + G$  is semi-ample for every  $\alpha \gg 0$ . The important thing is that  $G$  is not arbitrary but rather comes from the geometry involved. The existence of  $G$  is enough for proving Theorem 1.1. We will use the minimal model program in order to find  $G$ .

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## 2. PRELIMINARIES

**Pairs.** We work over  $k = \mathbb{C}$ . A *pair*  $(X, B)$  consists of a normal quasi-projective variety  $X$  over  $k$  and a  $\mathbb{Q}$ -divisor  $B$  on  $X$  with coefficients in  $[0, 1]$  (called a boundary) such that  $K_X + B$  is  $\mathbb{Q}$ -Cartier. A pair  $(X, B)$  is called *Kawamata log terminal* (klt) if for any projective birational morphism  $g: Y \rightarrow X$  from a normal variety  $Y$ , every coefficient of  $B_Y$  is less than one where  $K_Y + B_Y := g^*(K_X + B)$ . For basic properties of singularities and other aspects of birational geometry we refer the reader to [KM98].

**Minimal models.** Let  $X$  be a normal projective variety and  $D$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$ . A normal projective variety  $Y$  is called a *minimal model* of  $D$  if there is a birational map  $\phi: X \dashrightarrow Y$  such that  $\phi^{-1}$  does not contract divisors,  $Y$  is  $\mathbb{Q}$ -factorial,  $D_Y := \phi_*D$  is nef, and there is a common resolution  $g: W \rightarrow X$  and  $h: W \rightarrow Y$  such that  $E := g^*D - h^*D_Y$  is effective and  $\text{Supp } g_*E$  is equal to the union of the exceptional divisors of  $\phi$ . If  $D_Y$  is semi-ample, we call  $Y$  a *good minimal model* of  $D$ . A standard way of obtaining a minimal model of  $D$  is to run a *minimal model program* on  $D$ , if possible. Such a program can be defined exactly as in the case  $D = K_X + B$  for a klt pair  $(X, B)$ .

When  $(X, B)$  is klt and  $D = K_X + B$  the above definition of minimal model is equivalent to the usual definition of log minimal models. Moreover, if  $K_X + B$  is pseudo-effective and  $B$  is big, then  $K_X + B$  has a good log minimal model by [BCHM].

**Section rings.** For a  $\mathbb{Q}$ -divisor  $D$  on a normal projective variety  $X$ , the *section ring* of  $D$  is defined as

$$R(X, D) = \bigoplus_{0 \leq m \in \mathbb{Z}} H^0(X, \lfloor mD \rfloor)$$

If  $\phi: X \dashrightarrow Y$  is a partial minimal model program on  $D$ , i.e. a finite sequence of divisorial contractions and flips with respect to  $D$ , then there is a common resolution  $g: W \rightarrow X$  and  $h: W \rightarrow Y$  such that  $E := g^*D - h^*D_Y$  is effective and  $\text{Supp } g_*E$  is equal to the union of the exceptional divisors of  $\phi$ . In particular, this implies that there is a natural isomorphism

$$R(X, D) \simeq R(Y, D_Y)$$

**Kodaira dimension.** Let  $f: X \rightarrow Z$  be a contraction of normal varieties and  $D$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$ . By  $\kappa(D/Z)$  we mean the Kodaira dimension of  $D|_F$  where  $F$  is the generic fibre of  $f$ .

## 3. THE MODULI DIVISOR OF A KLT-TRIVIAL FIBRATION

**The semi-ampleness conjecture of moduli divisors.** Assume that  $f: X \rightarrow Z$  is a contraction of normal projective varieties, and that  $(X, B)$  is a Kawamata log terminal (klt) pair such that  $K_X + B \sim_{\mathbb{Q}} 0/Z$ , i.e.  $K_X + B \sim_{\mathbb{Q}} f^*N$  for some  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $N$ . Such a contraction is called a *klt-trivial fibration*. By a construction of Kawamata [Ka97],[Ka98] we have a decomposition

$$N \sim_{\mathbb{Q}} K_Z + B_Z + M_Z$$

where  $B_Z$  is defined using the singularities of  $(X, B)$  and of the fibres of  $f$  over the codimension one points of  $Z$ , and  $(Z, B_Z)$  is klt if  $K_Z + B_Z$  is  $\mathbb{Q}$ -Cartier. The part  $B_Z$  is called the *discriminant part* and the part  $M_Z$  is called the *moduli part*. More precisely,  $B_Z$  is defined as follows: for each prime divisor  $Q$  on  $Z$ , let  $t$  be the log canonical threshold of  $f^*Q$  over the generic point of  $Q$ , with respect to the pair  $(X, B)$ ; then let  $(1 - t)$  be the coefficient of  $Q$  in  $B_Z$ . The more complicated the singularities over the generic point of  $Q$  the larger the coefficient of  $Q$  would be. Except for finitely many  $Q$ ,  $t = 1$  hence  $B_Z$  has finitely many components.

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{\sigma} & Z \end{array}$$

in which  $X', Y$  are normal and projective,  $\sigma, \tau$  are birational, and  $f'$  is a contraction. Let  $K_{X'} + B' := \tau^*(K_X + B)$  where  $B'$  might have negative coefficients. Using the relation  $K_{X'} + B' \sim_{\mathbb{Q}} 0/Y$ , we can similarly define a decomposition

$$\sigma^*N \sim_{\mathbb{Q}} K_Y + B_Y + M_Y$$

where:  $B_Z = \sigma_*B_Y$  and  $M_Z = \sigma_*M_Y$ . This means that the moduli part is a b-divisor. In general,  $B_Y$  may have negative coefficients.

Kawamata [Ka98] showed that  $M_Y$  is nef if  $Y$  is a sufficiently high resolution. Ambro [Am04] proved that the moduli part is a b-Cartier divisor, i.e. we can find a resolution  $Y \rightarrow Z$  so that for any other resolution  $Y' \rightarrow Y$  the moduli part  $M_{Y'}$  is just the pullback of  $M_Y$ . Moreover, Ambro [Am05] proved that  $M_Y$  is actually the pullback of some nef and big divisor for some contraction  $g: Y \rightarrow T$ . These results rely on deep connections with Hodge theory.

**Conjecture 3.1** (cf. [Ka97], [Am99],[PSh09]). *If  $Y$  is a sufficiently high resolution, then  $M_Y$  is semi-ample.*

Only some very special cases are known: when  $M_Y \equiv 0$  [Am05], when  $\dim X = \dim Z + 1$  [Ka97][PSh09], and when the geometric generic fibre of  $f$  is birationally an abelian variety [F03].

**Our result concerning the moduli divisor.** We are not able to prove the semi-ampleness of  $M_Y$  but we prove semi-ampleness of a “nearby” divisor. To

state our main result we need to introduce one more notation. Since  $(Y, B_Y)$  is log smooth and each coefficient of  $B_Y$  is strictly smaller than 1, there is an exceptional/ $Z$   $\mathbb{Q}$ -divisor  $E \geq 0$  on  $Y$  such that  $(Y, \Delta_Y)$  is klt where  $\Delta_Y := B_Y + E \geq 0$ .

**Theorem 3.2.** *Under the above notation and assumptions, suppose that we have  $\kappa(K_Y + \Delta_Y/T) \geq 0$ , i.e. suppose that there is a rational number  $b \geq 0$  such that*

$$K_Y + \Delta_Y + bM_Y \sim_{\mathbb{Q}} D_Y \text{ for some } D_Y \geq 0$$

*Then, we can choose  $D_Y \geq 0$  in its  $\mathbb{Q}$ -linear equivalence class so that there exist a resolution  $\phi: W \rightarrow Y$  and a  $\mathbb{Q}$ -divisor  $G$  on  $W$  such that*

- $0 \leq G \leq \phi^*D_Y$ ,
- $\alpha M_W + G = \alpha\phi^*M_Y + G$  is semi-ample for any  $\alpha \gg 0$ , and
- $R(W, lG) \simeq R(Y, lD_Y)$  for some integer  $l > 0$ .

Here  $R(-)$  stands for the section ring (see Section 2 for precise definition). An interesting special case of the theorem is when  $K_X + B \sim_{\mathbb{Q}} 0$  in which case we take  $D_Y = E$ . This is exactly the situation which comes up in the proof of Theorem 1.1.

*Proof.* This follows immediately from Theorem 4.1 in the next section. □

#### 4. AN AUXILIARY THEOREM ON KLT PAIRS

In this section, we prove the following result which might be of independent interest. The statement is essentially identical to that of Theorem 3.2 except that we do not need to care where the data come from. The proof heavily relies on the minimal model program.

**Proposition 4.1.** *Let  $(Y, \Delta_Y)$  be a projective klt pair with  $\Delta_Y$  a  $\mathbb{Q}$ -divisor,  $g: Y \rightarrow T$  a contraction,  $M_T$  a nef and big  $\mathbb{Q}$ -divisor on  $T$ , and  $M_Y = g^*M_T$ . Assume that  $b \geq 0$  is a rational number such that*

$$K_Y + \Delta_Y + bM_Y \sim_{\mathbb{Q}} D_Y \text{ for some } D_Y \geq 0$$

*Then, we can choose  $D_Y \geq 0$  in its  $\mathbb{Q}$ -linear equivalence class such that there exist a resolution  $\phi: W \rightarrow Y$  and a  $\mathbb{Q}$ -divisor  $G$  on  $W$  such that*

- $0 \leq G \leq \phi^*D_Y$ ,
- $\alpha\phi^*M_Y + G$  is semi-ample for any  $\alpha \gg 0$ , and
- $R(W, lG) \simeq R(Y, lD_Y)$  for some integer  $l > 0$ .

*Proof.* For any morphism  $U \rightarrow T$ ,  $M_U$  will denote the pullback of  $M_T$ .

*Step 1.* Let  $\pi: Y' \rightarrow Y$  be any log resolution of  $(Y, \Delta_Y)$ . There is an exceptional/ $Y$   $\mathbb{Q}$ -divisor  $E' \geq 0$  and a boundary  $\Delta_{Y'}$  such that  $(Y', \Delta_{Y'})$  is klt and

$$K_{Y'} + \Delta_{Y'} = \pi^*(K_Y + \Delta_Y) + E'$$

Let  $D_{Y'} = \pi^*D_Y + E'$ . Then,

$$K_{Y'} + \Delta_{Y'} + bM_{Y'} \sim_{\mathbb{Q}} D_{Y'}$$

Now assume that the result holds on  $Y'$ , that is, assume that we can choose  $D_{Y'} \geq 0$  in its  $\mathbb{Q}$ -linear equivalence class such that there exist a resolution  $\psi: W \rightarrow Y'$  and a  $\mathbb{Q}$ -divisor  $G$  on  $W$  satisfying:

- $0 \leq G \leq \psi^* D_{Y'}$ ,
- $\alpha M_W + G$  is semi-ample for any  $\alpha \gg 0$ , and
- $R(W, lG) \simeq R(Y', lD_{Y'})$  for some integer  $l > 0$ .

We will show that the result also holds for  $Y$  by taking  $\phi = \pi\psi$ . Since  $D_{Y'} = \pi^* D_Y + E'$  and  $E' \geq 0$  is exceptional/ $Y$ ,

$$R(Y, lD_Y) \simeq R(Y', lD_{Y'}) \simeq R(W, lG)$$

It only remains to show that  $G \leq \phi^* D_Y$ . Put  $C = \phi^* D_Y - G$ . Since  $E'$  is exceptional/ $Y$ ,

$$\phi_* C = \phi_*(\phi^* D_Y - G) = \phi_*(\phi^* D_Y + \psi^* E' - G) = \phi_*(\psi^* D_{Y'} - G) \geq 0$$

On the other hand,  $\alpha M_W + G$  is semi-ample for any  $\alpha \gg 0$  and  $M_W \sim_{\mathbb{Q}} 0/Y$  hence  $G$  is semi-ample over  $Y$ . So,  $C$  is antinef over  $Y$  which implies that  $C \geq 0$  by the negativity lemma. Therefore,  $G \leq \phi^* D_Y$ .

*Step 2.* Since  $M_Y \sim_{\mathbb{Q}} 0/T$ ,  $K_Y + \Delta_Y \sim_{\mathbb{Q}} D_Y/T$  hence  $\kappa(K_Y + \Delta_Y/T) \geq 0$ . By taking a log resolution and applying Step 1 we could assume that the relative Iitaka fibration of  $K_Y + \Delta_Y$  over  $T$  is a morphism  $Y \rightarrow S'/T$  where  $S'$  is smooth. In particular, this means that  $\kappa(K_Y + \Delta_Y/S') = 0$ . By applying Fujino-Mori [FM00], we can find a commutative diagram

$$\begin{array}{ccc} & V & \\ \tau \swarrow & & \searrow h \\ Y & & S \\ g \searrow & & \swarrow \theta \\ & T & \end{array}$$

where  $\tau$  is a resolution, and  $h, \theta$  are contractions such that we have the following data:

- a klt  $(S, \Delta_S)$  where  $\Delta_S$  is a  $\mathbb{Q}$ -divisor,
- a nef  $\mathbb{Q}$ -divisor  $L_S$  on  $S$ ,
- $K_S + \Delta_S + L_S$  is big/ $T$ ,
- $\kappa(K_Y + \Delta_Y/T) = \kappa(K_S + \Delta_S + L_S/T) = \dim S - \dim T$ ,
- and

$$\tau^*(K_Y + \Delta_Y) + R^- \sim_{\mathbb{Q}} h^*(K_S + \Delta_S + L_S) + R^+$$

where  $R^+, R^-$  are effective  $\mathbb{Q}$ -divisors with  $R^-$  exceptional/ $Y$ ,  $h(R^-)$  has no codimension one components, and  $h_* \mathcal{O}_V([iR^+]) = \mathcal{O}_S$  for every  $i > 0$ .

The above properties imply that for any rational number  $\alpha \geq 0$ , we have

$$\begin{aligned}
 \tau^* D_Y + R^- + \alpha M_V &\sim_{\mathbb{Q}} \tau^*(K_Y + \Delta_Y + bM_Y) + R^- + \alpha M_V \\
 &= \tau^*(K_Y + \Delta_Y) + R^- + bM_V + \alpha M_V \\
 &\sim_{\mathbb{Q}} h^*(K_S + \Delta_S + L_S) + R^+ + bM_V + \alpha M_V \\
 &= h^*(K_S + \Delta_S + L_S + bM_S + \alpha M_S) + R^+
 \end{aligned}$$

*Step 3.* Fix a rational number  $\lambda \geq 0$  so that  $b + \lambda > 0$ . By construction

$$0 \leq \kappa(D_Y) = \kappa(K_Y + \Delta_Y + bM_Y) = \kappa(K_S + \Delta_S + L_S + bM_S)$$

so there is a  $\mathbb{Q}$ -divisor  $D_S \geq 0$  such that

$$D_S \sim_{\mathbb{Q}} K_S + \Delta_S + L_S + bM_S$$

Since  $D_S$  is big/ $T$  and  $M_T$  is big, there is  $a > 0$  such that  $D_S + aM_S$  is big. Thus, we have

$$D_S + aM_S \sim_{\mathbb{Q}} A + B$$

where  $A$  is ample and  $B \geq 0$ .

Pick  $\epsilon > 0$  sufficiently small. This ensures that

$$\beta_\lambda := \epsilon\lambda - \epsilon a + b + \lambda \geq 0$$

Moreover, since  $A$  is ample and  $L_S$  and  $M_S$  are nef, there is a big boundary  $\Gamma_{S,\lambda}$  such that  $(S, \Gamma_{S,\lambda})$  is klt and such that we can write

$$\begin{aligned}
 K_S + \Gamma_{S,\lambda} &\sim_{\mathbb{Q}} K_S + \Delta_S + L_S + \epsilon A + \epsilon B + \beta_\lambda M_S \\
 &\sim_{\mathbb{Q}} K_S + \Delta_S + L_S + \epsilon D_S + \epsilon a M_S + \beta_\lambda M_S \\
 &\sim_{\mathbb{Q}} K_S + \Delta_S + L_S + b M_S + \epsilon D_S + \epsilon \lambda M_S + \lambda M_S \\
 &\sim_{\mathbb{Q}} (1 + \epsilon)(K_S + \Delta_S + L_S + b M_S) + (1 + \epsilon)\lambda M_S \\
 &\sim_{\mathbb{Q}} (1 + \epsilon)(K_S + \Delta_S + L_S + b M_S + \lambda M_S) \\
 &\sim_{\mathbb{Q}} (1 + \epsilon)(D_S + \lambda M_S)
 \end{aligned}$$

By [BCHM], we can run an LMMP on  $K_S + \Gamma_{S,\lambda}$  which ends up with a good log minimal model of  $K_S + \Gamma_{S,\lambda}$ . By the relations above, this also produces a good minimal model of  $K_S + \Delta_S + L_S + bM_S + \lambda M_S$ .

For any  $\alpha \geq \lambda$ , we have

$$K_S + \Gamma_{S,\alpha} \sim_{\mathbb{Q}} K_S + \Gamma_{S,\lambda} + (1 + \epsilon)(\alpha - \lambda)M_S$$

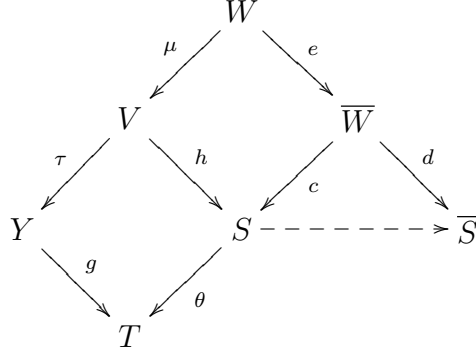
If  $\alpha \gg \lambda$  and if we run an LMMP on  $K_S + \Gamma_{S,\alpha}$ , then  $M_S$  is numerically trivial on each extremal ray contracted in the process: this follows from the boundedness of the length of extremal rays due to Kawamata [Ka91]; indeed if  $R$  is an extremal ray such that  $(K_S + \Gamma_{S,\alpha}) \cdot R < 0$ , then  $(K_S + \Gamma_{S,\lambda}) \cdot R < 0$  and there is a rational curve  $C$  generating  $R$  such that

$$-2 \dim S \leq (K_S + \Gamma_{S,\lambda}) \cdot C < 0$$

If  $\alpha \gg \lambda$  and if  $M_S \cdot C > 0$ , then  $(K_S + \Gamma_{S,\alpha}) \cdot C > 0$ , a contradiction. The same argument applies in each step of the LMMP because the Cartier index of  $M_S$  is preserved by the LMMP (by Cartier index of  $M_S$  we mean the smallest natural number  $n$  such that  $nM_S$  is Cartier). Therefore, the LMMP on  $K_S + \Gamma_{S,\alpha}$  is

also an LMMP on  $K_S + \Gamma_{S,\alpha'}$  for any  $\alpha' \geq \alpha$ . In particular, if  $\bar{S}$  is the good minimal model of  $K_S + \Gamma_{S,\alpha}$  obtained by the LMMP above, then  $\bar{S}$  is also a good minimal model of  $K_S + \Gamma_{S,\alpha'}$  for any  $\alpha' \geq \alpha$ .

*Step 4.* Let  $\bar{S}$  be the model constructed in Step 3. By construction, there is a commutative diagram



where  $\mu$  is a resolution,  $e$  is a contraction, and  $c, d$  are also resolutions. By Step 3, we have  $e^*d^*M_{\bar{S}} = e^*c^*M_S = M_W$  because  $M_S$  is numerically trivial on each step of the LMMP that produced  $S \dashrightarrow \bar{S}$ . Moreover,  $D_{\bar{S}} + \alpha M_{\bar{S}}$  is semi-ample for any  $\alpha \gg 0$ .

*Step 5.* In general,  $D_{\bar{S}}$  may not be nef. However, the LMMP on  $K_S + \Gamma_{S,\alpha}$  in Step 3 is a partial LMMP on  $D_S$ , that is, each step of the LMMP on  $K_S + \Gamma_{S,\alpha}$  is also a step of an LMMP on  $D_S$  but it may not be a full LMMP on  $D_S$ . In any case, there is a  $\mathbb{Q}$ -divisor  $\bar{P} \geq 0$  on  $\bar{W}$  which is exceptional/ $\bar{S}$  and

$$c^*D_S = \bar{P} + d^*D_{\bar{S}}$$

Now put  $G = e^*d^*D_{\bar{S}}$  and  $\phi := \tau\mu$ . Then, for some integer  $l > 0$ , we have natural isomorphisms

$$R(Y, lD_Y) \simeq R(S, lD_S) \simeq R(\bar{S}, lD_{\bar{S}}) \simeq R(W, lG)$$

By construction,  $\alpha M_W + G$  is semi-ample for any  $\alpha \gg 0$ . Since  $M_W \sim_{\mathbb{Q}} 0/Y$ , we deduce that  $G$  is semi-ample over  $Y$ . Moreover,

$$\begin{aligned}
 \phi^*D_Y + \mu^*R^- &\sim_{\mathbb{Q}} \mu^*h^*(K_S + \Delta_S + L_S + bM_S) + \mu^*R^+ \\
 &\sim_{\mathbb{Q}} e^*c^*D_S + \mu^*R^+ \\
 &= e^*\bar{P} + G + \mu^*R^+
 \end{aligned}$$

Put  $D'_Y := \phi_*(e^*\bar{P} + G + \mu^*R^+)$ . Since  $e^*\bar{P} + G + \mu^*R^+ \geq 0$ , we get  $D'_Y \geq 0$ , and since  $R^-$  is exceptional/ $Y$ , we get  $D'_Y \sim_{\mathbb{Q}} D_Y$ . So,

$$\phi^*D'_Y + \mu^*R^- \sim_{\mathbb{Q}} e^*\bar{P} + G + \mu^*R^+$$

and by the negativity lemma we have equality

$$\phi^*D'_Y + \mu^*R^- = e^*\bar{P} + G + \mu^*R^+$$



Let  $N = \phi^* D'_Y - G$ . Then,

$$\phi_* N = \phi_*(N + \mu^* R^-) = \phi_*(e^* \bar{P} + \mu^* R^+) \geq 0$$

On the other hand, we know that  $G$  is nef over  $Y$  so  $N$  is anti-nef over  $Y$ . Therefore, by the negativity lemma we deduce  $N \geq 0$  hence  $\phi^* D'_Y \geq G$ . Now replace  $D_Y$  with  $D'_Y$  and this completes the proof of the proposition.  $\square$

*Proof.* (of Theorem 3.2) This follows from Proposition 4.1 using the same notation.  $\square$

## 5. PROOF OF THEOREM 1.1

As mentioned earlier the smoothness assumption of  $f$  in Theorem 1.1 cannot be removed. If  $f : X \rightarrow Z$  is a surjective morphism between smooth projective varieties with  $-K_X$  nef, then Zhang [Zh96] proved that either  $Z$  is uniruled or  $\kappa(Z) = 0$ . In particular, if  $\dim Z = 1$ , then  $-K_Z$  is semi-ample. Therefore, a counter-example to Fujino-Gongyo's conjecture without the smoothness assumption on  $f$  is possible only when  $\dim Z \geq 2$ . We give few counter-examples of different flavours.

**Example 5.1** Let  $B'$  be the  $(4, 4)$ -divisor on  $Z' = \mathbb{P}^1 \times \mathbb{P}^1$  which consists of 4 lines vertical with respect to the first projection and 4 lines vertical with respect to the second projection. Let  $\sigma : Z \rightarrow Z'$  be the blow up of the 16 intersection points in  $B'$ . Let  $B$  be the strict transform of  $B'$ . Then  $\sigma^*(B') = B + \sum_{i=1}^{16} 2E_i$ , where  $E_i$  are the irreducible exceptional divisors over the 16 points. So  $B$  is an even divisor in  $Z$ , i.e.,  $B \sim 2L$  for some divisor  $L$  in  $Z$ . Let  $f : X \rightarrow Z$  be the double cover ramified over  $B$ . We will show that  $-K_X$  is semi-ample but  $-K_Z$  is not semi-ample.

By construction,  $K_{Z'}^2 = 8$  and from  $K_Z = \sigma^* K_{Z'} + \sum_{i=1}^{16} E_i$  we get  $K_Z^2 = -8$ . Therefore,  $-K_Z$  is not nef, and thus not semi-ample. Since  $f : X \rightarrow Z$  is the double cover ramified over  $B$ ,  $K_X \sim_{\mathbb{Q}} f^*(K_Z + L)$ . On the other hand,

$$K_Z + L \sim_{\mathbb{Q}} \sigma^* K_{Z'} + \frac{1}{2}(\sigma^*(B') - B) + \frac{1}{2}B = \sigma^*(K_{Z'} + \frac{1}{2}B') \sim_{\mathbb{Q}} 0$$

So  $-K_X \sim_{\mathbb{Q}} 0$  is semi-ample. In this example  $f$  is a flat morphism but not a contraction.

**Example 5.2** Let  $r$  and  $s$  be positive integers. Let  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{r+1}$  and let  $Z_{r,s}$  be the smooth  $(r + s + 1)$ -dimensional variety  $\mathbb{P}(\mathcal{E})$ . The projection  $\pi_{r,s} : Z_{r,s} \rightarrow \mathbb{P}^s$  has a section  $\sigma$  with image  $P_{r,s} \subset Z_{r,s}$  corresponding to the trivial quotient  $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^s}$ . Let  $f : X_{r,s} \rightarrow Z_{r,s}$  be the blow-up of  $P_{r,s}$ . One can check by calculating intersection numbers and using Kleiman's ampleness criterion that when  $r > s$  the divisor  $-K_{X_{r,s}}$  is ample but  $-K_{Z_{r,s}}$  is not nef. For details see [De01] Example 1.36 and Example 3.16 (2). In this example  $f$  is a contraction but not a flat morphism.

For the convenience of the reader we reproduce another counter-example due to Hiroshi Sato given in [FG11], Example 4.6. In this case  $f$  would be both a contraction and a flat morphism.

**Example 5.3** (H. Sato) Let  $\Sigma$  be the fan in  $\mathbb{R}^3$  whose rays are generated by

$$\begin{aligned} x_1 &= (1, 0, 1), x_2 = (0, 1, 0), x_3 = (-1, 3, 0), x_4 = (0, -1, 0), \\ y_1 &= (0, 0, 1), y_2 = (0, 0, -1), \end{aligned}$$

and their maximal cones are

$$\begin{aligned} &\langle x_1, x_2, y_1 \rangle, \langle x_1, x_2, y_2 \rangle, \langle x_2, x_3, y_1 \rangle, \langle x_2, x_3, y_2 \rangle, \\ &\langle x_3, x_4, y_1 \rangle, \langle x_3, x_4, y_2 \rangle, \langle x_4, x_1, y_1 \rangle, \langle x_4, x_1, y_2 \rangle. \end{aligned}$$

Let  $\Delta$  be the fan obtained from  $\Sigma$  by successive star subdivisions along the rays spanned by

$$z_1 = x_2 + y_1 = (0, 1, 1)$$

and

$$z_2 = x_2 + z_1 = 2x_2 + y_1 = (0, 2, 1).$$

We can see that  $V = X_\Sigma$ , the toric threefold corresponding to the fan  $\Sigma$  with respect to the lattice  $\mathbb{Z}^3 \subset \mathbb{R}^3$ , is a  $\mathbb{P}^1$ -bundle over  $Z = \mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ . The  $\mathbb{P}^1$ -bundle structure  $V \rightarrow Z$  is induced by the projection  $\mathbb{Z}^3 \rightarrow \mathbb{Z}^2 : (x, y, z) \mapsto (x, y)$ . The toric variety  $X = X_\Delta$  corresponding to the fan  $\Delta$  was obtained by successive blow-ups from  $V$ . The maximal cones of  $X_\Delta$  are:

$$\begin{aligned} \tau_1 &= \langle x_1, x_2, z_2 \rangle, \tau_2 = \langle x_1, z_1, z_2 \rangle, \tau_3 = \langle x_1, z_1, y_1 \rangle, \\ \tau_4 &= \langle x_3, y_1, z_1 \rangle, \tau_5 = \langle x_3, z_2, z_1 \rangle, \tau_6 = \langle x_3, z_2, x_2 \rangle, \\ \sigma_3 &= \langle x_1, x_2, y_2 \rangle, \sigma_4 = \langle x_2, x_3, y_2 \rangle, \sigma_5 = \langle x_3, x_4, y_1 \rangle, \\ \sigma_6 &= \langle x_3, x_4, y_2 \rangle, \sigma_7 = \langle x_4, x_1, y_1 \rangle, \sigma_8 = \langle x_4, x_1, y_2 \rangle. \end{aligned}$$

To check  $-K_X$  is semi-ample, we can either apply the base-point-free criterion on toric varieties (see [CLS] Theorem 6.1.7), or the fact that on toric varieties, semi-ampleness of a divisor is equivalent to nefness (see [CLS] Theorem 6.3.12). Here we apply the base-point-free criterion (see Remark 5.4).

We have  $-K_X = \sum_{i=1}^4 D_{x_i} + D_{y_1} + D_{y_2} + D_{z_1} + D_{z_2}$ , where  $D_u$  denotes the invariant Cartier divisor corresponding to the ray generated by the point  $u$ . The Cartier data associated to  $-K_X$  are as follows:

$$\begin{aligned} m_{\tau_1} &= m_{\tau_6} = m_{\sigma_3} = m_{\sigma_4} = (-2, -1, 1), & m_{\tau_2} &= m_{\tau_3} = (0, 0, -1), \\ m_{\tau_4} &= m_{\tau_5} = (1, 0, -1), & m_{\sigma_5} &= (4, 1, -1), & m_{\sigma_6} &= (4, 1, 1), \\ m_{\sigma_7} &= (0, 1, -1), & m_{\sigma_8} &= (-2, 1, 1). \end{aligned}$$

One can check that all  $m_{\tau_i}, m_{\sigma_j}$  are in  $P_{-K_X}$ . So  $-K_X$  is base point free. Moreover,  $-K_X$  is big as  $X$  is a projective toric variety. So  $X$  is a toric weak Fano manifold. The morphism  $f : X \rightarrow Z$  induced by the projection  $\mathbb{Z}^3 \rightarrow \mathbb{Z}^2$  is a flat morphism onto  $Z$  since every fiber of  $f$  is one-dimensional.

On the other hand,  $-K_Z$  is not nef since  $-K_Z \cdot C < 0$  where  $C \subset Z$  is the image of the section of  $Z \rightarrow \mathbb{P}^1$  corresponding to the trivial quotient  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3) \rightarrow \mathcal{O}_{\mathbb{P}^1}$ .

**Remark 5.4** Here we recall some basic facts about toric varieties.

(1) Base point freeness on toric varieties: We follow the notation of [CLS]. Let  $X_\Sigma$  be a complete toric variety of dimension  $n$  and let  $D = \sum_\rho a_\rho D_\rho$  be a torus-invariant Cartier divisor on  $X_\Sigma$ . We define a polytope

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \subset M_{\mathbb{R}}$$

Let  $\{m_\sigma\}_{\sigma \in \Sigma(n)}$  be the Cartier data associated to  $D$ , where  $m_\sigma \in M$  with  $\langle m_\sigma, u_\rho \rangle = -a_\rho$ , for all  $\rho \in \sigma(1)$ . Then,  $D$  is base point free if and only if  $m_\sigma \in P_D$  for all  $\sigma \in \Sigma(n)$ .

(2) Kodaira dimension: Now assume that  $D$  is nef. Then, the Kodaira dimension  $\kappa(D) = \dim P_D$ . In particular, if  $X_\Sigma$  is projective and  $D = -K_{X_\Sigma}$  is nef, then it is not difficult to see that  $\dim P_D = n$  hence  $D$  is nef and big.

*Proof.* (of Theorem 1.1) By Lemma 2.4 of [FG11], we can assume that  $f$  is a contraction: this is achieved by taking the Stein factorisation which preserves the smoothness assumption. We may assume that  $\dim X > \dim Z$  otherwise the theorem holds trivially. Pick a closed point  $z \in Z$ . Since  $-K_X$  is semi-ample, for some  $m > 1$  we can find a divisor  $D \in |-mK_X|$  such that if we put  $B = \frac{1}{m}D$  then we have:  $K_X + B \sim_{\mathbb{Q}} 0$ ,  $(X, B)$  is klt, and  $D = \text{Supp } B$  is smooth (note that it is also possible to have  $B = 0$  depending on the situation). Since  $f$  is smooth, we can choose  $D$  so that there is a neighbourhood  $U$  of  $z$  such that  $D = \text{Supp } B$  is relatively smooth over  $U$ . In particular, this ensures that the discriminant part  $B_Z = 0$  near  $z$ .

Let  $\sigma: Y \rightarrow Z$  be a sufficiently high resolution. Then, possibly after shrinking  $U$  around  $z$ , we have  $M_Y = \sigma^*M_Z$  over  $U$  by [FG11], Remark 4.3, or by [Ko07], Proposition 8.4.9 (3) and Theorem 8.5.1. Since  $K_X + B \sim_{\mathbb{Q}} 0$ ,

$$K_Z + B_Z + M_Z \sim_{\mathbb{Q}} 0 \text{ and } K_Y + B_Y + M_Y \sim_{\mathbb{Q}} 0$$

and

$$K_Y + \Delta_Y + M_Y \sim_{\mathbb{Q}} D_Y := E$$

where  $\Delta_Y$  and  $E$  are as in the Introduction. Since  $E$  is exceptional/ $Z$ ,  $D_Y$  is the only effective  $\mathbb{Q}$ -divisor in its  $\mathbb{Q}$ -linear equivalence class. Moreover, we have  $\kappa(K_Y + \Delta_Y/T) \geq 0$  since

$$\kappa(K_Y + \Delta_Y + M_Y) = \kappa(D_Y) \geq 0$$

where  $Y \rightarrow T$  is again as in the Introduction.

By Theorem 3.2, there is a resolution  $\phi: W \rightarrow Y$  and a  $\mathbb{Q}$ -divisor  $G \geq 0$  such that  $\alpha M_W + G$  is semi-ample for any  $\alpha \gg 0$  and  $G \leq \phi^*D_Y = \phi^*E$ . In particular, this means that  $\alpha M_W + G$  is semi-ample over  $U$  for  $\alpha \gg 0$ . Since over  $U$  we have  $M_W = \phi^*\sigma^*M_Z$ , we deduce that  $G$  is semi-ample over  $U$ . But  $G$  is effective and exceptional over  $U$  hence by the negativity lemma it is zero over  $U$ , that is,  $G$  is mapped into  $Z \setminus U$ . Thus, we can find a rational number  $\alpha \gg 0$  and a  $\mathbb{Q}$ -divisor  $P_W \geq 0$  such that  $\alpha M_W + G \sim_{\mathbb{Q}} P_W$  and that  $P_W$  does not contain any irreducible component of  $(\sigma\phi)^{-1}\{z\}$ . But  $P_W$  restricted to  $(\sigma\phi)^{-1}\{z\}$  is numerically trivial so  $P_W$  does not intersect  $(\sigma\phi)^{-1}\{z\}$ . Therefore,

$$\alpha M_Z \sim_{\mathbb{Q}} P_Z := \sigma_*\phi_*P_W \geq 0$$

and  $z$  does not belong to  $\text{Supp } P_Z$  which implies that the stable base locus of  $M_Z$  does not contain  $z$ . By construction,  $\text{Supp } B_Z$  also does not contain  $z$ . Therefore,  $z$  is not in the stable base locus of  $-K_Z$ . Since  $z$  was chosen arbitrarily, we deduce that  $-K_Z$  is semi-ample.  $\square$

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