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Fixed Points Results for Graphic Contraction on Closed Ball

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Abstract In this paper, we introduce a new class of ciric fixed point theorem of (α,ψ)-contractive mappings on a closed ball in complete metric space. As an application, we have derived some new fixed point theorems for ciric ψ-graphic contractions defined on a metric space endowed with a graph in metric space. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: *fixed point, $\alpha-$admissible, $(\alpha,\psi)-$contraction, closed ball*

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1. Introduction

In 2012, Samet et al. [18], introduced a concept of $\alpha-\psi$ - contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar and Samet [6], refined the notions and obtain various fixed point results. Hussain et al. [9], enlarged the concept of α-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [4] introduced pairs of $\alpha-$ admissible mappings satisfying new sufficient contractive conditions different from [9] and [18], and proved fixed point and common fixed point theorems. Lately, Salimi et al. [17], modified the concept of $\alpha-\psi$ - contractive mappings and established fixed point results. Mohammadi et al. [7] introduced a new notion of $\alpha-\psi$-contractive mappings and show that this is a real generalization for some old results. Arshad et al. [2] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Hussain et al. [8], introduced the concept of an α-admissible map with respect to η and modify the $\alpha-\psi$-contractive condition for a pair of mappings and established common fixed point results for two, three, and four mappings in a closed ball in complete dislocated metric spaces. Over the years, fixed point theory has been generlized in multi-directions by several mathematicians(see [1-18]).

Let Ψ be a family of nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{+\infty}\psi^n(t)<+\infty$, for each $t>0$.

Lemma 1. ([17]). If $\psi\in\Psi$, then $\psi(t)<t$ for all $t>0$.

Definition 2. ([18]). Let (X,d) be a metric space. A mapping $T:X\to X$ is an $(\alpha,\psi)-$contractive mapping if there exist two functions $\alpha:X\times X\to[0,+\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y)),$$

for all $x,y\in X$.

Definition 3. ([18]). Let $T:X\to X$ and $\alpha:X\times X\to[0,+\infty)$. We say that T is α-admissible if $x,y\in X$, $\alpha(x,y)\ge1$ implies that $\alpha(Tx,Ty)\ge1$.

Example 4. Let $X=(0,\infty)$ and T an identity mapping on X. Define $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}e^{\frac{y}{x}} & \text{if } x\ge y, x\ne 0\\ 0 & \text{if } \quad x<y.\end{cases}$$

Then T is $\alpha-$admissible.

Definition 5. ([17]). Let $T:X\to X$ and $\alpha,\eta:X\times X\to[0,+\infty)$ two functions. We say that T is α-admissible mapping with respect to η if $x,y\in X$, $\alpha(x,y)\ge\eta(x,y)$ implies that $\alpha(Tx,Ty)\ge\eta(Tx,Ty)$.

If $\eta(x,y)=1$, then above definition reduces to definition 3. If $\alpha(x,y)=1$, then T is called an η-subadmissible mapping.

Definition 6. ([7]). Let $T:X\to X$ and $\alpha_0:X\times X\to[0,+\infty)$ by

$$\alpha_0(x,y)=\begin{Bmatrix}1 & \alpha(x,y)\ge\eta(x,y)\\ 0 & \text{otherwise}\end{Bmatrix}.$$

We say that T is α_0-admissible. If $\alpha_0(x,y)\ge1$, then $\alpha(x,y)\ge\eta(x,y)$ and so $\alpha(Tx,Ty)\ge\eta(Tx,Ty)$. This implies $\alpha_0(Tx,Ty)=1$. Also $\alpha_0(x_0,Tx_0)=1$.

2. Main Results

We prove ciric fixed point results for $(\alpha,\psi)-$ contraction mappings on a closed ball in complete metric space.

Theorem 7. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping with respect to η. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq\eta(x,y)\ \Rightarrow d(Tx,Ty)\leq\psi\big(M(x,y)\big), \tag{1}$$

where

$$M(x,y)=\max\left\{\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty),\\ \dfrac{d(x,Ty)+d(y,Tx)}{2}\end{array}\right\},$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\leq r,\ \text{for all}\ j\in N. \tag{2}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq\eta(x_n,u)$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

Proof. Let x_1 in X be such that $x_1=Tx_0$, $x_2=Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that, $x_n=Tx_n$. By assumption $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$ and T is α -admissible mapping with respect to η. we have , $\alpha(Tx_0,Tx_1)\geq\eta(Tx_0,Tx_1)$ from which we deduce that $\alpha(x_1,x_2)\geq\eta(x_1,x_2)$ which also implies that $\alpha(Tx_1,Tx_2)\geq\eta(Tx_1,Tx_2)$. Continuing in this way we obtain $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$. First, we show that $x_n\in\overline{B(x_0,r)}$ for all $n\in N$. Using inequality (2), we have,

$$d(x_0,Tx_0))\leq r.$$

It follows that,

$$x_1\in\overline{B(x_0,r)}.$$

Let $x_2,\cdots,x_j\in\overline{B(x_0,r)}$ for some $j\in N$. Using inequality (1), we obtain,

$$\begin{aligned} d(x_i,x_{i+1}) &= d(Tx_{i-1},Tx_i)\\ &\leq\psi(M(x_{i-1},x_i))\end{aligned}$$

$$\begin{aligned} M(x_{i-1},x_i) &= \max\left\{\begin{array}{l} d(x_{i-1},x_i),d(x_i,x_{i+1}),\\ \dfrac{d(x_{i-1},x_{i+1})}{2}\end{array}\right\}\\ &\leq\max\left\{\begin{array}{l} d(x_{i-1},x_i),d(x_i,x_{i+1}),\\ \dfrac{d(x_{i-1},x_i)+d(x_i,x_{i+1})}{2}\end{array}\right\}.\end{aligned}$$

So

$$M(x_{i-1},x_i)\leq\max\{d(x_{i-1},x_i),d(x_i,x_{i+1})\}. \tag{3}$$

the case $M(x_{i-1},x_i)=d(x_i,x_{i+1})$ is impossible

$$d(x_i,x_{i+1})\leq\psi(d(x_i,x_{i+1}))<d(x_i,x_{i+1}).$$

Which is a contradiction. Otherwise, in other case $M(x_{i-1},x_i)=d(x_{i-1},x_i)$

$$\begin{aligned} d(x_i,x_{i+1}) &\leq\psi(d(x_{i-1},x_i))\leq\psi^2(d(x_{i-2},x_{i-1}))\\ &\leq\cdots\leq\psi^i(d(x_0,x_1)).\end{aligned}$$

Thus we have,

$$d(x_i,x_{i+1})\leq\psi^i(d(x_0,x_1)). \tag{4}$$

Now,

$$\begin{aligned} d(x_0,x_{j+1}) &\leq d(x_0,x_1)+d(x_1,x_2)\\ &\quad +d(x_2,x_3)+\ldots+d(x_j,x_{j+1})\\ &\leq\sum_{i=0}^{j}\psi^i(d(x_0,x_1))\\ &\leq r.\end{aligned}$$

Thus $x_{j+1}\in\overline{B(x_0,r)}$. Hence $x_n\in\overline{B(x_0,r)}$ for all $n\in N$. Now inequality (3.4) can be written as

$$d_l(x_n,x_{n+1})\leq\psi^n(d(x_0,x_1)),\text{for all}\ n\in N. \tag{5}$$

Fix $\varepsilon>0$ and let $N\in N$ such that ${}_{n\geq N}\psi^n(d_l(x_0,x_1))<\varepsilon$. Let $m,n\in N$ with $m>n>N$. Then, by the triangle inequality, we have

$$\begin{aligned} d(x_n,x_m) &\leq\sum_{k=n}^{m-1}d(x_k,x_{k+1})\leq\sum_{k=n}^{m-1}\psi^k(d_l(x_0,x_1))\\ &\leq\sum_{n\geq N}\psi^k(d_l(x_0,x_1))<\varepsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)},d)$. As every closed ball in a complete metric space is complete, so there exists $x^*\in\overline{B(x_0,r)}$ such that $x_n\to x^*$. Also

$$\lim_{n\to\infty}d(x_n,x^*)=0. \tag{6}$$

So by given assumption from (ii), we have $\alpha(x_n,x^*)\geq\eta(x_n,x^*)$, for all $n\in N\cup\{0\}$. Now from (1), we obtain

$$d(x_{n+1},Tx^*)\leq\psi\Big(M(x_n,x^*)\Big). \tag{7}$$

where

$$M(x_n,x^*)=\max\left\{\begin{array}{l} d(x_n,x^*),d(x_n,x_{n+1}),d(x^*,Tx^*),\\ \dfrac{d(x_n,Tx^*)+d(x^*,x_{n+1})}{2}\end{array}\right\}.$$

If $d(x^*,Tx^*)\neq 0,$ then $M(x_n,x^*)>0$ for every n. Thus

$$d(x_{n+1,}Tx^*)\le\psi\left(M(x_n,x^*)\right)<M(x_n,x^*). \tag{8}$$

which on taking limit as $n\to\infty$ gives

$$\begin{aligned} d(x^*,Tx^*) &= \lim_{n\to\infty} d(x_{n+1,}Tx^*)\\ &\le \lim_{n\to\infty} M(x_n,x^*)=d(x^*,Tx^*).\end{aligned}$$

Hence $d(x^*,Tx^*)=0.$ The result follows.

Example 8. Let $X=[0,\infty]$ with metric on X defined by $d(x,y)=|x-y|$. Let $T:X\to X$ be defined by,

$$Tx=\begin{cases} x4 \text{ if } x\in[0,1]\\ x-\dfrac{1}{4} \text{ if } x\in(1,\infty).\end{cases}$$

Consider $x_0=1,\ r=2,\ \psi(t)=\frac{t}{3}$ and

$$\alpha(x,y)=\begin{cases}1 \text{ if } x,y\in[0,1]\\ 0 \quad \text{otherwise}\end{cases}.$$

Now $\overline{B(x_0,r)}=[0,1].$ then

$$d(x_0,Tx_0)=d(1,T1)=d(1,\frac{1}{4})=\left|1-\frac{1}{4}\right|=\frac{3}{4}$$

$$\sum_{i=0}^{n}\psi^n(d(x_0,Tx_0))=\frac{3}{4}\sum_{i=0}^{n}\frac{1}{3^n}<\frac{3}{2}(\frac{3}{4})=\frac{9}{8}<2$$

Also if $x,y\in(1,\infty),$ then

$$\begin{gathered} |3x-3y|>|x-y|\\ |x-y|>\frac{|x-y|}{3}\\ \left|x-\frac{1}{4}-(y-\frac{1}{4})\right|>\psi(|x-y|)\\ d(Tx,Ty)>\psi(d(x,y))\\ d(Tx,Ty)>\psi(M(x,y))\end{gathered}$$

Then the contractive condition does not hold on X. Also if, $x,y\in\overline{B(x_0,r)},$ then

$$\begin{gathered} \left|\frac{3x}{4}-\frac{3y}{4}\right|\le|x-y|\\ \left|\frac{x}{4}-\frac{y}{4}\right|\le\frac{|x-y|}{3}\\ \frac{1}{4}|x-y|\le\psi(|x-y|)\\ d(Tx,Ty)\le\psi(d(x,y))\le\psi(M(x,y)).\end{gathered}$$

If $\eta(x,y)=1$ in the Theorem 7, we have the following corollary.

Corollary 9. Let (X,d) be a complete metric space and T is $\alpha-$admissible mapping. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$\begin{aligned} & x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\ge 1\\ & \Rightarrow d(Tx,Ty)\le\psi\left(M(x,y)\right).\end{aligned} \tag{9}$$

where

$$M(x,y)=\max\left\{\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty),\\ \dfrac{d(x,Ty)+d(y,Tx)}{2}\end{array}\right\}.$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\le r, \text{ for all } j\in N. \tag{10}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\ge 1;$
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\ge 1$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

If $\alpha(x,y)=1$ in the Theorem 7, we have the following corollary.

Corollary 10. Let (X,d) be a complete metric space and T is η-subadmissible mapping. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$\begin{aligned} & x,y\in\overline{B(x_0,r)},\ \eta(x,y)\le 1\\ & \Rightarrow d(Tx,Ty)\le\psi\left(M(x,y)\right).\end{aligned} \tag{11}$$

where

$$M(x,y)=\max\left\{\begin{array}{l} d(x,y),d(x,Tx),d(y,Ty),\\ \dfrac{d(x,Ty)+d(y,Tx)}{2}\end{array}\right\}.$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\le r, \text{ for all } j\in N. \tag{12}$$

If following assertions hold:

- $\eta(x_0,Tx_0)\le 1;$
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\eta(x_n,x_{n+1})\le 1$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\eta(x_n,u)\le 1$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

Corollary 11. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping with respect to η. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq\eta(x,y) \Rightarrow d(Tx,Ty)\leq\psi\left(N(x,y)\right), \tag{13}$$

where

$$N(x,y)=\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\leq r,\text{ for all } j\in N. \tag{14}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq\eta(x_n,u)$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

If $\eta(x,y)=1$ in the corollary 11, we have the following corollary.

Corollary 12. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq 1 \Rightarrow d(Tx,Ty)\leq\psi\left(N(x,y)\right) \tag{15}$$

where

$$N(x,y)=\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\leq r,\text{ for all } j\in N. \tag{16}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\geq 1$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq 1$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

If $N(x,y)=\frac{d(x,Tx)+d(y,Ty)}{2}$ in the corollary 11, we have the following corollary.

Corollary 13. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq 1 \Rightarrow d(Tx,Ty)\leq\psi\left(N(x,y)\right), \tag{17}$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\leq r,\text{ for all } j\in N. \tag{18}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\geq 1$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq 1$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

If $N(x,y)=\frac{d(x,Ty)+d(y,Tx)}{2}$ in the corollary 11, we have the following corollary.

Corollary 14. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq 1 \Rightarrow d(Tx,Ty)\leq\psi\left(N(x,y)\right), \tag{19}$$

and

$$\sum_{i=0}^{j}\psi^i(d(x_0,Tx_0))\leq r,\text{ for all } j\in N. \tag{20}$$

Suppose that the following assertions hold:

- $\alpha(x_0,Tx_0)\geq 1$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq 1$ for all $n\in N\cup\{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^*=x^*$.

If $N(x,y)=d(x,y)$, we obtain the following corollary.

Corollary 15. Let (X,d) be a complete metric space and T is $\alpha-$ admissible mapping with respect to η. For $r>0$, $x_0\in\overline{B(x_0,r)}$ and $\psi\in\Psi$, assume that,

$$x,y\in\overline{B(x_0,r)},\ \alpha(x,y)\geq\eta(x,y) \Rightarrow d(Tx,Ty)\leq\psi\left(d(x,y)\right), \tag{21}$$

and

$$\sum_{i=0}^{j} \psi^i(d(x_0, Tx_0)) \le r, \text{ for all } j \in N. \tag{22}$$

Suppose that the following assertions hold:

- $\alpha(x_0, Tx_0) \ge \eta(x_0, Tx_0)$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \ge \eta(x_n, u)$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

If $\eta(x, y) = 1$, $N(x, y) = d(x, y)$ in the corollary 11, we have the following corollary.

Corollary 16. Let (X, d) be a complete metric space and T is $\alpha-$ admissible mapping. For $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, assume that,

$$\begin{aligned} &x, y \in \overline{B(x_0, r)}, \alpha(x, y) \ge 1 \\ &\Rightarrow d(Tx, Ty) \le \psi\left(d(x, y)\right) \end{aligned} \tag{23}$$

and

$$\sum_{i=0}^{j} \psi^i(d(x_0, Tx_0)) \le r, \text{ for all } j \in N. \tag{24}$$

Suppose that the following assertions hold:

- $\alpha(x_0, Tx_0) \ge 1$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \ge 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

3. Fixed Point Results for Graphic Contractions

Consistent with Jachymski [13], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [13]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length m $(m \in N)$ is a sequence $\{x_i\}_{i=0}^{m}$ of $m+1$ vertices such that $x_0 = x, x_m = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, m$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected(see for details [1,5,12,13]).

Definition 17. ([13]). We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

and T decreases weights of edges of G in the following way:

$$\begin{aligned} &\exists\, k \in (0,1), \forall x, y \in X, (x, y) \in E(G) \\ &\Rightarrow d(Tx, Ty) \le kd(x, y). \end{aligned}$$

Now we extend concept of G-contraction as follows.

Definition 18. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be self-mappings. Assume that for $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, following conditions hold,

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow d(Tx, Ty) \le \psi(M(x, y).$$

where

$$M(x, y) = \max\left\{\begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \\ \dfrac{d(x, Ty) + d(y, Tx)}{2} \end{matrix}\right\},$$

Then the mappings T is called ciric ψ-graphic contractive mappings. If $\psi(t) = kt$ for some $k \in [0,1)$, then we say T is G-contractive mappings.

Definition 19. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be self-mappings. Assume that for $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, following conditions hold,

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow d(Tx, Ty) \le \psi(N(x, y).$$

where

$$N(x, y) = \max\left\{\begin{matrix} d(x, y), \dfrac{d(x, Tx) + d(y, Ty)}{2}, \\ \dfrac{d(x, Ty) + d(y, Tx)}{2} \end{matrix}\right\}.$$

Then the mappings T is called ciric ψ-graphic contractive mappings. If $\psi(t) = kt$ for some $k \in [0,1)$, then we say T is G-contractive mappings.

Theorem 20. Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be ciric ψ-graphic contractive mappings and $x_0 \in \overline{B(x_0, r)}$. Suppose that the following assertions hold:

- $(x_0, Tx_0) \in E(G)$ and $\sum_{i=0}^{j} \psi^i(d(x_0, Tx_0)) \le r$ for all $j \in N$;
- if $\{x_n\}$ is a sequence in $\overline{B(x_0, r)}$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then T has a fixed point.

Proof. Define, $\alpha : X^2 \to (-\infty, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise} \end{cases}.$$

First we prove that the mapping T is α-admissible. Let $x, y \in \overline{B(x_0, r)}$ with $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. As T is ciric ψ-graphic contractive mappings, we have, $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. Thus T is α-admissible mapping. From (i) there exists x_0 such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. If $x, y \in \overline{B(x_0, r)}$ with $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. Now, since T, is ciric ψ-graphic contractive mapping, so $d(Tx, Ty) \leq \psi(M(x,y))$. That is,

$$\alpha(x,y) \geq 1 \Rightarrow d(Tx, Ty) \leq \psi(M(x,y)).$$

Let $\{x_n\} \subset \overline{B(x_0, r)}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. Then, $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$. So by (ii) we have, $(x_n, x) \in E(G)$ for all $n \in N$. That is, $\alpha(x_n, x) \geq 1$. Hence, all conditions of Corollary 9 are satisfied and T has a fixed point.

Corollary 21. Let (X, d) be a complete metric space endowed with a graph G and and $T : X \to X$ be a mapping. Suppose that the following assertions hold:

- T is Banach G-contraction on $\overline{B(x_0, r)}$;
- $(x_0, Tx_0) \in E(G)$ and $d(x_0, Tx_0)) \leq (1-k)r$;
- if $\{x_n\}$ is a sequence in $\overline{B(x_0, r)}$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then T has a fixed point.

Corollary 22. Let (X, d) be a complete metric space endowed with a graph G and and $T : X \to X$ be a mapping. Suppose that the following assertions hold:

- T is Banach G-contraction on X and there is $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
- if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then T has a fixed point.

Conflict of Interests

The authors declare that they have no competing interests.

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