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A Location Problem of Obstacles in Population Dynamics

Sidy Ly[1,2], Fulgence Mansal[1,2], Diaraf Seck[1,2] & Moussa Balde[1,3]

[1] University Cheikh Anta Diop, Dakar, Senegal.

[2] Laboratory Decision Mathematics and numerical Analysis

[3] Laboratory of Geometry and Application

Correspondence: Sidy LY, Laboratory Decision Mathematics and numerical Analysis, University Cheikh Anta Diop, Dakar, Senegal. E-mail: lysidyly@gmail.com

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Abstract

The aim of this paper is to determine the optimal locations where Fish Aggregating Devices (F.A.D) or artificial traps must be placed in a given place of the sea and to preverse resources. Our work focuses on two parts: the first one is the study of static optimization problem with a functional taking into account the distance between the sites or F.A.D and the second one is devoted to solving an optimization problem with constraints expressed in classical model of fishery: Lagrange's method and Pontryagin's maximum principle the main mathematical tools to get characterization results of the location of artificial traps.

Keywords: Dynamical systems; fishery; optimization; Lagrange's problem; Pontryagin's maximum principle; numerical simulations.

1. Introduction

In this paper one supposes to follow one type of fish in a given place to capture it by using artificial traps or Artificial Habitats called Fish Aggregating Devices (FADS) see for example (Moussaoui, 2011) and references therein for more details. Let us recall that fishery activities involve costs (such as salaries of workers, equipment, the fuel logisties..) But it is important to note that if the resource of fishes is not preserved then the economic activities will no longer be profitable in this sector. That's why even there are tools, means and techniques to capture a lot of fishes, it is very important to preserve the fisheries resources. To take into account the economic profitability and the preserving resource, we propose to study geometrical optimization problems linking these two concerns. And we are going to use mainly the Lagrange's method, the Pontryagin's Maximum Principle and the basic tools of the control theory of system of Ordinary Differential Equations. A good understanding of the location of obstacles by geometrical optimization could give a good approximation on the number of artificial habitats to be placed and there locations in order to contribute significautly to the preservation of the fisheries resources.

The main concern is to find and to characterize a network, a shape of unknown domains with constraints of ordinary differential equations translating the population dynamics. For this we shall to introduce a criteria to be optimized, depending on the position of the obstacles (traps) and minimizing both economics costs and distances between the obstacles.

Our contribution can be summed up as follows:

Comparing to pioneering the work due to Auger et al, (Auger; Moussaoui, 2011) for a given number of obstacles, we introduce geometrical functionals to get sufficient conditions describing the optimal location of the obstacles or traps. And from these sufficient conditions, several geometric configurations are obtained.

Another intersting is a geometrc controllability.In fact, introducing control variables depending implicitly or explicitly on the obstacles. We get an optimal necessary and sufficient condition to get stable evolution of the resource during a given time interval $[0; T]$, $T > 0$. The optimal control results, that is the main result is given. And finally, it is followed by numerical simulations.

In the sequel of this work, we will consider the expression FADS if necessary to mean traps or obstacles or sites and the work is organized as follows:

In section 1, we study the proposed optimization without constraints . For these problems, we use a functional which takes into account the distance between FADS.

The section 2 is devoted to the optimization problem with constraints that are described by ordinary differentiel equations.

They derived from a classical fisheries model which is giving by the following system:

$$\begin{cases} \frac{dn}{dt} &= \left(rn\left(1-\frac{n}{K}\right)-QnE\right) \\ \frac{dE}{dt} &= (-c+aQn)\,E \end{cases}$$

where $n(t)$ and $E(t)$ are respectively fish biomass and fishing effort.The other parameters q, c and a are as follows:

q the fish catchability parameter on the FADS,

c is the cost per unit of fishing effort on the FADS,

a is the price per unit of fish on the sites.

Let us point out that two methods shall be explored: the Lagrange's and Pontryagin's methods.

2. Optimization without Constraints

The aim in this section is to study the location of FADs so as to minimize the distances between traps. Now let us introduce the following functionnal:

$$G_1(M_1,\cdots,M_L)=\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2-R_0^2\right)^2$$

where $M_i=(x_i\,;\,y_i)\in\mathbb{R}^2$, $M_j=\left(x_j\,;\,y_j\right)\in\mathbb{R}^2$ are the positions of the FADS to be determined, L is the number of FADS, $\left\|M_iM_j\right\|^2=(x_i-x_j)^2+(y_i-y_j)^2$ is the square euclidian distance of the points M_i and M_j. This functional is introduced in order to minimize the distance in a given region that is assimilated to the disc $D(O,R_0)$. Our aim is to solve the above minimization problem in $D(0,R_0)$ centered at the origin O with radius R_0. We have the following first order necessary optimality conditions for the functional G_1.

Theorem 1 *Let us consider the functional G_1 defined as above. Then a first order necessary optimality condition for location of FADS is given by:*

$$\sum_{i=1}^{L-1}\|M_iM_{i+1}\|^2+\sum_{i=1}^{L-2}\|M_iM_{i+2}\|^2+\cdots+\sum_{i=1}^{2}\|M_iM_{i+L-2}\|^2+\|M_1M_L\|^2-R_0^2=0$$

Before proving this first result, let us remark that:

$$G_1=\left(\sum_{i=1}^{L-1}\|M_iM_{i+1}\|^2+\sum_{i=1}^{L-2}\|M_iM_{i+2}\|^2+\cdots+\sum_{i=1}^{2}\|M_iM_{i+L-2}\|^2+\|M_1M_L\|^2-R_0^2\right)^2$$

Proof. Expanding the functional G_1, we have

$$G_1 \;=\; \left(\sum_{i=1}^{L-1}\|M_iM_{i+1}\|^2+\sum_{i=1}^{L-2}\|M_iM_{i+2}\|^2+\cdots+\sum_{i=1}^{2}\|M_iM_{i+L-2}\|^2+\|M_1M_L\|^2-R_0^2\right)^2$$

For $M_i=(x_i\,;\,y_i)$ and $M_j=\left(x_j\,;\,y_j\right)$ $i;j\in\{1...L\}$ then we have

$$G_1=\left(\sum_{i=1}^{L-1}(x_i-x_{i+1})^2+(y_i-y_{i+1})^2+\cdots+(x_1-x_L)^2+(y_1-y_L)^2-R_0^2\right)^2$$

Let us set

$$\begin{aligned} A &= \sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2-R_0^2 \\ &= \sum_{i=1}^{L-1}\|M_iM_{i+1}\|^2+\sum_{i=1}^{L-2}\|M_iM_{i+2}\|^2+\cdots+\sum_{i=1}^{2}\|M_iM_{i+L-2}\|^2+\|M_1M_L\|^2-R_0^2 \end{aligned}$$

A necessary optimality condition for location the obstacles is given by $\nabla G_1=0$. This is translated by:

$$\left\{\begin{array}{rcl} \left[Lx_1 - \sum_{i=1}^{L} x_i\right] A & = & 0 \\ \vdots & & \\ \left[Lx_L - \sum_{i=1}^{L} x_i\right] A & = & 0 \\ \left[Ly_1 - \sum_{i=1}^{L} y_i\right] A & = & 0 \\ \vdots & & \\ \left[Ly_L - \sum_{i=1}^{L} x_i\right] A & = & 0 \end{array}\right. \tag{1}$$

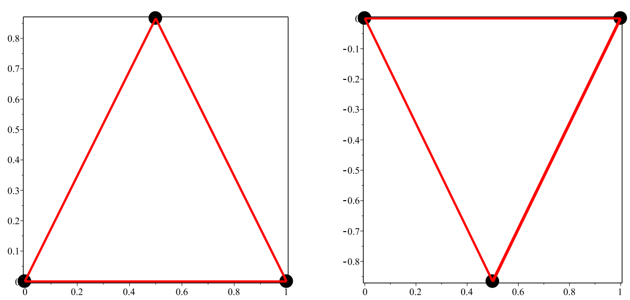
The solving of the system (1) is equivalent to solve 4^L systems. And each system corresponds to positions of FADS. Among there several possibilities of positions, we consider a particular case that is:

$$A = \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left\|M_i M_j\right\|^2 - R_0^2 = 0.$$

That prove the theorem.

Now let us proceed to some geometrical representations of FADS. For this we shall consider the cases given by **theorem 1** i.e the case where $A = 0$ for different values of the number of sites L. Here we plot the positions for $L = 3, 4, 5, 6$ and $L = 7$. For all representations M_1 is supposed to be given and fixed. We can assume that $M_1 = O$. The figures are obtained by solving equation $A = 0$ with additional data.

For example Figure1 is obtained by assuming that, $||M_1 M_2|| = ||M_1 M_3|| = ||M_2 M_1|| = 1$ and $x_2 = 0.5$.



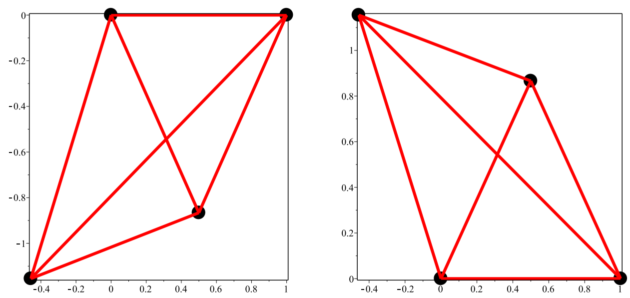
(a) (b)

Figure 1. Representations of FADS for $L = 3$.

For the cases of Figure2 and Figure3, we suppose that for $i = 1 \ldots 3$ and for $i = 1 \ldots 4$, $||M_i M_{i+1}|| = 1$, $x_2 = 1$ and $x_3 = 0.5$.

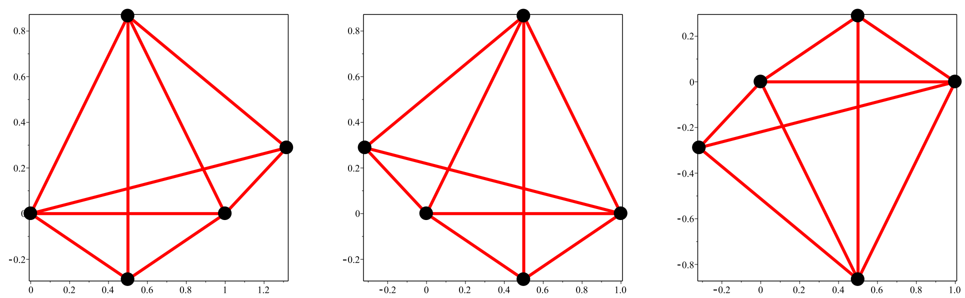
Figure4 is obtained by supposing for $= 1 \ldots 5$, $||M_i M_{i+1}|| = 1$, $x_2 = 1$, $x_3 = 0.5$, $x_4 = 0.8$ and $x_5 = 0.3$.

The last one is obtained by taking for $= 1 \ldots 6$, $||M_i M_{i+1}|| = 1$, $x_2 = 1$, $x_3 = 0.5$, $x_4 = 0.8$,$x_5 = 0.3$ and $x_5 = 0.2$.



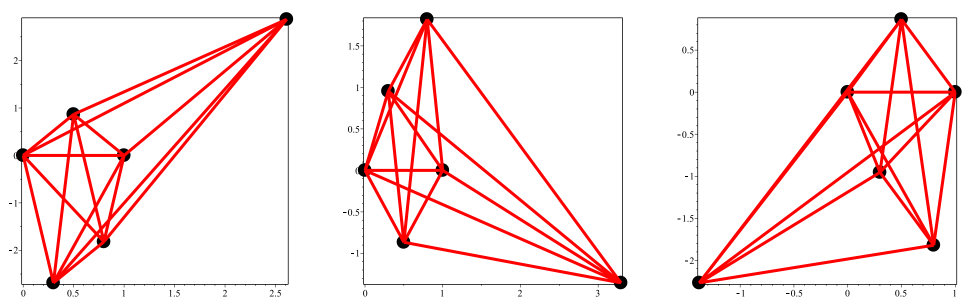
(*a*) (*b*)

Figure 2. Representation of FADS for $L = 4$.



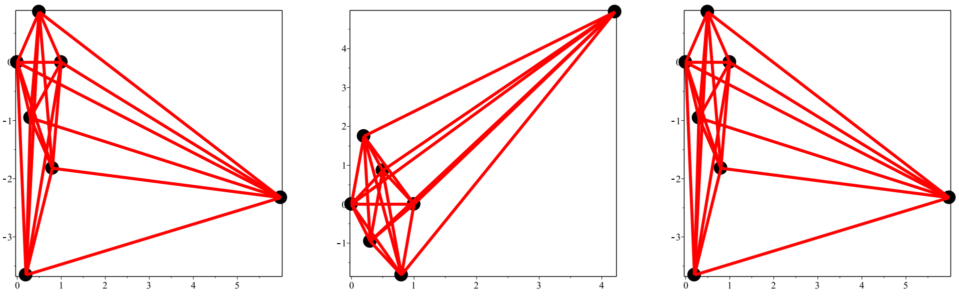
(*a*) (*b*) (*c*)

Figure 3. Representation of FADS for $L = 5$.



(*a*) (*b*) (*c*)

Figure 4. Representation of FADS for $L = 6$.



(*a*) (*b*) (*c*)

Figure 5. Representation of FADS for $L = 7$.

3. Optimization with Constraints

3.1 Lagrange's Method

In this part we shall introduce constraints and the economical dimension is translated by the payoff or the benefice related to fisheries activities. Let us introduce the functional defined by:

$H(t, M_1; \cdots; M_L)$ = catch - costs = $(anQ - c)E$. Our aim is to maximize H and to minimize G_1 under the constraints described by the aggregated model over a time interval $[0, T]$ where T is a given and fixed positive real. Let us consider the following dynamical optimization problem:

$$(\mathcal{P}) : min \int_0^T \left[-H(t, M_1; \cdots; M_L) + \frac{1}{T} G_1(M_1; \cdots; M_L) \right]$$

under the constraints of aggregated model:

$$\begin{cases} \frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - QnE \\ \frac{dE}{dt} = (-c + aQn)E \\ n(0) = n_0 \\ E(0) = E_0 \quad \text{where } M_i \in D(0;\ R_0) \text{ design the position of FAD } i \end{cases} \tag{2}$$

$(\mathcal{P})$ is nothing but a Lagrange's problem.

Remark 1 From the following inequality:

$$min \int_0^T \left[-H + \frac{1}{T} G \right] dt \geq min \int_0^T -H dt + minG$$

it is easy to see that another interesting optimization is
$max \int_0^T H + minG_1$. It should be interesting too to consider a multicriteria problem as follows: $minG_1$ and $max \int_0^T H$ under the constraints (2). It can be formulated in the following sense

$$min \int_0^T \gamma \left[-H(t, M_1; \cdots; M_L) \right] dt + (1 - \gamma) \int_0^T \left[\frac{1}{T} G_1(M_1; \cdots; M_L) \right] dt$$

where γ is an arbitrary constant, $\gamma \in [0; 1]$ under the constraints of aggregated model:

$$\begin{cases} \frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - QnE \\ \frac{dE}{dt} = (-c + aQn)E \\ n(0) = n_0 \\ E(0) = E_0 \quad \text{where } M_i \in D(0;\ R_0) \text{ is the position of FAD } i \end{cases}$$

Theorem 2 *The optimal conditions of Lagrange's problem is given by*

$$\begin{cases} \left[Lx_1 - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) &= 0 \\ \vdots & \\ \left[Lx_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) &= 0 \\ \left[Ly_1 - \sum_{i=1}^{L} y_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) &= 0 \\ \vdots & \\ \left[Ly_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) &= 0 \end{cases}$$

where $n(t)$, $E(t)$, $p_1(t)$ *and* $p_2(t)$ *satisfy the following system:*

$$\begin{cases} \dot{n} &= rn(1-\frac{n}{K}) - QnE \\ \dot{E} &= (-c + aQn)E \\ \dot{p_1} &= -aQE - p_1\left[r\left(1-\frac{2n}{K}\right) - QE\right] - p_2 aQE \\ \dot{p_2} &= (c - aQn) + p_1 Qn - p_2(-c + aQn) \end{cases} \tag{3}$$

$p_1(t)$ *and* $p_2(t)$ *be Lagrange multipliers.*

Proof. Let's set $X = (n\ ;\ E)$ and the control vector

$$U\begin{pmatrix} u_1 \\ \vdots \\ u_L \\ u_{L+1} \\ \vdots \\ u_{2L} \end{pmatrix}$$

with

$$\begin{array}{rcl} u_1 &=& x_1 \quad \dots \quad u_L = x_L \\ u_{L+1} &=& y_1 \quad \dots \quad u_{2L} = y_L \end{array}$$

and

$$f_0(t\ ;\ X;\ U) = (c - aQn)E + \frac{1}{T}\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right)^2$$

$$\varphi(t\ ;\ X;\ u) = \begin{pmatrix} rn\left(1-\frac{n}{K}\right) - QnE \\ (-c + aQn)E \end{pmatrix}.$$

Then, we can introduce the Lagrange's function defined by:

$$\mathcal{L}(t\ ;\ X;\ u;\ p;\ \lambda) = \int_0^T \left(\lambda_0 f_0(t\ ;\ X;\ u) + p(t)\left(\dot{X} - \varphi(t\ ;\ X;\ u)\right)\right)dt + \lambda_1 n(0) + \lambda_2 E(0)$$

$$= \int_0^T \left[\lambda_0\left((c - aQn)E + \frac{1}{T}\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right)^2\right) + p_1(\dot{n} - \varphi_1) + p_2(\dot{E} - \varphi_2)\right]dt$$

$$+\lambda_1 n(0) + \lambda_2 E(0).$$

• The Euler-Lagrange conditions are expressed as follows

$$\begin{cases} -\frac{d}{dt}F_{\dot{n}} + F_n = 0 \\ -\frac{d}{dt}F_{\dot{E}} + F_E = 0 \end{cases}$$

Where $F_{\dot{n}} = \frac{\partial F}{\partial \dot{n}}$, $F_{\dot{E}} = \frac{\partial F}{\partial \dot{E}}$, $F_n = \frac{\partial F}{\partial n}$, $F_E = \frac{\partial F}{\partial E}$ $F_{\dot{X}} = (\frac{\partial F}{\partial n}, \frac{\partial F}{\partial E})$ and $F = \lambda_0 f_0 + p_1 (\dot{n} - \varphi_1) + p_2 \left(\dot{E} - \varphi_2\right)$.

$$\begin{cases} \dot{p_1} = -\lambda_0 aQE - p_1 \left[r\left(1 - \frac{2n}{K}\right) - QE\right] - p_2 aQE \\ \dot{p_2} = \lambda_0 (c - aQn) + p_1 Qn - p_2 (-c + aQn). \end{cases}$$

• The transversality conditions are equivalent to the following systems:

$$\begin{cases} p_1 (0) = \lambda_1 \\ p_1 (T) = 0 \end{cases} ; \begin{cases} p_2 (0) = \lambda_2 \\ p_2 (T) = 0 \end{cases}$$

• The optimality conditions are given by the equations:

$$F_{x_i} = 0 \ ; \ F_{y_i} = 0 \quad \text{for} \quad i = 1; \ldots ; L$$

that are equivalent to:

$$\begin{cases} \left[Lx_1 - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\ \vdots \\ \left[Lx_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\ \left[Ly_1 - \sum_{i=1}^{L} y_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\ \vdots \\ \left[Ly_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \end{cases}$$

For $\lambda_0 = 1$, taking into account the equations of constraints and the Euler-Lagrange equations we obtain the following system:

$$\begin{cases} \dot{n} & = & rn(1 - \frac{n}{K}) - QnE \\ \dot{E} & = & (-c + aQn)E \\ \dot{p_1} & = & -aQE - p_1 \left[r\left(1 - \frac{2n}{K}\right) - QE\right] - p_2 aQE \\ \dot{p_2} & = & (c - aQn) + p_1 Qn - p_2 (c - aQn) \end{cases} \tag{4}$$

The solution of the (4) is given by the figure 6.

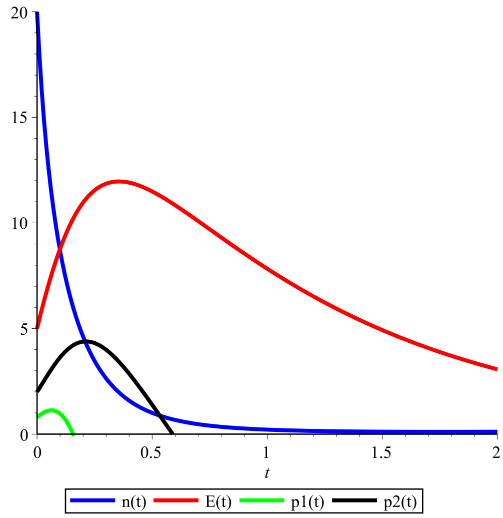


Figure 6. Representation of solutions of system (4). Figure 6 is obtained for $a = 1$, $r = 2$, Q=1/2 c=1 and K=4 with initial conditions given: $n(0) = 20$, $E(0) = 5$ $p_1(0) = 0.8$ and $p_2(0) = 2$

3.2 Pointryagin's Method

In this subsection, we aim study the following problem by using Pointryagin's method

$$(\mathcal{P}) : min \int_0^T \left[-H(t, M_1; \cdots ; M_L) + \frac{1}{T} G_1 (M_1; \cdots ; M_L)\right]$$

under the same constraints than those considered in Lagrange's problem. that is translated by:

$$(\mathcal{P}) : max \int_0^T \left[H(t, M_1; \cdots ; M_L) - \frac{1}{T} G_1 (M_1; \cdots ; M_L)\right]$$

under the constraints of aggregated model:

$$\begin{cases} \frac{dn}{dt} &=& rn\left(1 - \frac{n}{K}\right) - QnE \\ \frac{dE}{dt} &=& (-c + aQn) E \\ n(0) &=& n_0 \\ E(0) &=& E_0 \end{cases}$$

where

$$G_1(M_1, \cdots , M_L) = \left(\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left\|M_i M_j\right\|^2 - R_0^2\right)^2, \quad M_i \in D(0;\ R_0), \quad L \text{ number of FADS}$$

Theorem 3 *Assuming U^* the optimal control of above problem and X the corresponding trajectory. Then there exists a vector $P(p_1; p_2)$ such that the couple of vectors $(X; P)$ satisfies the following hamiltonian system:*

$$\begin{cases} \dot{n} &=& rn(1 - \frac{n}{K}) - QnE \\ \dot{E} &=& (-c + aQn)E \\ \dot{p_1} &=& -aQE - p_1\left[r\left(1 - \frac{2n}{K}\right) - QE\right] - p_2 aQE \\ \dot{p_2} &=& (c - aQn) + p_1 Qn - p_2 (-c + aQn) \end{cases} \quad (5)$$

and the maximization condition is given by

$$\left\{\begin{array}{rcl}
\left[Lx_1 - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\
& \vdots & \\
\left[Lx_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\
\left[Ly_1 - \sum_{i=1}^{L} y_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0 \\
& \vdots & \\
\left[Ly_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left\|M_iM_j\right\|^2 - R_0^2\right) & = & 0
\end{array}\right.$$

Proof. Let U be the control defined in $[-R_0; R_0]^{2L}$ by

$$U\begin{pmatrix} u_1 \\ \vdots \\ u_L \\ u_{L+1} \\ \vdots \\ u_{2L} \end{pmatrix}$$

with

$$\begin{array}{rcl} u_1 & = & x_1 \quad \dots \quad u_L = x_L \\ u_{L+1} & = & y_1 \quad \dots \quad u_{2L} = y_L \end{array}$$

and $X = \begin{pmatrix} n \\ E \end{pmatrix}$ Then the hamiltonian is given by

$$H(X, p, U) = \left(rn\left(1 - \frac{n}{K}\right) - QnE\right)p_1 + (-c + aQn)\,Ep_2 + (-c + aQn)\,E$$

$$-\frac{1}{T}\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 - R_0^2\right)^2$$

Then the equations $\dot{X} = \frac{\partial H}{\partial P}$ and $\dot{P} = -\frac{\partial H}{\partial X}$ imply that:

$$\left\{\begin{array}{rcl}
\dot{n} = \frac{\partial H}{\partial p_1} & = & rn(1 - \frac{n}{K}) - QnE \\
\dot{E} = \frac{\partial H}{\partial p_2} & = & (-c + aQn)E \\
\dot{p_1} = -\frac{\partial H}{\partial n} & = & -aQE - p_1\left[r\left(1 - \frac{2n}{K}\right) - QE\right] - p_2aQE \\
\dot{p_2} = -\frac{\partial H}{\partial p_1} & = & (c - aQn) + p_1Qn - p_2\,(-c + aQn)
\end{array}\right.$$

The maximization condition is given by differentiating the hamiltonian with respect to each variable u_i for $i = 1, \dots, 2L$, That is equivalent to

$$\begin{cases} \left[Lx_1 - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \|M_iM_j\|^2 - R_0^2\right) & = & 0 \\ \vdots & & \\ \left[Lx_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \|M_iM_j\|^2 - R_0^2\right) & = & 0 \\ \left[Ly_1 - \sum_{i=1}^{L} y_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \|M_iM_j\|^2 - R_0^2\right) & = & 0 \\ \vdots & & \\ \left[Ly_L - \sum_{i=1}^{L} x_i\right]\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \|M_iM_j\|^2 - R_0^2\right) & = & 0 \end{cases}$$

Remark 2 It is interesting to note that we find the same solution as in the Lagrange's method. This means that we have the same representations of solutions than in figure 6. The maximization condition corresponds to the geometrical optimization problem without constraints, developed in first section. We can claim that optimal control is given by the optimal location of FADS.

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