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Lectures on Cheeger-Gromov Theory of Riemannian manifolds

M. Boileau

Laboratoire Emile Picard
UFR MIG
Université Paul-Sabatier
118, route de Narbonne
F-31062 Toulouse, Cedex 4
France

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

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Michel Boileau

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The goal of these lectures were to introduce some fundamental tools and results in the theory of Gromov-Hausdorff convergence of Riemannian manifolds. Details of the proofs of the results presented here can be found in the following basic references : [BBI], [Fa], [Fu], [GLP], [G], [Pet1].

1 Gromov-Hausdorff distance between metric spaces

In the 1980's Gromov extended the classical Hausdorff distance between compact subspaces of a metric space to a distance between abstract metric spaces, called the Gromov-Hausdorff distance (G-H distance for short). However two metric spaces which are close for this distance generally can be topologically different.

All metric spaces in this section will be separable metric space (X, d).

For a a subset $A \subset X$ the ε-neighborhood around A is $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$.

The classical Hausdorff distance between two subsets A, B in a metrix space X is:

$$d_H^X(A, B) = \inf\{\varepsilon : A \subset B(B, \varepsilon) \text{ and } B \subset B(A, \varepsilon)\}$$

This metric is only a pseudo-metric since $d_H^X(A, B) = 0$ implies only that the closures $\overline{A} = \overline{B}$. However for closed subspaces of X it is a metric.

Gromov extended this notion to the setting of abstract metric spaces by getting ride of the role of the ambient space and found important applications to differential geometry.

Definition 1.1 (Gromov-Hausdorff distance) *Two metric spaces X and Y are ε-near in the Gromov-Hausdorff topology if there is a metric on the disjoint union $X \sqcup Y$ which extends the metrics on X and Y such that $d_H^{X \sqcup Y}(X,Y) \leq \varepsilon$.*

Then define : $d_{GH}(X,Y) = \inf\{\varepsilon$ such that X and Y are ε-near$\}$.

From the definition it follows that $d_{GH}(X,Y) = d_{GH}(Y,X)$ and that $d_{GH}(X,Y)$ is finite if X and Y are compact. We will show below that d_{GH} is a distance ont the set of compact metric spaces.

First we give some basic examples which show that often one can give upper bound for the G-H distance, even if it is usually very hard to compute it exactly.

Example 1.2 *Let X and Y be compact metric spaces with $diam(X) \leq D$ and $diam(Y) \leq D$. Then $d_{GH}(X,Y) \leq D/2$.*

Example 1.3 *Let $(X = \{x_1, \ldots, x_k\}, d)$ and $(Y = \{y_1, \ldots, y_k\}, d)$ be finite metric spaces with k points. If $|d(x_i,x_j) - d(y_i,y_j)| < \varepsilon$ for all $1 \geq i, j \leq k$, then $d_{GH}(X,Y) \leq \varepsilon$*

Example 1.4 *A map $f : X \to Y$ between two compact metric spaces is called a ε-Hausdorff approximation if the following holds:*

(i) Y is the ε-tube around $f(X)$.

(ii) $\forall u, v \in X, |d(f(u), f(v)) - d(u,v)| < \varepsilon$

If $f : X \to Y$ is a ε-Hausdorff approximation then $d_{GH}(X,Y) \leq 3\varepsilon$.

The following examples show that the Hausdorff dimension is not is not continuous with respect to the Gromov-Hausdorff topology.

Example 1.5 *Let X be a compact space with a metric g. Then $(X, \lambda d)$ converges to a point for the Gromov-Hausdorff distance when $\lambda \to 0$.*

Example 1.6 *Consider the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ with the standard S^1-action induced by $\mathbb{C}*$. The quotient $\pi : \mathbb{S}^3 \to \mathbb{S}^3/\mathbb{S}^1 = \mathbb{S}^2$ is the Hopf fibration, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the standard sphere with curvature 4. The finite cyclic subgroup $\mathbb{Z}_n \subset \mathbb{S}^1$ of order*

n acts freely and orthogonally on $\mathbb{S}^3$. One denotes by $\mathbb{S}^3/\mathbb{Z}_n = L_n$ the lens space obtained. As $n \to \infty$, the subgroup $\mathbb{Z}_n$ fills up $\mathbb{S}^1$ and the 3-dimensional lens spaces L_n converge for the Gromov-Hausdorff distance to the 2-dimensional base $\mathbb{S}^2$ of the Hopf fibration. This phenomenon is called a collapse because the dimension of the limit space is smaller than the dimension of the spaces in the sequence. We will come back to this phenomenon latter on.

Example 1.7 *Let $X_n = \{\frac{1}{n}(p,q) : p \in \mathbb{Z}, q \in \mathbb{Z}\}$, then X_n with the induced metric from $\mathbb{R}^2$ converges for the Hausdorff-Gromov metric to $\mathbb{R}^2$.*

In the unit cube $[0,1]^3 \subset \mathbb{R}^3$ consider the subspace $X_n = \{(x,y,z) \in [0,1]^3\}$, where at least two coordinates are of the form $\frac{p}{n}, p \in \mathbb{Z}$. Then $Y_n = \partial X_n$ is a surface in $\mathbb{R}^3$ which fills up $[0,1]^3$ as $n \to \infty$. This phenomenon is called an explosion, since the limit space has larger dimension than the spaces in the sequence.

Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces, then d_{GH} is a distance on $\mathcal{M}$ and :

Theorem 1.8 *$(\mathcal{M}, d_{GH})$ is a metric space which is separable and complete.*

2 Gromov's precompactness theorem

In order to state Gromov's precompactness criterion we need the following definitions:

Definition 2.1 *Let X be a compact metric space and $\varepsilon > 0$ a real number:*

A ε-net is a finite set of points Z^ε in X such thaT $X = B(Z, \varepsilon)$.

Define $Cov(X, \varepsilon)$ has the minimal number of points of a ε-net in X.

Lemma 2.2 *Let $X, Y \in \mathcal{M}$ such that $d_{GH}(X, Y) \leq \delta$. Show that $Cov(X, \varepsilon + 2\delta) \leq Cov(Y, \varepsilon)$.*

The following precompactness criterion for subset $\mathcal{C}$ in $\mathcal{M}$ is important and very useful :

Theorem 2.3 *A subset $\mathcal{C} \subset \mathcal{M}$ is precompact for the Gromov-Hausdorff topology iff there is a function $N : (0, \beta) \to (0, \infty)$ such that : $\forall\ \varepsilon > 0$ and $\forall\ X \in \mathcal{C}$ one has $Cov(X, \varepsilon) \leq N(\varepsilon)$.*

Exercise 2.4 *Let $N : (0, \beta) \to (0, \infty)$ be a function and let $\mathcal{C}_N \subset \mathcal{M}$ be the class of compact metric spaces X such that $Cov(X, \varepsilon) \leq N(\varepsilon), \forall\ \varepsilon \in (0, \beta)$. Show that $\mathcal{C}_N$ is compact.*

We present now two important applications of Gromov's precompactness criterion to Riemannian geometry.

2.1 Riemannian manifolds with a lower bound on the injectivity radius

The injectivity radius $inj(M, x)$ of a Riemannian manifold M at a point x is the maximal radius r so that the exponential $exp_x : B(0, r) \subset T_xM \to M$ is an embedding. The injectivity radius of M is $inj(M) = \inf_{x \in M} inj(M, x)$.

Denote by $\mathcal{R}(n, \delta, v)$ the set of closed, connected Riemannian manifolds of dimension n with injectivity radius $inj(M) \geq \delta > 0$ and volume $vol(M) \leq v$.

The following result is a consequence of croke's isoperimetric inequality [Cro], see also [Cha, 6.6]:

Proposition 2.5 *Let M be a closed Riemannian n-manifold. If $inj(M) \geq \delta$, there is a constant $c(n)$ depending only on the dimension n such that $vol(B(x,r) \geq c(n)r^n$ for any $0 < r \leq \delta/2$ and any $x \in M$.*

Given a Riemannian manifold $M \in \mathcal{R}(n, \delta, v)$ one chooses a maximal set $\{B(x_i, \varepsilon/2)\}$ of disjoint balls in M. Then the set of balls $\{B(x_i, \varepsilon)\}$ covers M and one gets that $Cov(M, \varepsilon) \leq \frac{2^n v}{c(n)} \varepsilon^{-n}$. Thus Gromov's criterion applies to show:

Corollary 2.6 *The set $\mathcal{R}(n, \delta, v)$ is precompact in $\mathcal{M}$ for the Gromov-Hausdorff topolopgy.*

2.2 Riemannian manifolds with a lower bound on the Ricci curvature

For our second application of Gromov's precompactness criterion we consider the set $\mathcal{R}(n, k, D)$ of closed, connected Riemannian manifolds of dimension n with Ricci curvature $Ric_M \geq k(n-1)$ and diameter $diam(M) \leq v$.

The Ricci curvature reflects important informations on the growth of the volume of the balls in M.

Theorem 2.7 (Bishop-Gromov) *Let M be a complete Riemannian n-manifold with $Ric_M \geq k(n-1)$. Then for every point $x \in M$ the quantity $\frac{vol(B(x,r))}{v_k(r)}$ is decreasing with respect to r, where $v_k(n,r)$ denotes the volume of a geodesic ball in the space form of contant sectionnal curvature k and of dimension n.*

In particular : $\frac{vol(M)}{vol(B(x,r))} \leq \frac{v_k(n,D)}{v_k(n,r)} \leq \frac{\int_{[0,r]} sinh^{n-1}(\sqrt{|k|}t)dt}{\int_{[0,D]} sinh^{n-1}(\sqrt{|k|}t)dt}$ for any $0 < r < D$. Therefore as above $Cov(M, \varepsilon) \leq \frac{v_k(n,D)}{v_k(n,r)} \leq c(n,k,D)\varepsilon^{-n}$. Gromov's criterion applies once more to show:

Corollary 2.8 *The set $\mathcal{R}(n, k, D)$ is precompact in $\mathcal{M}$ for the Gromov-Hausdorff topolopgy.*

2.3 Dimension of the limit space

We have shown that every sequence in $\mathcal{R}(n, \delta, v)$ or $\mathcal{R}(n, k, D)$ subconverges in $\mathcal{M}$, but a priori the limit space may not be a manifold and may have a dimension different from n. Here we show that in both cases the dimension of the limit space stays $\leq n$, so no explosion can occures like in examples 1.7. However the example 1.6 shows that collapse may occure in $\mathcal{R}(n, k, D)$, even with pinched sectionnal curvature. We will show that no collapse can occure in $\mathcal{R}(n, \delta, v)$, this points out the importance of controlling the injectivity radius.

We first recall the definition of the(covering) dimension of a topological space:

Definition 2.9 *The (covering) dimension of a topological space X is $\leq n$ if every locally finite, open covering of X admits a refinement such that no point in X belongs to more than $n+1$ open subsets. The dimension $dim(X)$ is the smallest integer n such that X has dimension $\leq n$.*

The (covering) dimension of a n-dimensional manifold is n.

For a metric space there is another concept of dimension which has a more metric flavour. Both concept coincides for a compact n-dimensional manifold

Definition 2.10 *For a compact metric space X the Hausdorff dimension is:*

$$dim_H(X) = limsup_{\varepsilon \to 0} \frac{\log(Cov(X, \varepsilon))}{-\log(\varepsilon)}$$

This metric dimension can take non integral values for Cantor sets. It is related to the usual (covering) dimension by the following inequality due to Pontriagin et Schnirelmann [PS]: for a compact metric space X, $dim(X) \leq dim_H(X)$.

Proposition 2.11 *Let $\{M_i\}$ be a sequence of closed Riemannian n-manifolds which converges in the Gromov-Hausdorff topology to a compact metric space X.*

(1) If $\{M_i\} \subset \mathcal{R}(n, \delta, v)$ then $dim(X) = dim_H(X) = n$. So no collapse, nor explosion occures.

(2) If $\{M_i\} \subset \mathcal{R}(n, k, D)$ then $dim(X) \leq dim_H(X) \leq n$. So no explosion occures, but collapses are possible.

The fact that in both cases (1) and (2) $dim_H \leq n$ follows immediatly from the bound $Cov(M, \varepsilon) \leq c\varepsilon^{-n}$, where the constant c depends only of the dimension n and the bounds given on the injectivity and the volume, or on the Ricci curvature and the diameter.

The fact that in case (1) the dimension cannot decrease is more subtle: it uses the notion of $(n-1)$-diameter and a local contractibility argument, see [Pet2].

As a consequence of Perelman's stability theorem for Alexandrov spaces with lower curvature bound (see [BBI,Chap. 10.10]) one obtains:

Corollary 2.12 *The set of closed, connected Riemannian n-manifolds M with* $K_M \geq -1$, $inj(M) \geq \delta > 0$ *and* $vol(M) \leq v$ *contains only finitely many homeomorphism types.*

Instead of working in the general class of compact metric spaces we could have work in the smaller class of length spaces:

Definition 2.13 *Let* X *be a compact metric space. A continuous map* $\ell : [0, a] \to X$ *is a minimizing geodesic if* $d(\ell(u) - \ell(v)) = |u - v|$ *holds for each* $0 \leq u \leq v \leq a$*. The space* X *is a length space if two points in* X *can be joined by a minimizing geodesic.*

An easy application of Ascoli-Arzela 's theorem shows that a Gromov-Hausdorff limit of length spaces is a length space.

3 Riemannian manifolds with pinched curvature

First we introduce a new topology on the set of metric spaces called the Lipschitz topology.

Definition 3.1 *Let X and Y be two metric spaces.*

For a map $f : X \to Y$ let $dil(f) = \sup_{x \neq y} \frac{d(f(x),f(y))}{d(x,y)}$ denote the dilatation of f. It is finite for a lipschitz map.

A homeomorphism $f : X \to Y$ is said bilipschitz if both $dil(f)$ and $dil(f^{-1})$ are finite

Exercise 3.2 *Let $f : M \to N$ be a $\mathcal{C}^1$-map between two compact Riemannian manifolds. Show that $dil(f) = \sup_{x \in M} \|df(x)\|$.*

Definition 3.3 (Lipschitz distance) *Let X and Y be metric spaces. Define the Lipschitz distance $d_L(X,Y)$ between X and Y as:*

$d_L(X,Y) = \inf\{|\log(dil(f))| + \log(dil(f^{-1})|, \ \forall$ *bilipschitz homeomorphism* $f : X \to Y\}$

$d_L(X,Y) = \infty$ *if X and Y are not bilipschitz homeomorphic.*

Proposition 3.4 *Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces, then d_L is a distance on $\mathcal{M}$.*

The next proposition shows that Lipschitz convergence implies Gromov-Hausdorff convergence. Hence the Gromov-Hausdorff topology is weaker than the Lipschitz topology on $\mathcal{M}$.

Proposition 3.5 *Let $\{X_n\}_{n\in\mathbb{N}}$ and X be compact metric spaces in $\mathcal{M}$.*

(i) Assume that $\lim_{n\to 0} d_{GH}(X_n, X) = 0$. Then given $\varepsilon > 0$ and $\eta > 0$, for any η-discrete ε-net $Z^\varepsilon \subset X$ there is a sequence of ε_n-nets $Z_n^{\varepsilon_n} \subset X_n$ such that $\lim_{n\to 0} d_L(Z_n^{\varepsilon_n}, Z^\varepsilon) = 0$ with $0 \geq \varepsilon_n - \varepsilon \to 0$.

(ii) Conversely assume that $\forall \varepsilon > 0$ there is a ε-net $Z^\varepsilon \subset X$ and a sequence of ε-nets $Z_n^\varepsilon \subset X_n$ such that $\lim_{n\to 0} d_L(Z - n^\varepsilon, Z^\varepsilon) = 0$. Then $\lim_{n\to 0} d_{GH}(X_n, X) = 0$.

Part (ii) of the above proposition immediatly implies the following:

Corollary 3.6 *Let Let $\{X_n\}_{n\in\mathbb{N}}$ and X be compact metric spaces in $\mathcal{M}$. If $\lim_{n\to 0} d_L(X_n, X) = 0$, then $\lim_{n\to 0} d_{GH}(X_n, X) = 0$.*

3.1 Rigidity theorem

In general the Lipschitz topology is stronger than the Gromov-Hausdorff topology. However in the setting of Riemannian manifolds with pinched sectional curvature both topology coincides.

More precisely, let $\mathcal{M}(n,\delta,v)$ be the set of Riemannian n-manifolds M with a pinched sectional curvature $|K_M| \le 1$, a lower bound on the injectivity radius $inj(M) \ge \delta > 0$ and an upperbound on the volume $vol(M) \le v$. Then two Riemannian manifolds in $\mathcal{M}(n,\delta,v)$ which are sufficiently nearby in the Gromov-Hausdorff topology are in fact bilipschitz homeomorphic: this the content of the following result due to Gromov[GLP], (see also [G], [Ka]).

Theorem 3.7 (Rigidity Theorem) *Given $\varepsilon > 0$ there is a constant $\eta(n,\delta,v,\varepsilon) > 0$ such that if $d_{GH}(M,N) \le \eta$ for M and N in $\mathcal{M}(n,\delta,v)$, then $d_L(M,N) \le \varepsilon$.*

The proof of this theorem goes back in fact to Cheeger's finiteness Theorem [Che], [Pe1] which now is a consequence of it and of Gromov's precompacness theorem:

Corollary 3.8 (Finiteness Theorem) *Up to diffeomorphism there are only finitely many manifolds in $\mathcal{M}(n,\delta,v)$.*

Another important corollary is the following convergence theorem due to Gromov [GLP], [Pe2]:

Corollary 3.9 (Convergence Theorem) *Every sequence $\{M_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}(n,\delta,v)$ subconverges in the Lipschitz topology to a smooth manifold M with a $\mathcal{C}^0$ metric tensor. Moreover M is diffeomorphic to M_k for k sufficiently large.*

The regularity of the metric tensor on the limit manifold M can be improved to obtain a $\mathcal{C}^{1,1}$-metric tensor on M see [GW], [Pe2], [Pu].

3.2 Pointed topologies

Gromov-Hausdorff or Lipschitz convergences, as defined above, are too restrictive, because one may be interested in sequences X_n with $diam X_n \to \infty$. Such a sequence cannot converge to a compact space in any reasonable sense. For instance, intuitively, a sequence of round 2-spheres of radius n should converge to $\mathbb{E}^2$. But if X_n is obtained by gluing a round 2-sphere of radius n to a round 3-sphere of radius n (the union occurring at a single point), then what should $\lim X_n$ be: $\mathbb{E}^2$ or $\mathbb{E}^3$?

This problem is solved by considering sequences of pointed spaces, i.e. pairs (X, x) where X is a metric space and x is a point of X. This works well when the spaces considered are proper (which means that metric balls are compact.)

Definition 3.10 *Let (X_n, x_n) be a sequence of pointed proper metric spaces and (X, x) be a pointed proper metric space. Then (X_n, x_n) converges to (X, x) for the pointed Gromov-Hausdorff topology if for every $R > 0$*

$$\lim_{n\to\infty} d_{GH}(B(x_n, R), B(x, R)) = 0.$$

If the limit exists, it is unique up to isometry. The next example illustrates the importance of the choice of basepoint in a hyperbolic context.

Example 3.11 *Let M be a noncompact hyperbolic manifold. Set $X_n = M$ and choose $x_n \in M$.*

- *When the sequence x_n stays in a compact subset of M, (X_n, x_n) subconverges to some (X_∞, x_∞) with X_∞ isometric to M.*
- *When x_n goes to infinity in a cusp of maximal rank ($\dim M - 1$), (X_n, x_n) converges to a line. The cusp is a warped product of a compact Euclidean manifold with a line, and the diameter of the Euclidean manifold containing x_n converges to zero as x_n goes to infinity.*
- *When x_n goes to infinity in a geometrically finite end of infinite volume, (X_n, x_n) converges to a hyperbolic space of dimension $\dim M$. This holds true because one can find metric round balls $B_{R_n}(x_n)$ with $R_n \to \infty$.*

Here is the version of Gromov's precompactness criterion for pointed metric spaces:

For a metric space X and for constants $R > \varepsilon > 0$, let $Cov(X, R, \varepsilon)$ denote the maximal number of disjoints balls of radius ε in a ball of radius R in X.

Theorem 3.12 (Precompactness criterion) *A sequence of pointed metric geodesic spaces (X_n, x_n) is precompact for the pointed Hausdorff-Gromov topology if and only if, for every $\varepsilon > 0$ and $R > 0$, $Cov(X_n, R, \varepsilon)$ is bounded on n.*

In an analogous way there is a notion of Lipschitz convergence for pointed proper metric spaces:

Definition 3.13 (Pointed Lipschitz convergence) *A sequence of pointed proper metric spaces* (X_n, x_x) *converges to a proper metric space* (X, x) *for the pointed Lipschitz topology if for every* $R > 0$

$$\lim_{n\to\infty} d_L(B(x_n, R), B(x, R)) = 0.$$

Remark 3.14 *When* $(X_n, x_n) \to (X, x)$ *for the pointed Lipschitz topology, if the limit* X *is compact, then for* n *large enough* X_n *is bilipschitz homeomorphic to* X.

Then one has the following compactness theorem:

Theorem 3.15 (Compactness Theorem) *The set* $\mathcal{M}_\delta$ *of complete Riemannian* n*-manifolds* M *with bounded sectional curvature* $|K_M| \geq 1$ *and lower bound on the injectivity radius* $inj(M) \geq \delta > 0$ *is compact for the bilipschitz topology.*

4 Thick/Thin decomposition

A phenomenon which has received much attention in all dimensions from geometers is the notion of collapse : we say that a family of Riemannian metrics on a manifold collapses with bounded geometry if all the sectional curvatures remain bounded while the injectivity radius goes uniformly everywhere to zero.

Example 4.1 (Berger spheres) *Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf fibration and g the standard metric on $\mathbb{S}^3$. Let g_ε be the metric obtained after rescaling by ε the metric g in the direction tangent to the fibres.*

It means that for a tangent vector $v \in T_x\mathbb{S}^3$, $g_\varepsilon(x)(v,v) = \varepsilon g(x)(v,v)$ if $d\pi_x(v) = 0$, while $g_\varepsilon(x)(v,v) = g(x)(v,v)$ if v is orthogonal to the Hopf fibre. Moreover $\sup_{(0,1]} |K_{g_\varepsilon}| \leq 1$.

If we put on $\mathbb{S}^2$ a Riemannian metric with constant curvature equal to 4, then $\lim_{\varepsilon\to 0} d_{GH}((\mathbb{S}^3, g_\varepsilon), \mathbb{S}^2) = 0$.

This example can be generalized to any isometric locally free S^1-action on a closed Riemannian manifolds. For example any flat torus T^n collapses to any small dimensional torus T^k with $k < n$ by rescaling the metric on some of the S^1 factors. These examplee turn out to be basic.

Cheeger and Gromov [CG1,CG2] have proved that a necessary and sufficient condition for a manifold to have such a collapse with bounded geometry is the existence of a "generalized torus action" which they call an F-structure. F stands for "flat" in this terminology. Intuitively an F-structure corresponds to different tori of varying dimension acting locally on finite coverings of open subsets of the manifold. Certain compatibility conditions on these local actions on intersections of these open subsets will insure that the manifold is partitioned into disjoint orbits of positive dimension.

Definition 4.2 *A pure F-structure $\mathcal{F}$ of positive rank $k > 0$ on a manifold M is a partition of M into compact submanifolds (leaves of variable dimension) which support an affine flat structure. Moreover M has an open covering $\{U_\alpha\}$ such that the partition induced on some regular finite covering $\pi_\alpha : \widetilde{U_\alpha} \to U_\alpha$ coincides with the orbits of a smooth affine action of the k-dimensional torus $\mathbb{T}^k$ on $\widetilde{U_\alpha}$.*

Two pure F-structures $\mathcal{F}_1$ and $\mathcal{F}_2$ are compatible if either $\mathcal{F}_1 \subset \mathcal{F}_2$ (i.e. every leaf of $\mathcal{F}_1$ is an affine submanifold of a leaf of $\mathcal{F}_2$) or $\mathcal{F}_2 \subset \mathcal{F}_1$.

Definition 4.3 *A F-structure $\mathcal{F}$ on a manifold M is an open covering $\{(U_\alpha, \mathcal{F}_\alpha)\}$ of M where $\mathcal{F}_\alpha$ is a pure F-structure on U_α such that $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are compatible on $U_\alpha \cap U_\beta$.*

The rank of $\mathcal{F}$ is the minimum rank of the local F-structures $\mathcal{F}_\alpha$.

A more precise definition of an F-structure can be given using the notion of sheaf of local groups actions.

A compact orientable 3-manifold M with an F-structure admits a partition into orbits which are circles and tori, such that each orbit has a saturated subset. It follows from the definition of F-structure that such a partition corresponds to a *graph structure* on M (see [Ro], [Wa]).

Another description of the family of all graph manifolds is that they are precisely those compact three manifolds which can be obtained, starting with the family of compact geometric *non-hyperbolic* three-manifolds, by the operations of connect sum and of glueing boundary tori together. Thus they arise naturally in both the Geometrization of 3-manifolds and in Riemannian geometry.

The following theorem is a precise version of Cheeger-Gromov's thick/thin decomposition (see [CFG, Thm.1.3 and 1.7] for a proof). We recall that the ε-thin part of a Riemannian n-manifold (M, g) is the set of points $\mathcal{F}(\varepsilon) = \{x \in M \,, inj(x, g) \ < \varepsilon\}$

Theorem 4.4 *For each n, there is a constant μ_n, depending only on the dimension n, such that for any $0 < \varepsilon \leq \mu_n$ and any complete Riemannian n-manifold (M, g) with $|K_g| \leq 1$, there exists a Riemannian metric g_ε on M such that:*

(1) The ε-thin part $\mathcal{F}(\varepsilon)$ of (M, g_ϵ) admits an F-structure compatible with the metric g_ε, whose orbits are all compact tori of dimension ≥ 1 and with diameter $< \varepsilon$.

(2) The Riemannian metric g_ε is ε-quasi-isometric to g and has bounded covariant derivatives of curvature, i.e. it verifies the following properties:

- $e^{-\varepsilon} g_\varepsilon \leq g \leq e^{\varepsilon} g_\varepsilon$.
- $\|\nabla^g - \nabla^{g_\varepsilon}\| \leq \varepsilon$, *where ∇ and ∇^{g_ε} are the Levi-Civita connections of g and g_ε respectively.*
- $\|(\nabla^{g_\varepsilon})^k R_{g_\varepsilon}\| \leq C(n, k, \varepsilon)$, *where the constant C depends only on ε, the dimension n and the order of derivative k.*

Using Cheeger-Gromov's chopping theorem [CG3, Thm.0.1] one can prove the following proposition which is the analogue in bounded curvature of Jørgensen's finiteness theorem [Th, Chap. 5], which states that all complete hyperbolic 3-manifolds of bounded volume can be obtained by surgery on a finite number of cusped hyperbolic 3-manifolds. The finiteness of hyperbolic manifolds with volume bounded above and injectivity radius bounded below is a particular case of Cheeger finteness theorem, while the Margulis lemma takes the place of the Cheeger-Gromov thick/thin decomposition [CG2, Thm.0.1].

Proposition 4.5 *Let M be a closed Riemannian n-manifold with $|K_M| \leq 1$ and $vol(M) \leq v$. Then M has a decomposition $M = N \cup G$ into two compact n-submanifolds such that:*

- *G admits an F-structure such that $\partial N = \partial G$ is an union of orbits.*
- *N belongs, up to diffeomorphism, to a finite set $\mathcal{N}(n, v)$ of smooth, compact, orientable n-manifolds.*

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Laboratoire Émile Picard, CNRS UMR 5580, Université Paul Sabatier, 118 Route de Narbonne, F-31062 Toulouse, France *e.mail address:* boileaupicard.ups-tlse.fr