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Numerical Solution of Nonlinear Stochastic Differential Delay Equation with Markovian Switching

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Abstract This paper is concerned with the Euler-Maruyama approximate solution of nonlinear stochastic delay differential equations with Markovian switching (SDDEwMSs). We establish the existence and uniqueness results for the global solution of SDDEwMSs under the polynomial growth and the local Lipschitz condition. we then introduce Euler-Maruyama approximate solution of this equation, and establish the convergence in probability of the numerical solution to the exact solution of the problem without the linear growth condition. As an application, we also give one example to demonstrate our results.

Keywords: *stochastic differential delay equation, Euler-Maruyama, convergence in probability, markovian switching*

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1. Introduction

Stochastic modelling has come to play an important part in many areas of science and engineering for a long time. Most of stochastic modelling cannot be solved explicitly. As a result, numerical and analytical techniques have been used to study such problems. Many numerical techniques designed to produce approximate solutions in the literature, see for example [2-7] and references therein. It is well known that the local Lipschitz and the linear growth condition are classical conditions in order to guarantee existence and uniqueness of the global solutions (see [1]). In [2,3,4], the authors investigate the convergence and stability of Euler-Maruyama numerical solution under these classical conditions. However, many stochastic equations do not satisfy the linear growth condition. Recently, In [5], the author consider an even more general Khasminskii-type test for nonlinear stochastic delay differential equations (SDDEs) that covers a wide class of highly nonlinear equations, and studied convergence in probability of the Euler-Maruyama solution for SDDEs. For the similar result of highly nonlinear neutral stochastic delay differential equations (NSDDEs) with time-dependent delay, please see Milosevic [6]. Specially, that Zhou and Fang [7] established new criteria of the existence-and-uniqueness of the global solution and the convergence in probability of Euler-Maruyama approximate solution for nonlinear NSFDEs under the polynomial growth conditions.

On the other hand, we also remark that a great deal of reseach for the stochastic differential equations (SDEs) are successfully extended to the stochastic differential equations with Markovian switching (SDEwMSs)and the stochastic delay differential equations with Markovian switching (SDDEwMSs) (see [8,9,10,11]). In [9], the authors introduce the Euler-Maruyama(EM) numerical solution which strong converge to the actual solution under the global Lipschitz condition, and the same problem have been discussed under the local Lipschitz condition and the linear growth condition, and furthermore describe the convergence in probability, instead of L^2, under some additional conditions in terms of Lyapunov-type functions. In [12], the numerical solution for NSDDEwMSs are discussed. And in [13], the authors consider the strong convergence in the sense of the L^p-norm when the drift and diffusion coefficients are Taylor approximations. However, to the best of our knowledge, few papers can be found in the literature on the numerical methods for nonlinear SDDEwMSs. So, being directly inspired by [7], the purpose of this paper to study the Numerical solution of nonlinear stochastic differential delay equation with Markovian switching.

The paper is organized as follows: Some necessary notations and the property of right-continuous Markov chain are in Section 2. In Section 3, we prove the existence and uniqueness of the solution of SDDEwMSs under the polynomial growth. In Section 4, the Euler-Maruyama approximate solution for SDDEwMSs are obtained, and establish the convergence in probability. Finally, in Section 5, we give one example to demonstrate our results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), ..., w_m(t))^T$ be an m-dimensional Brownian motion, and let $r(t)$, $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$. Set $|x|$ be the Euclidean norm in $x \in R^d$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. For $\tau > 0$, we shall denote by $C([-\tau, 0]; R^d)$ the family of continuous functions φ from $[-\tau, 0]$ to R^d with the norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Let $p > 0, \tau > 0$, denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; R^d)$ the family of all $\mathcal{F}_t$ -measurable and $C([-\tau, 0]; R^d)$ -valued random variables ξ such that $E\|\xi\|^p < +\infty$.

Set $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with the generator $\Gamma = (\gamma_{ij})_{(N \times N)}$ given by

$$p\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & if\ i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta), & if\ i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} > 0$ is transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$.

We assume that the Markov chain $r(t)$ is independent of the Brownian motion $w(t)$. It is well know that almost every sample path of $r(t)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $R_+ = [0,\infty)$. Moreover, for convenience, denoted by $y(t) = x(t - \tau)$.

Consider the d-dimensional Euler-Maruyama (EM) numerical solutions of nonlinear stochastic differential delay equation with Markovian switching (SDDEwMSs)

$$\begin{aligned} dx(t) = {} & f(x(t), x(t-\tau), r(t), t)dt \\ & + g(x(t), x(t-\tau), r(t), t)dw(t), \ t \geq 0, \end{aligned} \tag{2.1}$$

with the initial data $\xi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^d)$. Here

$$f : R^d \times R^d \times S \times R_+ \to R^d,$$
$$g : R^d \times R^d \times S \times R_+ \to R^{d \times m}.$$

Assumption (H_1). (Local Lipschitz Condition) For each integer $R \geq 1$ and $i = 1, 2, ..., N$, there exists a positive constant L_R, such that

$$\begin{aligned} & \left| \begin{matrix} f(x_1, y_1, i, t) \\ -f(x_2, y_2, i, t) \end{matrix} \right|^2 \vee \left| \begin{matrix} g(x_1, y_1, i, t) \\ -g(x_2, y_2, i, t) \end{matrix} \right|^2 \\ & \leq L_R \left(|x_1 - x_2|^2 + |y_1 - y_2|^2 \right), \end{aligned} \tag{2.2}$$

for $x_1, x_2, y_1, y_2 \in R^d$, with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$.

Remark 2.1. According to assumption (H_1), it is easy to obtain that

$$\begin{aligned} & |f(x, y, i, t)|^2 \\ & \leq 2|f(x, y, i, t) - f(0, 0, i, t)|^2 + 2|f(0, 0, i, t)|^2 \\ & \leq 2L_R\left(|x|^2 + |y|^2\right) + 2|f(0, 0, i, t)|^2 \\ & \leq K_R\left(1 + |x|^2 + |y|^2\right). \end{aligned} \tag{2.3}$$

Similarly, $|g(x, y, i, t)|^2 \leq K_R(1 + |x|^2 + |y|^2)$. Here, $K_R = 2\left(L_R \vee |f(0, 0, i, t)|^2 \vee |g(0, 0, i, t)|^2\right)$.

Assumption (H_2). (Polynomial Growth Condition) Assume that for some positive integer L, there exist positive constants a_1, a_2, a_3, b_1, b_2, b_3, α, β, such that

$$\begin{aligned} & x^T f(x, y, i, t) \leq -a_1|x|^2 - a_2|x|^{\alpha+2} + a_3|y|^{\beta+2}, \\ & |g(x, y, i, t)|^2 \leq b_1|x|^2 + b_2|x|^{\beta+2} + b_3|y|^{\beta+2}. \end{aligned} \tag{2.4}$$

Let $C^{2,1}(R^d \times S \times R_+; R_+)$ denote the family of all nonnegative functions $V(x, i, t)$ on $R^d \times S \times R_+$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^d \times S \times R_+; R_+)$, define an operator LV from $R^d \times R^d \times S \times R_+$ to R by

$$\begin{aligned} & LV(x(t), y(t), r(t), t) = V_t(x(t), r(t), t) \\ & + V_x(x(t), r(t), t) f(x(t), y(t), r(t), t) \\ & + \frac{1}{2} trace \begin{bmatrix} g^T(x(t), y(t), r(t), t) \\ V_{xx}(x(t), r(t), t) \\ g(x(t), y(t), r(t), t) \end{bmatrix} \\ & + \sum_j \gamma_{r(t)j} V(x(t), j), \end{aligned} \tag{2.5}$$

where

$$V_t = \frac{\partial V(x, i, t)}{\partial t},$$
$$V_x = \left(\frac{\partial V(x, i, t)}{\partial x_1}, \frac{\partial V(x, i, t)}{\partial x_2}, ..., \frac{\partial V(x, i, t)}{\partial x_d} \right)$$
$$V_{xx} = \left(\frac{\partial^2 V(x, i, t)}{\partial x_i \partial x_j} \right)_{d \times d}.$$

In particular, if V is independent of i, that is $V(x, i, t) = V(x, t)$, then

$$LV(x,y,i,t)=V_t(x,t)+V_x(x,t)f(x,y,i,t) +trace\left[g^T(x,y,i,t)V_{xx}(x,t)g(x,y,i,t)\right] \quad (2.6)$$

since $\sum_{j=1}^{N}\gamma_{ij}=0$.

Let by the generalized *Itô* formula, we obtain:

$$EV(x(\tau_2),r(\tau_2),\tau_2)=EV(x(\tau_1),r(\tau_1),\tau_1) +E\int_{\tau_1}^{\tau_2}LV(x(s),y(s),r(s),s)ds,$$

holds for any stopping times $0\le\tau_1\le\tau_2<\infty$ as long as the integrations involved exist and are finite.

3. Global Solution of SDDEwMSs

In this sections, we prove the existence and uniqueness of the solution of SDDEwMSs under the polynomial growth.

Theorem 3.1. Let Assumptions (H_1) and (H_2) hold, and assume that $T>0$, $p\ge1, a_1>\frac{p-1}{2}b_1$, $\alpha\ge\beta$, $a_2>a_3+\frac{p-1}{2}(b_2+b_3)$, then, for any initial condition $\xi\in L^p_{\mathcal{F}_t}([-\tau,0];R^d)$, there almost surely exists a unique global solution $x(t,\xi)$ to equation (2.1) on $t\ge-\tau$. Moreover, there exists a positive constant $\tilde{M}$ such that $E|x|^p\le\tilde{M}$.

Proof. Bearing in mind the local Lipschitz condition (2.2), it follows that for any given initial data $\xi\in L^p_{\mathcal{F}_t}([-\tau,0];R^d)$, there exists a unique maximal local solution $\{x(t),t\in[-\tau,\tau_e]\}$ to Eqs. (2.1), where τ_e is the explosion time. To show this solution is global, we only need to show that $\tau_e=\infty$ a.s.

Assume that there exists an integer k_0 such that $\max_{\theta\in[-\tau,0]}|\xi(\theta)|<k_0$. For each integer $k>k_0$, define the stopping time

$$\tau_k=inf\{t\in[0,\tau_e]:|x(t)|>k\}, \quad (3.1)$$

and $inf\,\phi=\infty$ (as usual, ϕ =the empty set). Define $\tau_\infty=\lim_{k\to\infty}\tau_k$, it is obvious that τ_∞ is an increasing function with k, so $\tau_\infty\le\tau_e$ a.s. Our goal is to prove that $\tau_\infty=\infty$ a.s, which implies that $\tau_e=\infty$. In other words, we only prove that $P(\tau_k\le t)\to(k\to\infty,t>0)$.

Define $V(x, i, t) = |x|_p$, we obtain

$$\begin{aligned}&LV(x(t),y(t),r(t),t)\\&=LV(x(t),y(t))\\&=p|x(t)|^{p-2}x^Tf(x(t),y(t),r(t),t)\\&+\frac{p(p-1)}{2}|x(t)|^{p-2}|x(t),y(t),r(t),t|^2.\end{aligned} \quad (3.2)$$

Using the Assumption (H_2) and the inequality $|x|^{p-2}|y|^{\beta+2}\le\frac{p-2}{\beta+p}|x|^{\beta+p}+\frac{\beta+2}{\beta+p}|y|^{\beta+p}$, we can obtain

$$\begin{aligned}&LV(x(t),y(t),r(t),t)\\&\le p|x(t)|^{p-2}\left[-a_1|x|^2-a_2|x|^{\alpha+2}+a_3|y|^{\beta+2}\right]\\&+\frac{p(p-1)}{2}|x(t)|^{p-2}\left[b_1|x|^2+b_2|x|^{\beta+2}+b_3|y|^{\beta+2}\right]\\&\le p\left(a_3+\frac{p-1}{2}b_3\right)\frac{\beta+2}{\beta+p}\left(|y|^{\beta+p}-|x|^{\beta+p}\right)-F(x),\end{aligned} \quad (3.3)$$

where

$$F(x)=p\left(a_1-\frac{p-1}{2}b_1\right)|x|^p+pa|x|^{\alpha+p} -p\left(a_3+\frac{p-1}{2}(b_2+b_3)\right)|x|^{\beta+p}.$$

Recalled that conditions of this theorem, $a_1>\frac{p-1}{2}b_1$, $\alpha\ge\beta$, $a_2>a_3+\frac{p-1}{2}(b_2+b_3)$, there exists a positive constant c_0, such that

$$F(x)\ge c_0|x|^p. \quad (3.4)$$

Substituting for this an (3.4) into (3.3), implies

$$\begin{aligned}&V(x(t),r(t),t)\\&=V(\xi(0),i_0)+\int_0^tLV(x(s),x(s-\tau),r(s),s)ds\\&+\int_0^tV_x(x(s),r(s),s)g(x(s),x(s-\tau),r(s),s)ds\\&\le V(\xi(0),i_0)\\&+p\left(a_3+\frac{p-1}{2}b_3\right)\frac{\beta+2}{\beta+p}\int_0^t\left(|x(s-\tau)|^{\beta+p}-|x(s)|^{\beta+p}\right)ds\\&-\int_0^tc_0|x(s)|^pds\\&+\int_0^tV_x(x(s),r(s),s)g(x(s),x(s-\tau),r(s),s)dw(s).\end{aligned}$$

Therefore

$$\begin{aligned}&EV(x(t),r(t),t)\\&=EV(\xi(0),i_0)+E\int_0^tLV(x(s),x(s-\tau),r(s),s)ds\\&\le EV(\xi(0),i_0)\\&+p\left(a_3+\frac{p-1}{2}b_3\right)\frac{\beta+2}{\beta+p}E\int_{-\tau}^0|x(s)|^{\beta+p}ds\\&-E\int_0^tc_0|x(s)|^pds.\end{aligned}$$

The Gronwall inequality implies

$$EV(x(t),r(t),t)=E|x|^p\le\tilde{M}.$$

Similar, we have $EV(x(t\wedge\tau_k),r(t\wedge\tau_k),t\wedge\tau_k)$ $=E|x(t\wedge\tau_k)|^p\le\tilde{M}_0$. By the definition of τ_k, we have that

$$p(\tau_k \le t)k^p \le p(\tau_k \le t)V(x(\tau_k),r(\tau_k),\tau_k)$$
$$\le EV(x(t\wedge\tau_k),r(t\wedge\tau_k),t\wedge\tau_k) \le \tilde{M}_0.$$

Clearly, we have that $P(\tau_k \le t) \to 0(k\to\infty, t>0)$.

4. Euler-Maruyama Method

In this section, we define the Euler-Maruyama approximate solution.

Lemma 4.1. ([12]) Given $r_k^\Delta = r(k\Delta)$ for $k \ge 0$ and $\Delta > 0$, then $\{r_k^\Delta, k=0,1,2,...\}$ is a discrete Markov chain with the one-step transition probability matrix

$$P(\Delta) = (P_{ij}(\Delta))_{N\times N} = e^{\Delta\tau}.$$

Since the γ_{ij} are independent of x, the paths of r can be generated independently of x and in fact, before computing x.

Let a stepsize $h\in(0,1)$, with satisfies $\tau = Mh$ for some positive integer M. Define $t_k = kh$ for $k=-M,-(M-1),...,0,1,2,...,$ the discrete markovian chain $\{r_k^h, k=0,1,2,...\}$ can be simulated as follows: Compute the one-step transition probability matrix $P(h)$.

Let $r_0^h = i_0$ and generate a random number $\zeta 1$ which is uniformly distributed in [0, 1]. Define

$$r_1^h = \begin{cases} N, \ if \ \sum_{j=1}^{N-1} P_{i_0,j}(h) \le \zeta_1, \\ i_1, \ if \ i_1 \in S-\{N\} \ such \ that \\ \qquad \sum_{j=1}^{i_1-1} P_{i_0,j}(h) \le \zeta_1 < \sum_{j=1}^{i_1} P_{i_0,j}(h), \end{cases}$$

where we set $\sum_{j=1}^{0} P_{i_0,j}(h) = 0$ as usual. Generate independently a new random number ζ_2 which is again uniformly distributed in [0, 1] and then define

$$r_2^h = \begin{cases} N, \ if \ \sum_{j=1}^{N-1} P_{r_1^h,j}(h) \le \zeta_2, \\ i_2, \ if \ i_2 \in S-\{N\} \ such \ that \\ \qquad \sum_{j=1}^{i_2-1} P_{r_1^h,j}(h) \le \zeta_2 < \sum_{j=1}^{i_2} P_{r_1^h,j}(h). \end{cases}$$

Repeating this procedure, a trajectory of $\{r_k^h, k=0,1,2,...\}$ can be generated. This procedure can be carried out independently to obtain more trajectory.

After explaining how to simulate the discrete Markov chain $\{r_k^h, k=0,1,2,...\}$, the Euler-Maruyama numerical scheme applied to the Eqs.(2.1) is to compute the discrete approximations $X_k \approx x(t_k)$ by setting $X_k = \xi(kh)$ for $k=-M,-(M-1),...,0$ and forming

$$X_{k+1} = X_k + f(X_k, X_{k-M}, r_k^h, t_k)h + g(X_k, X_{k-M}, r_k^h, t_k)\Delta w_k \quad for \ k=1,2,..., \tag{4.1}$$

where $\Delta w_k = w(t_{k+1}) + w(t_k)$.

For each $k>0$, define $\bar{X}(t) = X_k$, $\bar{X}(t-\tau) = X_{k-M}$, $\bar{r}(t) = r_k^h$ with the initial value $\bar{X}_0 = \xi_0$ on $[-\tau,0]$. That is

$$\bar{X}(t) = \sum_{k=-M}^{\infty} X_k I_{[kh,(k+1)h]}(t), \tag{4.2}$$

while the continuous-time Euler-Maruyama approximation process $X(t)$ on $t\in[-\tau,\infty]$ is to be interpreted as the stochastic integral.

$$X(t) = \begin{cases} \xi, \ if \ [-\tau,0], \\ \xi(0) + \int_0^t f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)ds \\ \quad + \int_0^t g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)dw(s), \\ \qquad if \ t\in[0,T]. \end{cases} \tag{4.3}$$

Therefore

$$X(t) = X(kh) + \int_{kh}^t f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)ds + \int_{kh}^t g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)dw(s), \ t\ge 0. \tag{4.4}$$

It is useful to know that $X(t_k) = X_k = \bar{X}(t_k)$. that is, $X(t)$ and $\bar{X}(t)$ concide with the discrete approximate solution at the gridpoints. For convenience, let $T > 0$ be arbitrary and define the sequence of stopping times

$$\sigma_R = \inf\{t\ge 0: |X(t)| > R\}, \ \tau_R = \inf\{t \ge 0: |x(t)| > R\},$$

and $\mu_R = \sigma_R \wedge \tau_R$.

For convenience, let C be a positive constant independent of h, and the product of C and other constants is still denoted by C.

Lemma 4.2 Under Assumptions (H_1),

$$E\left[\sup_{-\tau\le t\le T} |X(t\wedge\mu_R)|^2\right] \le C.$$

Proof. Recalling the Lemma 3.1 in [7]. By (4.3) we have

$$E\left[\sup_{0\le t\le T} |X(t\wedge\mu_R)|^2\right]$$
$$= E\left[\sup_{0\le t\le T}\left|\begin{matrix}\xi(0) \\ +\int_0^{t\wedge\mu_R} f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)ds \\ +\int_0^{t\wedge\mu_R} g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)dw(s)\end{matrix}\right|^2\right]$$
$$\le 3E|\xi(0)|^2 \tag{4.5}$$
$$+3E\left[\sup_{0\le t\le T}\left|\int_0^{t\wedge\mu_R} f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)ds\right|^2\right]$$
$$+3E\left[\sup_{0\le t\le T}\left|\int_0^{t\wedge\mu_R} g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)dw(s)\right|^2\right].$$

Using the Holder inequality and (2.3)

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}\left|\int_0^{t\wedge\mu_R} f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)ds\right|^2\right] \\
&\le TE\left[\sup_{0\le t\le T}\left|\int_0^{t\wedge\mu_R} f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)ds\right|^2\right] \\
&\le TE\left[\sup_{0\le t\le T}\int_0^{t\wedge\mu_R} K_R\left(1+\left|\bar{X}(s)\right|^2,+\left|\bar{X}(s-\tau)\right|^2\right)ds\right] \\
&\le TK_R\left(T+2\int_0^T E\left[\sup_{-\tau\le\eta\le s\wedge\mu_R}\left|X(\eta)\right|^2\right]ds\right).
\end{aligned} \tag{4.6}$$

By the BDG inequality

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}\left|\int_0^{t\wedge\mu_R} g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)dw(s)\right|^2\right] \\
&\le 4E\left[\left|\int_0^{t\wedge\mu_R} g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)dw(s)\right|^2\right] \\
&\le 4E\left[\int_0^{t\wedge\mu_R} K_R\left(1+\left|\bar{X}(s)\right|^2,+\left|\bar{X}(s-\tau)\right|^2\right)ds\right] \\
&\le 4K_R\left(T+2\int_0^T E\left[\sup_{-\tau\le\eta\le s\wedge\mu_R}\left|X(\eta)\right|^2\right]ds\right).
\end{aligned} \tag{4.7}$$

Using (4.7)and (4.6), the estimate becomes

$$\begin{aligned}
&E\left[\sup_{-\tau\le t\le T}\left|X(t\wedge\mu_R)\right|^2\right] \\
&\le 4E\|\xi\|^2+3K_RT(T+4) \\
&+6K_4(T+4)\int_0^T E\left[\sup_{-\tau\le\eta\le s\wedge\mu_R}\left|X(\eta)\right|^2\right]ds.
\end{aligned}$$

The Gronwall inequality gives

$$E\left[\sup_{-\tau\le t\le T}\left|X(t\wedge\mu_R)\right|^2\right]\le C.$$

Lemma 4.3 Let Assumption (H_1) hold, for any $t\in[0,T]$, there exist a positive constant C independent of h, such that

$$\int_0^{t\wedge\mu_R} E\left|X(s)-\bar{X}(s)\right|^2 ds\le Ch.$$

Proof. We have by definition of $X(t)$ and $\bar{X}(t)$, thus

$$\begin{aligned}
&\left|X(s)-\bar{X}(s)\right|^2 \\
&\le\left|\begin{matrix}\int_{kh}^t f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)ds \\ +\int_{kh}^t g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)dw(s)\end{matrix}\right|^2 \\
&\le 2\left|f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\right|^2(t-t_k)^2 \\
&+2\left|g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\right|^2\left(w(t)-w(t_k)\right)^2 \\
&\le 2K_R\left(1+\left|\bar{X}(s)\right|^2,+\left|\bar{X}(s-\tau)\right|^2\right)\left(h^2+\left|w(t)-w(t_k)\right|^2\right).
\end{aligned}$$

Then by Lemma 4.2, it is easy to obtain that $E\left[\sup_{-\tau\le t\le T\wedge\mu_R}\left|\bar{X}(t)\right|^2\right]\le C$, thus

$$\begin{aligned}
&\int_0^{t\wedge\mu_R} E\left|X(s)-\bar{X}(s)\right|^2 ds \\
&\le 2K_R\int_0^{t\wedge\mu_R} E\begin{pmatrix}1+\left|\bar{X}(s)\right|^2 \\ +\left|\bar{X}(s-\tau)\right|^2\end{pmatrix}\left(h^2+mh\right)ds \\
&\le 2K_RT(1+2C)\left(h^2+mh\right)\le C.
\end{aligned}$$

Lemma 4.4 Under Assumption (H_1) and (H_2), for any $\epsilon\in(0,1)$ and $T>0$ there exists a sufficiently large $R^*=R(\epsilon,T)$ and sufficiently small h^* such that

$$P\{\sigma_R\le T\}\le\epsilon, R\ge R^*, h<h^*.$$

Proof. Recalling that (3.2), the proof of this lemma is similar to the argument of Theorem 3.5 in [7]. Applying the generalized *Itô* formula to $V(X(t),r(t),t)=|x(t)|^p$ yields.

$$\begin{aligned}
&dV(X(t),r(t),t) \\
&=\left[\begin{matrix}V_x(X(t),r(t),t)f\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\ +\frac{1}{2}trace\left[\begin{matrix}g^T\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\ V_{xx}(X(t),r(t),t) \\ g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right)\end{matrix}\right] \\ +\sum_j\gamma_{r(t)j}V(X(t),j,t)\end{matrix}\right]dt \\
&+V_x(X(t),r(t),t)g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right)dw(s) \\
&=LV\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right)dt \\
&+V_x(X(t),r(t),t)g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right)dw(s) \\
&+H\left(X(t),\bar{X}(t),X(t-\tau),\bar{X}(t-\tau),r(t),\bar{r}(t),t\right)dt,
\end{aligned}$$

where

$$\begin{aligned}
&H\left(X(t),\bar{X}(t),X(t-\tau),\bar{X}(t-\tau),\bar{r}(t),t\right) \\
&=P|X(t)|^{p-2}X^T(t)\left[\begin{matrix}f\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\ -f\left(X(t),X(t-\tau),r(t),t\right)\end{matrix}\right] \\
&+\frac{p(p-2)}{2}|X(t)|^{p-2}\left[\begin{matrix}g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\ -g\left(X(t),X(t-\tau),r(t),t\right)\end{matrix}\right] \\
&\quad\times g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\
&+\frac{p(p-2)}{2}|X(t)|^{p-2}g\left(X(t),X(t-\tau),r(t),t\right) \\
&\quad\times\left[\begin{matrix}g\left(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t\right) \\ -g\left(X(t),X(t-\tau),\bar{r}(t),t\right)\end{matrix}\right]^T.
\end{aligned}$$

Let R be sufficiently large integer, if $|X(t)| \vee |X(t-\tau)| \vee |\bar{X}(t)| \vee |\bar{X}(t-\tau)| \le R$, then, the Assumption ($H_1$) implies there exists constant C which depends on R such that

$$\begin{aligned} &dV(X(t),r(t),t) \\ &\le LV(X(t),X(t-\tau),r(t),t) \\ &+V_x(X(t),r(t),t)g(\bar{X}(t),\bar{X}(t-\tau),\bar{r}(t),t)dw(s) \\ &+C\left[|X(t)-\bar{X}(t)|+|X(t-\tau)-\bar{X}(t-\tau)|\right]dt. \end{aligned}$$

Hence

$$\begin{aligned} &EV(X(t),r(t),t) \le EV(\xi(0),i_0) \\ &+E\int_0^t LV(X(s),X(s-\tau),r(s),s)ds \\ &+CE\int_0^t\left[|X(s)-\bar{X}(s)|+|X(s-\tau)-\bar{X}(s-\tau)|\right]ds. \end{aligned} \tag{4.8}$$

By the Lemma 4.2 and $s \in [0, t\wedge\sigma_R]$, we have that

$$\begin{aligned} &E|X(s)-\bar{X}(s)| \\ &\le E\left|\begin{matrix} f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)(s-t_k) \\ +g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)(w(s)-w(t_k)) \end{matrix}\right| \\ &\le \left[E\left|f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)\right|^2\right]^{\frac{1}{2}} h \\ &+\left[E\left|g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)\right|^2\right]^{\frac{1}{2}} \\ &\times\left[E(w(s)-w(t_k))^2\right]^{\frac{1}{2}} \\ &\le [K_R(1+2C)]^{\frac{1}{2}}h+[K_R(1+2C)]^{\frac{1}{2}}[mh]^{\frac{1}{2}} \\ &\le Ch^{\frac{1}{2}}. \end{aligned} \tag{4.9}$$

Recalling the proof of Theorem 3.1,

$$\begin{aligned} &EV(X(t\wedge\sigma_R),r(t\wedge\sigma_R),t\wedge\sigma_R) \\ &\le EV(\xi(0),i_0)+\tilde{M}_0 \\ &+CE\int_0^{t\wedge\sigma_R}\left[\begin{matrix}|X(s)-\bar{X}(s)| \\ +|X((s-\tau))-\bar{X}((s-\tau))|\end{matrix}\right]ds \\ &\le EV(\xi(0),i_0)+\tilde{M}_0+2CTh^{\frac{1}{2}}, \end{aligned}$$

here $\tilde{M}_0$ be a positive constant. Repeating the procedure from Theorem 3.1, we can prove that $P(\sigma_R < t) \to 0 (R\to\infty, T>0)$, which completes the proof.

Lemma 4.5 [12] Let the Assumption (H_1) hold, for every $t \in [0, T\wedge\mu_R]$, we have

$$\begin{aligned} &E\int_0^{t\wedge\sigma_R}\left|\begin{matrix} f(\bar{X}(s),\bar{X}(s-\tau),r(s),s) \\ -f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)\end{matrix}\right|^2 ds \\ &\le Ch+o(h), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &E\int_0^{t\wedge\sigma_R}\left|\begin{matrix} g(\bar{X}(s),\bar{X}(s-\tau),r(s),s) \\ -g(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)\end{matrix}\right|^2 ds \\ &\le Ch+o(h), \end{aligned} \tag{4.11}$$

C is a positive constant dependent on $\max_{0\le i\le N}(-\gamma_{ii})$, but independent of h.

Lemma 4.6 Under the condition of Theorem 3.1, the numerical solution convergence to the exact solution of Eqs.(2.1) in the sense

$$\lim_{h\to 0} E\left[\sup_{0\le t\le T}|x(t\wedge\mu_R)-X(t\wedge\mu_R)|^2\right]=0.$$

Proof. From Eqs.(2.1) and Eqs.(4.3), we have

$$\begin{aligned} &x(t\wedge\mu_R)-\xi(0) \\ &=\int_0^{t\wedge\mu_R} f(X(s),X(s-\tau),r(s),s)ds \\ &+\int_0^{t\wedge\mu_R} g(x(s),x(s-\tau),r(s),s)dw(s), \\ &X(t\wedge\mu_R)-\xi(0) \\ &=\int_0^{t\wedge\mu_R} f(\bar{X}(s),\bar{X}(s-\tau),r(s),s)ds \\ &+\int_0^{t\wedge\mu_R} g(\bar{X}(s),\bar{X}(s-\tau),r(s),s)dw(s). \end{aligned}$$

By the inequality $(a+b)^2 \le 2a^2+2b^2$, for any $t_1 \in [0,T]$,

$$\begin{aligned} &E\left[\sup_{0\le t\le t_1}|x(t\wedge\mu_R)-X(t\wedge\mu_R)|^2\right] \\ &\le 2E\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\mu_R}\left[f\begin{pmatrix}x(s),x(s-\tau), \\ r(s),s\end{pmatrix} - f\begin{pmatrix}\bar{X}(s),\bar{X}(s-\tau), \\ \bar{r}(s),s\end{pmatrix}\right]ds\right|^2\right] \\ &+2E\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\mu_R}\left[g\begin{pmatrix}x(s),x(s-\tau), \\ r(s),s\end{pmatrix} - g\begin{pmatrix}\bar{X}(s),\bar{X}(s-\tau), \\ \bar{r}(s),s\end{pmatrix}\right]dw(s)\right|^2\right]. \end{aligned} \tag{4.12}$$

By the Holder inequality, Lemma 4.3, Assumption (H_1), and (4.10), we have

$$\begin{aligned} &E\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\mu_R}\left[\begin{matrix} f(x(s),x(s-\tau),r(s),s) \\ -f(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s)\end{matrix}\right]ds\right|^2\right] \\ &\le E\sup_{0\le t\le t_1}\left[\int_0^{t\wedge\mu_R}ds\int_0^{t\wedge\mu_R}\left|\begin{matrix} f\begin{pmatrix}x(s),x(s-\tau) \\ ,r(s),s\end{pmatrix} \\ -f\begin{pmatrix}\bar{X}(s),\bar{X}(s-\tau), \\ \bar{r}(s),s\end{pmatrix}\end{matrix}\right|^2 ds\right] \end{aligned}$$

$$\le TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(x(s),x(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 2TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(x(s),x(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$+2TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 2TE\int_0^{t_1\wedge\mu_R}\left[\begin{matrix} \left|x(s)-\bar{X}(s)\right|^2 \\ +\left|x(s-\tau)-\bar{X}(s-\tau)\right|^2\end{matrix}\right] ds$$

$$+2TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 2TL_RE\int_0^{t_1\wedge\mu_R}\left[\begin{matrix} 2\left|x(s)-X(s)\right|^2 \\ +2\left|X(s)-\bar{X}(s)\right|^2 \\ +2\left|x(s-\tau)-X(s-\tau)\right|^2 \\ +2\left|X(s-\tau)-\bar{X}(s-\tau)\right|^2\end{matrix}\right] ds$$

$$+2TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 8TL_RE\int_0^{t_1\wedge\mu_R}\left|x(s)-X(s)\right|^2+\left|X(s)-\bar{X}(s)\right|^2 ds$$

$$+2TE\int_0^{t_1\wedge\mu_R}\left|\begin{matrix} f\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds \qquad (4.13)$$

$$\le 8TL_R\int_0^T E\left[\sup_{0\le\eta\le s}\left|x(\eta\wedge\mu_R)-X(\eta\wedge\mu_R)\right|^2\right]ds$$

$$+8TL_RCh+2T\left(Ch+o(h)\right).$$

Similarly, by the Burkholder-Davis-Gundy inequality , Lemma 4.3 and (4.11), we may obtain

$$E\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\mu_R}\begin{pmatrix} f\left(x(s),x(s-\tau),r(s),s\right) \\ -f\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{pmatrix}dw(s)\right|^2\right]$$

$$\le 4\int_0^{t_1\wedge\mu_R}E\left|\begin{matrix} g\left(x(s),x(s-\tau),r(s),s\right) \\ -g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 8\int_0^{t_1\wedge\mu_R}\left[\begin{matrix} E\left|\begin{matrix} g\left(x(s),x(s-\tau),r(s),s\right) \\ -g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 \\ +E\left|\begin{matrix} g\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2\end{matrix}\right] ds$$

$$\le 8L_R\int_0^{t_1\wedge\mu_R}E\left|x(s)-\bar{X}(s)\right|^2+E\left|\begin{matrix} x(s-\tau) \\ -\bar{X}(s-\tau)\end{matrix}\right|^2 ds$$

$$+8\int_0^{t_1\wedge\mu_R}E\left|\begin{matrix} g\left(\bar{x}(s),\bar{x}(s-\tau),r(s),s\right) \\ -g\left(\bar{X}(s),\bar{X}(s-\tau),\bar{r}(s),s\right)\end{matrix}\right|^2 ds$$

$$\le 8L_R\int_0^{t_1\wedge\mu_R}E\begin{pmatrix} 2\left|x(s)-X(s)\right|^2 \\ +2\left|X(s)-\bar{X}(s)\right|^2 \\ +2\left|x(s-\tau)-X(s-\tau)\right| \\ +2\left|X(s-\tau)-\bar{X}(s-\tau)\right|\end{pmatrix} ds$$

$$+8\left(Ch+o(h)\right)$$

$$\le 32L_R\int_0^{t_1\wedge\mu_R}\left[\begin{matrix} E\left|x(s)-X(s)\right|^2 \\ +E\left|X(s)-\bar{X}(s)\right|^2\end{matrix}\right] ds$$

$$+8\left(Ch+o(h)\right)$$

$$\le 32L_R\int_0^T E\left[\sup_{0\le\eta\le s}\left|x(\eta\wedge\mu_R)-X(\eta\wedge\mu_R)\right|^2\right]ds \qquad (4.14)$$

$$+32L_RCh+8\left(Ch+o(h)\right).$$

Substituting (4.13) − (4.14) into (4.12), yields

$$E\left[\sup_{0\le t\le t_1}\left|x(t\wedge\mu_R)-X(t\wedge\mu_R)\right|^2\right]$$

$$\le 2\left\{\begin{matrix} 8TL_R\int_0^T E\left[\sup_{0\le\eta\le s}\left|x(\eta\wedge\mu_R)-X(\eta\wedge\mu_R)\right|^2\right]ds \\ +8TL_RCh+2T\left(Ch+o(h)\right)\end{matrix}\right\}$$

$$+2\left\{\begin{matrix} 32L_R\int_0^T E\left[\sup_{0\le\eta\le s}\left|x(\eta\wedge\mu_R)-X(\eta\wedge\mu_R)\right|^2\right]ds \\ +32L_RCh+8\left(Ch+o(h)\right)\end{matrix}\right\}$$

$$\le 16(T+4)L_R\int_0^T E\left[\sup_{0\le t\le t_1}\left|x(T\wedge\mu_R)-X(T\wedge\mu_R)\right|^2\right]ds$$

$$+8L_RCh(T+4)+\left(Ch+o(h)\right)(2T+8).$$

The Gronwall inequality implies

$$E\left[\sup_{0\le t\le t_1}\left|x(t\wedge\mu_R)-X(t\wedge\mu_R)\right|^2\right]$$

$$\le\left[8L_RCh(T+4)+\left(Ch+o(h)\right)(2T+8)\right]e^{16(T+4)L_4T},$$

that is,

$$\lim_{h\to 0}E\left[\sup_{0\le t\le t_1}\left|x(\eta\wedge\mu_R)-X(\eta\wedge\mu_R)\right|^2\right]=0.$$

The proof is completed.

Theorem 4.7. Under Assumption ($H1$), ($H2$), for arbitrary $T > 0$.

$$\lim_{h\to 0}\sup_{t\in[0,T]}\left|x(t)-X(t)\right|=0 \text{ in probability.}$$

Proof. For arbitrary $\epsilon \in (0, 1)$, we define

$$B=\left\{\omega:\ \sup_{t\in[0,T]}\left|x(t)-X(t)\right|\geq\varepsilon\right\}.$$

If we can show that $P(B) \leq \epsilon$, then the Euler-Maruyama approximate solution converges to the exact solution of Eqs. (2.1).

Recalling the proof of Theorem 3.1 and Lemma 3.4, there exists a sufficient large $R^* = R(\epsilon, T)$ and sufficiently small $\tilde{h}$ such that

$$P\left(\tau_{R^*}\leq T\right)\leq\epsilon/3, P\left(\sigma_{R^*}\leq T\right)\leq\epsilon/3.$$

By Lemma 4.6, for the sufficiently small h, we have

$$\begin{aligned}
&\varepsilon^2 P\left\{B\cap\left\{\mu_{R^*}>T\right\}\right\}\\
&\leq E\left(I_{\mu_{R^*}>R}\sup_{-\tau\leq t\leq T}\left|x\left(t\wedge\mu_{R^*}\right)-X\left(t\wedge\mu_{R^*}\right)\right|^2\right)\\
&\leq E\left[\sup_{-\tau\leq t\leq T}\left|x\left(t\wedge\mu_{R^*}\right)-X\left(t\wedge\mu_{R^*}\right)\right|^2\right]\leq\epsilon/3.
\end{aligned}$$

Therefore

$$\begin{aligned}
&P(B)\leq P\left\{B\cap\left\{\mu_{R^*}>T\right\}\right\}+P\left\{\mu_{R^*}\leq T\right\}\\
&\leq P\left\{B\cap\left\{\mu_{R^*}>T\right\}\right\}+P\left\{\tau_{R^*}\leq T\right\}+P\left\{\sigma_{R^*}\leq T\right\}\\
&\leq\epsilon/3+\epsilon/3+\epsilon/3\leq\epsilon.
\end{aligned}$$

The proof is completed.

5. One Example

In this section, in order to illustrate our results, we consider an numerical example.

Example. Consider the following scalar nonlinear stochastic differential equation with Markovian switching

$$\begin{aligned}
dx(t)&=\begin{bmatrix}-a_1x(t)+a_2x^2(t)x^3(t-1)\\ -a_3x^5(t)+a\left(r(t)\right)x(t)\end{bmatrix}dt \qquad (5.1)\\
&+\left[b_1x^2(t)x(t-1)+b\left(r(t)\right)x^3(t-1)\right]dw(t),
\end{aligned}$$

on $t \geq 0$, where, $w(t)$ is a scalar Brownian motion, $r(t)$ is a right continuous Markov chain taking values in $s=\{1,2\}$ with the generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix},$$

of course $w(t)$ and $r(t)$ are assumed to be independent, and a_1, a_2, a_3, b_1 are positive constants. It is to obtain

$$\begin{aligned}
&x^T f\left(x(t),x(t-1),r(t),t\right)\\
&=-a_1x^2(t)+a_2x^3(t)x^3(t-1)\\
&\quad -a_3x^6(t)+a\left(r(t)\right)x^2(t)\\
&\leq-\left[a_1-a\left(r(t)\right)\right]x^2(t)-a_3x^6(t)\\
&\quad +a_2/2\left[x^6(t)+x^6(t-1)\right],\\
&\leq-\left[a_1-a\left(r(t)\right)\right]\left|x(t)\right|^2-\left(a_3-a_2/2\right)\left|x(t)\right|^6\\
&\quad +a_2/2\left|x(t-1)\right|^6,
\end{aligned}$$

$$\begin{aligned}
&\left|g\left(x(t),x(t-1),r(t),t\right)\right|^2\\
&=b_1^2x^4(t)x^2(t-1)+b^2\left(r(t)\right)x^6(t-1)\\
&\quad +2b_1b\left(r(t)\right)x^2(t)x^4(t-1)\\
&\leq 2/3b_1^2x^6(t)+1/3b_1^2x^6(t-1)+b^2\left(r(t)\right)x^6(t-1)\\
&\quad +2/3b_1b\left(r(t)\right)x^6(t)+4/3b_1b\left(r(t)\right)x^6(t-1)\\
&\leq 2/3b_1\left[b_1+b\left(r(t)\right)\right]x^6(t)\\
&\quad +\left[1/3b_1^2+4/3b_1b\left(r(t)\right)+b^2\left(r(t)\right)\right]x^6(t-1).
\end{aligned}$$

By the Theorem 3.1, we assume that

$$a_1>a\left(r(t)\right), b_1>b\left(r(t)\right), a_3>a_2/2,$$

and

$$1/3b_1^2+4/3b_1b\left(r(t)\right)+b^2\left(r(t)\right)>0$$

$$a_3-a_2>\frac{p-1}{2}\left(b_1+b\left(r(t)\right)\right),$$

then Eqs. (5.1) has unique global solution.

To carry out the numerical simulation we choose the step size h = 1/1024, and $a(1)$ = 0.1, $a(2)$ =0.05, $b(1)$ = 0.05, $b(2)$ = 0.02, the computer simulation result is shown in Figure 1 and Figure 2. And it is clear that the Euler-Maruyama method reveals the almost surely exponentially stable property of the solution.

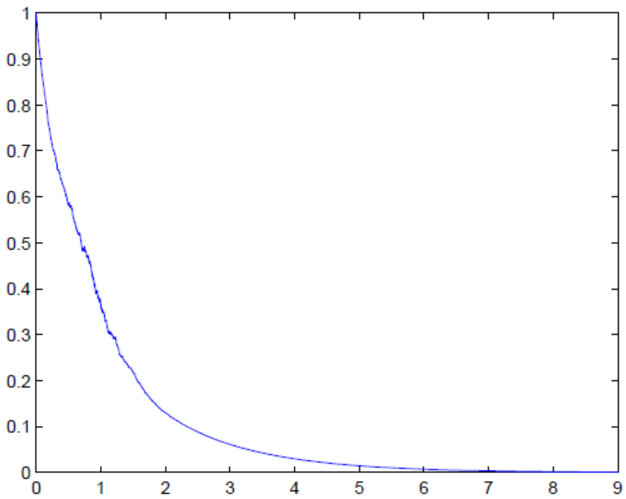

Figure 1. a_1 = 0.8, a_2 = 0.5, a_3 = 1.2, b_1 = 0.2

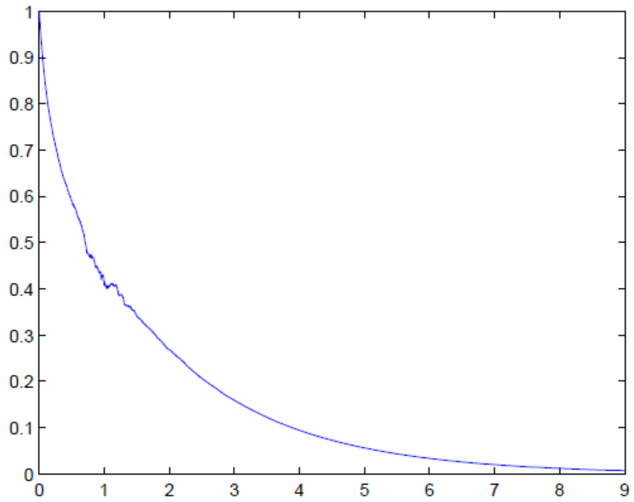

Figure 2. $a_1 = 0.6$, $a_2 = 1$, $a_3 = 1.8$, $b_1 = 0.1$

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