THE TETRAHEDRAL PROPERTY AND A NEW GROMOV-HAUSDORFF COMPACTNESS THEOREM

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ABSTRACT. We present the Tetrahedral Compactness Theorem which states that sequences of Riemannian manifolds with a uniform upper bound on volume and diameter that satisfy a uniform tetrahedral property have a subsequence which converges in the Gromov-Hausdorff sense to a countably $\mathcal{H}^m$ rectifiable metric space of the same dimension. The tetrahedral property depends only on distances between points in spheres, yet we show it provides a lower bound on the volumes of balls. The proof is based upon intrinsic flat convergence and a new notion called the sliced filling volume of a ball.

1. INTRODUCTION

We introduce the tetrahedral property which is an estimate on tetrahedra (see Figure 1):

Definition 1.1. *Given $C > 0$ and $\beta \in (0, 1)$, a metric space X has the m dimensional C,β-tetrahedral property at a point p for radius r if one can find points $p_1, ...p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that*

$$h(p, r, t_1, ..., t_{m-1}) \geq Cr \qquad \forall (t_1, ..., t_{m-1}) \in [(1 - \beta)r, (1 + \beta)r]^m \tag{1}$$

where $h(p, r, t_1, ..., t_{m-1}) = \inf \{d(x, y) : \ x \neq y, \ x, y \in P(p, r, t_1, ..., t_{m-1})\}$ *when*

$$P(p, r, t_1, ..., t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1}) \neq \emptyset \tag{2}$$

and $h(p, r, t_1, ..., t_{m-1}) = 0$ otherwise. In particular $P(p, r, t_1, ..., t_{m-1})$ is a discrete set of points.

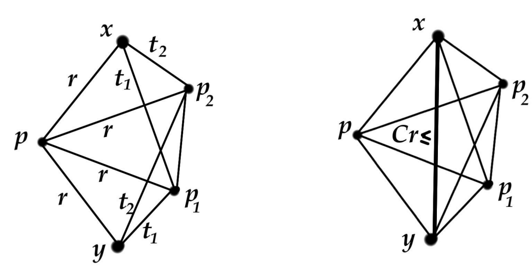

FIGURE 1. Tetrahedral Property in 3D

As this property is rather strong, we introduce the integral tetrahedral property:

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Definition 1.2. *Given $C > 0$ and $\beta \in (0,1)$, a metric space X is said to have the m dimensional integral C,β-tetrahedral property at a point p for radius r if $\exists\ p_1, ...p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that*

$$\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, ...t_{m-1}))\, dt_1 dt_2 ... dt_{m-1} \geq C(2\beta)^{m-1} r^m. \tag{3}$$

We can prove that both of these properties provide an estimate on volume:

Theorem 1.3. *If p_0 lies in a Riemannian manifold that has the m dimensional (integral) C,β-tetrahedral property at a point p for radius R then*

$$\mathrm{Vol}(B(p, r)) \geq C(2\beta)^{m-1} r^m \tag{4}$$

As a consequence of Gromov's Compactness Theorem we then have:

Theorem 1.4. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0$. If a sequence of compact Riemannian manifolds, M^m, has $\mathrm{Vol}(M^m) \leq V_0$, $\mathrm{Diam}(M^m) \leq D_0$, and the C,β (integral) tetrahedral property for all balls of radius $\leq r_0$, then a subsequence converges in the Gromov-Hausdorff sense. In particular they have a uniform upper bound on diameter depending only on these constants.*

Remark 1.5. *In fact we prove there is an intrinsic flat limit as well and the intrinsic flat and GH limit agree. Thus the limit space in Theorem 1.4 is a countably $\mathcal{H}^m$ rectifiable metric space.*

2. Examples

Example 2.1. *On Euclidean space, $\mathbb{E}^3$, taking $p_1, p_2 \in \partial B(p, r)$ to such that $d(p_1, p_2) = r$, then there exists exactly two points $x, y \in P(p, r, r, r)$ each forming a tetrahedron with p, p_1, p_2. As we vary $t_1, t_2 \in (r/2, 3r/2)$, we still have exactly two points in $P(p, r, t_1, t_2)$. By scaling we see that*

$$h(p, r, t_1, t_2) = rh(p, 1, t_1/r, t_2/r) \geq C_{\mathbb{E}^3} r \tag{5}$$

where $C_{\mathbb{E}^3} = \inf\{h(p, 1, s_1, s_2) : s_i \in (1/2, 3/2)\} > 0$. Taking $\beta = 1/2$, we see that $\mathbb{E}^3$ satisfies the $C_{\mathbb{E}^3}, \beta$ tetrahedral property.

Example 2.2. *On a torus, $M^3_\epsilon = S^1 \times S^1 \times S^1_\epsilon$ where S^1_ϵ has been scaled to have diameter ϵ instead of π, we see that M^3 satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all $r < \epsilon/4$. By taking $r < \epsilon/4$, we guarantee that the shortest paths between x and y stay within the ball $B(p, r)$ allowing us to use the Euclidean estimates. If r is too large, $P(p, r, t_1, t_2) = \emptyset$. So for a sequence M_ϵ with $\epsilon \to 0$ we fail to have a uniform tetrahedral property. There is a Gromov-Hausdorff limit but it is not three dimensional.*

Example 2.3. *Suppose one creates a Riemannian manifold M^3_ε, by gluing together two copies of Euclidean space with a large collection of tiny necks between corresponding points. That is,*

$$M^3_\varepsilon = \left(\mathbb{E}^3 \setminus \bigcup B_{z_i}(\varepsilon)\right) \sqcup \left(\mathbb{E}^3 \setminus \bigcup B_{z_i}(\varepsilon)\right) \tag{6}$$

where points on $\partial B_{z_i}(\varepsilon)$ in the first copy of Euclidean space are joined to corresponding points on $\partial B_{z_i}(\varepsilon_i)$ in the second copy of Euclidean space. We choose z_i such that $\mathbb{E}^3 \subset \bigcup_{i=1}^\infty B_{z_i}(10\varepsilon)$ and the balls $B_{z_i}(\varepsilon)$ are pairwise disjoint. Then for $r >> \varepsilon$, we will have an x and a y as in $\mathbb{E}^3$, but we will also have a nearby x' and y' in the second copy, with $d(x, x') < 20\varepsilon$. So for a sequence M_ε with $\varepsilon \to 0$ we fail to have a uniform tetrahedral property. If we create M^3_ε by joining increasingly many copies of Euclidean space together, this sequence wouldn't even have a subsequence converging in the Gromov-Hausdorff sense.

3. Intrinsic Flat Convergence

The Intrinsic Flat distance between Riemannian manifolds was introduced by the author and Stefan Wenger in [6]. It was defined using Gromov's idea of isometrically embedding two Riemannian manifolds into a common metric space. Rather than measuring the Hausdorff distance between the images as Gromov did when defining the Gromov-Hausdorff distance in [3], one views the images as integral currents in the sense of Ambrosio-Kirchheim in [1] and takes the flat distance between them. The author and Wenger proved that intrinsic flat limit spaces are countably $\mathcal{H}^m$ rectifiable metric spaces in [6].

Wenger has proven a compactness theorem for intrinsic flat convergence [7], but we do not need to apply that compactness theorem to prove the compactness theorems stated here. Instead our compactness theorem is based upon the Gromov-Hausdorff compactness theorem [3] and the fact that we obtain a uniform lower bound on the volumes of balls [Theorem 1.3]. Applying Ambrosio-Kirchheim's Compactness Theorem of [1], Wenger and the author proved that once a sequence of manifolds converges in the Gromov-Hausdorff sense to a limit space Y, then a subsequence converges in the Intrinsic Flat sense to a subset, X, of Y [6]. In [5], estimates on the filling volumes of spheres were applied to prove the two limit spaces were the same when the sequence of manifolds has nonnegative Ricci curvature. Recall that filling volumes were introduced by Gromov in [2].

Here we do not have strong estimates on the filling volumes of spheres. To prove Theorem 1.3 we first define the sliced filling volumes of balls and then prove a compactness theorem:

Definition 3.1. *Given points $q_1, ..., q_k \in M^m$, where $k < m$, with distance functions $\rho_i(x) = d(x, q_i)$, we define the sliced filling volume of a sphere $\partial B(p, r)$, to be*

(7)
$$\mathbf{SF}(p, r, q_1, ..., q_k) = \int_{t_1=m_1}^{M_1} \int_{t_2=m_2}^{M_2} \cdots \int_{t_k=m_k}^{M_k} \mathrm{FillVol}(\partial\, \mathrm{Slice}(B(p, r), \rho_1, ...\rho_k, t_1, ..., t_k))\, \mathcal{L}^k$$

where $m_i = \min\{\rho_i(x) : x \in \bar{B}_p(r)\}$ and $M_i = \max\{\rho_i(x) : x \in \bar{B}_p(r)\}$ and where the slice is defined as in Geometric Measure Theory so that it is supported on $B(p, r) \cap \rho_1^{-1}(t_1) \cap \cdots \cap \rho_k^{-1}(t_k)$.

Definition 3.2. *Given $p \in M^m$, then for almost every r, we can define the k^{th} sliced filling,*

$$\mathbf{SF}_k(p, r) = \sup\{\mathbf{SF}(p, r, q_1, ..., q_k) : q_i \in \partial B_p(r)\}. \tag{8}$$

Theorem 3.3. *Let $V_0, D_0, r_0 > 0$ and $C(r) > 0$. If M^m_i have $\mathrm{Vol}(M_i) \le V_0$, $\mathrm{Diam}(M_i) \le D_0$, and*

$$\mathbf{SF}_k(p, r) \ge C(r) > 0 \qquad \forall i \in \mathbb{N},\ \forall p \in M_i \text{ and almost every } r \in (0, r_0) \tag{9}$$

then a subsequence of the M_i converges in the Gromov-Hausdorff sense to a limit space which is also the intrinsic flat limit of the sequence and is thus a countably $\mathcal{H}^m$ rectifiable metric space.

This theorem is proven by the author in [4]. We first show that $\mathrm{Vol}(B(p,r)) \geq \mathbf{SF}_k(p,r)$, so that a subsequence has a Gromov-Hausdorff limit, M_∞, by Gromov's Compactness Theorem. We next observe that when the points $p_j \in M_j$ converge to a point p_∞ in the Gromov-Hausdorff limit, M_∞, their sliced fillings converge. Applying Ambrosio-Kirchheim's Slicing Theorem, we can then estimate the mass of the limit current and prove that the Gromov-Hausdorff and Intrinsic Flat limits agree.

4. Tetrahedral Property

The final step in the proof of the Theorem 1.4 is to relate the tetrahedral property to the $k = m - 1$ sliced filling volume. We first observe that

$$\mathrm{spt}\left(\partial\,\mathrm{Slice}\,(B(p,r), \rho_1, ...\rho_{m-1}, t_1, ..., t_{m-1})\right) = \partial B_p(r) \cap \bigcap_{i=1}^{m-1} \partial B_{q_i}(t_i) \tag{10}$$

which is a discrete collection of points for almost every value of $(t_1, ...t_{m-1})$. So we prove a theorem that the filling volume of a 0 dimensional integral current can be bounded below by the distance between the closest pair of points in the current's support. We then have:

Theorem 4.1. *If M^m is a Riemannian manifold with the m dimensional (integral) C,β-tetrahedral property at a point p for radius r then* $\mathrm{Vol}(B(p,r)) \geq \mathbf{SF}_{m-1}(p,r) \geq C(2\beta)^{m-1} r^m$

Combining this theorem with Theorem 3.3, we obtain both Theorem 1.4 and Remark 1.5. These theorems and related theorems are proven in [4] which is available on the arxiv. That paper will include many additional results before it is completed, as it explores many properties which are continuous under intrinsic flat convergence even in settings where there are no Gromov-Hausdorff limits and where the spaces are not Riemannian manifolds.

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