On the Number of Iterations to the Closest Quadratically Invariant Information Constraint

Michael C. Rotkowitz * **Nuno C. Martins** **

* *Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville VIC 3010 Australia,* mcrotk@unimelb.edu.au

** *Department of Electrical and Computer Engineering and the Institute for Systems Research, The University of Maryland, College Park MD 20740 USA,* nmartins@umd.edu

Abstract: Quadratic invariance is a condition which has been shown to allow for optimal decentralized control problems to be cast as convex optimization problems. The condition relates the constraints that the decentralization imposes on the controller to the structure of the plant. Recent work considered the problem of finding the closest subset and superset of the decentralization constraint which are quadratically invariant when the original problem is not. It was shown that this can itself be cast as a convex problem for the case where the controller is subject to delay constraints between subsystems, but that this fails when we only consider sparsity constraints on the controller. For that case, an algorithm was developed that finds the closest superset in a fixed number of steps, and it was shown to converge in n^2 iterations, where n is the number of subsystems. This paper studies the algorithm further and shows that it actually converges in $\log_2 n$ iterations.

1. INTRODUCTION

The design of decentralized controllers has been of interest for a long time, as evidenced in the surveys (Witsenhausen [1971], Sandell et al. [1978]), and continues to this day with the advent of complex interconnected systems. The counterexample constructed by Hans Witsenhausen in 1968 (Witsenhausen [1968]) clearly illustrates the fundamental reasons why problems in decentralized control are difficult.

Among the recent results in decentralized control, new approaches have been introduced that are based on algebraic principles, such as the work in (Rotkowitz and Lall [2006b,a], Voulgaris [2001]). Very relevant to this paper is the work in (Rotkowitz and Lall [2006b,a]), which classified the problems for which optimal decentralized synthesis could be cast as a convex optimization problem. Here, the plant is linear, time-invariant and it is partitioned into dynamically coupled subsystems, while the controller is also partitioned into subcontrollers. In this framework, the decentralization being imposed manifests itself as constraints on the controller to be designed, often called the *information constraint.*

The information constraint on the overall controller specifies what information is available to which controller. For instance, if information is passed between subsystems, such that each controller can access the outputs from other subsystems after different amounts of transmission time, then the information constraints are delay constraints, and may be represented by a matrix of these transmission delays. If instead, we consider each controller to be able to access the outputs from some subsystems but not from others, then the information constraint is a sparsity constraint, and may be represented by a binary matrix.

Given such pre-selected information constraints, the existence of a convex parameterization for all stabilizing controllers that satisfy the constraint can be determined via the algebraic test introduced in (Rotkowitz and Lall [2006b,a]), which is denoted as *quadratic invariance*. In contrast with prior work, where the information constraint on the controller is fixed beforehand, this paper addresses the design of the information constraint itself. More specifically, given a plant and a pre-selected information constraint that is not quadratically invariant, we give explicit algorithms to compute the quadratically invariant information constraint that is closest to the pre-selected one. We consider finding the closest quadratically invariant superset, which corresponds to relaxing the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain a lower bound on the original problem, as well as finding the closest quadratically invariant subset, which corresponds to tightening the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain upper bounds on the original problem.

In this paper we focus on sparsity constraints that represent which controllers can access which subsystem outputs, and represent such constraints with binary matrices. The distance between information constraints is then given by the hamming distance, applied to the binary sparsity matrices. We provide an algorithm that gives the closest superset; that is, the quadratically invariant constraint that can be obtained by way of *allowing* the least number of additional links, which was introduced in (Rotkowitz and Martins [2009]).

When the algorithm was introduced, it was shown to converge in at most n^2 iterations, where n is the number of subsystems. Here we examine it further, and show that it actually converges in $\log n$ iterations.

For the problem of finding a close subset, we propose a heuristic-based solution.

Paper organization: Besides the introduction, this paper has seven sections. Section 2 presents the notation and the basic concepts used throughout the paper. The sparsity constraints adopted in our work are described in detail in Section 3, while their characterization using quadratic invariance is given in Section 4. The main problem addressed in this paper is formulated and solved in Section 5. Section 6 briefly notes how this work also applies when assumptions of linear time-invariance are dropped, some numerical examples are provided in Section 7, and conclusions are given in Section 8.

2. PRELIMINARIES

Throughout the paper, we adopt a given causal linear time-invariant continuous-time plant P partitioned as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}$$

Here, $P \in \mathcal{R}_p^{(n_y+n_z)\times(n_w+n_u)}$, where $\mathcal{R}_p^{q\times r}$ denotes the set of matrices of dimension q by r, whose entries are proper transfer functions of the Laplace complex variable s. Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity. In addition, we indicate the number of rows and columns of G by n_y and n_u, respectively.

Given a causal linear time-invariant controller K in $\mathcal{R}_p^{n_u\times n_y}$, we define the ***closed-loop map*** by

$$f(P,K) \stackrel{def}{=} P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

where we assume that the feedback interconnection is well posed. The map $f(P,K)$ is also called the (lower) ***linear fractional transformation*** (LFT) of P and K. This interconnection is shown in Figure 1.

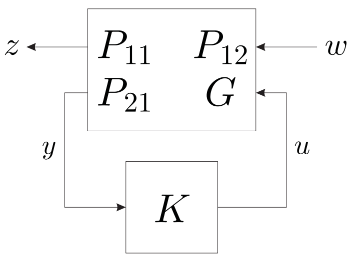

Fig. 1. Linear fractional interconnection of P and K

We suppose that there are n_y sensor measurements and n_u control actions, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_{n_y}^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_{n_u}^T \end{bmatrix}^T$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_yn_u} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_un_y} \end{bmatrix}$$

Given $A \in \mathbb{R}^{m\times n}$, we may write A in term of its columns as

$$A = [a_1 \ \dots \ a_n]$$

and then associate a vector $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ defined by

$$\mathrm{vec}(A) \stackrel{def}{=} \begin{bmatrix} a_1^T & \cdots & a_n^T \end{bmatrix}^T$$

Further notation will be introduced as needed.

2.1 Sparsity

We adopt the following notation to streamline our use of sparsity patterns and sparsity constraints.

Binary algebra. Let $\mathbb{B} = \{0,1\}$ represent the set of binary numbers. Given $x, y \in \mathbb{B}$, we define the following basic operations:

$$x + y \stackrel{def}{=} \begin{cases} 0, & \text{if } x = y = 0 \\ 1, & \text{otherwise} \end{cases}, \qquad x, y \in \mathbb{B}$$

$$xy \stackrel{def}{=} \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad x, y \in \mathbb{B}$$

Given $X, Y \in \mathbb{B}^{m\times n}$, we say that $X \le Y$ holds if and only if $X_{ij} \le Y_{ij}$ for all i, j satisfying $1 \le i \le m$ and $1 \le j \le n$, where X_{ij} and Y_{ij} are the entries at the i-th row and j-th column of the binary matrices X and Y, respectively.

Given $X, Y, Z \in \mathbb{B}^{m\times n}$, these definitions lead to the following immediate consequences:

$$Z = X + Y \ \Rightarrow \ Z \ge X \tag{1}$$
$$X + Y = X \ \Leftrightarrow \ Y \le X \tag{2}$$
$$X \le Y, \ Y \le X \ \Leftrightarrow \ X = Y \tag{3}$$

Given $X \in \mathbb{B}^{m\times n}$, we use the following notation to represent the total number of nonzero indeces in X:

$$\mathcal{N}(X) \stackrel{def}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}, \qquad X \in \mathbb{B}^{m\times n}$$

with the sum taken in the usual way.

Sparsity patterns. Suppose that $A^{\mathrm{bin}} \in \mathbb{B}^{m\times n}$ is a binary matrix. The following is the subspace of $\mathcal{R}_p^{m\times n}$ comprising the transfer function matrices that satisfy the sparsity constraints imposed by A^{bin}:

$$\mathrm{Sparse}(A^{\mathrm{bin}}) \stackrel{def}{=} \Big\{ B \in \mathcal{R}_p^{m\times n} \mid B_{ij}(j\omega) = 0 \text{ for all } i, j \text{ such that } A_{ij}^{\mathrm{bin}} = 0 \text{ for almost all } \omega \in \mathbb{R} \Big\}$$

Conversely, given $B \in \mathcal{R}_p^{m\times n}$, we define $\mathrm{Pattern}(B) \stackrel{def}{=} A^{\mathrm{bin}}$, where A^{bin} is the binary matrix given by:

$$A_{ij}^{\mathrm{bin}} = \begin{cases} 0, & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1, & \text{otherwise} \end{cases}$$

3. OPTIMAL CONTROL SUBJECT TO INFORMATION CONSTRAINTS

In this section, we give a detailed description of the problem setup with sparsity constraints, the main type of information constraint addressed in this paper.

3.1 Sparsity Constraints

We now introduce the other main class of constraints we will consider in this paper, where each control input may access certain sensor measurements, but not others.

We represent sparsity constraints on the overall controller via a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$. Its entries can be interpreted as follows:

$$K^{\text{bin}}_{kl} = \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l; \\ 0, & \text{if not.} \end{cases}$$

The subspace of admissible controllers can be expressed as:

$$S = \text{Sparse}(K^{\text{bin}})$$

From the quadratic invariance test introduced in (Rotkowitz and Lall [2006b,a]), we find that the relevant information about the plant is its sparsity pattern G^{bin}, obtained from:

$$G^{\text{bin}} = \text{Pattern}(G)$$

where G^{bin} is interpreted as follows:

$$G^{\text{bin}}_{ij} = \begin{cases} 1, & \text{if control input } j \\ & \text{affects sensor measurement } i; \\ 0, & \text{if not.} \end{cases}$$

3.2 Optimal Control Design Via Convex Programming

Given a generalized plant P and a subspace of appropriately dimensioned causal linear time-invariant controllers S, the following is a class of constrained optimal control problems:

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{4}$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The delays associated with dynamics propagating from one subsystem to another, or the sparsity associated with them not propagating at all, are embedded in P. The subspace of admissible controllers, S, has been defined to encapsulate the constraints on how quickly information may be passed from one subsystem to another (delay constraints) or whether it can be passed at all (sparsity constraints). We call the subspace S the ***information constraint***.

Many decentralized control problems may be expressed in the form of problem (4), including all of those addressed in (Siljak [1994], Qi et al. [2004], Rotkowitz and Lall [2006b]). In this paper, we focus on the case where S is defined by sparsity constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

4. QUADRATIC INVARIANCE

In this section, we define quadratic invariance, and we give a brief overview of related results, in particular, that if it holds then convex synthesis of optimal decentralized controllers is possible.

Definition 1. Let a causal linear time-invariant plant, represented via a transfer function matrix G in $\mathcal{R}_p^{n_y \times n_u}$, be given. If S is a subset of $\mathcal{R}_p^{n_u \times n_y}$ then S is called ***quadratically invariant*** under G if the following inclusion holds:

$$KGK \in S \qquad \text{for all } K \in S.$$

It was shown in (Rotkowitz and Lall [2006b]) that if S is a closed subspace and S is quadratically invariant under G, then with a change of variables, problem (4) is equivalent to the following optimization problem

$$\begin{aligned} \underset{Q}{\text{minimize}} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{5}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$. Here $\mathcal{RH}_\infty$ is used to indicate that T_1, T_2, T_3 and Q are proper transfer function matrices with no poles in $\mathbb{C}_+$ (stable).

The optimization problem in (5) is convex. We may solve it to find the optimal Q, and then recover the optimal K for our original problem as stated in (4). If the norm of interest is the $\mathcal{H}_2$-norm, it was shown in (Rotkowitz and Lall [2006b]) that the problem can be further reduced to an unconstrained optimal control problem and then solved with standard software. Similar results have been achieved (Rotkowitz and Lall [2006a]) for function spaces beyond $\mathcal{L}_e$ as well, also showing that quadratic invariance allows optimal linear decentralized control problems to be recast as convex optimization problems.

The main focus of this paper is thus characterizing information constraints S which are as close as possible to a pre-selected one, and for which S is quadratically invariant under the plant G.

4.1 QI - Sparsity Constraints

For the case of sparsity constraints, it was shown in (Rotkowitz and Lall [2006b]) that a necessary and sufficient condition for quadratic invariance is

$$K^{\text{bin}}_{ki}\, G^{\text{bin}}_{ij}\, K^{\text{bin}}_{jl}\, (1 - K^{\text{bin}}_{kl}) = 0 \tag{6}$$

This condition can be shown to be equivalent to one that arises for the quadratic invariance of a system with certain delays between subsystems. This formulation, discussed in (Rotkowitz and Martins [2009, 2011]), helps give some insight into where the difficulty arises in the problem that we address in the next section, but it is not necessary for understanding the algorithm presented, and we do not discuss it further here.

5. CLOSEST QI CONSTRAINT

We now address the main question of this paper, which is finding the closest sparsity constraints when the above conditions fail; that is, when the original problem is not quadratically invariant.

5.1 Sparsity Superset

Consider first finding the closest quadratically invariant superset of the original constraint set; that is, the sparsest quadratically invariant set for which all of the original connections $y_l \to u_k$ are still in place. We present an algorithm which achieves this and terminates in a fixed number of steps.

We can write the problem as

$$\begin{aligned} \underset{Z \in \mathbb{B}^{n_u \times n_y}}{\text{minimize}} \quad & \mathcal{N}(Z) \\ \text{subject to} \quad & Z G^{\text{bin}} Z \ \leq \ Z \\ & K^{\text{bin}} \ \leq \ Z \end{aligned} \tag{7}$$

where additions and multiplications are as defined for the binary algebra in the preliminaries, and where we will wish to use the information constraint $S = \text{Sparse}(Z)$. The objective is defined to give us the sparsest possible solution, the first constraint ensures that the constraint set associated with the solution is quadratically invariant with respect to the plant, and the last constraint requires the resulting set of controllers to be able to access any information that could be accessed with the original constraints. Let the optimal solution to this optimization problem be denoted as $Z^* \in \mathbb{B}^{n_u \times n_y}$.

Define a sequence of sparsity constraints $\{Z_m \in \mathbb{B}^{n_u \times n_y},\ m \in \mathbb{N}\}$ given by

$$Z_0 \ = \ K^{\text{bin}} \tag{8}$$

$$Z_{m+1} \ = \ Z_m + Z_m G^{\text{bin}} Z_m, \qquad m \geq 0 \tag{9}$$

again using the binary algebra.

This sequence was introduced in (Rotkowitz and Martins [2009]), where it was shown to converge in n^2 iterations. Our main result will be that this sequence converges to Z^*, and that it does so in $\log_2 n$ iterations. We first prove several preliminary lemmas, and start with a lemma elucidating which terms comprise which elements of the sequence.

Lemma 2.

$$Z_m \ = \ \sum_{s=0}^{2^m-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s \qquad \forall\ m \in \mathbb{N} \tag{10}$$

Proof. For $m = 0$, this follows immediately from (8). We then assume that (10) holds for a given $m \in \mathbb{N}$, and consider $m+1$. Then,

$$Z_{m+1} \ = \ \sum_{i=0}^{2^m-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^i$$

$$+ \left(\sum_{k=0}^{2^m-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^k \right) G^{\text{bin}} \left(\sum_{l=0}^{2^m-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^l \right)$$

All terms on the R.H.S. are of the form $K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s$ for various $s \in \mathbb{N}$. Choosing $0 \leq i \leq 2^m - 1$ gives $0 \leq s \leq 2^m - 1$, and choosing $k = 2^m - 1$ with $0 \leq l \leq 2^m - 1$ gives $2^m \leq s \leq (2^m - 1) + 1 + (2^m - 1) = 2^{m+1} - 1$. This last term is the highest order term, so we then have $Z_{m+1} = \sum_{s=0}^{2^{m+1}-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s$ and the proof follows by induction. ■

We now give a lemma showing how many of these terms need to be considered.

Lemma 3.

$$K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^r \ \leq \ \sum_{s=0}^{n-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s \qquad \forall\ \ r \in \mathbb{N} \tag{11}$$

where $n = \min\{n_y, n_u\}$.

Proof. Follows immediately from (1) for $r \leq n-1$. Now consider $r \geq n$, $k \in \{1, \ldots, n_u\}$, $l \in \{1, \ldots, n_y\}$. Then

$$[K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^r]_{kl} = \sum K^{\text{bin}}_{ki_1} G^{\text{bin}}_{i_1 j_1} K^{\text{bin}}_{j_1 i_2} G^{\text{bin}}_{i_2 j_2} \cdots G^{\text{bin}}_{i_r j_r} K^{\text{bin}}_{j_r l}$$

where the sum is taken over all possible $i_\alpha \in \{1, \ldots, n_y\}$ and $j_\alpha \in \{1, \ldots, n_u\}$. Consider an arbitrary such summand term that is equal to 1, and note that each component term must be equal to 1.

If $n = n_y$ (i), then by the pigeonhole principle either $\exists \alpha$ s.t. $i_\alpha = l$ (i.a), or $\exists \alpha, \beta$, with $\alpha \neq \beta$, s.t. $i_\alpha = i_\beta$ (i.b). In case (i.a), we have $K^{\text{bin}}_{ki_1} G^{\text{bin}}_{i_1 j_1} \cdots G^{\text{bin}}_{i_{\alpha-1} j_{\alpha-1}} K^{\text{bin}}_{j_{\alpha-1} l} = 1$, or in case (i.b), we have $K^{\text{bin}}_{ki_1} \cdots K^{\text{bin}}_{j_{\alpha-1} i_\alpha} G^{\text{bin}}_{i_\beta j_\beta} \cdots K^{\text{bin}}_{j_r l} = 1$. In words, we can bypass the part of the path that merely took y_{i_α} to itself, leaving a shorter path that still connects $y_l \to u_k$.

Similarly, if $n = n_u$ (ii), then either $\exists \alpha$ s.t. $j_\alpha = k$ (ii.a), or $\exists \alpha, \beta$ with $\alpha \neq \beta$ s.t. $j_\alpha = j_\beta$ (ii.b). In case (ii.a), we have $K^{\text{bin}}_{ki_{\alpha+1}} G^{\text{bin}}_{i_{\alpha+1} j_{\alpha+1}} \cdots K^{\text{bin}}_{j_r l} = 1$, or in case (ii.b), we have $K^{\text{bin}}_{ki_1} \cdots G^{\text{bin}}_{i_\alpha j_\alpha} K^{\text{bin}}_{j_\beta i_{\beta+1}} \cdots K^{\text{bin}}_{j_r l} = 1$, where we have now bypassed the part of the path taking u_{j_α} to itself to leave a shorter path.

We have shown that, $\forall\ r \geq n$, any non-zero component term of $K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^r$ has a corresponding non-zero term of strictly lower order, and the result follows. ■

We now prove another preliminary lemma showing that the optimal solution can be no more sparse than any element of the sequence.

Lemma 4. For $Z^* \in \mathbb{B}^{n_u \times n_y}$ and the sequence $\{Z_m \in \mathbb{B}^{n_u \times n_y},\ m \in \mathbb{N}\}$ defined as above, the following holds:

$$Z^* \ \geq \ Z_m, \qquad m \in \mathbb{N} \tag{12}$$

Proof. First, $Z^* \geq Z_0 = K^{\text{bin}}$ is given by the satisfaction of the last constraint of (7), and it just remains to show the inductive step.

Suppose that $Z^* \geq Z_m$ for some $m \in \mathbb{N}$. It then follows that

$$Z^* + Z^* G^{\text{bin}} Z^* \ \geq \ Z_m + Z_m G^{\text{bin}} Z_m$$

From the first constraint of (7) and (2) we know that the left hand-side is just Z^*, and then using the definition of our sequence, we get

$$Z^* \ \geq \ Z_{m+1}$$

and this completes the proof. ■

We now give a subsequent lemma, showing that if the sequence does converge, then it has converged to the optimal solution.

Lemma 5. If $Z_{m^*} = Z_{m^*+1}$ for some $m^* \in \mathbb{N}$, then $Z_{m^*} = Z^*$.

Proof. If $Z_{m^*} = Z_{m^*+1}$, then $Z_{m^*} = Z_{m^*}G^{\text{bin}}Z_{m^*}$, and it follows from (2) that $Z_{m^*}G^{\text{bin}}Z_{m^*} \leq Z_{m^*}$. Since $Z_{m+1} \geq Z_m$ for all $m \in \mathbb{N}$, it also follows that $Z_m \geq Z_0 = K^{\text{bin}}$ for all $m \in \mathbb{N}$. Thus the two constrains of (7) are satisfied for Z_{m^*}.

Since Z^* is the sparsest binary matrix satisfying these constraints, it follows that $Z^* \leq Z_{m^*}$. Together with Lemma 4 and equation (3), it follows that $Z_{m^*} = Z^*$. ■

We now give the main result, that the sequence converges, that it does so in $\log_2 n$ steps, and that it achieves the optimal solution to our problem.

Theorem 6. The problem specified in (7) has an unique optimal solution Z^* satisfying:

$$Z_{m^*} = Z^* \tag{13}$$

where $m^* = \lceil \log_2 n \rceil$ and where $n = \min\{n_u, n_y\}$.

Proof. $Z_{m^*} = \sum_{s=0}^{2^{m^*}-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s$ from Lemma 2, and then $Z_{m^*} = \sum_{s=0}^{n-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s$ from Lemma 3 since $2^{m^*} \geq n$. Similarly,

$$\begin{aligned} Z_{m^*+1} &= \sum_{s=0}^{2^{m^*+1}-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s \\ &= \sum_{s=0}^{n-1} K^{\text{bin}}(G^{\text{bin}}K^{\text{bin}})^s, \end{aligned}$$

and thus $Z_{m^*} = Z_{m^*+1}$ and the result follows from Lemma 5. ■

5.2 Sparsity Subset

We now notice an interesting asymmetry. While we have developed an algorithm which finds the closest quadratically invariant superset given by sparsity constraints, there is no clear way to flip the algorithm to find the closest subset. This does not arise with delay constraints (Rotkowitz and Martins [2009, 2011]), where the problem can be solved in the same manner either way.

This can be understood as follows. If there exist indeces i, j, k, l such that $K_{ki}^{\text{bin}} = G_{ij}^{\text{bin}} = K_{jl}^{\text{bin}} = 1$, but $K_{kl}^{\text{bin}} = 0$; that is, indeces for which condition (6) fails, then the above algorithm resets $K_{kl}^{\text{bin}} = 1$. In other words, if there is an indirect connection from $y_l \to u_k$, but not a direct connection, it hooks up the direct connection.

But now consider what happens if we try to develop an algorithm that goes in the other direction, that finds the least sparse constraint set which is more sparse than the original. If we again have indeces for which condition (6) fails, then we need to disconnect the indirect connection, but it's not clear if we should set K_{ki}^{bin} or K_{jl}^{bin} to zero, since we could do either. The goal is, in principle, to disconnect the link that will ultimately lead to having to make the fewest subsequent disconnections, so that we end up with the closest possible constraint set to the original.

We suggest a heuristic for dealing with this problem. Another one which builds off of the delay formulation is also given in (Rotkowitz and Martins [2009, 2011]). It is likely that they can be greatly improved upon, but are meant as a first cut at a reasonable polynomial time algorithm to find a close sparse subset.

The main idea is to keep track of the indirect connections that are associated with a direct connection. Define this weight as $w_{kl} = \sum_{i=1}^{n_y}\sum_{j=1}^{n_u} K_{ki}^{\text{bin}}G_{ij}^{\text{bin}}K_{jl}^{\text{bin}}$ thus giving the amount of 3-hop connections from $y_l \to u_k$. This is a crude measure of how many subsequent disconnections we will have to make to obtain quadratic invariance if we were to disconnect a direct path from $y_l \to u_k$. Then, given indeces for which condition (6) is violated, we set K_{ki}^{bin} to zero if $w_{ki} \leq w_{jl}$, and set K_{jl}^{bin} to zero otherwise.

Note that for this and the other heuristic, we have many options for how often to reset the guiding variables, that is, to re-solve the convex program or recalculate the weights, such as after each disconnection, or after each pass through all $n_u n_y$ indeces.

It has been noticed that some of the quadratically invariant constraints for certain classes of problems, including sparsity, may be thought of as partially ordered sets (Shah and Parrilo [2008]). This raises the possibililty that work in that area, such as (Cardinal et al. [2009]), may be leveraged to more efficiently find the closest sparse sets or subsets.

6. NONLINEAR TIME-VARYING CONTROL

It was shown in (Rotkowitz [2006]) that if we consider the design of possibly nonlinear, possibly time-varying (but still causal) controllers to stabilize possibly nonlinear, possibly time-varying (but still causal) plants, then while the quadratic invariance results no longer hold, the following condition

$$K_1(I \pm GK_2) \in S \qquad \text{for all } K_1, K_2 \in S$$

similarly allows for a convex parameterization of all stabilizing controllers subject to the given constraint.

This condition is equivalent to quadratic invariance when S is defined by delay constraints or by sparsity constraints, and so the algorithms in this paper may also be used to find the closest constraint for which this is achieved.

7. NUMERICAL EXAMPLES

We present some numerical examples of the algorithm developed in this paper.

We find the closest quadratically invariant superset, with respect to the following sparsity patterns of two plants with four subsystems each ($n = 4$):

$$G_I^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G_{II}^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{14}$$

The first sparsity pattern in (14) represents a plant where the first two control inputs effect not only their own subsystems, but also the subsequent subsystems, and where the last control input effects not only its own subsystem, but also the preceding subsystem. The second sparsity pattern represents a plant where each control input effects its own subsystem and the subsequent subsystem, which also corresponds to the open daisy-chain configuration. Now consider an initial proscribed controller configuration where the controller for each subsystem has access only to the measurement from its own subsystem:

$$K^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

that is, where the controller is block diagonal. Using the algorithm specified in (8)-(9) we arrive at:

$$Z_I^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Z_{II}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{16}$$

where Z_I^* and Z_{II}^* denote the optimal solution of (7) as applied to G_I^{bin} and G_{II}^{bin}, respectively, and thus represent the sparsity constraints of the closest quadratically invariant supersets of the set of block diagonal controllers. We see that a quadratically invariant set of controllers for the first plant (which contains all block diagonal controllers) has to have the same sparsity pattern as the plant, and an additional link from the first measurement to the third controller. We then see that any quadratically invariant set for the open daisy-chain configuration which contains the diagonal will have to be lower triangular.

8. CONCLUSIONS

The overarching goal of this paper is the design of linear time-invariant, decentralized controllers for plants comprising dynamically coupled subsystems. Given preselected constraints on the controller which capture the decentralization being imposed, we addressed the question of finding the closest constraint which is quadratically invariant under the plant. Problems subject to such constraints are amenable to convex synthesis, so this is important for bounding the optimal solution to the original problem.

We focused on a particular class of this problem where the decentralization imposed on the controller is specified by sparsity constraints; that is, each controller can access information from some subsystems but not others, and this is represented by a binary matrix. For this class of problems we showed that an algorithm which is guaranteed to give the closest quadratically invariant superset converges in at most $\log_2 n$ iterations, where n is the number of subsystems. We also provided a heuristic to give close quadratically invariant subsets.

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