SIAM REVIEW
Vol. 37, No. 4, pp. 531–551, December 1995
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# HISTORICAL DEVELOPMENT OF THE NEWTON–RAPHSON METHOD*

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**Abstract.** This expository paper traces the development of the Newton–Raphson method for solving nonlinear algebraic equations through the extant notes, letters, and publications of Isaac Newton, Joseph Raphson, and Thomas Simpson. It is shown how Newton's formulation differed from the iterative process of Raphson, and that Simpson was the first to give a general formulation, in terms of fluxional calculus, applicable to nonpolynomial equations. Simpson's extension of the method to systems of equations is exhibited.

**Key words.** nonlinear equations, iteration, Newton–Raphson method, Isaac Newton, Joseph Raphson, Thomas Simpson

**AMS subject classifications.** 01A45, 65H05, 65H10

## 1. Introduction.

The iterative algorithm

$$x_{i+1} = x_i - f(x_i)/f'(x_i) \tag{1.1}$$

for solving a nonlinear algebraic equation $f(x) = 0$ is generally called Newton's method. Occasionally it is referred to as the Newton–Raphson method. The method (1.1), and its extension to the solution of systems of nonlinear equations, forms the basis for the most frequently used techniques for solving nonlinear algebraic equations. In this expository paper we trace the development of the method (1.1) by exhibiting and analyzing relevant extracts from the preserved notes, letters, and publications of Isaac Newton, Joseph Raphson, and Thomas Simpson. Much of the sequence of events recounted here is familiar to historians, and the materials on which this paper is based are fairly readily available; hence we make no claims to originality. Our purpose is simply to provide a comprehensive account of the historical roots of the ubiquitous process (1.1), assembling a number of previously published accounts into a readily accessible whole.

In §§2 and 3 we show that methods which may be viewed as replacing the term $f'(x_i)$ in (1.1) by a finite difference approximation of the form

$$f'(x_i) \approx h_i^{-1}[f(x_i + h_i) - f(x_i)], \tag{1.2}$$

and also the secant method in which

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, \tag{1.3}$$

were precursors to the method (1.1). From a modern perspective each of the methods described by (1.1)–(1.3) arises naturally from a linearization of the equation $f(x) = 0$. In §4 we review Newton's original presentation of his method, contrasting this with the current formulation (1.1) and with Raphson's iterative formulation for polynomial equations discussed in §7. There is no clear evidence that Newton used any fluxional calculus in deriving his method, though we show in §6 that in the *Principia Mathematica* Newton applied his technique in an iterative manner to a nonpolynomial equation. Simpson's general formulation for nonlinear equations in terms of the fluxional calculus is presented in §8, and we discuss there Simpson's extension of the process to systems of nonlinear equations.

*Received by the editors October 6, 1993; accepted for publication (in revised form) January 12, 1995.
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In the absence of direct access to source materials we have based this paper on a recent translation of the relevant works of Viète [19], the invaluable volumes of Newton's mathematical papers collected and annotated by Whiteside [21], Newton's correspondence reproduced by Rigaud [14] and Turnbull [18], the third edition of Newton's *Principia Mathematica* in the edition by Cajori [4] with commentary by Koyré and Cohen [11], and copies of Raphson's text [12] and Simpson's book [16]. For references to the work and influences of other mathematicians we have relied on the notes in [13], [21] and related commentary in the concise summary of this material by Goldstine [6, pp. 64-68]. In our comments on the work of Newton we are heavily indebted to the notes in [21], while the recent papers [17] and [7] provide much insight into the contributions of Raphson and Simpson, respectively. The references, particularly [6] and [21], provide detailed bibliographic references to relevant source materials.

The biography by Westfall [20] includes incisive discussions of Newton's general mathematical development in addition to his many significant achievements—scientific and otherwise. The few verifiable facts concerning the life of Joseph Raphson are presented in [17]. The life of Thomas Simpson is described in [5], while a severely critical commentary on his character and achievements appears in [11].

**2. The method of Viète.** By late 1664, soon after his interest had been drawn to mathematics, Isaac Newton was acquainted with the work of the French algebraist François Viète (1540-1603) (often latinized to "Vieta"). Viète's work concerning the numerical solution of nonlinear algebraic equations, *De numerosa potestatum*, published in Paris in 1600, subsequently reappeared in a collection of Viète's works assembled and published as *Francisci Vietae Opera Mathematica* by Frans van Schooten in Leiden in 1646. A summary of that material, incorporating notational simplifications and some additional material, appeared in various editions of William Oughtred's *Clavis Mathematicae* from 1647 onward. A recent translation of the relevant work appears in [19], which includes a biography of Viète. Newton had access to both Schooten's collection and the third Latin edition of Oughtred's book, published in Oxford in 1652, and made extensive notes from them. These notes constitute a first sign of Newton's interest in the numerical solution of nonlinear equations.

Viète restricted his attention to monic polynomial equations. In modern functional notation we may write Viète's equations in the form

$$p(x) = N \tag{2.1}$$

in which the constant term $N$ appears on the right of the equation. Viète's technique can be regarded as yielding individual digits of the solution $x_*$ of (2.1) one by one as follows. Let the successive significant decimal digits of $x_*$ be $a_0, a_1, a_2, \ldots$, so that $x_* = a_0 10^k + a_1 10^{k-1} + a_2 10^{k-2} + \cdots$, and let the $i$th estimate $x_i$ of $x_*$ be given by $x_i = a_0 10^k + a_1 10^{k-1} + \cdots + a_i 10^{k-i}$. Assuming $x_i$ is given, we have $x_{i+1} = x_i + a_{i+1} 10^{k-(i+1)}$, and Viète's technique amounts to computing $a_{i+1}$ as the integer part of

$$\frac{N - .5\,(p(x_i + 10^{k-i-1}) + p(x_i))}{p(x_i + 10^{k-i-1}) - p(x_i) - 10^{(k-i-1)n}}, \tag{2.2}$$

where $n$ is the degree of the polynomial. Occasionally the quantity $.5\,(p(x_i + 10^{k-i-1}) + p(x_i))$ in the numerator of (2.2) is replaced by the value of $p(x_i)$. As noted by Maas [10], the integer $a_{i+1}$ may in fact be negative or have several digits, thus permitting the correction of earlier estimates $x_i$ of $x_*$.

Rewriting (2.1) as $f(x) = 0$ where $f(x) = p(x) - N$, the expression (2.2) can be reformulated as

$$\frac{-.5(f(x_i + 10^{k-i-1}) + f(x_i))}{f(x_i + 10^{k-i-1}) - f(x_i) - 10^{(k-i-1)n}},$$

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hence Viète's method is almost equivalent to

$$x_{i+1} = x_i - 10^{k-i-1}\left\lfloor \frac{f(x_i)}{f(x_i + 10^{k-i-1}) - f(x_i)} \right\rfloor. \tag{2.3}$$

This closely parallels the expression produced by substitution of (1.2) into (1.1) with $h_i = 10^{k-i-1}$. In this sense the method of Viète is a forerunner of the finite difference scheme (1.1)-(1.2), which is often presented as a modification of (1.1). The subtraction of the term $10^{(k-i-1)n}$ in the denominator of (2.2) can be motivated in the case that $p(x) = x^n$ by considering the binomial expansion of $(x_i + 10^{k-i-1})^n$; for monic quadratic equations the resulting method corresponds exactly to the Newton-Raphson method (1.1), but in practice this term is negligible and was often omitted [21, I]. An interesting comparison between this technique and the Newton-Raphson method for the case $p(x) = x^n$ is given in [10]. This technique was widely used until supplanted by the Newton-Raphson method.

In a portion of an unpublished notebook tentatively dated to late 1664, reproduced in [21, I, pp. 63-71], Newton made extensive notes on Viète's method. Figure 1 is a reproduction from [21] of Newton's modified transcript of Viète's solution of $x^3 + 30x = 14356197$. The exact solution $x_* = 243$ is computed. Here Newton used the "modified cossic" notation of Oughtred, in which $A$ and $E$ represent algebraic variables, while their $n$th powers for $n = 1, 2, 3, 4, 5, \ldots$ are represented by adjoining the symbols $l, q, c, qq, qc, \ldots$ respectively to the variables; thus $A^5$ is represented as $Aqc$. The line-by-line analysis below, based on (2.2) with $p(x) = x^3 + 30x$ and $N = 14356197$, amplifies the discussion of this extract in [21, I, pp. 66-67] and [6, p. 67]. Binomial expansions are used repeatedly to simplify the computational task.

(1) Line 1. $x = L$, $30 = C^2$, $N = 14356197 = P^3$, hence the phrases on the next two lines, a relic of Viète's insistence that all terms in a given equation must be of the same degree.

(2) Line 3. "pointing" (the lower dots), a technique developed by Oughtred [21], is used to obtain the initial estimate $x_0 = 200$; subsequently computed digits are adjoined to the initial 2.

(3) Lines 4-6. $p(x_0) = 8006000$ evaluated termwise as $200^3 + 30(200)$ .

(4) Lines 7, 9. $N - p(x_0) = 6350197$.

(5) Lines 8, 10-12. evaluation of $p(210) - p(200) - 10^3 = 1260300$ (cf. (2.2) with $k = 2, i = 0, n = 3$) using binomial expansions: $[(200 + 10)^3 + 30(200 + 10)] - [(200)^3 + 30(200)] - 10^3 = 3(200)^2 10 + 3(200)10^2 + 30(10) = 12(10^5) + 6(10^4) + 3(10^2) = 1260300$.

(6) In an unrecorded computation the next digit of $x_*$ is computed as 4, being the integer part of $(N - .5(p(210) + p(200)))/1260300 = 5719547/1260300 = 4.538\ldots$, producing $x_1 = 240$.

(7) Lines 13-17, 19. evaluation of $N - p(240) = 524997$: $N - p(240) = [N - p(200)] - [p(240) - p(200)] = 6350197 - [p(240) - p(200)]$ and the latter term is evaluated by binomial expansion as $[(200 + 40)^3 + 30(200 + 40)] - [(200)^3 + 30(200)] = 3(200)^2 40 + 3(200)40^2 + 40^3 + 30(40) = 48(10^5) + 96(10^4) + 64(10^3) + 12(10^2) = 5825200$.

(8) Lines 18, 20-22. $p(241) - p(240) - 1 = 173550$ (cf. (2.2) with $k = 2, i = 1, n = 3$) computed similarly to lines 8, 10-12.

(9) In another unrecorded computation $(N - p(240))/173550 = 524997/173550 = 3.025\ldots$ has integer part 3 so $x_2 = 243$.

(10) Line 28. $N - p(243) = 0$ is computed from $N - p(240)$ just as $N - p(240)$ was computed from $N - p(200)$ in lines 13-19.

In the final paragraph of his notes on the method of Viète [21, I, p. 71] Newton uses the Cartesian notation of lower-case symbols for algebraic variables and superscripts to represent powers (for example $x^5$); he used the latter notation in most of his subsequent work. He also briefly shifts the constant term in the equation to the left of the equality symbol, leaving zero on

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*The analysis of Cubick Equations.*

The equation supposed $Lc * + 30L = 14356197$. $Lc + CqL = Pc$.

| | | | | |
|---|---|---|---|---|
| The square coëfficient | | 3 | 0 | |
| The cube affected to be | 14 | 356 | 197 | (243 |
| Sollids to be substracted | 8 | | | $=Ac$ |
| | | 6 | 0 | $=ACq$ |
| Theire sum̄e | 8 | 006 | 0 | |
| Rests | 6 | 350 | 197 | for finding y<sup></sup>e 2d side. |
| The extraction of yᵉ seacond side | | | | |
| Coëfficient | | | 30 | or superior divisor. |
| The rest of yᵉ cube to be | 6 | 350 | 197 | resolved |
| The inferior divisors | 1 | 2 | | $3Aq$ |
| | | 6 | | |
| Their sum̄e | 1 | 260 | 30 | |
| Sollids to be subtracted | 4 | 8 | | $=3AqE$ |
| | | 96 | | $=3AEq$ |
| | | 64 | | $=Ec$ |
| | | 1 | 20 | $=ECq$ |
| Their sum̄e | 5 | 825 | 20 | |
| [The extraction of yᵉ 3d side] | | | | |
| The superior part of yᵉ divisor | | | 30 | or yᵉ square coefficient |
| The remainder for finding | | 524 | 997 | yᵉ third side |
| The inferior part of yᵉ divisor | | 172 | 8 | $3Aq$ that is $3 \times 24 \times 24$ |
| | | | 72 | $3A$ or $3 \times 24$ |
| The sum̄e of yᵉ divisors | | 173 | 550 | |
| Sollids to be taken away | | 518 | 4 | $3AqE$ |
| | | 6 | 48 | $3AEq$ |
| | | | 27 | $Ec$ |
| | | | 90 | $Ecq$ |
| Theire sum̄e | | 524 | 997 | |
| Remaines | | 000 | 000 | |

FIG. 1. *Newton's transcript of Viète's solution of* $x^3 + 30x - 14356197 = 0$.

the right, thus adopting the now conventional zero-finding formulation for solving nonlinear equations.

The general idea of solving an equation by improving an estimate of the solution by the addition of a correction term had been in use in many cultures for millenia prior to this time [6], [10]. Certain ancient Greek and Babylonian methods for extracting roots have this form, as do some methods of Arabic algebraists from at least the time of al-Khayyām (1048–1131). The precise origins of Viète's method are not clear, although its essence can be found in the work of the 12th century Arabic mathematician Sharaf al-Dīn al-Ṭūsī [13]. It is possible that the Arabic algebraic tradition of al-Khayyām, al-Ṭūsī, and al-Kashī survived and was known to 16th century algebraists, of whom Viète was the most important.

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**3. The secant method.** In a collection of unpublished notes termed "Newton's Waste Book" ([21, I, pp. 489-491] and there tentatively dated to early 1665) Newton demonstrates an iterative technique that we can identify as the "secant method" for solving nonlinear equations. In modern notation, this method for solving an equation $f(x) = 0$ is (1.1) with $f'(x_i)$ replaced by (1.3), that is

$$x_{i+1} = x_i - f(x_i)/\left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]. \tag{3.1}$$

An interesting historical discussion of this technique, and its relationship to the closely related method now commonly referred to as Regula Falsi, is given by Maas [10]. Several different arguments, all from a modern perspective essentially based on a linearization of the underlying function $f(x)$, yield this technique. The approach based on Fig. 2 below is consistent with Newton's computations. In both of the instances shown in Fig. 2 we note that, by similarity of the labelled triangles, $a/b = c/d$, that is $(f(x_i) - f(x_{i-1}))/(x_i - x_{i-1}) = \mp f(x_i)/d$; thus with $d \approx \pm(x_* - x_i)$, we get (with the appropriate choice of signs in each case) the formula (3.1) for $x_{i+1} \approx x_*$.

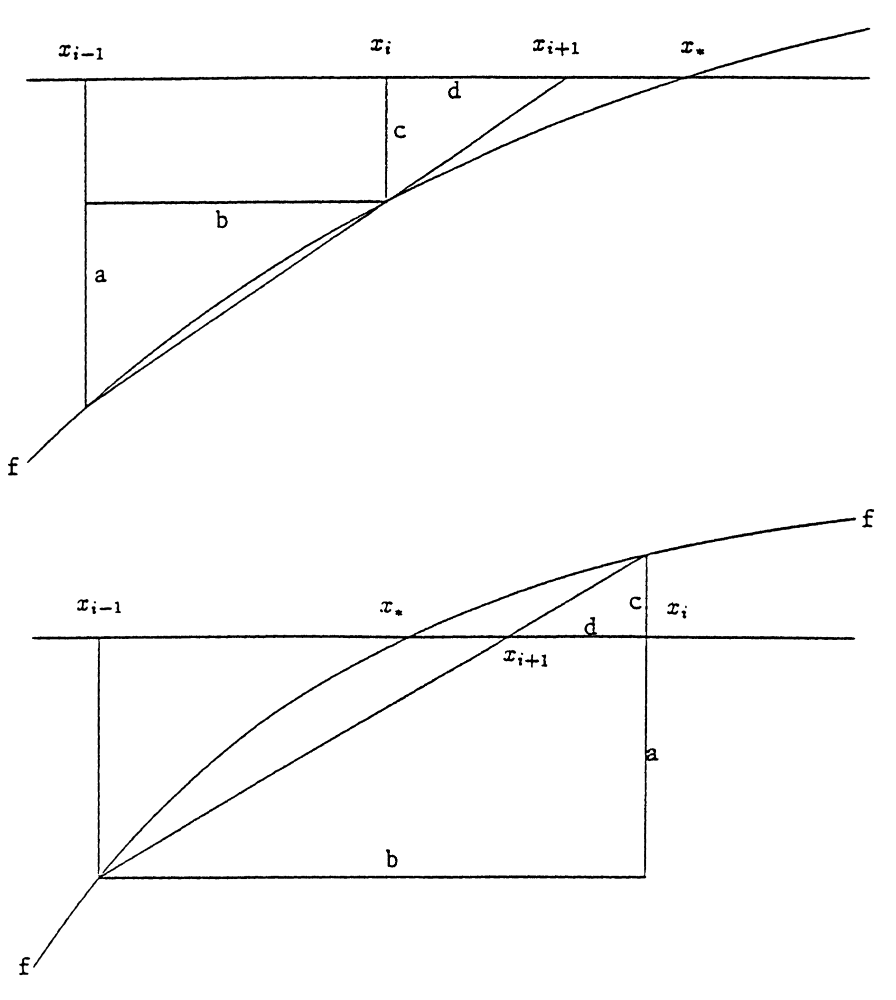


FIG. 2. *The secant method using similar triangles.*

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*The resolusion of $y^e$ affected Equation* $x^3+pxx+qx+r=0$. *Or* $x^3+10x^2-7x=44$.

First having fôund two or 3 of $y^e$ first figures of $y^e$ desired roote viz $2\lfloor 2$ ($w^{ch}$ may bee done either by rationall or Logarithmical tryalls as $M^r$ Oughtred hath tought, or Geometrically by descriptions of lines, or by an instrument consisting of 4 or 5 or more lines of numbers made to slide by one another $w^{ch}$ may be oblong but better circular) this knowne pte of $y^e$ root I call $g$, $y^e$ other unknowne pte I call $y$[,] then is $g+y=x$. Then I prosecute $y^e$ Resolution after this manner (making $x+p$ in $x=a$. $a+q$ in $x=b$. $b+r$ in $x=c$. &c.)

$$\frac{\times}{x=2,}\left|\begin{array}{l}12=x+p\\ \hline 24=a\end{array}\right. . \quad \begin{array}{r}a+q=17\\ \hline b=34\end{array}\left|\frac{\times}{2=x}\right. . \quad r\ -b=10=h. \text{ by supposing } x=2.$$

Againe supposing $x=2\lfloor 2$.

$$\begin{array}{r|l} x+p=12,2 & \\ \hline 244 & 2\\ 244\ & 2\\ \hline 26,84=a & \end{array} \qquad \begin{array}{r|l} a+q=19,84 & \\ \hline 3968 & 2\\ 3968\ & 2\\ \hline 43,648=b & \end{array}$$

$r\ -b=00,352=k$. $h-k=9,648$. That is $y^e$

$$\left.\begin{cases}\text{latter } r-b \text{ substracted from the former } r-b \text{ there remaines}\\ \text{difference twixt this \& } y^e \text{ former valor of } r-b \text{ is}\end{cases}\right\}9,648.$$

& $y^e$ difference twixt this & $y^e$ former valor of $x$ is 0,2. Therefore make

$$9,648:0,2::0,352:y.$$

Then is $y=\dfrac{0,0704}{9,648}=0,00728$ &c. the first figure of $w^{ch}$ being added to $y^e$ last valor of $x$ makes $2,207=x$. Then $w^{th}$ this valor of $x$ prosecuting $y^e$ $\text{oper}\overline{\text{acon}}$ as before tis

$$\begin{array}{r|l} x+p=12,207 & \\ \hline 85449 & 7\\ 2\,44140 & 02\\ 24\,414\ \ & 2\\ \hline 26,940849=a & \end{array} \qquad \begin{array}{r|l} a+q=19,94084 & \\ \hline 13958588 & 7\\ 39881680 & 02\\ 3988168\ \ & 2\\ \hline 44,00943388=b & \end{array} \qquad r\ -b=-0,00943388.$$

$w^{ch}$ valor of $r-b$ substracted from $y^e$ precedent valor of $r-b$ $y^e$ diff: is $+0,36143388$. Also $y^e$ diff[:] twixt this & $y^e$ precedent valor of $x$ is 0,007. Therefore I make $0,36143388:0,007::-0,00943388:y$. That is

$$y=\frac{-5,903716}{36143388}=-0,0001633 \text{ \&c.}$$

2 figures of $w^{ch}$ (because negative) I substract from $y^e$ former value of $x$ & there rests $x=2,20684$. And so might $y^e$ Resolution be prosecuted.

FIG. 3. *Newton uses the secant method to solve* $x^3+10x^2-7x-44=0$.

We now give a detailed analysis of Newton's computations, reproduced in Fig. 3 from [21], to solve $x^3+10x^2-7x=44$ using this method. We write $p(x)=x^3+10x^2-7x$, $N=44$, and $f(x)=p(x)-N$.

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Assume given (by one of a variety of means) two initial estimates $x_0 = 2$ and $x_1 = 2.2$ of the exact solution $x_* = 2.2068173\ldots$. First $p(2) = 34$ is evaluated (by nested multiplication), producing $N - p(2) = 10 = -f(2)$. Similarly $p(2.2) = 43.648$ and $-f(2.2) = .352$ are computed. Thus $f(2.2) - f(2) = 9.648$, hence from the proportionality $9.648/.2 = .352/y$, that is $(f(2.2) - f(2))/(2.2 - 2) = -f(2.2)/y$, where $y$ is the correction to be added to the current estimate of $x_*$, we obtain the exact value $y = .0072968\ldots$, given by Newton as .00728 and truncated to .007, thus producing $x_2 = 2.207$. Then $p(2.207) = 44.00945374$ is evaluated (with truncation of one intermediate digit, using nested and long multiplication) as 44.00943388, producing $-f(2.207) = -.00943388$. Hence $f(2.207) - f(2.2) = .36143388$, so that from $.36143388/0.007 = -.00943388/y$, that is $(f(2.207) - f(2.2))/(2.207 - 2.2) = -f(2.207)/y$, we obtain (apparently with an error in transcription) $y = (.007)(-.00843388)/(.36143388) = -5.903716/36143.388 = -.0001633\ldots$, truncated to $-.00016$. Thus $x_3 = 2.207 - .00016 = 2.20684$.

Clearly Newton had progressed beyond Viète's method in no longer constructing $x_*$ digitwise, but the influence of that process is still evident in that Newton truncates $y$ to only its first one or two significant digits to obtain the next correction term.

**4. Newton's method—first formulation.** Newton's tract *De analysi per aequationes numero terminorum infinitas* (henceforth *De analysi* for brevity), probably dating from mid 1669, is noted chiefly for its initial announcement of the principle of fluxions. The extract from that tract quoted below, in the translation into English of [21, II, pp. 218-223], is the first recorded discussion by Newton of what we may recognize as an instance of the Newton-Raphson method (1.1), although the formulation differs considerably from the now conventional form, the computations are much more tedious than in the current formulation, and the method is given only in the context of solving a polynomial equation. No calculus is used in the presentation, and references to fluxional derivatives first appear later in that tract, suggesting that Newton regarded this as a purely algebraic procedure. In several other instances Newton is known to have used more traditional methods and notations in an effort to make his ideas more accessible to a wider audience, but there is no clear evidence that at that time he perceived this particular technique as an application of the calculus or derived it using the techniques of calculus. The general role of calculus in the historical development of (1.1) is surveyed in [7].

Newton's technique may be described in modern functional notation as follows. Let $x_0$ be a given first estimate of the solution $x_*$ of $f(x) = 0$. Write $g_0(x) = f(x)$, and suppose $g_0(x) = \sum_{i=0}^{n} a_i x^i$. Writing $e_0 = x_* - x_0$ we obtain by binomial expansion about the given $x_0$ a new polynomial equation in the variable $e_0$:

$$(4.1)\quad 0 = g_0(x_*) = g_0(x_0 + e_0) = \sum_{i=0}^{n} a_i (x_0 + e_0)^i = \sum_{i=0}^{n} a_i \left[ \sum_{j=0}^{i} \binom{i}{j} x_0^j e_0^{i-j} \right] = g_1(e_0).$$

Neglecting terms involving higher powers of $e_0$ (effectively linearizing the explicitly computed polynomial $g_1$) produces

$$0 = g_1(e_0) \approx \sum_{i=0}^{n} a_i [x_0^i + i x_0^{i-1} e_0] = \left[ \sum_{i=0}^{n} a_i x_0^i \right] + e_0 \left[ \sum_{i=0}^{n} a_i i x_0^{i-1} \right]$$

from which we deduce that

$$e_0 \approx c_0 = - \left[ \sum_{i=0}^{n} a_i x_0^i \right] \Bigg/ \left[ \sum_{i=0}^{n} a_i i x_0^{i-1} \right]$$

and set $x_1 = x_0 + c_0$. Formally this correction can be written $c_0 = -g_0(x_0)/g_0'(x_0) = -f(x_0)/f'(x_0)$. Now repeat the process, but instead of expanding the original polynomial

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$g_0$ about $x_1$ expand the polynomial $g_1$ obtained explicitly in (4.1) about the point $c_0$, i.e., $c_0$ is considered to be a first estimate of the solution $e_0$ of the new equation $g_1(x) = 0$. Thus similarly obtain $0 = g_1(e_0) = g_1(c_0 + e_1) = g_2(e_1)$, where the polynomial $g_2$ is explicitly computed. Linearizing again produces as before a correction formally equivalent to $e_1 \approx c_1 = -g_1(c_0)/g_1'(c_0)$, corresponding to $c_1 = -g_0(x_0 + c_0)/g_0'(x_0 + c_0) = -f(x_1)/f'(x_1)$ and $x_2 = x_1 + c_1$ . The process continues by expanding $g_2$ about $c_1$, and so on.

In Fig. 4 we reproduce the English translation of [21, II, pp. 219–220] of Newton's solution of $g(x) = x^3 - 2x - 5 = 0$ (a standard test problem of the era [6, p. 64]) by this method. Our analysis of this passage uses the notation introduced above.

(1) Line 1. take $x_0 = 2$; the successive estimates of the solution $x_* = 2.09455148\ldots$ are accumulated here.

(2) Lines 2–5. expand $g_0$ as $g_0(x_*) = g_0(2 + p) = p^3 + 6p^2 + 10p - 1 = g_1(p) = 0$ using the binomial expansion; omitting higher order terms leaves $10p - 1 \approx 0$ hence $p \approx .1$ and $x_1 = 2.1$.

(3) Lines 6–10. expand $g_1(p) = g_1(.1 + q) = q^3 + 6.3q^2 + 11.23q + .061 = g_2(q)$; truncated to $11.23q + .061 \approx 0$ this gives $q \approx -.00543\ldots$, which is rounded to $-.0054$, hence $x_2 = 2.0946$.

(4) Lines 11–14. expand $g_2(q) = g_2(-.0054 + r) = 6.3r^2 + 11.16196r + .000541708 = g_3(r)$ (where the term $q^3$ in $g_2$, being small, has been omitted), which produces after linearization $r \approx -.00004853\ldots$ and hence $x_3 = 2.09455147$ after truncation.

The process described by Newton requires the explicit computation of the successive polynomials $g_1, g_2, \ldots$, which makes it laborious. Clearly calculus is not used by Newton in his presentation, which is based entirely on retention of the lowest order terms in a binomial expansion. Also note that the final estimate of $x_*$ is only computed at the end of the process as $x_* = x_0 + c_0 + c_1 + \cdots$ instead of successive estimates $x_i$ being updated and used successively. Clearly this process is significantly different from the iterative technique currently in use. This extract hints that Newton was already aware of the quadratic convergence of the technique, as characterized by the approximate doubling of the number of correct significant digits in successive steps: observe that the number of digits retained by Newton in successive steps doubles.

Newton gave no further explanation of his method, though he used it in an analytic form a few pages later in his tract and, as we shall see in §§5 and 6, it reappears in a later letter and in his *Principia Mathematica*. A slightly revised version of the passage quoted above was incorporated in the opening pages of Newton's more comprehensive tract *De methodis fluxionum et serierum infinitarum* written in 1671. The unfinished manuscript of *De methodis* was initially intended for publication, but the unprofitable nature of mathematical publishing combined with Newton's reluctance to publish following the controversy surrounding his "New Theory about Light and Colors" suppressed the work at the time. *De analysi* was not published until 1711, in *Analysis per quantitatum series, fluxiones, ac differentias...* by William Jones, by which time its status was largely historical. *De methodis* was not published until 1736 in translation by John Colson [21, III, p. 13]. Nevertheless various copies of Newton's manuscripts circulated among leading mathematicians. Newton gave a copy of *De analysi* to Isaac Barrow, who sent a copy to John Collins, who circulated news of the work among his international correspondents, some of whom were privileged to make their own copies of portions of the work. Among the most interesting of these are the extracts made by Leibniz in 1676 during a visit to London, reproduced in [21, II, pp. 248–259], which include the passage analyzed above almost verbatim while omitting much of the material on calculus. The earliest printed account of Newton's method, including essentially the content of Fig. 4, is in chapter 94 of John Wallis' *A Treatise of Algebra both Historical and Practical*, London, 1685.

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EXAMPLES BY THE RESOLUTION OF AFFECTED EQUATIONS. Since the difficulty here lies wholly in the resolution technique, I will first elucidate the method I use in a numerical equation. The numerical resolution of affected equations.

Suppose $y^3-2y-5=0$ is to be resolved: and let 2 be the number which differs from the root sought by less than its tenth part. Then I set $2+p=y$ and substitute this value for it in the equation, and in consequence there arises the new equation

$$p^3+6p^2+10p-1=0$$

whose root $p$ must be sought for it to be added to the quotient: specifically (when $p^3+6p^2$ are neglected on account of their smallness)

$$10p-1=0$$

or $p=0{\cdot}1$ very nearly true; and so I write 0·1 in the quotient and suppose $0{\cdot}1+q=p$,

$$\begin{cases} +2{\cdot}10000000 \\ -0{\cdot}00544853 \end{cases}$$
$$2{\cdot}09455147$$

| | | |
|---|---|---|
| $2+p=y$ | $y^3$ | $+8+12p+6p^2+p^3$ |
| | $-2y$ | $-4-2p$ |
| | $-5$ | $-5$ |
| | Sum | $-1+10p+6p^2+p^3$ |
| $0{\cdot}1+q=p$ | $+p^3$ | $+0{\cdot}001+0{\cdot}03q+0{\cdot}3q^2+q^3$ |
| | $+6p^2$ | $+0{\cdot}06\quad+1{\cdot}2\quad+6{\cdot}0$ |
| | $+10p$ | $+1\quad\quad+10$ |
| | $-1$ | $-1$ |
| | Sum | $+0{\cdot}061+11{\cdot}23q+6{\cdot}3q^2+q^3$ |
| $-0{\cdot}0054+r=q$ | $6{\cdot}3q^2$ | $+0{\cdot}000183708-0{\cdot}06804r+6{\cdot}3r^2$ |
| | $+11{\cdot}23q$ | $-0{\cdot}060642\quad+11{\cdot}23$ |
| | $+0{\cdot}061$ | $+0{\cdot}061$ |
| | Sum | $+0{\cdot}000541708+11{\cdot}16196r+6{\cdot}3r^2$ |
| $-0{\cdot}00004853$ | | |

and on substituting this value for it, as before, there arises in consequence

$$q^3+6{\cdot}3q^2+11{\cdot}23q+0{\cdot}061=0.$$

And, since $11{\cdot}23q+0{\cdot}061[=0]$ approaches the truth closely or there is almost $q=-0{\cdot}0054$ (by dividing, that is, until as many figures are elicited as the number of places by which the first figures of this and of the principal quotient are distant one from the other), I write $-0{\cdot}0054$ in the lower part of the quotient since it is negative. Again, supposing $-0{\cdot}0054+r=q$, I substitute this as before, and in this way continue the operation as far as I please. But if I desire to continue working merely to twice as many figures, less one, as are now found in the quotient, in place of $q$ in this equation

$$6{\cdot}3q^2+11{\cdot}23q+0{\cdot}061[=0]$$

I substitute $-0{\cdot}0054+r$, neglecting its first term $q^3$ by reason of its insignificance, and there arises $6{\cdot}3r^2+11{\cdot}16196r+0{\cdot}000541708=0$ nearly, or (when $6{\cdot}3r^2$ is rejected) $r=\dfrac{-0{\cdot}000541708}{11{\cdot}16196}=-0{\cdot}00004853$ nearly. This I write in the negative part of the quotient. Finally, on taking the negative portion of the quotient from the positive part, I have the required quotient 2·09455147.

FIG. 4. *Newton's method for solving $x^3-2x-5=0$.*

A method algebraically equivalent to Newton's method was known to the 12th century algebraist Sharaf al-Dīn al-Ṭūsī [13], and the 15th century Arabic mathematician Al-Kāshī used a form of it in solving $x^p-N=0$ to find roots of $N$. In western Europe a similar method was used by Henry Briggs in his *Trigonometria Britannica*, published in 1633, though Newton appears to have been unaware of this [21, II, pp. 221–222].

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But yet I conceive these roots may be easilier extracted by logarithmes. Suppose $c$ as nearly as you can guesse equall to ye root $z$: & if ye æquation be

$z^n=bz+R$ make $\sqrt{(n)}\,\overline{bc+R}$ $\left(\text{or } \sqrt{(n)}\,\overline{\overline{c+\frac{R}{b}\times b}}\right)=d.$ $\sqrt{(n)}\,\overline{\overline{d+\frac{R}{b}\times b}}=e$

$\sqrt{(n)}\,e+\frac{R}{b}\times b=f.$ $\sqrt{(n)}\,f+\frac{R}{b}\times b=g$ &c. Or if ye æquation be $z^n\pm bz^p=R$,

make $\sqrt{(p)}\,\frac{R}{b+c}=d.$ $\sqrt{(p)}\,\frac{R}{b+d}=e$ $\sqrt{(p)}\,\frac{R}{b+e}=f$ &c. And so shall ye last found terme $e, f$, or $g$ &c be ye desired root $z$.

For instance if $z^{30}=8z-5$: suppose $c=1$, & ye work will be this $\sqrt{(30)}\,8\times 1-5$, or $\sqrt{(30)}\,3=1\lfloor\underline{0373}=d.$ $\sqrt{(30)}\,3\lfloor\underline{2984}=1\lfloor\underline{0407}=e$,
$\sqrt{(30)}\,3\lfloor\underline{3256}=1\lfloor\underline{040894}=f$ etc

Therefore ye root $z$ is $1\lfloor\underline{040894}$

So if $z^{22}+120z^{21}=3748000$ suppose $c=1$, & ye work will be

$$\sqrt{(21)}\,\frac{3748000}{121}=1\lfloor\underline{6363}=d. \quad \sqrt{(21)}\,\frac{3748000}{121\lfloor\underline{6363}}=1\lfloor\underline{63587}=e \text{ \&c.}$$

Therefore ye root $z$ is $1\lfloor\underline{63587}$

FIG. 5. *Newton's use of fixed point iterations.*

**5. Further methods for nonlinear equations.** Newton displayed a continued interest in methods for the numerical solution of nonlinear equations during the years following 1671. Some of his activities are summarized in [6, pp. 65-66]. For example, in a letter to Michael Dary dated 6 October 1674 [18, I, pp. 319-322] Newton proposed schemes, which we may write as

$$x_{i+1} = (bx_i + c)^{(1/n)} \quad \text{and} \quad x_{i+1} = \left(\frac{c}{b \pm x_i}\right)^{(1/(n-1))},$$

for solving the equations $x^n = bx + c$ and $x^n \pm bx^{n-1} = c$, respectively. Equations of this sort arise in the analysis of annuities. We now recognize these schemes as examples of fixed point (or functional) iteration. Similar schemes were previously proposed by James Gregory and communicated in letters to John Collins to solve the equations $b^n c + x^{n+1} = b^n x$ (8 November 1672) and $b^n c + x^{n+1} = b^{n-1}(b+c)x$ (2 April 1674) [18, I, pp. 321-322] and [6, pp. 65-66].

In Fig. 5 we reproduce from [18] a portion of Newton's letter to Michael Dary, giving these iterative schemes and applying them to the equations $x^{30} = 8x - 5$ and $x^{22} + 120x^{21} = 3748000$, respectively, in each case with $x_0 = 1$. The text is largely self-explanatory once it is understood that the notation $\sqrt{\odot}$ denotes taking the indicated root of the subsequent quantity, and that the symbol $\lfloor$ is used instead of the decimal point. There are minor errors in the fifth and subsequent significant digits in Newton's computations.

Also of interest are two letters by Newton addressed to John Smith, dated 24 July and 27 August 1675. Smith was preparing a table of square, cube, and quartic roots of all the integers from 1 through 10000; in a letter dated 8 May 1675 [18, I, pp. 342-345] Newton suggested that he do this by computing only the roots of every hundredth integer to a sufficient number of digits (10), followed by interpolation to produce all the other desired quantities to the required accuracy (8 digits). Newton presented in his letters to Smith several schemes, which we may synthesize as

$$x_1 = (1/n)[(n-1)x_0 + a/x_0^{n-1}], \qquad n = 2, 3, 4, \tag{5.1}$$

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1. When you have extracted any ℞ by common Arithmetick to 5 Decimal places, you may get the figures of the other 6 places by Dividing only the Residuum by

$$\left\{\begin{array}{l}\text{double the Quotient}\\ \text{triple the } q \text{ of Quotient}\\ \text{quadruple the } c \text{ of Quotient}\end{array}\right\} \text{ for the ℞ } \left\{\begin{array}{l}\text{square}\\ \text{cube}\\ \text{square square}\end{array}\right.$$

Suppose $B$. the Quotient or ℞ extracted to 5 Decimal places, and $C$. the last Residuum, by the Division of wch you are to get the next figure of the Quotient, and $D$ the Divisor (that is $2B$ or $3BB$ or $4B^{\cdot c \cdot} = D$) & $B + \frac{C}{D}$ shall be the ℞ desired. That is, the same Division, by wch you would finde the 6th decimal figure, if prosecuted, will give you all to the 11th decimal figure.

2. You may seek the ℞ if you will, to 5 Decimal places by the logarithm's, But then you must finde the rest thus. Divide the propounded number

$$\left.\begin{array}{l}\text{once}\\ \text{twice}\\ \text{thrice}\end{array}\right\} \text{ by yt ℞ prosecuting the Division alwayes to 11 Decimal places, and}$$

to the Quotient add

$$\text{ye said ℞ } \left\{\begin{array}{l}\text{once, \& halfe}\\ \text{twice, \& a third part}\\ \text{thrice, \& a quarter}\end{array}\right\} \begin{array}{l}\text{of the summ}\\ \text{shall be the}\end{array} \left\{\begin{array}{l}\text{square}\\ \text{Cube}\\ \text{square square}\end{array}\right\} \text{ ℞ desired.}$$

For instance

Let $A$ be the number, and $B$. its $\left\{\begin{array}{l}Q\\ C\\ QQ\end{array}\right.$ ℞ extracted by Logarithms unto 5 decimal places:

$$\text{and} \quad \left.\begin{array}{l}2)\ B + \frac{A}{B},\\[2ex] 3)\ 2B + \frac{A}{B^2}, \text{ shall be the}\\[2ex] 4)\ 3B + \frac{A}{B^3},\end{array}\right\} \begin{array}{l}Q\\[2ex] C \quad \text{root desired}\\[2ex] QQ\end{array}$$

FIG. 6. *Newton's method for extracting roots of numbers.*

for finding the second, third, and fourth roots of a given positive scalar $a$, intending them to be used to compute the required roots of every hundredth integer to the required accuracy. Formula (5.1) is (1.1) with $i = 0$ used to solve $f(x) = x^n - a = 0$ for arbitrary $n$. This technique had previously been used by the Arabic mathematician al-Kāshī [13], and forms of it appear in earlier writings. This formula may also be obtained by applying just the first step of the technique of §4 to the polynomial equation $x^n - a = 0$. We reproduce in Fig. 6, and analyze below, an extract from [18, I, pp. 348–349] of the letter dated 24 July 1675 in which Newton discusses this technique for finding roots. Further discussion of this material appears in [21, IV, pp. 663–665].

Here Newton is solving the equation $x^n - A = 0$, for $n = 2, 3, 4$ starting from an initial estimate $x_0 = B$ and writing $C = A - B^n = a - x_0^n$. He partly reverts to the modified cossic notation described in §2. In the first paragraph he gives the method in a form corresponding

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to $x_1 = x_0 + c_0$, where

$$c_0 = \frac{C}{nB^{n-1}} = \frac{a - x_0^n}{nx_0^{n-1}} = \frac{-f(x_0)}{f'(x_0)}$$

and notes that one should retain all the computed digits for use as the correction term rather than truncate after the first few significant digits. In the next paragraph the method is reformulated corresponding to (5.1), with the symbol $n)$ denoting division by $n$. Newton shows no sign of using his formulae iteratively, given his instructions to select a first estimate of the solution with five correct significant digits to get eleven correct digits after one application of the formula, rather than indicating that any desired accuracy can be obtained from an appropriate initial estimate by simply applying the formula repeatedly. This passage again shows that Newton was aware of the rough doubling of the number of correct significant digits in one step of his process, characteristic of the quadratic convergence of the process (1.1).

For completeness we mention also a letter from Newton to John Collins, dated 20 August 1672 [18, I, pp. 229–234] in which he describes the use of "Gunters line" (that is, a logarithmically graduated ruler) to solve polynomial equations. Since this ingenious method is not iterative, we refer the interested reader to [17] and [21, III, pp. 559–561] for further details.

**6. Newton's method.** The first published use by Newton of the method (1.1) in an iterative form and applied to a nonpolynomial equation is in the second and third editions of his *Philosophiae Naturalis Principia Mathematica*, whose first edition was published in London in 1687. In each successive edition he described techniques for the solution of Kepler's equation

$$x - e\sin(x) = M. \tag{6.1}$$

To understand Newton's geometrical technique and relate it to the conventional analytic form of the Newton–Raphson method it is necessary to have an understanding of the terms involved in (6.1). Our description below is based on [15, pp. 72–85].

The origins of the problem lie in determining the position of a planet moving in an elliptical orbit around the sun, given the length of time since perihelion passage. With reference to Fig. 7, let the ellipse (the orbit) $ABA'B'$ centered at the origin $O$ be defined by the canonical equation

$$(y^2/a^2) + (z^2/b^2) = 1. \tag{6.2}$$

Then the ellipse has a semimajor axis $AO$ of length $a$ and a focus (the sun) at $S = -b = -ae$, where $e = b/a$ is the *eccentricity* of the ellipse. Let $ACA'C'$ be a circle centered on the origin with radius $a$ circumscribing the ellipse. Let $P$ (the planet) be a point on the ellipse whose Cartesian coordinates are to be determined, and let $QPR$ be a line perpendicular to $AO$ passing through $P$ and intercepting the circumscribed circle and the line $AO$ at the points $Q$ and $R$, respectively. Then the Cartesian coordinates of $P$ are defined by the lengths $|PR|$ and $|OR|$. Given the readily proved fact that $|PR|/|QR| = e$, so that we need find only $|QR|$ and $|OR|$, it is easy to see that knowledge of the *eccentric anomaly*, i.e., the angle $x = \angle AOQ$, is sufficient to locate $P$: $\sin(x) = |QR|/|QO| = |QR|/a$ and $\cos(x) = |RO|/|QO| = |RO|/a$, thus $|PR| = ea\sin(x)$ and $|OR| = a\cos(x)$. The eccentric anomaly $x$ is to be computed.

Suppose that $P$ represents the position of a planet in an elliptic orbit about the sun $S$, at a time $t$ after passing through the point $A$ (*perihelion passage*) at time 0. If the planet has orbital period $T$, then since the radius vector $SP$ turns through an angle of $2\pi$ radians in the course of one orbit, the *mean angular velocity* of the planet is $n = 2\pi/T$. During the time $t$ the angle swept out by a radius vector rotating about $S$ with angular velocity $n$ is the *mean anomaly* $M = nt$. A clever argument [15, pp. 83–85] exploiting Kepler's laws of planetary

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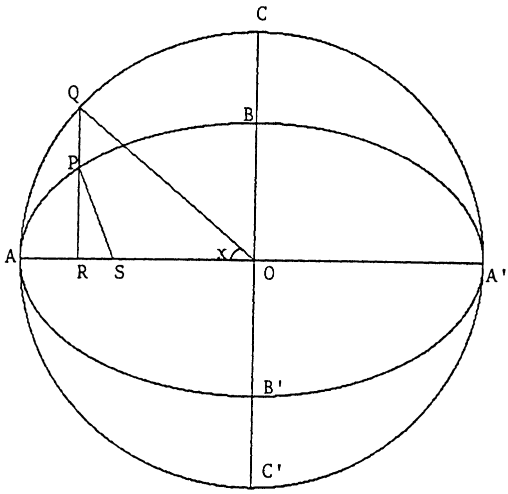


FIG. 7. *Locating the position of a planet in an elliptical orbit.*

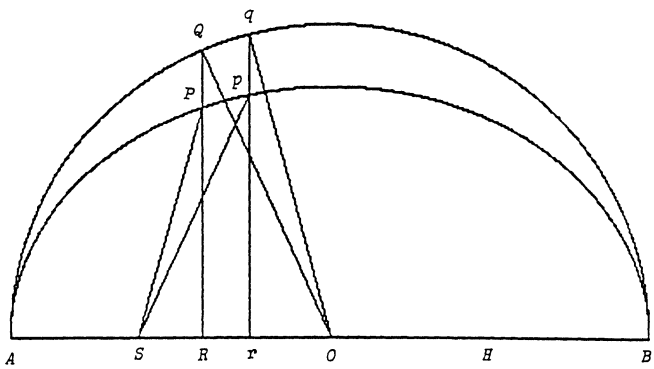


FIG. 8. *Diagram to accompany Newton's solution of Kepler's equation.*

motion reveals that the eccentric anomaly $x$, the mean anomaly $M$ and the eccentricity of the ellipse $e$ are related through (6.1), where both $M$ and $x$ are in radian measure. Thus in (6.1) we are to solve for $x$, being the angle $\angle AOQ$, given $M$ and $e$, where $M$ is computed from $t$ as $M = 2\pi t/T$.

The historical origins of this problem are discussed in [21, IV, pp. 668–669]. Newton's interest in Kepler's problem is first revealed in a letter to Henry Oldenburg dated 13 June 1676 [18, II, pp. 20–47] in which he derives a series expansion for the quantity $|PR|$ but gives no numerical algorithm for solving (6.1).

We reproduce here the text and accompanying figure (Fig. 8) of Book 1, Proposition 31, Scholium, from Cajori's edition of 1934 [4, pp. 113–114] of the third edition of the *Principia*, published in Latin in 1726 and translated by Andrew Motte into English in 1729. Here Newton presents his technique for solving (6.1) numerically.

> "But since the description of this curve is difficult, a solution by approximation will be preferable. First, then, let there be found a certain angle B which may be to an angle of 57.29578 degrees, which an arc equal to the

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> radius subtends, as SH, the distance of the foci, to AB, the diameter of the ellipse. Secondly, a certain length L, which may be to the radius in the same ratio inversely. And these being found, the Problem may be solved by the following analysis. By any construction (or even by conjecture), suppose we know P the place of the body near its true place *p*. Then letting fall on the axis of the ellipse the ordinate PR from the proportion of the diameters of the ellipse, the ordinate RQ of the cicumscribed circle AQB will be given; which ordinate is the sine of the angle AOQ, supposing AO to be the radius, and also cuts the ellipse in P. It will be sufficient if that angle is found by a rude calculus in numbers near the truth. Suppose we also know the angle proportional to the time, that is, which is to four right angles as the time in which the body described the arc A*p* to the time of one revolution in the ellipse. Let this angle be N. Then take an angle D, which may be to the angle B as the sine of the angle AOQ to the radius; and an angle E which may be to the angle N − AOQ + D as the length L to the same length L diminished by the cosine of the angle AOQ, when that angle is less than a right angle, or increased thereby when greater. In the next place, take an angle F that may be to the angle B as the sine of the angle AOQ + E to the radius, and an angle G, that may be to the angle N − AOQ − E + F as the length L to the same length L diminished by the cosine of the angle AOQ + E when that angle is less than a right angle, or increased thereby when greater. For the third time take an angle H, that may be to the angle B as the sine of the angle AOQ + E + G to the radius; and an angle I to the angle N − AOQ − E − G + H, as the length L is to the same length L diminished by the cosine of the angle AOQ + E + G, when that angle is less than a right angle, or increased thereby when greater. And so we may proceed *in infinitum*. Lastly, take the angle AO*q* equal to the angle AOQ + E + G + I +, etc., and from its cosine O*r* and the ordinate *pr*, which is to its sine *qr* as the lesser axis of the ellipse to the greater, we shall have *p* the correct place of the body. When the angle N − AOQ + D happens to be negative, the sign + of the angle E must be everywhere changed into −, and the sign − into + . And the same thing is to be understood of the signs of the angles G and I, when the angles N − AOQ − E + F, and N − AOQ − E − G + H come out negative. But the infinite series AOQ + E + G + I + etc., converges so very fast, that it will be scarcely ever needful to proceed beyond the second term E. And the calculus is founded upon this Theorem, that the area APS varies as the difference between the arc AQ and the right line let fall from the focus S perpendicularly upon the radius OQ."

This passage may be understood as follows. Implicitly assume that the horizontal axis has been scaled so that "the radius" $a = 1$. Let the angle $\angle B$ equal $e$ radians (the ratio of the angle $\angle B$ to 57.29578 degrees is set equal to $2b/2a = e$, and 57.29578 is approximately one radian, as Newton notes, since the "angle ... which an arc equal to the radius subtends" (in degrees on the circle) is $360a/(2\pi a) = 360/(2\pi) = 1$ radian). Let $L$ be such that $L/a = 1/e$ (thus $L^{-1} = e$). Let $p$ be the position of the planet, and $P$ be a first estimate of $p$. Note (as above) that to locate $p$ it suffices to compute the angle $\angle AOq$, the eccentric anomaly. Let the angle $\angle AOQ = x_0$ be our first estimate of $\angle AOq$. Determine $\angle AOq$ as follows. Let the angle $\angle N$ be such that $\angle N/(2\pi) = t/T$ (thus $\angle N = 2\pi t/T = M$, the mean anomaly). Let the angle $\angle D$ be such that $\angle D/\angle B = \sin(\angle AOQ)/a$ (thus $\angle D = e\sin(x_0)$), and let the angle $\angle E$ (the correction $c_0$ to $x_0 = \angle AOQ$) be such that

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$$\frac{\angle E}{\angle N - \angle AOQ + \angle D} = \frac{L}{L - \cos(\angle AOQ)}$$

equivalently

$$c_0 = \angle E = \frac{L(\angle N - x_0 + \angle D)}{L - \cos(x_0)} = \frac{M - x_0 + e\sin(x_0)}{1 - e\cos(x_0)}, \tag{6.3}$$

which produces $x_1 = x_0 + c_0 = \angle AOQ + \angle E$. Similarly determine, using the same formula, successively $\angle G = c_1$, $\angle I = c_2$, etc. from $x_1 = \angle AOQ + \angle E$ and $x_2 = \angle AOQ + \angle E + \angle G$, etc., respectively, and set $\angle AOq = \angle AOQ + \angle E + \angle G + \angle I + \cdots$, i.e., $x_* = x_0 + c_0 + c_1 + c_2 + \cdots$.

Equation (6.3) is precisely $c_0 = -f(x_0)/f'(x_0)$ with $f(x) = x - e\sin(x) - M$, and the subsequent corrections $c_i$ are similarly equivalent to applications of (1.1).

Apparently the first to recognize this passage as an instance of the Newton-Raphson method was John Couch Adams in 1882 [1]. Newton's method is used here for the first time in an iterative process to solve a nonpolynomial equation. This contrasts with the frequently repeated claim [3] that Newton used his method only for rational integral polynomial equations, with the extension to irrational and transcendental equations being made first by Thomas Simpson as described in §8 below. However, given the geometrical obscurity of the argument, it seems unlikely that this passage exerted any influence on the historical development of the Newton-Raphson technique in general.

There is again no clear evidence that Newton associated his technique with the use of the calculus. There are numerous ways to derive this process that do not require the use of calculus; for example, a purely geometric derivation is given in the third of Simpson's *Essays* [16]. The passage just quoted from the third edition of the *Principia* duplicates the corresponding passage from the second edition of 1713. This passage is a modification of the corresponding passage from the first edition of 1687. These different versions are discussed in [3], [21, VI, pp. 314-318] and [8, I, pp. 191-196] and suggest a derivation of this method consistent with the approach previously presented by Newton in *De analysi* discussed in §4. Following [21, VI, pp. 314-317], set $x_* = x_i + e_i$, so that (6.1) is rewritten

$$\begin{aligned} M &= x_i + e_i - e\sin(x_i + e_i) = x_i + e_i - e(\sin(x_i)\cos(e_i) + \cos(x_i)\sin(e_i)) \\ &= x_i + e_i - e(\sin(x_i)[1 - \tfrac{1}{2}e_i^2 \ldots] + \cos(x_i)[e_i - \ldots]) \end{aligned}$$

from which

$$M - x_i + e\sin(x_i) = e_i(1 - e[\cos(x_i) - \tfrac{1}{2}e_i\sin(x_i)\ldots]) \approx e_i(1 - e\cos(x_i + \tfrac{1}{2}e_i)),$$

which leads to the iteration $x_{i+1} = x_i + c_i$, where

$$e_i \approx c_i = \frac{M - x_i + e\sin(x_i)}{1 - e\cos(x_i + \frac{1}{2}c_{i-1})}. \tag{6.4}$$

The latter form was apparently intended by Newton in the corresponding passage in the first edition of the *Principia*, but as pointed out by Fatio de Duillier in an annotation dating from about 1690 [21, VI, pp. 315-316] Newton's form there was flawed. Rather than correct the passage to reflect the form (6.4), Newton adopted the simpler alternative of omitting the term $\frac{1}{2}c_{i-1}$ to arrive at the form (6.3) in the subsequent editions. It is not clear what role, if any, was played by calculus in this revision.

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```
                          PROBLEMA. IX.

        Proponatur  a a a - b a = c  Aequatio secundae Formulae
           Numeris  a a a - 2 a = 5

                    Theor. x = c + bg - ggg
                               ------------
                                 3gg - b

           2 = g      5 = c   2,1 = g                2,1000
         -----        4 = bg  2,1                    -,0054
         -8 = ggg    ---     -----                  -------
 3gg = 12  +9         9        21                   2,0946 = g
   b = -2  ---                 42                   2,0946
     -------                  -----                --------
     10)+1,0(+,1 = x          4,41 = gg             125676
                                2,1                 83784
                              ------              188514
                              441                418920
                              882               ---------
                             ------              4,38734916 = gg
          3gg = 13,23         -9,261 = ggg           2,0946
            b = -2,           +9,200           -------------
               -------        -------            2632409496
              +11,23)         -,06100(-,0054 = x 1754939664
                                               3948614244
 13,16204748 = 3gg    4,1892 = bg             8774698320
 -2, = b              5, = c                 ---------------
 -----------          ---------              -9,189741550536 = ggg
 +11,16204748         9,1892                 +9,1892 = bg + c
                                             ---------------
                               +11,16205)-,000541550536
                                               (-,000048517 = x

                                             2,0946
                                             -,000048517
                                             ------------
                                             2,094551483 = g
```

FIG. 9. *Raphson's method for solving* $x^3 - 2x - 5 = 0$.

**7. Raphson's formulation.** In 1690 Joseph Raphson (1648–1712 ?) published a tract *Analysis aequationum universalis* [12] in which he presented a new method for solving polynomial equations. A second edition of this tract was published as a book in 1697, with an appendix with references to Newton, but omitting a preface with different references to Newton that had appeared in the original tract.

A copy of Raphson's tract, including a handwritten dedication from the author to John Wallis, corrections to the text that appear to be in the same handwriting, and including the preface but without the appendix described in [7] and [17], is available on microfilm [12] and is the basis for our comments. A general description of the book is given in [17], while [2] contains a reproduction of the title page and Raphson's Problem IX. The latter passage is reproduced in Fig. 9.

Here Raphson considers equations of the form $a^3 - ba - c = 0$ in the unknown $a$, and indicates that, if $g$ is an estimate of the solution $x_*$, a better estimate can be obtained as $g + x$, where

$$x = \frac{c + bg - g^3}{3g^2 - b}. \tag{7.1}$$

Formally this is of the form $g + x = g - f(g)/f'(g)$ with $f(a) = a^3 - ba - c$. Raphson then applies this formula iteratively to the equation $a^3 - 2a - 5 = 0$. Starting from an initial estimate $g = 2$, Raphson computes successively the corrections $x = 0.1, -0.0054, -0.000048517$ and $-.0000000014572895859$ and the corresponding estimates $g = 2, 2.1, 2.0946, 2.094551483$ and $2.0945514815427104141$ of $x_*$.

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The equation $x^3 - 2x - 5 = 0$ was previously discussed in §4, where we analyzed Newton's technique for its solution. The two methods are mathematically equivalent; the distinction between Raphson's computations and those of Newton is that Raphson uses (7.1) repeatedly, applied to successively more accurate estimates $g$ of the solution and without the need to generate intermediate polynomials as Newton did. The discrepancies between the numbers computed by Newton and those computed by Raphson are partly due to the deliberate omission by Newton of a term in the polynomial expansion for $g_3$ defined in §4. Raphson also retains all the significant digits computed in successive iterations, while Newton uses only the first few significant digits generated by each step of his method.

Raphson presented more than 30 examples and formulae in his book. In each case they involve polynomials, up to degree 10. His derivation of the expressions for the correcting terms $x$, such as that given in (7.1), is described in the opening passages of the tract, and is precisely that used by Newton described in §4 above, using binomial expansions, but with the first correction formula derived for a particular equation being used iteratively. Thus Raphson essentially used (1.1), but as in Newton's presentation, Raphson proceeded purely algebraically rather than using the rules of calculus to form a derivative term, and in every case he wrote out the expressions corresponding to $f(x)$ and $f'(x)$ in full as polynomials. The book concludes with a long set of tables giving the appropriate correction formula for a variety of polynomial equations. Despite Raphson's subsequent extensive work concerning fluxions, it is convincingly argued in [7] that he never associated the calculus with his iterative technique for polynomial equations, and he never extended it to other classes of equations. It is nevertheless appropriate to consider Raphson's formulation to be a significant development of Newton's method, with the iterative formulation substantially improving the computational convenience.

The following comments on Raphson's technique, recorded in the Journal Book of the Royal Society and quoted from [17], are noteworthy.

> "30 July 1690: Mr Halley related that Mr Ralphson [*sic*] had Invented a method of Solving all sorts of Aquations, and giving their Roots in Infinite Series, which Converge apace, and that he had desired of him an Equation of the fifth power to be proposed to him, to which he return'd Answers true to Seven Figures in much less time than it could have been effected by the Known methods of Vieta."

> "17 December 1690: Mr Ralphson's Book was this day produced by E Halley, wherein he gives a Notable Improvemt of ye method of Resolution of all sorts of Equations Shewing, how to Extract their Roots by a General Rule, which doubles the known figures of the Root known by each Operation, So yt by repeating 3 or 4 times he finds them true to Numbers of 8 or 10 places."

Thus Raphson's technique is compared to that of Viète, while Newton's method is not mentioned although it had now appeared in Wallis' *Algebra*. The significance of the reference to the solution of a polynomial equation of degree 5 is that while analytic solutions in radicals for all polynomial equations up to degree 4 were known, no general formula was known for degree 5 (indeed none exists); Raphson demonstrated that numerical solutions were nevertheless attainable. Finally it is remarkable that the property of quadratic convergence was again noted from the outset.

The few verifiable details of the life of Joseph Raphson are discussed in [17]. Contact between Newton and Raphson seems to have been very limited, although it appears that Newton exploited the circumstances of Raphson's death to attach a self-serving appendix to

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Raphson's last book, the *Historia fluxionum* [17]. In the Preface to his tract of 1690, Raphson refers to Newton's work but states that his own method is "not only, I believe, not of the same origin, but also, certainly, not with the same development" [2], [12]. References to Newton in the Appendix to his book of 1697 apparently refer to Newton's use of the binomial expansion, rather than his method for solving equations [7], [17]. The two methods were long regarded by users as distinct, though in 1798 Lagrange [9] observed that "ces deux méthodes ne sont au fond que le même présentée différemment" although Raphson's technique was "plus simple que celle de Newton" because "on peut se dispenser de faire continuellement de nouvelles transformées." Further historical details, particularly concerning comparisons between the methods of Newton and Raphson and the failure to recognize the role of calculus in these methods, appear in [7].

**8. Simpson's contributions.** The life and some of the significant mathematical achievements of Thomas Simpson (1710-1761) are described in his biography [5] and with much critical commentary in [11]. In his *Essays ... in ... Mathematicks*, published in London in 1740 [16], Simpson describes "A new Method for the Solution of Equations in Numbers." He makes no reference to the work of any predecessors, and in the Preface (p. vii) contrasts his technique based on the use of calculus with the algebraic methods then current:

> "The Sixth [Essay], contains a new Method for the Solution of all Kinds of Algebraical Equations in Numbers; which, as it is more general than any hitherto given, cannot but be of considerable Use, though it perhaps may be objected, that the Method of Fluxions, whereon it is founded, being a more exalted Branch of the Mathematicks, cannot be so properly applied to what belongs to common Algebra."

Simpson's instructions are as follows [16, p. 81]:

> CASE I
>
> *When only one Equation is given, and one Quantity $(x)$ to be determined.*
>
> "Take the fluxion of the given Equation (be it what it will) supposing $x$, the unknown, to be the variable Quantity; and having divided the whole by $\dot{x}$, let the Quotient be represented by $A$. Estimate the value of $x$ pretty near the Truth, substituting the same in the Equation, as also in the Value of $A$, and let the Error, or resulting Number in the former, be divided by this numerical Value of $A$, and the Quotient be subtracted from the said former Value of $x$; and from thence will arise a new Value of that Quantity much nearer to the Truth than the former, wherewith proceeding as before, another new Value may be had, and so another, *etc.* 'till we arrive to any Degree of Accuracy desired."

In addition to applying his technique to a polynomial equation, Simpson gives an example of this technique applied to the nonpolynomial equation $\sqrt{1-x}+\sqrt{1-2x^2}+\sqrt{1-3x^3}-2=0$ [16, pp. 83-84]:

> "This in Fluxions will be $\frac{-\dot{x}}{2\sqrt{1-x}}-\frac{2x\dot{x}}{\sqrt{1-2xx}}-\frac{9x^2\dot{x}}{2\sqrt{1-3x^3}}$, and therefore $A$, here, $=-\frac{1}{2\sqrt{1-x}}-\frac{2x}{\sqrt{1-2xx}}-\frac{9x^2}{2\sqrt{1-3x^3}}$; wherefore if $x$ be supposed $=.5$, it will become $-3.545$: And, by substituting 0.5 instead of $x$ in the given Equation, the Error will be found .204; therefore $\frac{.204}{-3.545}$ (equal $-.057$) subtracted from .5, gives .557 for the next Value of $x$; from whence, by proceeding as before, the next following will be found .5516, *etc.*."

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Following [7], in these instructions fluxions were to be taken as described by Newton in now-lost letters to Wallis in August 1692 and reported by Wallis in 1693. In modern terms $\dot{x}$ is essentially equivalent to $dx/dt$; implicit differentiation is used to obtain $dy/dt$, subsequently dividing through by $dx/dt$ as instructed produces the derivative $A = dy/dx$ of the function. Thus Simpson's instructions closely resemble, and are mathematically equivalent to, the use of (1.1). This is the first formulation of the iterative method for general nonlinear equations, based on the use of fluxional calculus. Simpson's application of fluxions in the context of solving general nonlinear equations was certainly highly innovative. This significant contribution by Simpson received little recognition, except in [3] and [6], until the recent publication of [7].

The formulation of the method using the now familiar $f'(x)$ calculus notation of (1.1) was published by Lagrange in 1798 [9], though it probably appeared in earlier lectures; it is given in Note XI of that book (*Sur les formules d'approximation pour les racines des équations*), rather than Note V (*Sur la Méthode d'Approximation donnée par Newton*). By this time the familiar geometric motivation for the method was well known. Lagrange makes no reference to Simpson's work, though Newton and Raphson are both mentioned. In Fourier's 1831 *Analyse des Équations Determinées* the method is described as "le méthode newtonienne," and no mention is made of the contributions of either Raphson or Simpson. This attribution in the influential book by Fourier is probably a major source of the subsequent lack of recognition given to the contributions of either Raphson or Simpson [3], [7], [17].

The previous extract from Simpson's Sixth Essay referred to CASE I. Simpson's CASE II is equally remarkable [16, p. 82]:

CASE II

*When there are two Equations given, and as many Quantities ($x$ and $y$) to be determined.*

"Take the Fluxions of both the Equations, considering $x$ and $y$ as variable, and in the former collect all the Terms, affected with $\dot{x}$, under their proper Signs, and having divided by $\dot{x}$, put the Quotient = $A$; and let the remaining Terms, divided by $\dot{y}$, be represented by $B$: In like manner, having divided the Terms in the latter, affected with $\dot{x}$, by $\dot{x}$, let the Quotient be put =$a$, and the rest, divided by $\dot{y}$, = $b$. Assume the Values of $x$ and $y$ pretty near the Truth, and substitute in both the Equations, marking the Error in each, and let these Errors, whether positive or negative, be signified by $R$ and $r$ respectively: Substitute likewise in the values of $A$, $B$, $a$, $b$, and let $\frac{Br-bR}{Ab-aB}$ and $\frac{aR-Ar}{Ab-aB}$ be converted into Numbers, and respectively added to the former Values of $x$ and $y$; and thereby new Values of those Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad libitum*."

Simpson is here describing the technique now generally referred to as "Newton's Method" for systems of nonlinear equations, restricted to the case of two such equations. In modern terms, this involves solving the system of equations $F(x) = 0$; $F : \mathbb{R}^n \to \mathbb{R}^n$ by the iterative process $x_{i+1} = x_i - d_i$ where $d_i$ is the solution of the system of linear equations $F'(x_i)d_i = F(x_i)$, in which $F'(x_i)$ is the Jacobian matrix of $F$ evaluated at $x_i$. In the above passage Simpson describes the construction of the entries of the matrix $F'(x)$ for the case $n = 2$ and then gives the explicit formula (Cramer's Rule) for the solution of the resulting system of linear equations. His book contains three examples of this method; we quote the first [16, p. 84]:

"Let there be given the Equations $y+\sqrt{y^2 - x^2}-10 = 0$ and $x+\sqrt{yy + x}-12 = 0$; to find $x$ and $y$. The Fluxions here being $\dot{y} + \frac{y\dot{y}-x\dot{x}}{\sqrt{yy-xx}}$ and $\dot{x} + \frac{y\dot{y}+\frac{1}{2}\dot{x}}{\sqrt{yy+x}}$

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or $\dot{y}+\frac{y\dot{y}}{\sqrt{yy-xx}}-\frac{x\dot{x}}{\sqrt{yy-xx}}$, and $\dot{x}+\frac{\frac{1}{2}\dot{x}}{\sqrt{yy+x}}+\frac{y\dot{y}}{\sqrt{yy+x}}$, we have $A$ equal $-\frac{x}{\sqrt{yy-xx}}$, $B$ equal $1+\frac{y}{\sqrt{yy+xx}}$ [*sic*], $a=1+\frac{\frac{1}{2}}{\sqrt{yy+x}}$, and $b=\frac{y}{\sqrt{yy+x}}$ *Case II.* Let $x$ be supposed equal 5, and $y$ equal 6; then will $R$ equal -.68, $r$ equal -.6, $A$ equal -1.5, $B$ equal 2.8, $a$ equal 1.1, $b$ equal 9 [*sic*]; therefore $\frac{Br-bR}{Ab-aB}=.23$, and $\frac{aR-Ar}{Ab-aB}=.37$, and the new Values of $x$ and $y$ equal to 5.23, and 6.37 respectively; which are as near the Truth as can be exhibited in three places only, the next Values coming out 5.23263 and 6.36898."

This extension by Simpson of the familiar technique for a single equation to the equally fundamental technique for systems of equations appears to have been overlooked in the literature. It is a significant achievement.

In passing it seems appropriate to note also a contribution by Simpson to the closely related problem of multivariable unconstrained optimization. In *A New Treatise of Fluxions*, published in 1737, he gives what may be the first example of maximizing a function of several variables, obtaining the maximum by essentially setting the gradient of the function equal to zero. We quote from that text.

"*Note*, When in any Expression representing the Value of a *Maximum*, or a *Minimum* there are two or more variable Quantities, flowing independent on each other, the Value of those Quantities may be determined, by making them to flow one by one, whilst the rest are considered as invariable, according to the Methods used in this and the following Examples.

EXAMPLE XVII

*Required to find three such Values of* $x$, $y$, $z$ *as shall make the given Expression* $(b^3-x^3)(x^2z-z^3)(xy-yy)$ *the greatest possible.*

First considering $y$ as a variable, we have $x\dot{y}-2y\dot{y}=0$, or $y=\frac{x}{2}$, therefore $xy-yy=\frac{xx}{4}$. By making $z$ variable, we have $x^2\dot{z}-3z^2\dot{z}=0$, or $z=\frac{x}{\sqrt{3}}$ therefore $x^2z-z^3=\frac{2x^3}{3\sqrt{3}}$, and substituting these Values in the given Expression it will become $\left(\frac{xx}{4}\text{x}\frac{2x^3}{3\sqrt{3}}\right)\text{x}\left(b^3-x^3\right)=\left(\frac{b^3x^5-x^8}{6\sqrt{3}}\right)$; therefore $5b^3x^4\dot{x}-8x^7\dot{x}=0$, or $x=\frac{1}{2}b\sqrt[3]{5}$, therefore $y=\frac{1}{4}b\sqrt[3]{5}$ and $z=b\frac{\sqrt[3]{5}}{2\sqrt{3}}$.

*N.B.* The Reason for this Process is evident, for unless the Fluxion of the given Expression, when any of the three Quantities $(x, y, z)$ be made variable, be equal to Nothing, the same expression may become greater, without varying the Values of the other two, which are considered as constant; therefore when it is the greatest possible, each of those Fluxions must then become equal to Nothing."

The idea of setting the gradient equal to zero, combined with the use of "Newton's Method" to solve the resulting system of nonlinear equations, is a key ingredient of many techniques for solving unconstrained optimization problems.

**9. Conclusions.** We have traced the evolution of the method (1.1) from the appearance of forerunners in the work of the Arabic algebraists and Viète to its formulation in the modern functional form by Lagrange, focusing on the contributions of Isaac Newton, Joseph Raphson, and Thomas Simpson. In the light of this historical development it would seem that the Newton-Raphson-Simpson method is a designation more nearly representing the facts of history in reference to this method which "lurks inside millions of modern computer programs, and is printed with Newton's name attached in so many textbooks" [17].

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**Acknowledgments.** I thank Cambridge University Press for permission to reproduce Figs. 1 and 3-6. I am indebted to the reviewers of early drafts of this paper and the anonymous referees for numerous helpful suggestions and references.

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