# TWO 1935 QUESTIONS OF MAZUR ABOUT POLYNOMIALS IN BANACH SPACES: A COUNTER-EXAMPLE 

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#### Abstract

We construct a continuous scalar-valued 2-polynomial, $W$, on the separable Hilbert space $l_{2}$ and an unbounded set $R \subset l_{2}$ such that (i) $W$ is bounded on an $\varepsilon$-neighbourhood of $R$; (ii) $W$ is unbounded on $\frac{1}{2} R$; (iii) consequently, $W$ does not factor through any bounded 1-polynomial on $l_{2}$ sending $R$ to a bounded set. This answers in the negative two 1935 questions asked by Mazur (problems 55 and 75 in the Scottish Book). The construction is valid both over $\mathbb{R}$ and $\mathbb{C}$. (In finite dimensions the questions were answered in the positive by Auerbach soon after being asked.)


1. The following two problems were entered up in the Scottish Book by Mazur in 1935. (We quote Ulam's translation from the 1981 edition [3].)

55 (Mazur). There is given, in an n-dimensional space $E$ or, more generally, in a space of type $(B)$, a polynomial $W(x)$ bounded in an $\varepsilon$ neighbourhood of a certain nonbounded set $R \subset E$ (an $\varepsilon$-neighbourhood of $a$ set $R$ is the set of all points which are distant by less than $\varepsilon$ from $R$ ). Does there exist a polynomial $V(x)$ and a polynomial of first degree $\phi(x)$ such that
(1) $W(x)=V(\phi(x))$;
(2) The set $\phi(R)$, that it to say the image of the set $R$ under the mapping $\phi(x)$, is bounded?

75 (Mazur). In the Euclidean n-dimensional space E or, more generally, in a space of type $(B)$ there is given a polynomial $W(x) . \alpha$ is a number $\neq 0$. If a polynomial $W(x)$ is bounded in an $\varepsilon$-neighbourhood of a certain set $R \subset E$ is it then bounded in a $\delta$-neighbourhood of the set $\alpha R$ (which is the set composed of elements $\alpha x$ for $x \in R$ )? (See problem 55.)

Clearly, an affirmative answer to problem 55 implies an affirmative answer to problem 75. Auerbach wrote addenda to both problems (ibid.)
informing that he solved in the positive problem 55 and, ipso facto, problem 75 in the case of a (both real and complex) finite-dimensional space $E$ [1]. The general case where $E$ is a Banach space seems to have been open ever since.

In this note we answer the two questions in the negative by presenting a counter-example where $W$ is a 2 -polynomial from $l_{\infty}$ to itself. It can be easily converted into a scalar-valued 2-polynomial on a separable Hilbert space, and thus the answer remains in general negative even in the most favourable infinite-dimensional setting. The construction is the same in either real or complex case.
2. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. For each $n \in \mathbb{N}$, define homogeneous polynomials of degree 1 and 2 , respectively:

$$
\begin{aligned}
W_{n}^{(1)}(x) & =-\frac{n x}{n+1} \\
W_{n}^{(2)}(x) & =\frac{x^{2}}{n+1}
\end{aligned}
$$

Since for each $n \in \mathbb{N}$ and $x \in \mathbb{K}$ one has $\left|W_{n}^{(1)}(x)\right| \leq|x|$ and $\left|W_{n}^{(2)}(x)\right| \leq$ $|x|^{2}$, the formula

$$
l_{\infty} \ni x \equiv\left(x_{n}\right)_{n \in \mathbb{N}} \mapsto\left(W_{n}^{(i)}\left(x_{n}\right)\right)_{n \in \mathbb{N}}, i=1,2
$$

correctly determines homogeneous polynomials $W^{(1)}, W^{(2)}: l_{\infty} \rightarrow l_{\infty}$ of degree 1 and 2 , respectively. Both $W^{(1)}$ and $W^{(2)}$ are continuous, being bounded on the unit ball $B_{1}(0)_{l_{\infty}}$. (Cf. [2], Th. 1 and a remark on p. 61.) Define a continuous 2-polynomial mapping $W: l_{\infty} \rightarrow l_{\infty}$ via

$$
W=W^{(1)}+W^{(2)},
$$

that is,

$$
l_{\infty} \ni x \equiv\left(x_{n}\right)_{n \in \mathbb{N}} \stackrel{W}{\mapsto}\left(W_{n}\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in l_{\infty}
$$

where for all $n \in \mathbb{N}$ and $x \in \mathbb{K}$

$$
W_{n}(x)=W_{n}^{(1)}(x)+W_{n}^{(2)}(x) \equiv \frac{x(x-n)}{n+1}
$$

The following is obvious.
Claim 1. If either $|x| \leq 1$ or $|x-n| \leq 1$, then $\left|W_{n}(x)\right| \leq 1$.
Let $e_{n}$ is the ' $n$-th basic vector' in $l_{\infty},\left(e_{n}\right)_{k}=\delta_{n, k}$. Define an unbounded subset

$$
R=\left\{n e_{n}: n \in \mathbb{N}\right\} \subset l_{\infty}
$$

Claim 2. The polynomial $W$ is bounded on the 1-neighbourhood of $R$.
$\triangleleft$ Let $x \in B_{1}(R)$, that is, for some $k \in \mathbb{N}, x \in B_{1}\left(k e_{k}\right)$. It means $\left|x_{k}-k\right| \leq 1$ and $\left|x_{n}\right| \leq 1$ for all $n \neq k$. By Claim 1, $\left|W_{n}\left(x_{n}\right)\right| \leq 1$ for all

Claim 3. The polynomial $W$ is unbounded on $(1 / 2) R$.
$\triangleleft$ For each $k \in \mathbb{N}$, one has $(k / 2) e_{k} \in(1 / 2) R$ and

$$
\left\|W\left(\frac{k}{2} e_{k}\right)\right\|=\left|W_{k}\left(\frac{k}{2}\right)\right|=\frac{k^{2}}{2(k+2)} \rightarrow \infty \text { as } k \rightarrow \infty
$$

3. It is easy to further modify the example so as to obtain a $\mathbb{K}$-valued 2-polynomial on $l_{2}$ in possession of the same properties. Applying the Banach-Steinhaus theorem to the unbounded set $W((1 / 2) R) \subset l_{\infty}$, choose a bounded linear functional $\varphi: l_{\infty} \rightarrow \mathbb{K}$ with $\varphi[W((1 / 2) R)]$ unbounded. Denote by $\widetilde{W}$ the composition of the three mappings:

$$
l_{2} \stackrel{i}{\hookrightarrow} l_{\infty} \xrightarrow{W} l_{\infty} \xrightarrow{\varphi} \mathbb{K},
$$

where $i: l_{2} \hookrightarrow l_{\infty}$ stands for the canonical contractive injection. The mapping $\widetilde{W}: l_{2} \rightarrow \mathbb{K}$ is continuous 2-polynomial. Set $\widetilde{R}=i^{-1}(R)$. Since $i\left(B_{1}(R)_{l_{2}}\right) \subset B_{1}(R)_{l_{\infty}}$, the polynomial $\widetilde{W}$ is bounded on the 1-neighbourhood of $\widetilde{R}$, and since $i(\widetilde{R})=R$, it is unbounded on $(1 / 2) \widetilde{R}$.

## References

[1] H. Auerbach, Un théorème sur les polynomes à $n$ variables, Studia Math. 5 (1934), 171-173.
[2] J. Bochnak and J. Siciak, Polynomials and multilinear mappings in topological vector spaces, Studia Math. 39 (1971), 59-76.
[3] R.D. Mauldin (ed.), The Scottish Book, Birkhäuser, Boston-Basel-Stuttgart, 1981.

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