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COM205T Discrete Structures for Computing

Assignment 2 - Predicate Logic

Source: Discrete Mathematics and its applications, Sixth Edition - Kenneth H.Rosen
Chapter 1, Pages 58-62

Question 1 Translate these statements into English, where the domain for each variable consists of all real numbers.

- $\forall x \exists y (x < y)$.
- $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$.
- $\forall x \forall y \exists z (xy = z)$.

Solution:

- For every real number x there exists a real number y such that x is less than y .
- For every real number x and real number y , if x and y are both non-negative then, there product is non-negative.
- For every real number x and real number y there exists a real number z such that $xy = z$.

Question 2 Translate these statements into English, where the domain for each variable consists of all real numbers.

- $\exists x \forall y (xy = y)$
- $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
- $\forall x \forall y \exists z (x = y + z)$

Solution:

- There exist a real number x such that for every real number y , $xy = y$
- For every real number x and real number y , if x is nonnegative and y is negative, then the difference $x - y$ is positive.
- For every real numbers x, y , there exist a real number z such that $x = y + z$

Question 3 Let $Q(x, y)$ be the statement “ x has sent an e-mail message to y ,” where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

- a. $\exists x \exists y Q(x, y)$.
- b. $\exists x \forall y Q(x, y)$.
- c. $\forall x \exists y Q(x, y)$.

- d. $\exists y \forall x Q(x, y)$.
- e. $\forall y \exists x Q(x, y)$.
- f. $\forall x \forall y Q(x, y)$.

Solution:

- a. There is some student in your class who has sent a message to some student in your class.
- b. There is some student in your class who has sent a message to every student in your class.
- c. Every student in your class has sent a message to at least one student in your class.
- d. There is a student in your class who has been sent a message by every student in your class.
- e. Every student in your class has been sent a message from at least one student in your class.
- f. Every student in the class has sent a message to every student in the class.

Question 4 Let $P(x, y)$ be the statement “student x has taken class y ,” where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

- a. $\exists x \exists y P(x, y)$
- b. $\exists x \forall y P(x, y)$
- c. $\forall x \exists y P(x, y)$
- d. $\exists y \forall x P(x, y)$
- e. $\forall y \exists x P(x, y)$
- f. $\forall x \forall y P(x, y)$

Solution:

- a. At least one student has taken class y

- b. At least one student has taken all classes
- c. Every student has taken at least one class
- d. At least one class has all students
- e. Every class has at least one student
- f. All students are taking all classes.

Question 5 Let $W(x, y)$ mean that student x has visited website y , where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

- a. $W(\text{Sarah Smith}, \text{www.att.com})$.
- b. $\exists x W(x, \text{www.imdb.com})$.
- c. $\exists y W(\text{Jose Orez}, y)$.
- d. $\exists y (W(\text{Ashok Puri}, y) \wedge W(\text{Cindy yoon}, y))$.
- e. $\exists y \forall z (y \neq (\text{David Belcher}) \wedge (W(\text{David Belcher}, z) \rightarrow W(y, z)))$.
- f. $\exists x \exists y \forall z ((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$.

Solution:

- a. Sarah Smith has visited www.att.com.
- b. At least one person has visited www.imdb.com.
- c. Jose Orez has visited at least one website.
- d. There is a website that both Ashok Puri and Cindy yoon has visited.
- e. There is a person besides David Belcher who has visited all the websites that David Belcher has visited.
- f. There are two different people who have visited exactly the same websites.

Question 6 Let $C(x, y)$ mean that student x is enrolled in class y , where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

- a. $C(\text{Randy Goldberg}, \text{CS 252})$

- b. $\exists x C(x, \text{Math 695})$
- c. $\exists y C(\text{Carol Sitea}, y)$
- d. $\exists x (C(x, \text{Math 222}) \wedge C(x, \text{CS 252}))$
- e. $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
- f. $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

Solution:

- a. Randy Goldberg is enrolled in CS 252
- b. At least one student enrolled in Math 695
- c. Carol Sitea is enrolled in at least one class
- d. At least one student enrolled in both Math 222 and CS 252
- e. There exist at least two students such that if one is enrolled in every courses, then the other
- f. There exist at least two students such that one is enrolled in every courses, if and only if the other

Question 7 Let $T(x, y)$ mean that student x likes cuisine y , where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

- a. $\neg T(\text{Abdallah Hussein}, \text{Japanese})$.
- b. $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$.
- c. $\exists y T(\text{Monique Arsenault}, y) \wedge T(\text{Jay Johnson}, y)$
- d. $\forall x \forall z \exists y ((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
- e. $\exists x \exists z \forall y (T(x, y) \leftrightarrow T(z, y))$
- f. $\forall x \forall z \exists y (T(x, y) \leftrightarrow T(z, y))$

Solution:

- a. Abdallah Hussein does not like Japanese cuisine.

- b. Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.
- c. There is some cuisine that either Monique Arsenault or Jay Johnson likes.
- d. For every pair of distinct students at your school, there is some cuisine that at least one of them does not like.
- e. There are two students at your school who like exactly the same set of cuisine.
- f. For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

Question 8 Let $Q(x, y)$ be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

- a. There is a student at your school who has been a contestant on a television quiz show.
- b. No student at your school has ever been a contestant on a television quiz show.
- c. There is a student at your school who has been a contestant on Jeopardy and on Wheel of Fortune.
- d. Every television quiz show has had a student from your school as a contestant.
- e. At least two students from your school have been contestants on Jeopardy.

Solution:

- a. $\exists x \exists y Q(x, y)$
- b. $\neg \exists x \exists y Q(x, y)$
- c. $\exists x (Q(x, Jeopardy) \wedge Q(x, WheelofFortune))$
- d. $\forall y \exists x Q(x, y)$
- e. $\exists x \exists y ((x \neq y) \wedge Q(x, Jeopardy) \wedge Q(y, Jeopardy))$

Question 9 Let $L(x, y)$ be the statement " x loves y ," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

- a. Everybody loves Jerry.

- b. *Everybody loves somebody.*
- c. *There is somebody whom everybody loves.*
- d. *Nobody loves everybody.*
- e. *There is somebody whom Lydia does not love.*
- f. *There is somebody whom no one loves.*
- g. *There is exactly one person whom everybody loves.*
- h. *There are exactly two people whom Lynn loves.*
- i. *Everyone loves himself or herself.*
- j. *There is someone who loves no one besides himself or herself.*

Solution:

- a. $\forall x L(x, \text{Jerry})$
- b. $\forall x \exists y L(x, y)$
- c. $\exists y \forall x L(x, y)$
- d. $\forall x \exists y \neg L(x, y)$
- e. $\exists x \neg L(\text{Lydia}, x)$
- f. $\exists x \forall y \neg L(y, x)$
- g. $\exists x (\forall y L(y, x) \wedge \forall z ((\forall w L(w, z)) \rightarrow z = x))$
- h. $\exists x \exists y (x \neq y \wedge L(\text{Lynn}, x) \wedge L(\text{Lynn}, y) \wedge \forall z (L(\text{Lynn}, z) \rightarrow (z = x \vee z = y)))$
- i. $\forall x L(x, x)$
- j. $\exists x \forall y (L(x, y) \leftrightarrow x = y)$

Question 10 *Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.*

- a. *Everybody can fool Fred.*
- b. *Evelyn can fool everybody.*

- c. *Everybody can fool somebody.*
- d. *There is no one who can fool everybody.*
- e. *Everyone can be fooled by somebody.*
- f. *No one can fool both Fred and Jerry.*
- g. *Nancy can fool exactly two people.*
- h. *There is exactly one person whom everybody can fool.*
- i. *No one can fool himself or herself.*
- j. *There is someone who can fool exactly one person besides himself or herself.*

Solution:

- a. $\forall x F(x, \text{Fred})$
- b. $\forall y F(\text{Evelyn}, y)$
- c. $\forall x \exists y F(x, y)$
- d. $\neg \exists x \forall y F(x, y)$
- e. $\forall y \exists x F(x, y)$
- f. $\neg \exists x (F(x, \text{Fred}) \wedge F(x, \text{Jerry}))$
- g. $\exists x \exists y ((x \neq y) \wedge F(\text{Nancy}, x) \wedge F(\text{Nancy}, y)) \wedge \forall z [((z \neq y) \wedge (z \neq x)) \rightarrow \neg F(\text{Nancy}, z)]$
- h. $\forall x \exists! y F(x, y)$
- i. $\neg \exists x F(x, x)$
- j. $\exists x \exists! y (F(x, x) \wedge F(x, y))$

Question 11 Let $S(x)$ be the predicate “ x is a student,” $F(x)$ the predicate “ x is a faculty member,” and $A(x, y)$ the predicate “ x has asked y a question,” where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

- a. *Lois has asked Professor Michaels a question.*
- b. *Every student has asked Professor Gross a question.*

- c. Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
- d. Some student has not asked any faculty member a question.
- e. There is a faculty member who has never been asked a question by a student.
- f. Some student has asked every faculty member a question.
- g. There is a faculty member who has asked every other faculty member a question.
- h. Some student has never been asked a question by a faculty member.

Solution:

- a. $A(\text{Lois}, \text{Professor Michael})$
- b. $\forall x(S(x) \rightarrow A(x, \text{Professor Gross}))$
- c. $\forall x(F(x) \rightarrow (A(x, \text{Professor Miller}) \vee A(\text{Professor Miller}, x)))$
- d. $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(x, y)))$
- e. $\exists x(F(x) \wedge \forall y(S(y) \rightarrow \neg A(y, x)))$
- f. $\forall y(F(y) \rightarrow \exists x(S(x) \vee A(x, y)))$
- g. $\exists x(F(x) \wedge \forall y((F(y) \wedge (y \neq x)) \rightarrow A(x, y)))$
- h. $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(y, x)))$

Question 12 Let $I(x)$ be the statement " x has an Internet connection" and $C(x, y)$ be the statement " x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

- a. Jerry does not have an Internet connection.
- b. Rachel has not chatted over the Internet with Chelsea.
- c. Jan and Sharon have never chatted over the Internet.
- d. No one in the class has chatted with Bob.
- e. Sanjay has chatted with everyone except Joseph.
- f. Someone in your class does not have an Internet connection.

- g. Not everyone in your class has an Internet connection.
- h. Exactly one student in your class has an Internet connection.
- i. Everyone except one student in your class has an Internet connection.
- j. Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
- k. Someone in your class has an Internet connection but has not chatted with anyone else in your class.
- l. There are two students in your class who have not chatted with each other over the Internet.
- m. There is a student in your class who has chatted with everyone in your class over the Internet.
- n. There are at least two students in your class who have not chatted with the same person in your class.
- o. There are two students in the class who between them have chatted with everyone else in the class.

Solution:

- a. $\neg I(\text{Jerry})$
- b. $\neg C(\text{Rachel}, \text{Chelsea})$
- c. $\neg C(\text{Jan}, \text{Sharon})$
- d. $\neg \exists x C(x, \text{Bob})$
- e. $\forall x ((x \neq \text{Joseph}) \leftrightarrow C(\text{Sanjay}, x))$
- f. $\exists x \neg I(x)$
- g. $\neg \forall x I(x)$
- h. $\exists! x I(x)$
- i. $\forall x \exists! y ((x \neq y) \wedge I(x))$
- j. $\forall x (I(x) \rightarrow \exists y C(x, y))$
- k. $\exists x (I(x) \wedge \forall y \neg C(x, y))$
- l. $\exists x \exists y ((x \neq y) \wedge \neg C(x, y))$

- m. $\exists x \forall y C(x, y)$
- n. $\exists x \exists y [(x \neq y) \wedge \forall z ((z \neq x) \wedge (z \neq y) \wedge \neg C(x, z) \wedge \neg C(y, z))]$
- o. $\exists x \exists y [(x \neq y) \wedge \forall z ((z \neq x) \wedge (z \neq y) \wedge C(x, z) \wedge C(y, z))]$

Question 13 Let $M(x, y)$ be “ x has sent y an e-mail message” and $T(x, y)$ be “ x has telephoned y ,” where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)

- a. Chou has never sent an e-mail message to Koko.
- b. Arlene has never sent an e-mail message to or telephoned Sarah.
- c. Jose has never received an e-mail message from Deborah.
- d. Every student in your class has sent an e-mail message to Ken.
- e. No one in your class has telephoned Nina.
- f. Everyone in your class has either telephoned Avi or sent him an e-mail message.
- g. There is a student in your class who has sent everyone else in your class an e-mail message.
- h. There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.
- i. There are two different students in your class who have sent each other e-mail messages.
- j. There is a student who has sent himself or herself an e-mail message.
- k. There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.
- l. Every student in the class has either received an e-mail message or received a telephone call from another student in the class.
- m. There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.
- n. There are two different students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

Solution:

- a. $\neg M(\text{Chou}, \text{Koko})$

- b. $\neg M(\text{Arlene, Sarah}) \vee \neg T(\text{Arlene, Sarah})$
- c. $\neg M(\text{Deborah, Jose})$
- d. $\forall x M(x, \text{Ken})$
- e. $\forall x \neg T(x, \text{Nina})$
- f. $\forall x (T(x, \text{Avi}) \vee M(x, \text{Avi}))$
- g. $\exists x \forall y (y \neq x \rightarrow M(x, y))$
- h. $\exists x \forall y (y \neq x \rightarrow (M(x, y) \vee T(x, y)))$
- i. $\exists x \exists y (x \neq y \wedge M(x, y) \wedge M(y, x))$
- j. $\exists x M(x, x)$
- k. $\exists x \forall y (x \neq y \rightarrow (\neg M(x, y) \wedge \neg T(y, x)))$
- l. $\forall x (\exists y (x \neq y \wedge (M(y, x) \vee T(y, x))))$
- m. $\exists x \exists y (x \neq y \wedge M(x, y) \wedge T(y, x))$
- n. $\exists x \exists y (x \neq y \wedge \forall z ((z \neq x) \wedge (z \neq y) \rightarrow (M(x, z) \vee M(y, z) \vee T(x, z) \vee T(y, z))))$

Question 14 Use quantifiers and predicates with more than one variable to express these statements.

- a. There is a student in this class who can speak Hindi
- b. Every student in this class plays some sport.
- c. Some student in this class has visited Alaska but has not visited Hawaii.
- d. All students in this class have learned at least one programming language.
- e. There is a student in this class who has taken every course offered by one of the departments in this school.
- f. Some student in this class grew up in the same town as exactly one other student in this class.
- g. Every student in this class has chatted with at least one other student in at least one chat group.

Solution:

- a. $\exists x \text{ speak}(x, \text{hindi})$ UOD(x): Students speak(x,y): x speaks y

- b. $\forall x \exists s \text{ Playing}(x, s)$ UOD(x): Students, UOD(s): Sports Playing(x,s): x plays s
- c. $\exists x [\text{Visited}(x, \text{Alaska}) \wedge \neg \text{Visited}(x, \text{Hawaii})]$ Visited(x,y): x visited y
- d. $\forall x \exists p \text{ Learned}(x, p)$ UOD(p): Prog. Lang. Learned(x,p): x learned prog. lang. p
- e. $\exists x \exists z [\forall y (\text{Offer}(y, z) \rightarrow C(x, y))]$ UOD(x): students, UOD(z): dept, UOD(y): courses, Offer(y,z): course z offered by dept z, C(x,y): student x takes course y
- f. $\exists x \exists! y [(x \neq y) \wedge \exists t (G(x, t) \wedge G(y, t))]$ UOD(x,y): Students, UOD(t): town, G(x,t): x is grown up in town t
- g. $\forall x \exists y \exists z (\text{chatgroup}(y) \wedge \text{stud}(z) \wedge \text{stud}(x) \wedge (x \neq z) \wedge \text{participatechat}(z, y) \wedge c(x, y))$
stud(x): x is a student, chatgroup(y): y is a chatgroup, participatechat(z,y): student z is in chatgroup y, c(x,y): student x is chatting with student y.

Question 15 Use quantifiers and predicates with more than one variable to express these statements.

- a. Every computer science student needs a course in discrete mathematics.
- b. There is a student in this class who owns a personal computer.
- c. Every student in this class has taken at least one computer science course.
- d. There is a student in this class who has taken at least one course in computer science.
- e. Every student in this class has been in every building on campus.
- f. There is a student in this class who has been in every room of at least one building on campus.
- g. Every student in this class has been in at least one room of every building on campus.

Solution:

- a. $\forall x P(x)$, where $P(x)$ is “x needs a course in discrete mathematics” and the domain for x consists of all computer science students.
- b. $\exists x P(x)$, where $P(x)$ is “x owns a personal computer” and the domain for x consists of all students in this class.
- c. $\forall x \exists y P(x, y)$, where $P(x, y)$ is “x has taken y” and the domain for x consists of all students in this class and the domain for y consists of all computer science classes.
- d. $\exists x \exists y P(x, y)$, where $P(x, y)$ is “x has taken y” and the domain for x consists of all students in this class and the domain for y consists of all computer science classes.

- e. $\forall x \forall y P(x, y)$, where $P(x, y)$ is “ x has been in y ” and the domain for x consists of all students in this class and the domain for y consists of all buildings on campus.
- f. $\exists x \exists y \forall z (P(z, y) \rightarrow Q(x, z))$, where $P(z, y)$ is “ z is in y ” and $Q(x, z)$ is “ x has been in z ”; the domain for x consists of all students in this class, the domain for y consists of all buildings on campus and the domain for z consists of all rooms.
- g. $\forall x \forall y \exists z (P(z, y) \wedge Q(x, z))$, where $P(z, y)$ is “ z is in y ” and $Q(x, z)$ is “ x has been in z ”; the domain for x consists of all students in this class, the domain for y consists of all buildings on campus and the domain for z consists of all rooms.

Question 16 *A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.*

- a. There is a student in the class who is a junior.
- b. Every student in the class is a computer science major.
- c. There is a student in the class who is neither a mathematics major nor a junior.
- d. Every student in the class is either a sophomore or a computer science major.
- e. There is a major such that there is a student in the class in every year of study with that major.

Solution:

Freshman: First year graduate school.

Sophomore: Second year graduate school.

Junior: Third year graduate school.

Senior: Fourth year graduate school.

Study(a,b,c): Student a who is in year b is majoring in c, Year(a,b): Student a is in year b,

Major(a,c): Student a majors in c

- a. $\exists x \text{Year}(x, \text{junior})$. **TRUE**
- b. $\forall x \text{Major}(x, \text{computer science})$. **FALSE**
- c. $\exists x [\neg \text{Major}(x, \text{Mathematics}) \wedge \neg \text{Year}(x, \text{junior})]$ **TRUE**
- d. $\forall x [\text{Year}(x, \text{sophomore}) \vee \text{Major}(x, \text{computer science})]$ **FALSE**
- e. $\exists m \forall y \exists x \text{Study}(x, y, m)$ **FALSE**

Question 17 Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

- a. Every user has access to exactly one mailbox.
- b. There is a process that continues to run during all error conditions only if the kernel is working correctly.
- c. All users on the campus network can access all websites whose url has a.edu extension.
- d. There are exactly two systems that monitor every remote server.

Solution:

- a. $\forall u \exists m (A(u, m) \wedge \forall n (n \neq m \rightarrow \neg A(u, n)))$, where $A(u, m)$ means that user u has access to mailbox m .
- b. $\exists p \forall e (H(e) \wedge S(p, \text{running})) \rightarrow S(\text{kernel}, \text{working Correctly})$, where $H(e)$ means that error condition e is in effect and $S(x, y)$ means that the status of x is y .
- c. $\forall u \forall s (E(s, .edu) \rightarrow A(u, s))$, where $E(s, x)$ means that website s has extension x , and $A(u, s)$ means that user u can access website s .
- d. $\exists x \exists y (x \neq y \wedge \forall z ((\forall s M(z, s)) \leftrightarrow (z = x \vee z = y)))$, where $M(a, b)$ means that system a monitors remote server b

Question 18 Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

- a. At least one console must be accessible during every fault condition.
- b. The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.
- c. For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.
- d. There are at least two paths connecting every two distinct endpoints on the network.
- e. No one knows the password of every user on the system except for the system administrator, who knows all passwords.

Solution:

- a. $\exists c \forall f \text{ Access}(c, f) \text{ UOD}(c)$: console, UOD(f): faults, Access(c, f): console c must be accessible during fault f

- b. $\exists m \forall u (Archive(m, u) \rightarrow retemailid(u))$ UOD(m): message, UOD(u): user Archive(m,u): Archive contains message m sent by user u
- c. $\forall x S(x) \rightarrow [\exists y (M(y) \wedge D(x, y)) \leftrightarrow \exists z (P(z) \wedge \neg C(y, z))]$ S(x): x is a Security breach, M(y): y is a mechanism, D(x,y): x detects y, C(y,z): process z compromise y
- d. $\forall u \forall v [((u \neq v) \wedge node(u) \wedge node(v)) \rightarrow \exists p_1 \exists p_2 ((p_1 \neq p_2) \wedge path(p_1) \wedge path(p_2)) \wedge connect(p_1, u, v) \wedge connect(p_2, u, v)]$
node(u): u is a node, path(p): p is a path, connect(p,u,v): p is a path from u to v
- e. $\exists!x \forall y ((x = sysadmin) \leftrightarrow know(x, y))$ UOD(x,y): users, know(x,y): x knows password of y

Question 19 Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.

- a. The sum of two negative integers is negative.
- b. The difference of two positive integers is not necessarily positive.
- c. The sum of the squares of two integers is greater than or equal to the square of their sum.
- d. The absolute value of the product of two integers is the product of their absolute values.

Solution:

- a. $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (x + y < 0))$
- b. $\neg \forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x - y > 0))$
- c. $\forall x \forall y ((x^2 + y^2) \geq (x + y)^2)$
- d. $\forall x \forall y (|xy| = |x| |y|)$

Question 20 Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

- a. The product of two negative integers is positive.
- b. The average of two positive integers is positive.
- c. The difference of two negative integers is not necessarily negative.
- d. The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

Solution:

- a. $\forall x \forall y ((x < 0) \wedge (y < 0)) \rightarrow (xy > 0)$
- b. $\forall x \forall y ((x > 0) \wedge (y > 0)) \rightarrow (xy/2 > 0)$
- c. $\forall x \forall y ((x > 0) \wedge (y > 0)) \rightarrow ((x - y < 0) \vee (x - y \geq 0))$
- d. $\forall x \forall y \text{abs}(x + y) \leq \text{abs}(x) + \text{abs}(y)$

Question 21 Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

Solution:

$\forall x \exists a \exists b \exists c \exists d ((x > 0) \rightarrow x = a^2 + b^2 + c^2 + d^2)$, where the domain consists of all integers.

(OR)

$\forall x ((x > 0) \rightarrow \exists a \exists b \exists c \exists d (x = a^2 + b^2 + c^2 + d^2))$

Question 22 Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.

Solution:

$\exists x [\text{posint}(x) \wedge \forall u \forall v \forall w (\text{int}(u) \wedge \text{int}(v) \wedge \text{int}(w) \wedge x \neq u^2 + v^2 + w^2)]$

UOD(x,u,v,w): real, posint(x): x is positive integer, int (x): x is integer

Question 23 Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

- a. The product of two negative real numbers is positive.
- b. The difference of a real number and itself is zero.
- c. Every positive real number has exactly two square roots.
- d. A negative real number does not have a square root that is a real number.

Solution:

- a. $\forall x \forall y ((x < 0) \wedge (y < 0)) \rightarrow (xy > 0)$.
- b. $\forall x (x - x = 0)$.
- c. $\forall x \exists a \exists b ((a \neq b) \wedge \forall c (c^2 = x \leftrightarrow (c = a \vee c = b)))$.
- d. $\forall x ((x < 0) \rightarrow \neg \exists y (x = y^2))$.

Question 24 Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

- a. $\exists x \forall y (x + y = y)$
- b. $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
- c. $\exists x \exists y (((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
- d. $\forall x \forall y ((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$

Solution:

- a. There exists a real number x that when added to every number y will sum to y .
- b. Subtracting a negative number from positive number results in a positive number
- c. There exist at least two non positive numbers when subtracted gives a positive number
- d. Product of two numbers are non zero if and only if both are non zero.

Question 25 Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

- a. $\exists x \forall y (xy = y)$.
- b. $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$.
- c. $\exists x \exists y ((x^2 > y) \wedge (x < y))$.
- d. $\forall x \forall y \exists z (x + y = z)$.

Solution:

- a. There is a multiplicative identity for real numbers.
- b. The product of two negative real numbers is always a positive real number.
- c. There exists real numbers x and y such that x^2 exceeds y but x is less than y .
- d. The real numbers are closed under the operation of addition.

Question 26 Let $Q(x, y)$ be the statement " $x + y = x - y$ ". If the domain for both variables consists of all integers, what are the truth values?

- a. $Q(1, 1)$
- b. $Q(2, 0)$
- c. $\forall y Q(1, y)$
- d. $\exists x Q(x, 2)$
- e. $\exists x \exists y Q(x, y)$
- f. $\forall x \exists y Q(x, y)$
- g. $\exists y \forall x Q(x, y)$
- h. $\forall y \exists x Q(x, y)$
- i. $\forall x \forall y Q(x, y)$

Solution:

- a. FALSE
- b. TRUE
- c. FALSE
- d. FALSE
- e. TRUE
- f. TRUE
- g. TRUE
- h. FALSE
- i. FALSE

Question 27 Determine the truth value of each of these statements if the domain for all variables consists of all integers.

- a. $\forall n \exists m (n^2 < m)$.

- b. $\exists n \forall m (n < m^2)$.
- c. $\forall n \exists m (n + m = 0)$.
- d. $\exists n \forall m (nm = m)$.
- e. $\exists n \exists m (n^2 + m^2 = 5)$.
- f. $\exists n \exists m (n^2 + m^2 = 6)$.
- g. $\exists n \exists m (n + m = 4 \wedge n - m = 1)$.
- h. $\exists n \exists m (n + m = 4 \wedge n - m = 2)$.
- i. $\forall n \forall m \exists p (p = (m + n)/2)$.

Solution:

- a. TRUE.
- b. TRUE.
- c. TRUE.
- d. TRUE.
- e. TRUE.
- f. FALSE.
- g. FALSE.
- h. TRUE.
- i. FALSE.

Question 28 Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

- a. $\forall x \exists y (x^2 = y)$
- b. $\forall x \exists y (x = y^2)$
- c. $\exists x \exists y (xy = 0)$
- d. $\exists x \exists y (x + y \neq y + x)$
- e. $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$

- f. $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$
- g. $\forall x \exists y (x + y = 1)$
- h. $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$
- i. $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$
- j. $\forall x \forall y \exists z (z = (x + y)/2)$

Solution:

- a. TRUE
- b. FALSE
- c. TRUE
- d. FALSE
- e. TRUE
- f. FALSE
- g. TRUE
- h. FALSE
- i. FALSE
- j. TRUE

Question 29 Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y , where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

- a. $\forall x \forall y P(x, y)$.
- b. $\exists x \exists y P(x, y)$.
- c. $\exists x \forall y P(x, y)$.
- d. $\forall y \exists x P(x, y)$.

Solution:

- a. $P(1, 1) \wedge P(1, 2) \wedge P(1, 3) \wedge P(2, 1) \wedge P(2, 2) \wedge P(2, 3) \wedge P(3, 1) \wedge P(3, 2) \wedge P(3, 3)$.
- b. $P(1, 1) \vee P(1, 2) \vee P(1, 3) \vee P(2, 1) \vee P(2, 2) \vee P(2, 3) \vee P(3, 1) \vee P(3, 2) \vee P(3, 3)$.
- c. $(P(1, 1) \wedge P(1, 2) \wedge P(1, 3)) \vee (P(2, 1) \wedge P(2, 2) \wedge P(2, 3)) \vee (P(3, 1) \wedge P(3, 2) \wedge P(3, 3))$.
- d. $(P(1, 1) \vee P(1, 2) \vee P(1, 3)) \wedge (P(2, 1) \vee P(2, 2) \vee P(2, 3)) \wedge (P(3, 1) \vee P(3, 2) \vee P(3, 3))$.

Question 30 Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

- a. $\neg \exists y \exists x P(x, y)$
- b. $\neg \forall x \exists y P(x, y)$
- c. $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$
- d. $\neg \exists y (\exists x R(x, y) \vee \forall x S(x, y))$
- e. $\neg \exists y (\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$

Solution:

- a. $\forall y \forall x \neg P(x, y)$
- b. $\exists x \forall y \neg P(x, y)$
- c. $\forall y (\neg Q(y) \vee \exists x R(x, y))$
- d. $\forall y (\forall x \neg R(x, y) \wedge \exists x \neg S(x, y))$
- e. $\forall y (\exists x \forall z \neg T(x, y, z) \wedge \forall x \exists z \neg U(x, y, z))$

Question 31 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

- a. $\forall x \exists y \forall z T(x, y, z)$.
- b. $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$.
- c. $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$.
- d. $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$.

Solution:

- a. $\exists x \forall y \exists z \neg T(x, y, z)$.

- b. $\exists x \forall y \neg P(x, y) \wedge \exists x \forall y \neg Q(x, y)$.
- c. $\exists x \forall y (\neg P(x, y) \vee \forall z \neg R(x, y, z))$.
- d. $\exists x \forall y (P(x, y) \wedge \neg Q(x, y))$.

Question 32 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

- a. $\exists z \forall y \forall x T(x, y, z)$
- b. $\exists x \exists y P(x, y) \wedge \forall x \exists y Q(x, y)$
- c. $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$
- d. $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

Solution:

- a. $\forall z \exists y \exists x \neg T(x, y, z)$
- b. $\forall x \forall y \neg P(x, y) \vee \exists x \exists y \neg Q(x, y)$
- c. $\forall x \forall y (\neg Q(x, y) \leftrightarrow Q(y, x))$ or
 $\forall x \forall y (Q(x, y) \leftrightarrow \neg Q(y, x))$
- d. $\exists y \forall x \forall z (\neg T(x, y, z) \wedge \neg Q(x, y))$

Question 33 Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

- a. $\neg \forall x \forall y P(x, y)$.
- b. $\neg \forall y \exists x P(x, y)$.
- c. $\neg \forall y \forall x (P(x, y) \vee Q(x, y))$.
- d. $\neg (\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$.
- e. $\neg \forall x (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$.

Solution:

- a. $\exists x \exists y \neg P(x, y)$.

- b. $\exists y \forall x \neg P(x, y)$.
- c. $\exists y \exists x (\neg P(x, y) \wedge \neg Q(x, y))$.
- d. $(\forall x \forall y P(x, y) \vee \exists x \exists y \neg Q(x, y))$.
- e. $\exists x (\forall y \exists z \neg P(x, y, z) \vee \forall z \exists y \neg P(x, y, z))$.

Question 34 Find a common domain for the variables x, y , and z for which the statement $\forall x \forall y ((x \neq y) \rightarrow \forall z ((z = x) \vee (z = y)))$ is true and another domain for which it is false.

Solution:

If $\text{UOD}(x, y, z)$ is a set of prime numbers less than 5, then the statement is **TRUE**, UOD contains two elements $\{2, 3\}$.

If $\text{UOD}(x, y, z)$ is a set of prime numbers less than or equals 5, then the statement is **FALSE**, UOD contains at least three elements $\{2, 3, 5, \dots\}$

Question 35 Find a common domain for the variables x, y, z , and w for which the statement $\forall x \forall y \forall z \exists w ((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

Solution:

Any domain with four or more members makes the statement **TRUE**; any domain with three or fewer members makes the statement **FALSE**.

Question 36 Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

- a. No one has lost more than one thousand dollars playing the lottery.
- b. There is a student in this class who has chatted with exactly one other student.
- c. No student in this class has sent e-mail to exactly two other students in this class.
- d. Some student has solved every exercise in this book.
- e. No student has solved at least one exercise in every section of this book.

Solution:

- a. Let $L(x, y)$ mean that person x has lost y dollars playing the lottery. The original statement is then $\neg \exists x \exists y (y > 1000 \wedge L(x, y))$. Its negation of course is $\exists x \exists y (y > 1000 \wedge L(x, y))$; someone has lost more than 1000 dollars playing the lottery.

- b. Let $C(x, y)$ mean that person x has chatted with person y . The given statement is $\exists x \exists y (y \neq x \wedge \forall z (z \neq x \rightarrow (z = y \leftrightarrow C(x, z))))$. The negation is therefore $\forall x \forall y (y \neq x \rightarrow \exists z (z \neq x \wedge \neg(z = y \leftrightarrow C(x, z))))$. In English, everybody in this class has either chatted with no one else or has chatted with two or more others.
- c. Let $E(x, y)$ mean that person x has sent e-mail to person y . The given statement is $\neg \exists x \exists y \exists z (y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w (w \neq x \rightarrow (E(x, w) \leftrightarrow (w = y \vee w = z))))$. The negation is obviously $\exists x \exists y \exists z (y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w (w \neq x \rightarrow (E(x, w) \leftrightarrow (w = y \vee w = z))))$. In English, some student in this class has sent e-mail to exactly two other students in this class.
- d. Let $S(x, y)$ mean that student x has solved exercise y . The statement is $\exists x \forall y S(x, y)$. The negation is $\forall x \exists y \neg S(x, y)$. In English, for every student in this class, there is some exercise that he or she has not solved.
- e. Let $S(x, y)$ mean that student x has solved exercise y , and let $B(y, z)$ mean that exercise y is in section z of the book. The statement is $\neg \exists x \forall z \exists y (B(y, z) \wedge S(x, y))$. The negation is of course $\exists x \forall z \exists y (B(y, z) \wedge S(x, y))$. In English, some student has solved at least one exercise in every section of this book.

Question 37 Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

- a. Every student in this class has taken exactly two mathematics classes at this school.
- b. Someone has visited every country in the world except Libya.
- c. No one has climbed every mountain in the Himalayas.
- d. Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

Solution:

- a. $\forall x (\text{countMath}(x) = 2)$. Its negation is $\exists x (\text{countMath}(x) \neq 2)$. Alternatively, There is someone in this class such that for every two different math courses, these are not the two and only two math courses this person has taken.
- b. $\exists x \forall y (\text{Visited}(x, y) \leftrightarrow y \neq \text{libya})$. Its negation is $\forall x \exists y (\text{Visited}(x, y) \leftrightarrow y = \text{libya})$. Alternatively, Every person has either visited Libya or has not visited a country other than Libya.
- c. $\neg \exists x \forall y (\text{climb}(x, y))$. Its negation is $\exists x \forall y (\text{climb}(x, y))$. Someone has climbed every mountain in the Himalayas.
- d. $\forall x (\text{actedWith}(x, \text{KevinBacon}) \vee \exists y (\text{actedWith}(x, y) \wedge \text{actedWith}(y, \text{KevinBacon})))$. Its negation is $\exists x \neg (\text{actedWith}(x, \text{KevinBacon}) \wedge \forall y (\neg \text{actedWith}(x, y) \vee \neg \text{actedWith}(y, \text{KevinBacon})))$. There is someone who has neither been in a movie with Kevin Bacon nor has been in a movie with someone who has been in a movie with Kevin Bacon.

Question 38 Express the negations of these propositions using quantifiers, and in English.

- a. Every student in this class likes mathematics.
- b. There is a student in this class who has never seen a computer.
- c. There is a student in this class who has taken every mathematics course offered at this school.
- d. There is a student in this class who has been in at least one room of every building on campus.

Solution:

- a. $\forall x(Likes(x, mathematics))$. Its negation: $\exists x(\neg Likes(x, mathematics))$. i.e., Some student in this class do not like mathematics.
- b. $\exists x(Notseen(x, computer))$. Negation: $\forall x\neg(Notseen(x, computer))$. i.e., Every student in this class has seen a computer.
- c. $\exists x\forall y(taken(x, y))$. Negation: $\forall x\exists y\neg(taken(x, y))$. i.e., Every student in this class has not taken a mathematics course offered at this school.
- d. $\exists x\forall y\exists z(within(z, y)\wedge attended(x, z))$. Negation: $\forall x\exists y\forall z\neg(within(z, y)\wedge attended(x, z))$. i.e., Every students in this class has not been in all the rooms of a building on campus.

Question 39 Find a counter example, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

- a. $\forall x\forall y(x^2 = y^2 \rightarrow x = y)$.
- b. $\forall x\exists y(y^2 = x)$.
- c. $\forall x\forall y(xy \geq x)$.

Solution:

- a. $x = 2, y = -2$.
- b. $x = -4$.
- c. $x = 17, y = -1$.

Question 40 Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

- a. $\forall x \exists y (x = 1/y)$
- b. $\forall x \exists y (y^2 - x < 100)$
- c. $\forall x \forall y (x^2 \neq y^3)$

Solution:

- a. if $x \neq 1$ or $x \neq -1$, then y does not exist.
- b. If $x < -100$, then y does not exist
- c. Statement is not true for $x = 8, y = 4$

Question 41 Use quantifiers to express the associative law for multiplication of real numbers.

Solution:

$$\forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

Question 42 Use quantifiers to express the distributive laws of multiplication over addition for real numbers.

Solution:

$$\forall x \forall y \forall z (x \wedge (y \vee z)) \leftrightarrow \forall x \forall y \forall z ((x \wedge y) \vee (x \wedge z))$$

Question 43 Use quantifiers and logical connectives to express the fact that every linear polynomial (that is, polynomial of degree 1) with real coefficients and where the coefficient of x is nonzero, has exactly one real root.

Solution:

$$\forall m \forall b (m \neq 0 \rightarrow \exists x (mx + b = 0 \wedge \forall w (mw + b = 0 \rightarrow w = x)))$$

Question 44 Use quantifiers and logical connectives to express the fact that a quadratic polynomial with real number coefficients has at most two real roots.

Solution:

$$\forall a \forall b \forall c ((a \neq 0) \rightarrow \forall x_1 \forall x_2 \forall x_3 (ax_1^2 + bx_1 + c = 0) \wedge (ax_2^2 + bx_2 + c = 0) \wedge (ax_3^2 + bx_3 + c = 0) \rightarrow (x_1 = x_2 \vee x_2 = x_3 \vee x_1 = x_3))$$

Question 45 Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of

- a. the non-zero real numbers.
- b. the non-zero integers.

c. the positive real numbers.

Solution:

- a. TRUE.
- b. FALSE.
- c. TRUE.

Question 46 Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of

- a. the positive real numbers.
- b. the integers.
- c. the nonzero real numbers.

Solution:

- a. FALSE
- b. TRUE
- c. TRUE

Question 47 Show that the two statements $\neg \exists x \forall y P(x, y)$ and $\forall x \exists y \neg P(x, y)$, where both quantifiers over the first variable in $P(x, y)$ have the same domain, and both quantifiers over the second variable in $P(x, y)$ have the same domain, are logically equivalent.

Solution:

$$\neg(\exists x \forall y P(x, y)) \leftrightarrow \forall x (\neg \forall y P(x, y)) \leftrightarrow (\forall x \exists y \neg P(x, y)).$$

SECTION B (Pg. 72 - 74)

Question 48 (Q: 01, pg: 72) Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

\therefore Socrates is mortal

Solution:

Conclusion is TRUE and it follows by applying modus ponens to the premise.

Question 49 (Q: 02, pg: 72) Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not an insect.

George is an insect.

\therefore George has eight legs.

Solution:

This is modus tollens. The first statement is $p \rightarrow q$, where p is “George does not have eight legs” and q is “George is not an insect”. The second statement is $\neg q$. The third is $\neg p$. Modus tollens is valid. We can therefore conclude that the conclusion of the argument (third statement) is true, given that the hypothesis (the first two statements) are true.

Question 50 (*Q: 03, pg: 72*) *What rule of inference is used in each of these arguments?*

- Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
- Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
- If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
- If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
- If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

Solution:

- Addition
- Simplification
- Modus ponens
- Modus tollens
- Hypothetical syllogism

Question 51 (*Q: 04, pg: 72*) *What rule of inference is used in each of these arguments?*

- Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.*
- It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.*
- Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.*

- d. Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.
- e. If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

Solution:

- a. Given the conjunction of two propositions and asserted one of them. This is simplification $((P \wedge Q) \rightarrow P)$.
- b. Given the disjunction of two propositions and the negation of one of them, and asserted the other. This is disjunctive syllogism $((\neg P \wedge (P \vee Q)) \rightarrow Q)$.
- c. Modus ponens: $(P \wedge (P \rightarrow Q)) \rightarrow Q$.
- d. Addition: $P \rightarrow (P \vee Q)$.
- e. Hypothetical syllogism: $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$.

Question 52 (Q: 05, pg: 72) Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Solution:

p: Randy works hard.
 q: Randy is a dull boy.
 r: Randy will get the job.

p
 $p \rightarrow q$
 $q \rightarrow \neg r$
 $\therefore \neg r$

i.e., Randy will not get the job.

Question 53 (Q: 06, pg: 72) Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the life saving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Solution:

Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \rightarrow (s \wedge l)$, $s \rightarrow t$, and $\neg t$. We want to conclude r . Let us prove this by logical implications.

Question 54 (Q: 07, pg: 72) What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal"

| Step | Reasons |
|--|-------------------------------------|
| 1. $\neg t$ | Hypothesis |
| 2. $s \rightarrow t$ | Hypothesis |
| 3. $\neg s$ | Modus tollens using (1) and (2) |
| 4. $(\neg r \vee \neg f) \rightarrow (s \wedge l)$ | Hypothesis |
| 5. $(\neg(s \wedge l)) \rightarrow \neg(\neg r \vee \neg f)$ | Contrapositive of (4) |
| 6. $(\neg s \vee \neg l) \rightarrow (r \wedge f)$ | De Morgan's law and double negative |
| 7. $\neg s \vee \neg l$ | Addition, using (3) |
| 8. $r \wedge f$ | Modus ponens using (6) and (7) |
| 9. r | Simplification using (8) |

Solution:

Universal instantiation is used to conclude that "If Socrates is a man, then Socrates is mortal."
Modus ponens is then used to conclude that Socrates is mortal.

Question 55 (Q: 08, pg: 72) *What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."*

Solution:

First we use universal instantiation to conclude from "For all x , if x is a man, then x is not an island". The special case of interest, "If Manhattan is a man, then Manhattan is not an island". Then we form the contrapositive (using also double negative): "If Manhattan is an island, then Manhattan is not a man." Finally, we use modus ponens to conclude that Manhattan is not a man.

Question 56 (Q: 09, pg: 72) *For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.*

- "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."
- "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."
- "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."
- "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."
- "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."
- "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

Solution:

- I took off on Thursday and thursday it rained

- b. I did not eat spicy foods and it did not thunder
- c. I am clever
- d. Ralph is not a CS major
- e. Buy lots of stuff is good for the United States. Buy lots of stuff is good for you
- f. Mice gnaw their food and Rabbits are not rodents

Question 57 (Q: 10, pg: 72) For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

- a. *"In play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."*
- b. *"In work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."*
- c. *"All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."*
- d. *"Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."*
- e. *"All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."*
- f. *"I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."*

Solution:

- a. If we use modus tollens on the last two premises, then we conclude that I am not sore. Another application of modus tollens then tells us that I did not play hockey.
- b. We really can't conclude anything specific here.
- c. By universal instantiation, we conclude from the first conditional statement by modus ponens that dragonflies have six legs, and we conclude by modus tollens that spiders are not insects. We could say using existential generalization that, for example, there exists a non-six-leg creature that eats six-leg creature, and that there exists a non-insect that eats an insect.
- d. We can apply universal instantiation to the conditional statement and conclude that if Homer (respectively, Maggie) is a student, then he (she) has an Internet account. Now modus tollens tells us that Homer is not a student. No conclusion can be drawn about Maggie.

- e. The first conditional statement is that if x is healthy to eat, then x does not taste good. Universal instantiation and modus ponens therefore tell us that tofu does not taste good. The third sentence says that if you eat x , then x tastes good. Therefore the fourth hypothesis already follows (by modus tollens) from the first three. No conclusions can be drawn about cheeseburgers from these statements.
- f. By disjunctive syllogism, the first two hypotheses allow us to conclude that I am hallucinating. Therefore by modus ponens we know that I see elephants running down the road.

Question 58 (*Q: 11, pg: 72*) Show that the argument form with premises P_1, P_2, \dots, P_n and conclusion $q \rightarrow r$ is valid if the argument form with premises P_1, P_2, \dots, P_n, q , and conclusion r is valid.

Solution:

Suppose that p_1, p_2, \dots, p_n are true. We want to establish that $q \rightarrow r$ is true. If q is false, then we are done, vacuously. Otherwise, q is true, so by the validity of the given argument form (that whenever p_1, p_2, \dots, p_n, q are true, then r must be true), we know that r is true.

Question 59 (*Q.12, pg.72*) Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p$, and $\neg s$ and conclusion $q \rightarrow r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

Solution:

Applying Exercise 11, we want to show that the conclusion r follows from the five premises $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, \neg s$ and q . From q and $q \rightarrow (u \wedge t)$ we get $u \wedge t$ by modus ponens. From there we get both u and t by simplification (and the commutative law). From u and $u \rightarrow p$ we get p by modus ponens. From p and t we get $p \wedge t$ by conjunction. From that and $(p \wedge t) \rightarrow (r \vee s)$ we get $r \vee s$ by modus ponens. From that and $\neg s$ we finally get r by disjunctive syllogism.

Question 60 (*Q: 13, pg: 72*) For each of these arguments, explain which rules of inference are used for each step.

- a. "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."
- b. "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
- c. "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."
- d. "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."

Solution:

- a. Universal instantiation, Modus ponens and Existential generalisation
- b. Existential Instantiation, Simplification, Universal instantiation, Modus Ponens, Conjunction, Existential generalization.
- c. Universal instantiation, Modus Ponens, Universal instantiation, Modus Ponens.
- d. Existential Instantiation, Simplification, Universal instantiation, Modus Ponens, Simplification, Existential generalization.

Question 61 (Q.14, pg:72) For each of these arguments, explain which rules of inference are used for each step.

- a. "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."
- b. "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."
- c. "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."
- d. "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

Solution:

- a. Let $c(x)$ be " x is in this class," let $r(x)$ be " x owns a red convertible," and let $t(x)$ be " x has gotten a speeding ticket." We are given premises $c(Linda), r(Linda), \forall x(r(x) \rightarrow t(x))$, and we want to conclude $\exists x(c(x) \wedge t(x))$.

- 1. $\forall x(r(x) \rightarrow t(x))$ Hypothesis
- 2. $r(Linda) \rightarrow t(Linda)$ Universal instantiation using (1)
- 3. $r(Linda)$ Hypothesis
- 4. $t(Linda)$ Modus ponens using (2) and (3)
- 5. $c(Linda)$ Hypothesis
- 6. $c(Linda) \wedge t(Linda)$ Conjunction using (4) and (5)
- 7. $\exists x(c(x) \wedge t(x))$ Existential generalization using (6)

- b. Let $r(x)$ be " r is one of the five room mates listed," let $d(x)$ be " x has taken a course in discrete mathematics," and let $a(x)$ be " x can take a course in algorithms." We are given premises $\forall x(r(x) \rightarrow d(x))$ and $\forall x(d(x) \rightarrow a(x))$, and we want to conclude $\forall x(r(x) \rightarrow a(x))$. In what follows y represents an arbitrary person.

1. $\forall x(r(x) \rightarrow d(x))$ Hypothesis
 2. $r(y) \rightarrow d(y)$ Universal instantiation using (1)
 3. $\forall x(d(x) \rightarrow a(x))$ Hypothesis
 4. $d(y) \rightarrow a(y)$ Universal instantiation using (3)
 5. $r(y) \rightarrow a(y)$ Hypothetical syllogism using (2) and (4)
 6. $\forall x(r(x) \rightarrow a(x))$ Universal generalization using (5)
- c. Let $s(x)$ be “ x is a movie produced by Sayles,” let $c(x)$ be “ x is a movie about coal miners,” and let $w(x)$ be “movie x is wonderful.” We are given premises $\forall x(s(x) \rightarrow w(x))$ and $\exists x(s(x) \wedge c(x))$, and we want to conclude $\exists x(c(x) \wedge w(x))$. In our proof, y represents an unspecified particular movie.
1. $\exists x(s(x) \wedge c(x))$ Hypothesis
 2. $s(y) \wedge c(y)$ Existential instantiation using (1)
 3. $s(y)$ Simplification using (2)
 4. $\forall x(s(x) \rightarrow w(x))$ Hypothesis
 5. $s(y) \rightarrow w(y)$ Universal instantiation using (4)
 6. $w(y)$ Modus ponens using (3) and (5)
 7. $c(y)$ Simplification using (2)
 8. $w(y) \wedge c(y)$ Conjunction using (6) and (7)
 9. $\exists x(c(x) \wedge w(x))$ Existential generalization using (8)
- d. Let $c(x)$ be “ x is in this class,” let $f(x)$ be “ x has been to France,” and let $l(x)$ be “ x has visited the Louvre.” We are given premises $\exists x(c(x) \wedge f(x))$, $\forall x(f(x) \rightarrow l(x))$, and we want to conclude $\exists x(c(x) \wedge l(x))$. In our proof, y represents an unspecified particular person.
1. $\exists x(c(x) \wedge f(x))$ Hypothesis
 2. $c(y) \wedge f(y)$ Existential instantiation using (1)
 3. $f(y)$ Simplification using (2)
 4. $c(y)$ Simplification using (2)
 5. $\forall x(f(x) \rightarrow l(x))$ Hypothesis
 6. $f(y) \rightarrow l(y)$ Universal instantiation using (5)
 7. $l(y)$ Modus ponens using (3) and (6)
 8. $c(y) \wedge l(y)$ Conjunction using (4) and (7)
 9. $\exists x(c(x) \wedge l(x))$ Existential generalization using (8)

Question 62 (*Q: 15, pg: 72*) For each of these arguments determine whether the argument is correct or incorrect and explain why.

- a. All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
- b. Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
- c. All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
- d. Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

Solution:

- a. Correct using Universal instantiation and Modus Ponens
- b. Invalid. fallacy of affirming the conclusion.
- c. Invalid. fallacy of denying the hypothesis.
- d. Correct using Universal instantiation and Modus Tollens

Question 63 (Q.16, pg:72) For each of these arguments determine whether the argument is correct or incorrect and explain why.

- a. *Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.*
- b. *A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.*
- c. *Quincy likes all action movies. Quincy likes the movie Eight Men Out. Therefore, Eight Men Out is an action movie.*
- d. *All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.*

Solution:

- a. This is correct, using universal instantiation and modus tollens.
- b. This is not correct. After applying universal instantiation, it contains the fallacy of denying the hypothesis.
- c. This is not correct. After applying universal instantiation, it contains the fallacy of affirming the conclusion.
- d. This is correct, using universal instantiation and modus ponens.

Question 64 (Q: 17, pg: 72) What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.

Solution:

x is some x and need not be Lola.

Question 65 (Q.18, pg:72) What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

Solution:

We know that some s exists that makes $S(s, Max)$ true, but we cannot conclude that Max is one such s . Therefore this first step is invalid.

Question 66 (Q: 19, pg: 72) Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

- If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.
- If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.
- If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

Solution:

- Conclusion is False. Given statement is implication and not bi-implication
- True using Modus tollens.
- Conclusion is False. Given statement is implication $p \rightarrow q$ and $\neg p$ which can not be concluded as $\neg q$.

Question 67 (Q.20, pg:72) Determine whether these are valid arguments.

- If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.
- If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

Solution:

- This is invalid. It is the fallacy of affirming the conclusion. Letting $a = -2$ provides a counterexample.
- This is valid; it is modus ponens.

Question 68 (Q: 21, pg: 72) Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 26 of Section 1.3?

Solution:

p: All lions are fierce.

q: Some lions do not drink coffee.

\therefore Some fierce creatures do not drink coffee.

By the second premise, there is some lion that does not drink coffee. Let Leo be such a creature.

By simplification we know that Leo is a lion. By modus ponens we know from the first premise that Leo is fierce. Hence, Leo is fierce and does not drink coffee. By the definition of the existential quantifier, there exist fierce creatures that do not drink coffee, that is, some fierce creatures do not drink coffee.

Question 69 (Q.22, pg:72) Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 27 of Section 1.3?

Solution:

We will give an argument establishing the conclusion. We want to show that all hummingbirds are small. Let Tweety be an arbitrary hummingbird. We must show that Tweety is small. The first premise implies that if Tweety is a hummingbird, then Tweety is richly colored. Therefore by (universal) modus ponens we can conclude that Tweety is richly colored. The third premise implies that if Tweety does not live on honey, then Tweety is not richly colored. Therefore by (universal) modus tollens we can now conclude that Tweety does live on honey. Finally, the second premise implies that if Tweety is a large bird, then Tweety does not live on honey. Therefore again by (universal) modus tollens we can now conclude that Tweety is not a large bird, i.e., that Tweety is small, as desired. Notice that we invoke universal generalization as the last step.

Question 70 (Q: 23, pg: 72) Identify the error or errors in this argument that supposedly shows that if $\exists xP(x) \wedge \exists xQ(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

- 1 $\exists x P(x) \wedge \exists xQ(x)$... Premise
- 2 $\exists x P(x)$... Simplification from (1)
- 3 $P(c)$... Existential instantiation from (2)
- 4 $\exists x Q(x)$... Simplification from (1)
- 5 $Q(c)$... Existential instantiation from (4)
- 6 $P(c) \wedge Q(c)$... Conjunction from(3) and (5)
- 7 $\exists x (P(x) \wedge Q(x))$... Existential Generalisation

Solution:

The error occurs in step (5), because we cannot assume, as is being done here, that the c that makes P true is the same as the c that makes Q true.

Question 71 (Q.24, pg:72) Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall xP(x) \vee \forall xQ(x)$ is true.

1. $\forall x(P(x) \vee Q(x))$ Premise
2. $P(c) \vee Q(c)$ Universal instantiation from (1)
3. $P(c)$ Simplification from (2)
4. $\forall xP(x)$ Universal generalization from (3)
5. $Q(c)$ Simplification from (2)
6. $\forall xQ(x)$ Universal generalization from (5)
7. $\forall x(P(x) \vee \forall xQ(x))$ Conjunction from (4) and (6)

Solution:

Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

Question 72 (Q: 25, pg: 72) Justify the rule of universal modus tollens by showing that the premises $\forall x (P(x) \rightarrow Q(x))$ and $\neg Q(a)$ for a particular element a in the domain, imply $\neg P(a)$.

Solution:

We are given the premises $\forall x(P(x) \rightarrow Q(x))$ and $\neg Q(a)$. We want to show $\neg P(a)$. Suppose, to the contrary, that $\neg P(a)$ is not true. Then $P(a)$ is true. Therefore, by universal modus ponens, we have $Q(a)$. But this contradicts the given premise $\neg Q(a)$. Therefore our supposition must have been wrong and so $\neg P(a)$ is true, as desired.

Question 73 (*Q:26, pg.72*) Justify the rule of universal transitivity, which states that if $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$ are true, then $\forall x(P(x) \rightarrow R(x))$ is true, where the domains of all quantifiers are the same.

Solution:

We want to show that the conditional statement $P(a) \rightarrow R(a)$ is true for all a in the domain; the desired conclusion then follows by universal generalization. Thus we want to show that if $P(a)$ is true for a particular a , then $R(a)$ is also true. For such an a , by universal modus ponens from the first premise we have $Q(a)$, and then by universal modus ponens from the second premise we have $R(a)$, as desired.

Question 74 (*Q: 27, pg: 72*) Use rules of inference to show that if $\forall x (P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x (P(x) \wedge R(x))$ are true, then $\forall x (R(x) \wedge S(x))$ is true.

Solution:

| | |
|---|--|
| 1 $\forall x P(x) \wedge R(x)$ | ... Premise |
| 2 $P(a) \wedge R(a)$ | ... Universal instantiation from (1) |
| 3 $P(a)$ | ... Simplification from (2) |
| 4 $\forall x (P(x) \rightarrow (Q(x) \wedge R(x)))$ | ... Premise |
| 5 $Q(a) \wedge S(a)$ | ... Universal instantiation from (4) and Modus Ponens (3), (4) |
| 6 $S(a)$ | ... Simplification from (5) |
| 7 $R(a)$ | ... Simplification from (2) |
| 8 $R(a) \wedge S(a)$ | ... Conjunction from (7), (6) |
| 9 $\forall x (R(x) \wedge S(x))$ | ... Universal Generalisation from (8) |

Question 75 (*Q.28, pg:72*) Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

Solution:

We want to show that the conditional statement $\neg R(a) \rightarrow P(a)$ is true for all a in the domain; the desired conclusion then follows by universal generalization. Thus we want to show that if $\neg R(a)$ is true for a particular a , then $P(a)$ is also true. For such an a , universal modus tollens applied to the second premise gives us $\neg(\neg P(a) \wedge Q(a))$. By rules from propositional logic, this gives us $P(a) \vee \neg Q(a)$. By universal generalization from the first premise, we have $P(a) \wedge Q(a)$. Now by resolution we can conclude $P(a) \wedge P(a)$, which is logically equivalent to $P(a)$, as desired.

Question 76 (*Q: 29, pg: 72*) Use rules of inference to show that if $\forall x (P(x) \vee Q(x))$, $\forall x (\neg Q(x) \vee S(x))$, $\forall x (R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

Solution:

- 1 $\exists x \neg P(x)$... *Premise*
- 2 $\neg P(c)$... *Existential instantiation from (1)*
- 3 $\forall x (P(x) \vee Q(x))$... *Premise*
- 4 $P(c) \vee Q(c)$... *Universal instantiation from (3)*
- 5 $Q(c)$... *Disjunctive Syllogism from (2), (4)*
- 6 $\forall x (\neg Q(x) \vee S(x))$... *Premise*
- 7 $\neg Q(c) \vee S(c)$... *Universal instantiation from (6)*
- 8 $S(c)$... *Disjunctive Syllogism from (5), (7)*
- 9 $\forall x (R(x) \rightarrow \neg S(x))$... *Premise*
- 10 $R(c) \rightarrow \neg S(c)$... *Universal instantiation from (9)*
- 11 $\neg R(c)$... *Modus tollens from (9)*
- 12 $\exists x \neg R(x)$... *Existential Generalisation from (11)*