Fall 2016 Math 2B
Suggested homework problems solutions

	Section	Problems
7.5	Strategy for integration	2, 3, 7, 18, 22, 25, 27, 28, 30, 42
8.1	Arc length	10, 12, 14, 34
11.1	Sequences	4, 10, 14, 26, 31-36 all, 47, 50, 55

Strategy for integration

Problem 2 : We make the substitution $u = 3x+1$, $du = 3\,dx$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 4$. We get

$$\int_0^1 (3x+1)^{\sqrt{2}}\,dx = \frac{1}{3}\int_1^4 u^{\sqrt{2}}\,du = \frac{4^{\sqrt{2}+1}-1}{3(\sqrt{2}+1)}.$$

Problem 3 : We integrate by parts to get rid of $\ln y$,with

$$u = \ln y, \qquad du = \frac{dy}{y},$$
$$v = \frac{2}{3}y^{3/2}, \quad dv = \sqrt{y}\,dy.$$

We get

$$\begin{aligned}\int_1^4 \sqrt{y}\ln y\,dy &= \left[\frac{2}{3}y^{3/2}\ln y\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{y}\,dy \\ &= \left[\frac{2}{3}y^{3/2}\ln y - \frac{2^2}{3^2}y^{3/2}\right]_1^4 = \frac{32}{3}\ln 2 - \frac{28}{9}.\end{aligned}$$

Problem 7 : We do the substitution $u = \arctan y$, $du = \dfrac{dy}{1+y^2}$. When $y = -1$, $u = \arctan(-1) = -\arctan 1 = -\pi/4$. When $y = 1$, $u = \arctan 1 = \pi/4$.

$$\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$$

Problem 18 : We do the substitution $u = \sqrt{t}$, $du = \dfrac{dt}{2\sqrt{t}}$. When $t = 1$, $u = 1$. When $t = 4$, $u = 2$.

$$\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = 2\int_1^2 e^u\,du = 2\,[e^u]_1^2 = 2(e^2 - e).$$

Problem 22 : We do the substitution $u = \ln x$, $du = \dfrac{dx}{x}$.

$$\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx = \int \frac{u}{\sqrt{1 + u^2}}\,du.$$

We make the substitution $v = 1 + u^2$, $dv = 2u\,du$.

$$\int \frac{u}{\sqrt{1+u^2}}\,du = \int \frac{dv}{2\sqrt{v}} = \sqrt{v} = \sqrt{1 + (\ln x)^2}.$$

Problem 25 : We have $\deg P = 1$ and $\deg Q = 1$. We do a long division and have

$$1 + 12t = 4(1 + 3t) - 3.$$

Thus we have

$$\int_0^1 \frac{1 + 12t}{1 + 3t}\,dt = \int_0^1 (4 - \frac{3}{1 + 3t})\,dz = [4t - \ln(3t + 1)]_0^1 = 4 - \ln 4.$$

Problem 27 : We make the substitution $u = e^x$, $du = e^x\,dx$.

$$\int \frac{dx}{1 + e^x} = \int \frac{du}{u(1+u)} = \int \frac{du}{u} - \int \frac{du}{1+u} = \ln|u| - \ln|1+u| + C = x - \ln(e^x + 1) + C.$$

Problem 28 : We make the substitution $u = \sqrt{at}$, $du = \dfrac{a}{2\sqrt{at}}\,dt = \dfrac{a}{2u}\,dt$.

$$\int \sin\sqrt{at}\,dt = \int (\sin u)\frac{2u}{a}\,du = \frac{2}{a}\int u\sin u\,du.$$

We perform an integration by parts with

$$\begin{aligned} v &= u, & dv &= du, \\ w &= -\cos u, & dw &= \sin u\,du. \end{aligned}$$

$$\begin{aligned} \frac{2}{a}\int u\sin u\,du &= \frac{2}{a}\left[-u\cos u + \int \cos u\,du\right] = \frac{2}{a}\left[-u\cos u + \sin u\right] + C \\ &= \frac{2}{a}\left[-\sqrt{at}\cos\sqrt{at} + \sin\sqrt{at}\right] + C. \end{aligned}$$

Problem 30 : $e^x \geq 1 \Leftrightarrow x \geq 0$. We thus have

$$|e^x - 1| = \begin{cases} 1 - e^x, & \text{for } -1 \leq x \leq 0, \\ \\ e^x - 1, & \text{for } 0 \leq x \leq 2. \end{cases}$$

Therefore

$$\int_{-1}^{2} |e^x - 1|\, dx = \int_{-1}^{0} (1 - e^x)\, dx + \int_{0}^{2} (e^x - 1)\, dx = [x - e^x]_{-1}^{0} + [e^x - x]_0^2 = e^2 + e^{-1} - 3.$$

Problem 42 : We integrate by parts to get rid of $\tan^{-1} x$,with

$$\begin{aligned} u &= \tan^{-1} x, \quad & du &= \frac{dx}{1+x^2}, \\ v &= -\frac{1}{x}, & dv &= \frac{dx}{x^2}. \end{aligned}$$

We get

$$\int \frac{\tan^{-1} x}{x^2}\, dx = -\frac{\tan^{-1} x}{x} + \int \frac{dx}{x(1+x^2)}.$$

$\frac{1}{x(1+x^2)}$ is a proper rational function whose denominator is already factored. We want to find constants A, B, and C such that

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

By putting the two fractions on the right on the same denominator, we get the equality

$$1 = A(x^2 + 1) + (Bx + C)x \Leftrightarrow 1 = (A + B)x^2 + Cx + A.$$

We identify the different powers of x and obtain the system

$$\begin{cases} A + B = 0 \\ \\ C = 0 \\ \\ A = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1, \\ \\ B = -1, \\ \\ C = 0. \end{cases}$$

We can now compute our integral

$$\begin{aligned} \int \frac{1}{x(x^2+1)}\, dx &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\, dx \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 1| + C. \end{aligned}$$

Therefore

$$\int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{\tan^{-1} x}{x} + \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

Arc length

Problem 10 : Let f be the function defined by

$$f(x) = \frac{1}{6}\sqrt{(x^2-4)^3} = \frac{1}{6}(x^2-4)^{3/2}.$$

We have $f'(x) = \frac{x}{2}(x^2-4)^{1/2}$, and so

$$\sqrt{1+f'(x)^2} = \sqrt{1+(\frac{x(x^2-4)^{1/2}}{2})^2} = \sqrt{1+\frac{x^2(x^2-4)}{4}} = \sqrt{\frac{(x^2-2)^2}{4}} = \frac{x^2-2}{2}.$$

We want the length of the curve defined by $y = f(x)$, $2 \le x \le 3$.

$$\frac{1}{2}\int_2^3 x^2 - 2\,dx = \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_2^3 = \frac{13}{6}.$$

Problem 12 : Let g be the function defined by

$$g(y) = \frac{y^4}{8} + \frac{1}{4y^2}$$

We have $g'(y) = \frac{y^3}{2} - \frac{1}{2y^3} = \frac{1}{2}(y^3 - y^{-3})$, and so

$$\begin{aligned}\sqrt{1+g'(y)^2} &= \sqrt{1+(\frac{1}{2}(y^3-y^{-3}))^2} = \sqrt{1+\frac{1}{4}(y^6-2+y^{-6})} = \sqrt{\frac{4y^6+y^{12}-2y^{-6}+1}{4y^6}} \\ &= \frac{\sqrt{(y^6+1)^2}}{2y^3} = \frac{y^6+1}{2y^3} = \frac{1}{2}(y^3+y^{-3}).\end{aligned}$$

We want the length of the curve defined by $x = g(y)$, $1 \le y \le 2$.

$$\frac{1}{2}\int_1^2 (y^3+y^{-3})\,dy = \frac{1}{2}\left[\frac{y^4}{4} - \frac{1}{2y^2}\right]_1^2 = \frac{33}{16}.$$

Problem 14 : Let f be the function defined by

$$f(x) = \ln(\cos x).$$

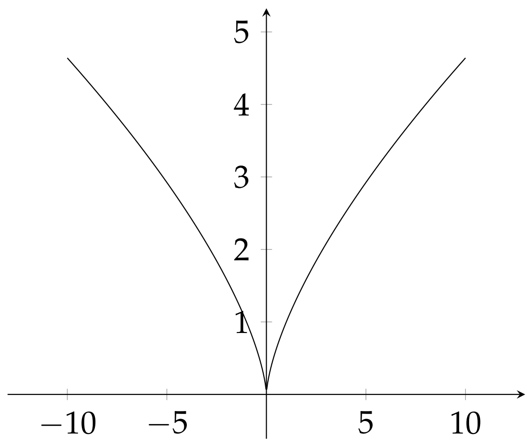
We have $f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x$, and so

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \tan^2 x} = \sec x.$$

We want the length of the curve defined by $y = f(x)$, $0 \le x \le \dfrac{\pi}{3}$.

$$\int_0^{\pi/3} \sec x \, dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln(2 + \sqrt{3}).$$

Problem 34 : (a)



(b) Let f be the function defined by

$$f(x) = x^{2/3}.$$

We have $f'(x) = \dfrac{2}{3}x^{-1/3}$, and so

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{4}{9}x^{-2/3}} = \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}.$$

We want the length of the curve defined by $y = f(x)$, $0 \le x \le 1$.

$\sqrt{1 + f'(x)^2}$ has an infinite discontinuity at 0.

Let $0 < t \le 1$.

We do the substitution $u = x^{2/3}$, $du = \dfrac{2}{3}x^{-1/3}\,dx$. When $x = t$, $u = t^{2/3}$. When $x = 1$, $u = 1$.

$$\int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} \, dx = \frac{1}{2} \int_{t^{2/3}}^1 \sqrt{9u + 4} \, du = \frac{1}{27} \left[(9u + 4)^{3/2} \right]_{t^{2/3}}^1 = \frac{1}{27} \left(13^{3/2} - (9t^{2/3} + 4)^{3/2} \right).$$

As $t \to 0$, $t^{2/3} \to 0$ and hence $(9t^{2/3} + 4)^{3/2} \to 4^{3/2} = 8$. So

$$\int_0^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} \, dx = \frac{1}{27} \left(13^{3/2} - 8 \right).$$

Let g be the function defined by

$$g(y) = y^{3/2}.$$

We have $g'(y) = \frac{3}{2}y^{1/2}$, and so

$$\sqrt{1 + g'(y)^2} = \sqrt{1 + \frac{9}{4}y} = \frac{1}{2}\sqrt{4 + 9y}.$$

We want the length of the curve defined by $x = g(y)$, $0 \le y \le 1$.

$$\frac{1}{2}\int_0^1 \sqrt{4+9y}\, dy = \frac{1}{27}\left[(4+9y)^{3/2}\right]_0^1 = \frac{1}{27}\left(13^{3/2} - 8\right).$$

(c)

$$\begin{aligned}\frac{1}{2}\int_0^1 \sqrt{4+9y}\, dy + \frac{1}{2}\int_0^4 \sqrt{4+9y}\, dy &= \frac{1}{27}\left(13^{3/2} - 8\right) + \frac{1}{27}\left[(4+9y)^{3/2}\right]_0^4 \\ &= \frac{1}{27}\left(13^{3/2} + 80\sqrt{10} - 16\right).\end{aligned}$$

Sequences

Problem 4 : $a_1 = 0, a_2 = \frac{3}{5}, a_3 = \frac{4}{5}, a_4 = \frac{15}{17}, a_5 = \frac{12}{13}$.

Problem 10 : $a_1 = 6, a_2 = 6, a_3 = 3, a_4 = 1, a_5 = \frac{1}{4}$.

Problem 14 : $a_n = \frac{(-1)^{n+1}}{4^{n-2}}$.

Problem 26 : $-1 < 0.86 < 1$ so $\lim_{n\to+\infty} (0.86)^n = 0$, and $\lim_{n\to+\infty} a_n = 2$.

Problem 31 : We factor n^2 in the numerator and denominator of the rational function

$$\frac{1+4n^2}{1+n^2} = \frac{\frac{1}{n^2} + 4}{\frac{1}{n^2} + 1}.$$

Hence

$$\lim_{n\to+\infty} \frac{1+4n^2}{1+n^2} = \lim_{n\to+\infty} \frac{\frac{1}{n^2} + 4}{\frac{1}{n^2} + 1} = \frac{4}{1} = 4.$$

The square root function is continuous at 4, so $\lim_{n\to+\infty} a_n = \sqrt{4} = 2$.

Problem 32 : We factor n in the numerator and denominator of the rational function

$$\frac{n\pi}{n+1} = \frac{\pi}{1+\frac{1}{n}}.$$

Hence

$$\lim_{n\to+\infty} \frac{n\pi}{n+1} = \lim_{n\to+\infty} \frac{\pi}{1+\frac{1}{n}} = \frac{\pi}{1} = \pi.$$

cos is continuous at π, so $\lim\limits_{n\to+\infty} a_n = \cos(\pi) = -1$.

Problem 33 : We factor by n^2 the numerator and denominator, because it is the highest power of n.

$$\frac{n^2}{\sqrt{n^3+4n}} = \frac{n^2}{n^2\sqrt{\frac{1}{n}+\frac{4}{n^3}}} = \frac{1}{\sqrt{\frac{1}{n}+\frac{4}{n^3}}}.$$

$\lim\limits_{n\to+\infty} \sqrt{\frac{1}{n}+\frac{4}{n^3}} = 0^+$, so $\lim\limits_{n\to+\infty} a_n = +\infty$.

Problem 34 : We factor by n the numerator and denominator, and take the limit.

$$\lim_{n\to+\infty} \frac{2n}{n+2} = \lim_{n\to+\infty} \frac{2}{1+\frac{2}{n}} = \frac{2}{1} = 2.$$

$x \mapsto e^x$ is continuous at 2, so $\lim\limits_{n\to+\infty} a_n = e^2$.

Problem 35 : We take the absolute value of a_n to get rid of $(-1)^n$.

$$\lim_{n\to+\infty} |a_n| = \lim_{n\to+\infty} \frac{1}{2\sqrt{n}} = 0.$$

Hence $\lim\limits_{n\to+\infty} a_n = 0$.

Problem 36 : We take the absolute value of a_n to get rid of $(-1)^{n+1}$.

$$|a_n| = \frac{n}{n+\sqrt{n}}.$$

We factor by n the numerator and denominator, and take the limit.

$$\lim_{n\to+\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to+\infty} \frac{1}{1+\sqrt{\frac{1}{n}}} = \frac{1}{1} = 1.$$

Hence a_n has odd-numbered terms that approach 1 and even-numbered terms that approach -1. Therefore a_n is divergent.

Problem 47 : $a_n = (1+\dfrac{2}{n})^n = e^{n\ln(1+\frac{2}{n})}$. We define $f(x) = x\ln(1+\dfrac{2}{x})$. We have $\lim\limits_{n\to+\infty} n\ln(1+\dfrac{2}{n}) = \lim\limits_{x\to+\infty} f(x)$. We apply l'Hospital rule

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to+\infty} \frac{2}{1+\frac{2}{x}} = \frac{2}{1} = 2.$$

Then $\lim\limits_{n\to+\infty} n\ln(1+\dfrac{2}{n}) = 2$. $x \mapsto e^x$ is continuous at 2, so $\lim\limits_{n\to+\infty} a_n = e^2$.

Problem 50 : We define $f(x) = \dfrac{(\ln x)^2}{x}$. We have $a_n = f(n)$ and $\lim\limits_{n\to+\infty} a_n = \lim\limits_{x\to+\infty} f(x)$. We apply l'Hospital rule two times in a row and get

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{2\cdot\frac{1}{x}\ln x}{1} = \lim_{x\to+\infty} \frac{2\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to+\infty} \frac{2}{x} = 0.$$

Thus $\lim\limits_{n\to+\infty} a_n = 0$.

Problem 55 : Let $n > 1$.

$$a_n = \frac{n!}{2^n} = \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdots\frac{n-1}{2}\cdot\frac{n}{2}.$$

Each fraction between $\frac{2}{2}$ and $\frac{(n-1)}{2}$ is larger than 1.
We thus have $a_n \geq \dfrac{1}{2}\cdot\dfrac{n}{2}$. And $\lim\limits_{n\to+\infty} \dfrac{n}{4} = +\infty$. Therefore $a_n \to +\infty$, as $n \to +\infty$.