Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #1

February 17-20, 2020

- This is an *Open Notes* Exam. You can use materials, homeworks problems, lecture notes, etc. that you manually worked on.
- There is **NO** *Open Internet* allowed. You can only access our Main Course Webpage.
- You are not allowed to work on or discuss these problems with other people. You can ask a few small, clarifying, questions in Office Hours, but these problems will not be solved.
- Submit your final work in Gradescope in the Exam 1 entry.
- Please *show* all of your work and *justify* all of your answers.

Problem	Score	Possible Points
1		8
2		22
3		30
4		40
Total		100

1. [8 Points] Use Trig Substitution to **PROVE** that $\displaystyle\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4+4\tan^2\theta}\cdot 2\sec^2\theta\,d\theta$$

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}}$$

$$= \int \frac{1}{\underbrace{4(1+\tan^2\theta)}_{\cancel{4}\,\cancel{\sec^2\theta}\ \ 2}}\cdot \cancel{2}\cancel{\sec^2\theta}\,d\theta$$

$$\tan\theta = \frac{x}{2}$$

$$\theta = \arctan\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\int 1\,d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

2. [22 Points] Evaluate the following **limit**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $\lim\limits_{x\to 0} \dfrac{\cosh(3x) - \arctan(2x) + 2x - 1}{\arcsin x - e^x + 1}$ $\quad 0/0$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{3\sinh(3x) - \frac{2}{1+(2x)^2} + 2}{\frac{1}{\sqrt{1-x^2}} - e^x} \quad 0/0$$

$-2(1+4x^2)^{-1}$

$(1-x^2)^{-1/2}$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{9\cosh(3x) + 2(1+4x^2)^{-2}(8x)}{-\frac{1}{2}(1-x^2)^{-3/2}(-2x) - e^x}$$

$$= \lim_{x\to 0} \frac{9\cosh(3x) + \frac{16x}{(1+4x^2)^2}}{\frac{x}{(1-x^2)^{3/2}} - e^x} = \frac{9}{-1} = \boxed{-9}$$

(b) $\lim\limits_{x\to\infty} \left(1 - \dfrac{2}{x^3}\right)^{x^3}$ $\quad 1^\infty$

$$= e^{\lim_{x\to\infty} \ln\left[\left(1-\frac{2}{x^3}\right)^{x^3}\right]} = e^{\lim_{x\to\infty} x^3 \ln\left(1-\frac{2}{x^3}\right)} = e^{\lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x^3}\right)}{\frac{1}{x^3}}} \quad 0/0$$

Flip

$-2x^{-3} \to 6x^{-4}$

$$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-\frac{2}{x^3}} \cdot \left(\frac{6}{x^4}\right)}{\frac{-3}{x^4}}} = \boxed{e^{-2}}$$

cancel

3. [30 Points] Compute each of the following **definite integrals**. Please simplify your answer.

(a) Show that $\displaystyle\int_1^{e^2} \sqrt{x}\cdot \ln x\, dx = \frac{8e^3+4}{9}$

$$\boxed{\begin{array}{ll} u=\ln x & dv=\sqrt{x} \\ du=\frac{1}{x}dx & v=\frac{2}{3}x^{3/2}\end{array}}$$

$$=\frac{2}{3}x^{3/2}\ln x\Big|_1^{e^2}-\frac{2}{3}\int_1^{e^2}x^{1/2}dx=\frac{2}{3}x^{3/2}\ln x\Big|_1^{e^2}-\frac{4}{9}x^{3/2}\Big|_1^{e^2}$$

$$=\frac{2}{3}\big(\underbrace{\cancel{\sqrt{e^2}}}_{e}\big)^3\underbrace{\ln(e^2)}_{2}-\frac{2}{3}\underbrace{\cancel{\ln 1}}_{0}-4/9\Big[\big(\underbrace{\cancel{\sqrt{e^2}}}_{e}\big)^3-1\Big]$$

$$=\underbrace{\frac{4}{3}}_{12/9}e^3-\frac{4}{9}e^3+\frac{4}{9}=\frac{8}{9}e^3+\frac{4}{9}=\boxed{\frac{8e^3+4}{9}}$$ Match!

(b) Show that $\displaystyle\int_3^9 \frac{1}{\sqrt{x}\,(x+9)}\,dx = \frac{\pi}{18}$

$$=\int_3^9\frac{1}{\sqrt{x}\left((\sqrt{x})^2+9\right)}dx = 2\int_{\sqrt{3}}^{3}\frac{1}{w^2+9}dw=\frac{2}{3}\arctan\left(\frac{w}{3}\right)\Big|_{\sqrt{3}}^{3}$$

$$\boxed{\begin{array}{l} w=\sqrt{x}\\ dw=\frac{1}{2\sqrt{x}}dx \\ 2dw=\frac{1}{\sqrt{x}}dx\end{array}} \quad \boxed{\begin{array}{l} x=3\Rightarrow w=\sqrt{3}\\ x=9\Rightarrow w=\sqrt{9}=3\end{array}}$$

$$=\frac{2}{3}\Big[\underbrace{\cancel{\arctan}\left(\tfrac{3}{3}\right)}_{\pi/4}-\underbrace{\cancel{\arctan}\left(\tfrac{\sqrt{3}}{3}\right)}_{\pi/6}\Big]$$

$$=2/3\left[\pi/4-\pi/6\right]=\frac{2}{3}\left[\frac{3\pi}{12}-\frac{2\pi}{12}\right]=\frac{\cancel{2}}{3}\left(\frac{\pi}{\cancel{12}_{6}}\right)=\boxed{\frac{\pi}{18}}$$ Match!

(c) Show that $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \frac{\pi}{6}$

$$=\int_0^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-(e^x)^2}}dx=\int_1^{\sqrt{3}}\frac{1}{\sqrt{4-w^2}}dw=\arcsin\left(\frac{w}{2}\right)\Big|_1^{\sqrt{3}}$$

$$=\underbrace{\cancel{\arcsin}\left(\tfrac{\sqrt{3}}{2}\right)}_{\pi/3}-\underbrace{\cancel{\arcsin}\left(\tfrac{1}{2}\right)}_{\pi/6}=\frac{\pi}{3}-\frac{\pi}{6}=\boxed{\frac{\pi}{6}}$$ Match!

$$\boxed{\begin{array}{l} w=e^x\\ dw=e^x dx\end{array}}$$

$$\boxed{\begin{array}{l} x=0\Rightarrow w=e^0=1\\ x=\ln\sqrt{3}\Rightarrow w=\cancel{e^{\ln}}^{\sqrt{3}}=\sqrt{3}\end{array}}$$

4. [40 Points] Compute the following **indefinite integral**. Please simplify your answer.

(a) $\int \frac{x^2+x+1}{x^2+3}\,dx$ Split-Split-Split

Slip-in|Slip-out

split-split

$$= \int \frac{x^2+3-3}{x^2+3} + \frac{x}{x^2+3} + \frac{1}{x^2+3}\,dx$$

$$= \int 1 - \frac{3}{x^2+3} + \frac{x}{x^2+3} + \frac{1}{x^2+3}\,dx$$

$$= \int 1 + \frac{x}{x^2+3} - \frac{2}{x^2+3}\,dx$$

w-sub.

$$= x + \int \frac{x}{x^2+3}\,dx - \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$w = x^2+3$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$$= x + \underbrace{\frac{1}{2}\int \frac{1}{w}\,dw}_{\frac{1}{2}\ln|w|} - \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$$\boxed{= x + \frac{1}{2}\ln|x^2+3| - \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$

OR $\int \frac{x^2+x+1^{+2-2}}{x^2+3}\,dx = \int \frac{x^2+3+x-2}{x^2+3}\,dx = \ldots$

4. (Continued) Compute the following **indefinite integral**. Please simplify your answer.

split | split

(b) $\int \overset{P}{(x+1)}\, \overset{I}{\arcsin x}\, dx = \left(\frac{x^2}{2}+x\right)\arcsin x - \int \frac{\frac{x^2}{2}+x}{\sqrt{1-x^2}}\, dx$

IBP

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x+1\, dx \\ du = \frac{1}{\sqrt{1-x^2}} & v = \frac{x^2}{2} + x \end{array}}$$

$$= -\frac{1}{2}\underbrace{\int \frac{x^2}{\sqrt{1-x^2}}\, dx}_{\text{Trig. Sub. } (*)} - \underbrace{\int \frac{x}{\sqrt{1-x^2}}\, dx}_{\text{u-sub } (**)}$$

$$\boxed{= \left(\frac{x^2}{2}+x\right)\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right) + \sqrt{1-x^2} + C}$$

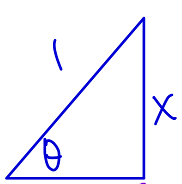
↳ OR split $\int x \arcsin x\, dx + \int \arcsin x\, dx$

? ?

(*) Trig. Sub.

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{array}}$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \int \frac{\sin^2\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta} = \cos\theta}} \cdot \cancel{\cos\theta}\, d\theta = \int \sin^2\theta\, d\theta = \int \frac{1-\cos(2\theta)}{2}\, d\theta$$



$\Rightarrow \sqrt{1-x^2}$

$$= \frac{1}{2}\left[\theta - \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right] + C = \frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right] + C$$

(**) u-sub

$$\int \frac{x}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \cancel{-\frac{1}{2}\,\frac{u^{1/2}}{1/2}} + C = -\sqrt{1-x^2} + C$$

($\frac{1}{\sqrt{u}} \to u^{-1/2}$)

$$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\, dx \\ -\frac{1}{2} du = x\, dx \end{array}}$$

4. (Continued) Compute the following **indefinite integral**. Please simplify your answer.

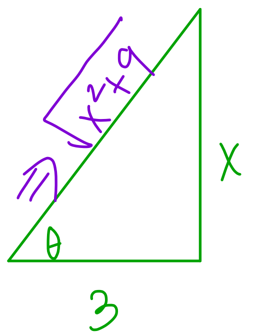
(c) $\displaystyle\int \frac{1}{(9+x^2)^{\frac{7}{2}}}\,dx$ Hint: $3^6 = 729$

$$= \int \frac{1}{\left(\sqrt{9+x^2}\right)^7}\,dx = \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^7}\cdot 3\sec^2\theta\,d\theta$$

$$\left(\sqrt{9+9\tan^2\theta}\right)^7 = \left(\sqrt{9(1+\tan^2\theta)}\right)^7 = \left(\sqrt{9\sec^2\theta}\right)^7 = (3\sec\theta)^7$$

$$\boxed{\begin{aligned} x &= 3\tan\theta \\ dx &= 3\sec^2\theta\,d\theta \end{aligned}}$$

$\hookrightarrow \tan\theta = \frac{x}{3}$



$$= \int \frac{3\sec^2\theta}{3^7\sec^7\theta}\,d\theta = \frac{1}{3^6}\int \frac{1}{\sec^5\theta}\,d\theta$$

$$= \frac{1}{729}\int \cos^5\theta\,d\theta = \frac{1}{729}\int \underbrace{\cos^4\theta}_{(\cos^2\theta)^2}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{729}\int (1-\sin^2\theta)^2\cos\theta\,d\theta$$

$$\boxed{\begin{aligned} w &= \sin\theta \\ dw &= \cos\theta\,d\theta \end{aligned}}$$

$$= \frac{1}{729}\int (1-w^2)^2\,dw = \frac{1}{729}\int 1-2w^2+w^4\,dw$$

$$= \frac{1}{729}\left[w - \frac{2w^3}{3} + \frac{w^5}{5}\right] + C = \frac{1}{729}\left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\right] + C$$

$$= \boxed{\frac{1}{729}\left[\frac{x}{\sqrt{x^2+9}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+9}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+9}}\right)^5\right] + C}$$