A LEAST SQUARE PROCEDURE FOR SOLVING COMPLEX INTEGRAL EQUATIONS OF A SINGLE REAL VARIABLE BY POLYNOMIAL APPROXIMATION

by

Jerome Wenker

Submitted in partial fulfillment
of the requirements for the
degree of Bachelor of Science
at the
Massachusetts Institute of Technology
1956

Signature of Author: Signature redacted

Department of Mathematics
May 23, 1956

Certified by: Signature redacted

Professor P.D. Crout

TABLE OF CONTENTS

INTRODUCTION

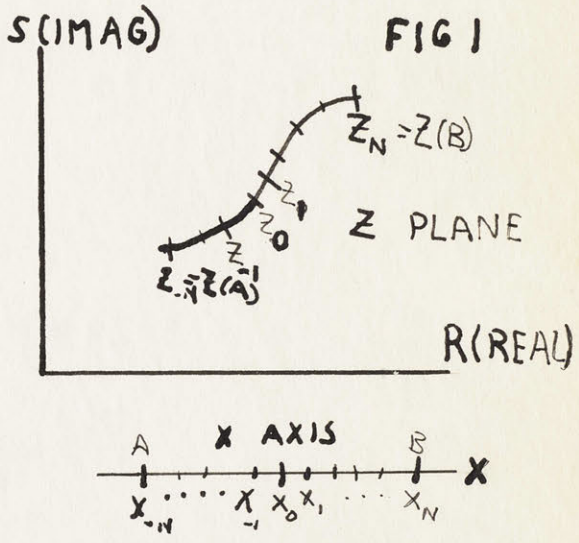
This paper develops and uses a least squares procedure for solving complex integral equations in a single real variable. The procedure is developed in parts 2 and 3 and the problem is solved in parts 4 and 5. The problem that is solved is that of a string of length L rotating with an angular velocity w in a viscous fluid and having a traveling wave as a forcing function.

APPROXIMATION BY POLYNOMIALS.

Let us consider the algebraic polynomial

$$(1) \qquad Z = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2N} x^{2N} + j\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_{2N} x^{2N}\right)$$

where $j = \sqrt{-1}$. Choosing any 2N+1 equally spaced points on the x axis, the midpoints being x_0 and the spacing being h. Let the corresponding values of Z be $Z_{-N}, \ldots, Z_{-1}$, $Z_0, Z_1, \ldots, Z_N$. The graph of the polynomial thus passes through the points $(x_0-Nh, Z_{-N}), \ldots, (x_0-h, Z_{-1})$, $(x_0, Z_0), (x_0+h, Z_1), (x_0+2h, Z_2), \ldots, (x_0+Nh, Z_N)$. (See Fig. 1) For convenience, we place



$$(2) \qquad x = x_0 + hq \quad \text{OR} \quad q = \frac{x - x_0}{h}$$

Hence substituting in Eq. (1) we obtain a polynomial of the same degree in q; thus

$$(3)\quad Z = D_0 + D_1 q + D_2 q^2 + \cdots + D_{2N} q^{2N} + j\left(E_0 + E_1 q + \cdots + E_{2N} q^{2N}\right)$$

Where the D's and G's are new constant coefficients. This polynomial is evidently satisfied by the following pairs of values of q and Z:

$(-N, Z_{-N}) \ldots (-1, Z_{-1}), (0, Z_0), (1, Z_1), (2, Z_2) \ldots, (N, Z_N)$

Substituting each of these in (3) we obtain a set of 2N+1 linear equations, thus

$$(4)\quad Z_i = R_i + jS_i = D_0 + D_1 i + \cdots + D_{2N} i^{2N} + j\left(E_0 + E_1 i + \cdots + E_{2N} i^{2N}\right) \qquad i = -N, \cdots, -1, 0, 1, \cdots, N$$

Separating these equations into real and imaginary parts, we get

$$(5)\quad R_i = D_0 + D_1 i + \cdots + D_{2N} i^{2N} \qquad i = -N, \cdots, N$$

and

$$S_i = E_0 + E_1 i + \cdots E_{2N} i^{2N} \qquad i = -N, \cdots, N$$

These equations completely determine the D's and E's as linear combinations of the R's and S's: Thus

$$(6)\quad D_m = B_{m,-N} R_{-N} + \cdots + B_{m,-1} R_{-1} + B_{m,0} R_0 + B_{m,1} R_1 + \cdots B_{m,N} R_N \qquad m = 0, 1, 2, \cdots, 2N$$

where the B's are constants dependent only upon N[1]. Similarly,

$$(7)\quad E_P = B_{P,-N} S_{-N} + \cdots + B_{P,-1} S_{-1} + B_{P,0} S_0 + B_{P,1} S_1 + \cdots + B_{P,N} S_N \qquad P = 0, 1, 2, \cdots, 2N$$

Multipling the relationships (6) by q^M and the relationships (7) by jq^P, adding, and comparing the sum on the left-hand sides with (3), we obtain the polynomial expressed in the form

$$Z = K_{-N} R_{-N} + \cdots + K_{-1} R_{-1} + K_0 R_0 + K_1 R_1 + \cdots + K_N R_N$$
$$(8)\quad + j(K_{-N} S_{-N} + \cdots + K_{-1} S_{-1} + K_0 S_0 + K_1 S_1 + \cdots + K_N S_N)$$

or recombining (and remembering that $Z_i = R_i + jS_i$) we get

$$(9)\quad Z = K_{-N} Z_{-N} + \cdots + K_{-1} Z_{-1} + K_0 Z_0 + K_1 Z_1 + \cdots + K_N Z_N$$

Here the K's are polynomials of degree 2N in q: Thus

$$(10)\quad K_R = B_{0,R} + B_{1,R} q + B_{2,R} q^2 + \cdots + B_{2N,R} q^{2N} \qquad R = -N, \cdots, -1, 0, 1, \cdots, N$$

1. See Crout "An Application of Polynomial Approximation to the Solution of Integral Equations Arising in Physical Problems", Joun. of Math. and Physics. Vol. XIX. No. p.36, January 1940. Henceforce this will be referred to as reference 2.

Equation (9) is the same as Eq. (6) in reference 2. The rest of the material covered in pp. 36-40 of reference 2 can also be used in the material covered in this paper.

LEAST SQUARE TECHNIQUE FOR COMPLEX EQUATIONS OF A SINGLE REAL VARIABLE.

If the unknown function in an integral equation is replaced by the algebraic polynomial approximation $Z(x) = \sum_{i=-N}^{N} K_i(q) Z_i$ [See Eq. (9)] and if the order of integration and summation is then inverted, the result is an expression of the form

$$\sum_{i=-N}^{N} Z_i I_i(x) \cong H(x) \tag{11}$$

Here the Z's are the complex values of 2N + 1 equally spaced ordinates which serve to specify the polynomial, the K's are Lagrange coefficients [polynomials of degree 2N obtained before in Eq. (10)]. Tables of which are available and $q = \frac{x - x_o}{h}$ is the distance in h units by which x lies to the right of the midpoint, x_o, h being the distance between adjacent ordinates. (See Fig. 1.)

We shall determine the Z's so that the integral of the square of the absolute value of the two sides of (11) is a minimum. Thus

$$\int_A^B \left[H(x) - \sum_{i=-N}^{N} Z_i I_i(x) \right] \left[\overline{H(x)} - \sum_{i=-N}^{N} \overline{Z_i I_i(x)} \right] dx = \text{MINIMUM} \tag{12}$$

where $\overline{}$ denotes complex conjugation of the symbol beneath it. Equating the corresponding partial derivatives to zero, we have

$$(13)\quad \int_A^B \left\{ \left(H(x) - \sum_{i=-N}^{N} Z_i I_i(x) \right) \overline{I_p(x)}\, \partial \overline{Z_p} + \left(\overline{H(x)} - \sum_{i=-N}^{N} \overline{Z_i I_i(x)} \right) I_p(x)\, \partial Z_p \right\} \partial x = 0$$

$$p = -N, \cdots, -1, 0, 1, \cdots, N$$

Let $\partial Z \equiv \partial R + j\partial S$. Hence $\partial \overline{Z} = \overline{\partial Z} = \partial R - j\partial S$ and

$$(14)\quad \left\{ \left[\int_A^B H(x) \overline{I_p(x)}\, \partial x - \sum_{i=-N}^{N} Z_i \int_A^B I_i(x) \overline{I_p(x)}\, \partial x \right] + \left[\int_A^B \overline{H(x)} I_p(x)\, \partial x - \sum_{i=-N}^{N} \overline{Z_i} \int_A^B \overline{I_i(x)}\, I_p(x)\, \partial x \right] \right\} \partial R$$

$$-j \left\{ \left[\int_A^B H(x) \overline{I_p(x)}\, \partial x - \sum_{i=-N}^{N} Z_i \int_A^B I_i(x) \overline{I_p(x)}\, \partial x \right] - \left[\int_A^B \overline{H(x)} I_p(x)\, \partial x - \sum_{i=-N}^{N} \overline{Z_i} \int_A^B \overline{I_i(x)}\, I_p(x)\, \partial x \right] \right\} \partial S = 0$$

$$p = -N, \cdots, -1, 0, 1, \cdots, N$$

∂r and ∂S are arbitrarily; hence

$$\left[\int_A^B H(x)\,\overline{I_p}(x)\,\partial x - \sum_{i=-N}^{N} Z_i \int_A^B I_i(x)\,\overline{I_p}(x)\,\partial x\right] +$$

(15)

$$\left[\int_A^B \overline{H(x)}\,I_p(x)\,\partial x - \sum_{i=-N}^{N} \overline{Z_i} \int_A^B \overline{I_i(x)}\,I_p(x)\,\partial x\right] = 0$$

$$P = -N, \cdots, N$$

and

$$\left[\int_A^B H(x)\,\overline{I_p}(x)\,\partial x - \sum_{i=-N}^{N} Z_i \int_A^B I_i(x)\,\overline{I_p}(x)\,\partial x\right] +$$

(16)

$$\left[\int_A^B \overline{H(x)}\,I_p(x)\,\partial x - \sum_{i=-N}^{N} \overline{Z_i} \int_A^B \overline{I_i(x)}\,I_p(x)\,\partial x\right] = 0$$

$$P = -N, \cdots, N$$

It follows that

$$\int_A^B H(x)\,\overline{I_p}(x)\,\partial x - \sum_{i=-N}^{N} Z_i \int_A^B I_i(x)\,\overline{I_p}(x)\,\partial x = 0 \qquad (17)$$

Equation (17) is necessary and sufficient for equation (12) to be satisfied.[2]

Replacing the exact by an approximate integration and using the formular (Simpson's Rule) $\int f\,\partial x = \sum_{k=1}^{M} D_k f(x_k)$ where the D_ks are $\frac{h}{3}$, $\frac{2h}{3}$, or $\frac{4h}{3}$, depending on k. k is an even integer and h is the spacing of the x_k's. Equation (17) becomes

$$\sum_{k=1}^{M} D_k\,H(x_k)\,\overline{I_p}(x_k) = \sum_{i=-N}^{N} Z_i \sum_{k=1}^{M} D_k\,\overline{I_i}(x_k)\,\overline{I_p}(x_k) \qquad (18)$$

$$P = -N, \cdots, N$$

[2] Eqs. (11)-(17) were given to the author by Professor Crout.

The matrix of the coefficients of the Z's is hence

$$(19) \quad \| D_k I_p(x_k) \| \cdot \| I_i(x_k) \|$$

where the row and column indices are P and k for the first matrix, and k and i for the second, respectively. We thus see that the matrix of the system of Eqs. (18) can be obtained by forming the approximate equations

$$(20) \quad \sum_{i=-N}^{N} Z_i I_i(x_k) = H(x_k)$$

in the usual manner, and then multiplying the augmented matrix of this system by the matrix obtained by multiplying the columns of the conjugated, transposed (unaugmented) matrix by the integration coefficients $D_1, D_2, \ldots, D_M$ respectively. This gives as a final solution (expressed in matrix form)

$$(21) \quad \| D_k \overline{I_p(x_k)} \| \cdot \| I_i(x_k) \| \cdot \| Z_i \| = \| D_k \overline{I_p}(x_k) \| \cdot \| H(x_k) \|$$

where the row and column indices are the same as given before and $\| Z_i \|$ and $\| H(x_k) \|$ are column vectors.

It is evidently permissible to multiply all of the D's by any constant r, since this merely multiplies both sides of the resulting linear equations (18) by r, and hence does not alter the final results. A given set of integration coefficients may therefore be arbitrarily magnified before being used as a set of D's.

It is evident that this least square process does not require the parameters Z_i in (11) to arise from the Lagrangean form of the algebraic polynomial, although the letter Z and the range -N to +N for i were chosen with this in mind; in fact, it does not even require (11) to arise from an integral equation. Any approximating function of the form $Z(x) = a_1W_1(x) + a_2W_2(x)+,...a_NW_n(x)$ could have been used.

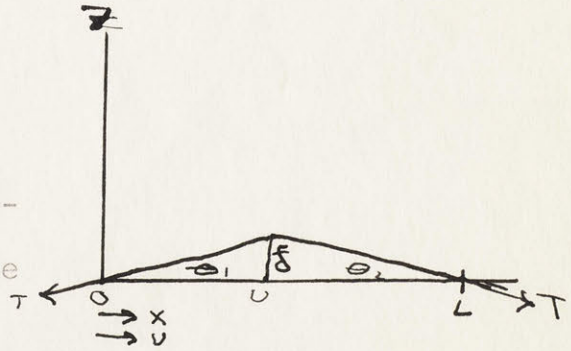
GENERAL SOLUTION TO THE PROBLEMS

We now illustrate the least square technique by solving the following problem.

A string of length L and density P(x) is rotating with an angular velocity w in a viscous fluid with a coefficient of friction c. The string is stretched with a uniform tension T. A forcing function w(x,t) is impressed upon the string. Find the displacement of the string as a function of x where x is the distrance from one end of the string.

THE GREEN'S FUNCTION

This problem will be solved by finding the deflection of x due to a unit force at u, integrating the product of the actual force times the deflection due to a point force over the length of the string.



We suppose the string is fixed at both ends and initially so tightly stretched that nonuniformity of the tension, due to small deflections can be neglected. If a unit concentrated load is applied in the Z-direction at an arbitrary point u (See Fig. 2) the string will then be deflected into 2 linear parts with a corner at the point x = u. If we denote the (approximately) uniform tension by T, the requirements of force equilibrium in the Z-direction leads to the condition

$$T \sin\theta_1 + T \sin\theta_2 = 1 \tag{22}$$

with the notation of Fig. 2. For small deflections (and slopes) we have the approximations

$$\sin\theta_1 \approx \tan\theta_1 \approx \frac{\delta}{u}$$
$$\sin\theta_2 \approx \tan\theta_2 \approx \frac{\delta}{L-u} \tag{23}$$

where δ is the maximum deflection of the string, at the loaded point u. The introduction of these approximations into (22) leads to the relation

$$T\left(\frac{\delta}{u} + \frac{\delta}{L-u}\right) = 1$$

and hence determines the deflection in the form

$$\delta = \frac{u}{TL}\{L-u\} \tag{24}$$

The equations of the corresponding deflection curve is readily obtained in the form

$$Z = \begin{cases} \delta \frac{x}{U} & \text{WHEN } x < U \\ \delta \left(\frac{L-x}{L-U}\right) & \text{WHEN } U < x \end{cases} \tag{25}$$

Where δ is given by(24), so that the influence function (for small deflections) is given by the expression[3]

$$G(x,U) = \begin{cases} \frac{x}{TL}(L-U) & \text{WHEN } x < U \\ \frac{U}{TL}(L-x) & \text{WHEN } U < x \end{cases} \tag{26}$$

For a string rotating uniformly above the x-axis with angular velocity w, with a frictional force of c times the velocity, a loading function w(x) imposed in the direction outward from the axis of revolution the deflection takes the form

$$Z(x) = \int_0^L G(x,U) R(U) dU \tag{27}$$

where G(x,u) is described in Eq. (25) and

$$R(x) = (w^2 - jwc) P(x) Z(x) + W(x) \tag{28}$$

3 See Ref. 5, Hildebrand "Methods of Applied Mathematics pp.402-404.

is the total force acting on the string at point x.

Equation (27) then takes the form[4]

$$Z(x) = \int_0^L G(x,v)\, w(v)\, dv + (\omega^2 - j\omega c)\int_0^L G(x,v)\, P(v)\, Z(v)\, dv \tag{29}$$

SET

$$\int_0^L G(x,v)\, w(v)\, dv = H(x) \tag{30}$$

It seems physically evident that the deflection curve is smooth enough to permit good approximation by a fourth degree polynomial $Z(x)$ passing through the fixed end points. Let $x_{-2}=0$, $x_{-1} = \frac{L}{4}$, $x_0 = \frac{L}{2}$, $x_1 = \frac{3L}{4}$, $x_2 = L$ and let the values of $Z(x)$ at these points be $Z_{-2} = 0$, Z_1, Z_0, Z_1, $Z_2 = 0$. Then using Equation (9) equations (29) and (30) becomes

$$H(x) = \sum_{i=-2}^{2} Z_i \left\{ K_i[q(x)] - (\omega^2 - j\omega c)\int_0^L G(x,v)\, P(v)\, K_i[q(v)]\, dv \right\} \tag{31}$$

Setting

$$\left\{ K_i[q(x)] - (\omega^2 - j\omega c)\int_0^L G(x,v)\, P(v)\, K_i[q(v)]\, dv \right\} = I_i(x) \tag{32}$$

we get

$$\sum_{i=-2}^{2} Z_i\, I_i(x) = H(x) \tag{33}$$

[4] Equations (28) and (29) obtained from Professor Crout

which is the same as equation (11). Using this equation the least squares method of solution will be applied.

For the problems in this paper $P(x)$ will be held to the constant[5] P and equation (32) becomes

$$(34) \qquad I_i(x) = K_i[q(x)] - P\omega(\omega - jc)\int_0^L G(x,u)\, K_i[q(x)]\, \partial u$$

Because of the large amounts of calculations necessary, the problem will be solved in segments.

THE MATRIX $\left[\int_0^L G(x_k,u)\, K_i[q(u)]\, \partial u\right]$

While it is possible to integrate the integrals in this matrix it is simpler to proceed as follows. This equation is a rearrangement of $\int_0^L G(x_k,u)Z(u)\partial u$. Since $Z(u)$ is a smooth curve and since $G(x_k u)$ is smooth on either side of the corner at $u = x_k$ the integrand is smooth o n each side of this point, at which it too has a corner. Approximate integration using Simpson's Rule should give good results if no one interpolation polynomial is required to approximate both sides of the corner.

Let us divide the interval $0 \leq u \leq L$ into 16 equal subintervals and apply Simpson's Rule. Each of the intervals $0 < u < \frac{1}{8} L$, $\frac{1}{8}$ - $< u < \frac{1}{4} L$ etc. is hence approximated by a second degree polynomial. Simpson's Rule is used so we can apply the same integration formulas regardless of whether

[5]Notice that if $P(x)$ were a variable it could be included as a weighing function on one of the matrices that would be obtained this this problem.

$x_k = \frac{1}{8}L,\ \frac{1}{4}L,\ \frac{3}{8}L,\ \frac{1}{2}L,\ \frac{5}{8}L,\ \frac{3}{4}L$ or $\frac{7}{8}L$.

This formula is

$$\int_0^L P(v)\,dv = \sum_{j=1}^{17} D_j f_j \tag{35}$$

where the 17 values of j refer to the 17 points bounding the 16 equal subintervals and the f's are the ordinates at these points; Also

$$D_1 = D_{17} = \frac{L}{48}, \quad D_2 = D_4 = D_6 = D_8 = D_{10} = D_{12} = D_{14} = D_{16} = \frac{4L}{48}$$
$$D_3 = D_5 = D_7 = D_9 = D_{11} = D_{13} = D_{15} = \frac{2L}{48} \tag{36}$$

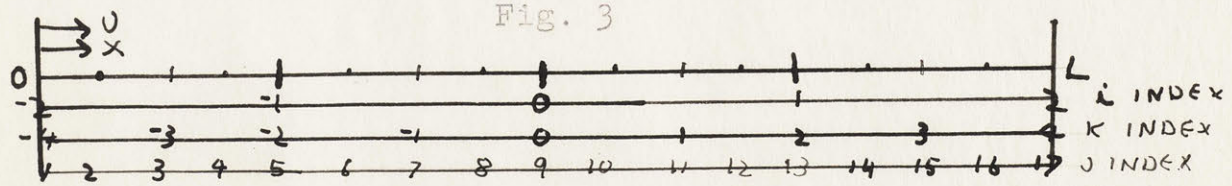
The $q(v_j)$ is defined by $q(v_j) = 4\frac{u_j}{L} - 2$ where u_j is found from the j-index in Fig. 3.

The matrix of the terms in equation (35) may be written as the (row times column) product of two matrices; thus[6]

$$\| G(x_k, v_j) \| \cdot \| D_j K_i(q(v_j)) \| \tag{37}$$

where the row and column indices are k and j for the first matrix and j and i for the second matrix respectfully.

Fig. 3



6. If P(x) were a (given) variable, it could be absorbed into the second matrix so that this matrix takes the form $\| D_j P(v_j) K_i(q(v_j)) \|$

As the value of $G(x_{kj}u_j)$ is zero when $x_k = 0$ or L; or when $u_j = 0$ or L this matrix will have the same numerical value if its first and last columns and its top and bottom rows are removed. This means that the matrix $\{G(x_k u_j)\}$ has 7 rows and 15 columns. In order that matrix multiplication remain defined, the top and bottom rows of $\{D_j K_i[q(u_j)]\}$ must also be removed[7]. This does not change the value of the sum as the only multiplications that involve these quantities are multiplications by zero. Also as $Z_{-2} = Z_2 = 0$ the first and last colums of this matrix can be removed. This means that the matrx $[D_j K_i[q(u_j)]]$ has 15 rows and 3 columns. The row and column indices of the first matrix are K and j respectively while the row and column indices of the second matrix are j and i respectively. The final result of this is that the matrix product of these two matrices has 7 rows and 3 columns, the row index being k and the column index being i.

As usual for problems of this nature the first matrix depend only on the Green's Function, and hence contains the physical parts for this particular problem. The second matrix depends only on the method of solution, and can be computed once and for all and used on a variety of problems[8]

---

[7] Note after the first and last colums of this matrix have been removed these rows consist of nothing but zero's anyhow so nothing is lows when they are removed. This does not change this set of equations as $H(x_k) = \int G(x_k u) w(u) \partial u$ and this quantity is zero for $x_k = 0$ or $x_k = L$ (as $G(0,u) = G(L,u) = 0$) unless $w(0)$ or $w(L)$ is infinite. This possibility is excluded.

[8] See Ref. 2, pg. 44

Using Equation (27) $G(x_k, u_j)$ becomes:

(38) $$\{G(x_k, u_j)\} = \frac{L}{128T} \begin{bmatrix} 7 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 18 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 20 & 25 & 30 & 27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 28 & 24 & 20 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & 25 & 20 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 18 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 7 \end{bmatrix}$$

As the top and bottom rows are removed a $\frac{L}{24}$ can be factored out of the second matrix and (using Table 3 in reference No. 2 and Table 6 in reference No. 3) $\{D_j, K_i[q(u)]\}$ becomes;

(39) $$\{D_j K_i[q(u_j)]\} = \frac{L}{24} \begin{bmatrix} 1.50390\ 625 & -.96679\ 6875 & .41015\ 625 \\ 1.093750 & -.54687\ 5 & .21875\ 0 \\ 2.28515\ 625 & -.68554\ 6875 & .25390\ 625 \\ 1 & 0 & 0 \\ 1.50390\ 625 & .75195\ 31250 & -.21484\ 375 \\ .46875\ 0 & .70312\ 5 & -.15625\ 0 \\ .41015\ 625 & 1.84570\ 3125 & -.24609\ 375 \\ 0 & 1 & 0 \\ -.24609\ 375 & 1.84570\ 3125 & .41015\ 625 \\ -.15625 & .70312\ 5 & .46875 \\ -.21484\ 375 & .75195\ 3125 & 1.50390\ 625 \\ 0 & 0 & 1 \\ .25390\ 625 & -.68554\ 6875 & 2.28515\ 625 \\ .21875 & .54687\ 5 & 1.09375\ 0 \\ .41015\ 625 & -.96679\ 6875 & 1.50390\ 625 \end{bmatrix}$$

Both equations (38) and (3) are exact.

Carrying out the matrix multiplication of $\{G(x_k,u_j)\}\{D_j k_i[q(u_j)]\}$ to get (approximately the matrix $\{\int_0^L G(x_k,u)k_i[q(u)]\partial u\}$ the results are:

$$(40)\quad \{\int_0^L G(x_k,u)k_i[q(u)]\partial u\} = F \begin{bmatrix} 90.34375 & 33.35937\,5 & 30.84375 \\ 132.875 & 88.6875 & 52.875 \\ 129.09375 & 143.48437\,5 & 74.59375 \\ 102.5 & 166.25 & 102.5 \\ 74.543750 & 143.48437\,5 & 129.09375 \\ 52.875 & 88.6875 & 132.875 \\ 30.84375 & 33.35937\,50 & 90.34375 \end{bmatrix}$$

$$F = \frac{L^2}{3072\,T}$$

Equation (40) is the exact matrix product of Equation (38) and Equation (39).

Now having found the numerical values of $\{\int_0^L G(x_k u)k_i[q(u)]\partial u\}$ return to the problem of finding $\{I_i(x)\}$.

$\{I_i(x_k)\}$

An approximate integration will be used in the least squares procedure, this is reduced (by use of an approximate integration using Simpson's Rule) to the problem of finding the numbers to go into a matrix of the form $\{k_i(q_k)\}$ As the answer is expected to be a relatively smooth curve a forth degree polynomial approximation will again be used. This means that i takes on the values -2,-1,0,1,2 and q_k takes on the values $-2,-\frac{3}{2},-1,-\frac{1}{2},0,+\frac{1}{2},1\frac{3}{2},2$. This is a tabulated function (the Lagrangean interpolation polynomial) and is here copied from Reference No. 2. Table 3, p. 85.

$$(41)\quad \{K_i(q_k)\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .2734375 & 1.09375 & -.546875 & .21875 & -.0390625 \\ 0 & 1 & 0 & 0 & 0 \\ .0390625 & .46875 & .703125 & -.15625 & .0234375 \\ 0 & 0 & 1 & 0 & 0 \\ .0234375 & -.15625 & .703125 & .46875 & .0390625 \\ 0 & 0 & 0 & 1 & 0 \\ .0390625 & .21875 & -.546875 & 1.09375 & .2734375 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where k and i are the row and column indices respectively.

As $Z_2=Z_2=0$ the first and last columns of the matrix $\{k_i(q_k)\}$ can be removed without changing the problem. With these columns removed the top and bottom rows become all zeros and can be removed[9]. With these simplifications addition of the two matrices $\{k_i(q_k)\}$ and $\{\int_0^L G(x_k,u)k_i[q(u)]\partial u\}$ is defined.

Using the equations (34) we get(for set values of x_k)

$$(42)\quad \{I_c(x_k)\} = \left[\{k_i(q_k)\} - PW(w-jc)\{\int_0^L G(x_k,u)k_i[q(u)]\partial u\}\right]$$

Using equations (40), (41), and (42) we get

---

9 SEE COMMENTS ON THIS SIMPLIFICATION ON PAGE 14.

(43) $$\{I_i(x)\} = \begin{bmatrix} 1.09375-90.34375\frac{PWL^2(W-JC)}{3072T} & -.546875-33.359375\frac{PWL^2(W-JC)}{3072T} & .21875-30.84375\frac{PWL^2(W-JC)}{3072T} \\ 1-132.875\frac{PWL^2(W-JC)}{3072T} & -88.6875\frac{PWL^2(W-JC)}{3072T} & -52.875\frac{PWL^2(W-JC)}{3072T} \\ .46875-129.09375\frac{PWL^2(W-JC)}{3072T} & .703125-143.484375\frac{PWL^2(W-JC)}{3072T} & -.15625-74.59375\frac{PWL^2(W-JC)}{3072T} \\ -102.5\frac{PWL^2(W-JC)}{3072T} & 1-16625\frac{PWL^2(W-JC)}{3072T} & -102.5\frac{PWL^2(W-JC)}{3072T} \\ -.15625-74.59375\frac{PWL^2(W-JC)}{3072T} & 703125-143.484375\frac{PWL^2(W-JC)}{3072T} & .46875-129.09375\frac{PWL^2(W-JC)}{3072T} \\ -52.875\frac{PWL^2(W-JC)}{3072T} & -88.6875\frac{PWL^2(W-JC)}{3072T} & 1-132.875\frac{PWL^2(W-JC)}{3072T} \\ .21875-30.84375\frac{PWL^2(W-JC)}{3072T} & -.546875-33.359375\frac{PWL^2(W-JC)}{3072T} & 1.09375-90.34375\frac{PWL^2(W-JC)}{3072T} \end{bmatrix}$$

This corresponds to a system of 7 equations in 3 unknowns Z_{-1}, Z_o, Z_1. The second equation of this system is

(44) $$\left\{1-132.875\frac{PWL^2(W-JC)}{3072T}\right\}Z_{-1} + \left\{-88.6875\frac{PWL^2(W-JC)}{3072T}\right\}Z_o - \left\{52.875\frac{PWL^2(W-JC)}{3072T}\right\}Z_1 = H_{-2}$$

As this set of equations would be very difficult to solve as they are, they definitions will be made

(45) $$C = gW \quad \text{WHERE } g<1 \qquad \frac{PW^2L^2}{3072T} = \frac{1}{D} \qquad DH_k = Q_k$$

then equation (43) becomes:

(46)

$$\{I_2(x_2)\} = \frac{1}{D}\begin{bmatrix} 1.09375D-90.34375(1-jg) & -.546875D-33.359375(1-jg) & .21875D-30.84375(1-jg) \\ D-132.875(1-jg) & -88.6875(1-jg) & -52.875(1-jg) \\ .46875D-129.09375(1-jg) & .703125D-143.484375(1-jg) & -.15625D-74.59375(1-jg) \\ -102.5(1-jg) & D-166.25(1-jg) & -102.5(1-jg) \\ -.15625D-74.59375(1-jg) & .703125D-143.484375(1-jg) & .46875D-129.09375(1-jg) \\ -52.875(1-jg) & -88.6875(1-jg) & D-132.875(1-jg) \\ .21875D-30.84375(1-jg) & -.546875D-33.359375(1-jg) & 1.09375D-90.34375(1-jg) \end{bmatrix}$$

Equation (44) then becomes:

$$(D-132.875 + 132.875jg)Z_{-1} + (88.6875jg-88.6875)Z_0 + (52.875jg-52.875)Z_1 = Q_{-2}$$

where Q_{-2} is the numerical value of $D\int_0^L G(x_{-2},u)w(u)\partial u$. (The top row of (44) corresponds to a k index of-3 instead of a k index of -4 because the top row of the original system consists of zero's and hence has been ommited. [See note on page 19]

If $Z(x)$ were actually a fourth degree polynomial in the line from x = 0 to x = L then the equations obtainable from (43) would be exact instead of approximate except for small errors introduced by the approximate integrations; however, $Z(x)$ is not a polynomial, to our problem is to determine the 3 Z's of the i index so that for a given set of H's all seven of these equations will be satisfied closely, though none will be satisfied exactly except by accident. Using the least squares procedure obtained in part 3 we proceed as follows.

$\{D_k I_\nu(x_k)\}$ The quantities D_k are, except for an arbitrary constant multiplier, the coefficients in an approximate integration formula for the 9 points of the k index. The equations for $k = \pm 4$ have been removed because the equations would be of the form 0=0 as shown above on page 19.

A consideration of those two equations would have led to the addition of a row of zeros at the top and bottom of (43) and (46). Choosing Simpson's rule as the source of the D's leaving all comment for Section 6 , we have

(47) $D_{-4} = D_4 = \frac{1}{2}$, $D_{-3} = D_{-1} = D_1 = D_3 = 2$, $D_{-2} = D_0 = D_2 = 1$

D_{-4} and D_4 will, of course, not be used.

We now take the complex conjugate of equation (46), multiply each now by the corresponding D from (47) [K^{th} row multiplied by D_k] and finally interchange the rows and columns of the resulting matrix, thereby obtaining the "derived matrix" (48). (This matrix has been transposed)

(48)

$$\mathcal{L} = \begin{bmatrix} 2.1875D-180.6875(1+jg) & -1.09375D-66.71875(1+jg) & .43750-61.6875(1+jg) \\ D-132.875(1+jg) & -88.6875\ (1+jg) & -52.875(1+jg) \\ .9375D-258.1875(1+jg) & 1.40625D-286.96875(1+jg) & -.3125D-149.1875 \\ & & -149.1875(1+jg) \\ -102.5\ (1+jg) & D-166.25(1+jg) & -102.5(1+jg) \\ -.3125D-149.1875(1+jg) & 1.40625D-286.96875(1+jg) & .9375D- \\ & & 258.1875(1+jg) \\ -52.875(1+jg) & -88.6875(1+jg) & D-132.875(1+jg) \\ .4375D-61.6875(1+jg) & -1.09375D-66.71875(1+jg) & 2.1875D \\ & & -180.6875(1+jg) \end{bmatrix}^T$$

$$\mathcal{L} = D\left\{D_k \overline{T_p(x_k)}\right\}$$

THE LINEAR EQUATIONS

Combining equations (47),(48) and 21) we get 3 linear equations in 3 unknowns (where $g = \frac{c}{w}$ and $D = \frac{3072T}{PW^2L^2}$ from Equation (45)] these equations are also exact.

$$(49)\ \{3.97656\ 25\ D^2-883.421875D+93,643.2890625(1+g^2)\}\ Z_{-1}$$
$$+\{-.99609\ 375D^2-522.3203125D+9.546875jgD$$
$$+100,051.75390\ 625(1+g^2)\}\ Z_0$$
$$+\{.66406\ 25D^2-378.921875D+74,222.28906\ 25(1+g^2)\}\ Z_1$$
$$=\{2.1875D-180.6875(1+jg)\}\ Q_{-3}+\{D-132.875(1+jg)\}\ Q_{-2}$$
$$+\{.9375D-258.1875(1+jg)\}\ Q_{-1}+\{-102.5(1+jg)\}\ Q_0$$
$$+\{-.3125D-149.1875(1+jg)\}\ Q_1+\{-52.875(1+jg)\}\ Q_2$$
$$+\{.4375D-61.6875(1+jg)\}\ Q_3$$

~~$\{-.99609375D^2-522.3203125D-9.546875jgD$~~
~~$+100,051.75390625(1+g^2)\}\ Z_{-1}$~~

$$+\{-.99609375D^2-522.3203125D-9.546875jgD$$
$$(50)\qquad +100,051.75390\ 625(1+g^2)\}\ Z_{-1}$$
$$+\{4.173828125D^2-993.65234375D+130,172.462890625(1+g^2)\}\ Z_0$$
$$+\{-.99609375D^2-522.3203125D-9.546875jgD$$
$$+100,051.75390625(1+g^2)\}\ Z_1$$
$$=\{-1.09375D-66.71875(1+jg)\}\ Q_{-3}+\{-88.6875(1+jg)\}\ Q_{-2}$$
$$+\{1.40625D-286.96875(1+jg)\}\ Q_{-1}+\{D-166.25(1+jg)\}\ Q_0+$$
$$\{1.40625D-286.96875(1+jg)\}\ Q_1$$
$$+\{-88.6875(1+jg)\}\ Q_2+\{-1.09375D-66.71875(1+jg)\}\ Q_3$$

51) $\{.6640625D^2-378.921875D+74,222.2890625(1+g^2)\}Z_{-1}$

$+\{-.99609375D^2-522.3203125D+9.546875jgD$

$+100,051.75390625(1+g^2)\}Z_0$

$+\{3.9765625D^2-883.421875D+93,643.2890625(1+g^2)\}Z_1$

$=\{.4375D-61.6875(1+jg)\}Q_{-3}+\{-52.875(1+jg)\}Q_{-2}$

$+\{-.3125D-149.1875(1+jg)\}Q_{-1}+\{-102.5(1+jg)\}Q_0$

$+\{.9375D-258.1875(1+jg)\}Q_1+\{D-132.875(1+jg)\}Q_{+2}$

$+\{2.1875D-180.6875(1+jg)\}Q_3$

These 3 equations are symmetric in that if in the 3 equations Q_k is replaced by Q_{-k} and Z_i is replaced by Z_{-i} equation (49) becomes equation (51), equation (51) becomes equation (49) and equation (50) remains unchanged. Therefore if $Q_k=Q_{-k}$ then $Z_1=Z_{-1}$.

As these equations would become too difficult to solve as they are, values of D & g must be assumed. Before doing this the form of H(x) (and hence Q(x)) will be determined.

This solution will work for any forcing function w(x,t) SUCH THAT THE t PART CAN BE REMOVED BY COMPLEX NOTATION AND that will produce a reasonable curve when integrated with G(x,u) to obtain $H(x) = \int_0^L G(x,u)w(u)\partial u$.[10]

THE SOLUTION OF THE SPECIFIC PROBLEM.

The specific problem to be solved in this part is that

10. This is necessary so that this function can be reasonably approximated at the 7 points ($\frac{x}{L}=\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4}$, AND $\frac{7}{8}$) that are used in equations (47)to(51).The function will be zero at x=0, L as G(x, u) = 0 for x =0, L for any u and any finite w(u).

of a string rotating in a viscous mediuam with a traveling wave as its forcing function.

Chapter 1. Q(x)

Let $$w(x,t) = A_1 \sin\left[\frac{2\pi}{w_0}(x - V_0 t)\right]$$

where w_0 is the wave length of the forcing function, t is time and v_0 is the velocity of the travelling waves as they travel down the line.

$$(52)\quad w(x,t) = \left\{A_1 \cos\frac{2\pi V_0 t}{w_0}\right\} \sin\frac{2\pi x}{w_0} - \left\{A_1 \sin\frac{2\pi V_0 t}{w_0}\right\} \cos\frac{2\pi x}{w_0}$$

As complex notation has been used throughout to remove the time varing component of the problem, equation (52) can be rewriteen as[11]

$$(53)\quad w(x,t) = A \sin\frac{2\pi x}{w_0} - jA\cos\frac{2\pi x}{w_0}$$

Now $$H(x) = \int_0^L G(x,u)\, w(u)\, du \quad \text{AND}$$

$$G(x,u) = \frac{x}{TL}(L-u) \quad \text{FOR } x \leq u \text{ AND}$$

$$G(x,u) = \frac{u}{TL}(L-x) \quad \text{FOR } x \geq u, \text{ SO}$$

$$(54)\quad H(x) = \frac{L-x}{TL}\int_0^x u\, w(u)\, du + \frac{x}{TL}\int_x^L (L-u)\, w(u)\, du$$

$$= \frac{1}{T}\int_0^x u\, w(u)\, du - \frac{x}{TL}\int_0^x u\, w(u)\, du$$

$$+ \frac{x}{T}\int_x^L w(u)\, du - \frac{x}{TL}\int_x^L u\, w(u)\, du$$

11. Equations (51)-(53) were given to the author by Professor Crout

or

$$(55)\quad H(x) = \frac{1}{T}\int_0^x U\,w(U)\,dU + \frac{x}{T}\int_x^L w(U)\,dU - \frac{x}{TL}\int_0^L U\,w(U)\,dU$$

$$(56)\quad \frac{1}{T}\int_0^x U\,w(U)\,dU = \frac{A}{T}\int_0^x U\sin\frac{2\pi U}{W_0}\,dU - \frac{jA}{T}\int_0^x U\cos\frac{2\pi U}{W_0}\,dU$$

$$= \frac{W_0 A}{2\pi T}\left\{\frac{W_0}{2\pi}\sin\frac{2\pi x}{W_0} - x\cos\frac{2\pi x}{W_0}\right\}$$

$$- \frac{j W_0 A}{2\pi T}\left\{\frac{W_0}{2\pi}\cos\frac{2\pi x}{W_0} + x\sin\frac{2\pi x}{W_0} - \frac{W_0}{2\pi}\right\}$$

$$(57)\quad \frac{x}{T}\int_x^L w(U)\,dU = \frac{Ax}{T}\int_x^L \sin\frac{2\pi U}{W_0}\,dU - j\frac{Ax}{T}\int_x^L \cos\frac{2\pi U}{W_0}\,dU$$

$$= \frac{W_0 A x}{2\pi T}\left\{\cos\frac{2\pi x}{W_0} - \cos\frac{2\pi L}{W_0}\right\}$$

$$- \frac{j W_0 A x}{2\pi T}\left\{\sin\frac{2\pi L}{W_0} - \sin\frac{2\pi x}{W_0}\right\}$$

$$(58)\quad \frac{x}{TL}\int_0^L U\,w(U)\,dU = \frac{Ax}{TL}\int_0^L U\sin\frac{2\pi U}{W_0}\,dU - \frac{jAx}{TL}\int_0^L U\cos\frac{2\pi U}{W_0}\,dU$$

$$= \frac{W_0 A x}{2\pi TL}\left\{\frac{W_0}{2\pi}\sin\frac{2\pi L}{W_0} - L\cos\frac{2\pi L}{W_0}\right\}$$

$$- \frac{j W_0 A x}{2\pi TL}\left\{\frac{W_0}{2\pi}\cos\frac{2\pi L}{W_0} + L\sin\frac{2\pi L}{W_0} - \frac{W_0}{2\pi}\right\}$$

By use of Equations (55)-(58) we get

$$(59)\quad H(x) = \frac{w_0 A}{2\pi T}\left\{\frac{w_0}{2\pi}\sin\frac{2\pi x}{w_0} - x\cos\frac{2\pi x}{w_0}\right\}$$

$$- \frac{jw_0 A}{2\pi T}\left\{\frac{w_0}{2\pi}\cos\frac{2\pi x}{w_0} + x\sin\frac{2\pi x}{w_0} - \frac{w_0}{2\pi}\right\}$$

$$+ \frac{w_0 A}{2\pi T}\left\{x\cos\frac{2\pi x}{w_0} - x\cos\frac{2\pi L}{w_0}\right\}$$

$$- \frac{jw_0 A}{2\pi T}\left\{x\sin\frac{2\pi L}{w_0} - x\sin\frac{2\pi x}{w_0}\right\}$$

$$- \frac{w_0 A}{2\pi T}\left\{\frac{x}{L}\frac{w_0}{2\pi}\sin\frac{2\pi L}{w_0} - x\cos\frac{2\pi L}{w_0}\right\}$$

$$+ \frac{jw_0 A}{2\pi T}\left\{\frac{x}{L}\frac{w_0}{2\pi}\cos\frac{2\pi L}{w_0} + x\sin\frac{2\pi L}{w_0} - \frac{w_0 x}{2\pi L}\right\}$$

After cancelling terms we get

$$H(x) = \frac{A w_0}{2\pi T}\left\{\frac{w_0}{2\pi}\sin\frac{2\pi x}{w_0} - \frac{w_0 x}{2\pi L}\sin\frac{2\pi L}{w_0}\right\}$$

$$+ j\frac{A w_0}{2\pi T}\left\{\frac{-w_0}{2\pi}\cos\frac{2\pi x}{w_0} - \frac{w_0}{2\pi} + \frac{w_0 x}{2\pi L}\cos\frac{2\pi L}{w_0} - \frac{w_0 x}{2\pi L}\right\}$$

Extracting $\frac{w_0}{2\pi}$ we get

$$H(x)\quad \frac{A w_0^2}{4\pi^2 T}\left\{\sin\frac{2\pi x}{w_0} - \frac{x}{L}\sin\frac{2\pi L}{w_0}\right\}$$

$$+ \frac{jA w_0^2}{4\pi^2 T}\left\{1 - \frac{x}{L} + \frac{x}{L}\cos\frac{2\pi L}{w_0} - \cos\frac{2\pi x}{w_0}\right\}$$

(60) $$Q(x) = DH(x) = \frac{3072\,T}{P w^2 L^2} H(x) =$$
$$= \frac{768 A w_o^2}{P \pi^2 L^2 w^2} \left\{ \left(\sin \frac{2\pi x}{w_o} - \frac{x}{L} \sin \frac{2\pi L}{w_o} \right) + 3\left(1 - \frac{x}{L} + \frac{x}{L} \cos \frac{2\pi L}{w_o} - \cos \frac{2\pi x}{w_o}\right) \right\}$$

Equation (60) is the equation needed in equations (49)-(51). It vanishes identically at x=0 and x=L as is necessary due to the fact that G(x,u)=0 whenever x=0,L or u=0,L. (In this equation w_o=the wave length of the forcing function.)

Chapter 2. Numerical calculations In obtaining the solution to the problem numerical values must be given for w, c, A, and w_o.

Let $w = \frac{\pi}{L}\sqrt{\frac{T}{P}}$. . This is the first critical angular velocity for the case of a string rotating in a frictionless medium[12] and using this $D = \frac{3072}{\pi^2}$.

Let $w_o = 2L$. This quantity which can be defined independently of w is, however, defined such that the real part of the forcing function is of the same shape as the curve of the string would take if it were rotating in a frictionless medium at its first critical speed.

As in equations (49)-(51) c is set equal to gw, c must be determined in terms of w. By proper choice of $\frac{\pi}{L}\sqrt{\frac{T}{P}}$ w will have a numerical value less than 10. Using this, c can be set equal to $\frac{w}{10}$ giving a value of $g = \frac{1}{10}$

[12] SEE HILDEBRAND "ADVANCED CALCULUS FOR ENGINEERS" PAGE 204.

Let $A = \frac{10{,}000\ \pi^4 T}{3072L^2}$. Using these calculations equation (60) becomes

$$(61)\quad Q(x_i) = 10{,}000 \left\{ \sin \frac{\pi x_i}{L} - j\left(\frac{2x_i}{L} - 1 + \cos \frac{\pi x_i}{L}\right)\right\}$$

where x_i takes on these values as i goes from -3 to + 3:

$$(62)\quad x_{-3} = \frac{L}{8},\ x_{-2} = \frac{L}{4},\ x_{-1} = \frac{3L}{8},\ x_0 = \frac{L}{2}$$
$$x_1 = \frac{5L}{8},\ x_2 = \frac{3L}{4} \text{ and } x_3 = \frac{7L}{8}$$

Using these above found value of $D = \frac{3072}{\pi^2}$, the above obtained value of $g = \frac{1}{10}$, and equation (61) equations (49)-(51) becomes:

$$(63)\quad 204{,}864.16Z_{-1} + \{-158{,}027.96 + 297.15476j\} Z_0 + 21{,}357.469Z_{+1} = 43{,}589.067 - 2{,}293{,}736j$$

$$(64)\quad \{-158{,}027.96 + 297.15476j\} Z_{-1} + 226{,}559.92Z_0 + \{-158{,}027.96 + 297.15476j\} Z_{+1} = -135{,}106.88 + 872{,}986.49j$$

$$(65)\quad 21{,}357.469Z_{-1} + \{-158{,}027.96 + 297.15476j\} Z_0 + 204{,}864.16Z_{+1} = 147{,}034.47 + 887{,}809.66j$$

These equations were obtained using 8 place tables of sines, cosines and $\frac{1}{\pi^2}$ [13]. The calculations were done treating these 8 place values as they were exact and then after all

[13] The tables of sines and cosines were obtained from "8 Place tables of trigonometric functions" by Dr. J. Peters while the 8 place value of $\frac{1}{\pi^2}$ was obtained from Burington's "Handbook Of Mathematical Tables and Formulas."

calculations were compleated these equations were rounded off to 8 significant places.

The solution to these equations is

(66)
$$Z_{-1} = +.102\ 69144 - 24.941\ 463\ j$$
$$Z_{o} = -.017\ 200\ 14 - 18.846\ 552j$$
$$Z_{+1} = +.666\ 409\ 73 - 7.603\ 970j.$$

When the solution is applied to equations (63)-(65), the maximum error was numerically less than one. However, it takes a large change in the variables to cause a small change in the error forms. This is partially due to the fact that the angular velocity is set to be the first critical angular velocity. The equations (63)-(65) involve an approximate integration as well as approximate numerical calculations. Due to these considerations equation (66) probably has no more than 6 significant places.

Using these answers with equation (9) and table 3 of reference 2. The values of Z(x) are calculated for 21 equally placed points that divide the length L into 20 equal segments. These values are printed in the table on page 31 ~~32~~. They were calculated treating Z_{-1}, Z_{o} and Z_{+1} as if these values were exact and then rounding off to 4 figures. These graphs of the curves (see Fig. 5) ~~is~~ ARE drawn from the data of this table.

Table 1

x	q	z(x)
0	-2	0
$\frac{L}{20}$	-1.8	+1944 - 9.289j
$\frac{2L}{20}$	-1.6	+.2657-16.069j
$\frac{3L}{20}$	-1.4	+.2526-20.714j
$\frac{4L}{20}$	-1.2	+.1888-23.566j
$\frac{5L}{20}$	-1	+.1027-24.941j
$\frac{6L}{20}$	-.8	+.0179-25.131j
$\frac{7L}{20}$	-.6	-.0472-24.395j
$\frac{8L}{20}$	-.4	-.0794-22.969j
$\frac{9L}{20}$	-.2	-.0705-21.060j
$\frac{10L}{20}$	0	-.0172-18.847j
$\frac{11L}{}$	.2	+.0784-16.482j
$\frac{12L}{20}$	.4	+.2092-14.092j
$\frac{13L}{20}$	.6	+.3631-11.773j
$\frac{14L}{20}$	.8	+.5230-9.596j
$\frac{15L}{20}$	1	+.6664-7.604j
$\frac{16L}{20}$	1.2	+.7660-5.812j
$\frac{17L}{20}$	1.4	+.7893-4.208j

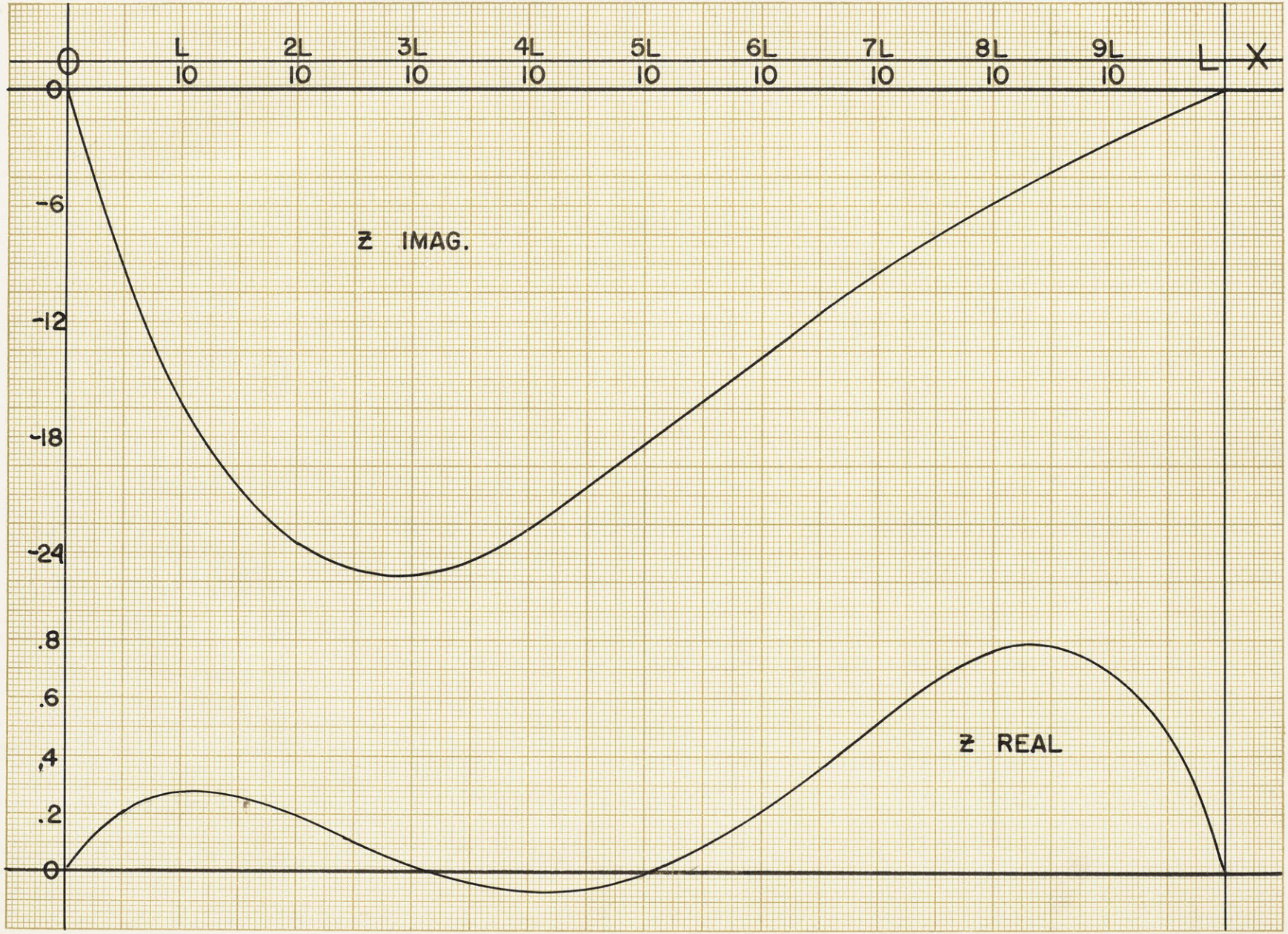

0
L/10
2L/10
3L/10
4L/10
5L/10
6L/10
7L/10
8L/10
9L/10
L
X
0
-6
-12
-18
-24
.8
.6
.4
.2
0
Z IMAG.
Z REAL

Table 1 continued

$\frac{18L}{20}$	1.6	+.6987 -2.754j
$\frac{19L}{20}$	1.8	+.4515 -1.383j
$\frac{20L}{20}$	2	0
x	q	z(x)

CONCLUSIONS.

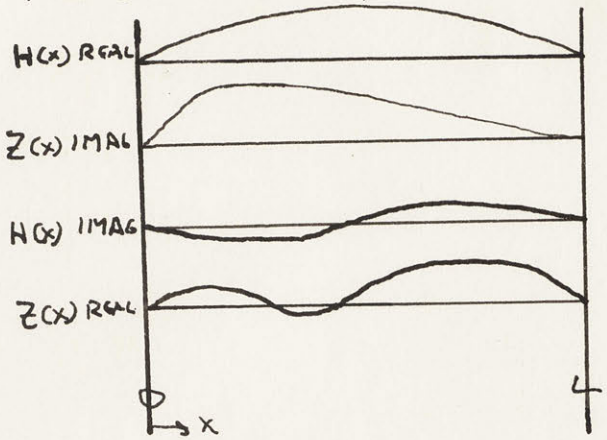
As this particular problem has been solved before there is no other solution with which to compare this solution. in order to get a measure of the air. The air is due partially to polynomial approximation, partially to the approximate integration, and partially due to numerical errors that arise from the use of approximate values of the sines and cosines. Due to the value chosen for w, the values of the real part of Z(x) vary greatly with only a slight change in the error. However, the changes were such that the real part never became much greater than one.

The problem was solved for a forcing function that has a sine curve as its real part and a cosine curve for its imaginary part. After the integration required to produce H(x) the real part remains a sine curve but the cosine curve becomes $1-\cos\frac{\pi x}{L} - \frac{2x}{L}$. This function is 0 at $x = 0, \frac{L}{2}$, AND L . At all other points this function is less than $\frac{3}{10}A$. The deflection curve is such that the deflection of the real part is almost zero, while the deflection of the imaginary part is large. This deflection of the imaginary part appears to be similar to a distorted sine curve with a phase lage. The small values for the real part show that the string's rotation is essentially in a plane. Comparing the real part of the deflection with the imaginary part of H(x) and the imaginary part of the deflection with the real part of H(x)[14] similarity

[14] See Figure 6

it can be seen that there is an apparent similarity between the curves. This can be accounted for by considering a phase shift of -90° between the force and the resulting deflection.

FIG 6



RATIO $\frac{H(x)\ \text{REAL (MAX)}}{H(x)\ \text{IMAG (MAX)}} \simeq \frac{1A}{.3A} \simeq 3.3$

$\frac{Z(x)\ \text{IMAG (MAX)}}{Z(x)\ \text{REAL (MAX)}} \simeq \frac{-25}{.8} = -32.$

The general conclusions are that this complex least square method can be used on any integral equation that can be solved by the regular method of solving integral equations by polynomial approximations. The specific conclusions above this problem including the fact that equations (49)-(51) are set up for all time and can be used for any specified forcing function that can be integrated (approximated if necessary) with respect to the Green's function. If the mass of the string is nonuniform the problem can be solved by solved by multiplying each row of equation (39) by the mass associated with that point to get the matrix $\{D_j P(u_j) k_i [q(u_j)]\}$ and using this matrix

in place of the matrix $\{D_j k_i [q(u_j)]\}$ in all subsequent calculations. If the forcing function is identically zero the equations can be solved to find the value of D and hence of the value of w.

BIBLIOGRAPHY

1. Crout, P.D:"A short method for evaluating determinants and solving systems of linear equations with real or complex coefficients". Trans. AIEE, vol. 69,pp.1235-1241 (1941)

2. Crout, P.D! "An Application of Polynomial Approximation for the solution of Integral Equations Arising in Physicsl Problems". J. Math. Phys., Vol. 19, no. 1 (1940)

3. Crout, P.D. and F.B. Hildebrand: "A Least Square Procedure for solving integral equations by polynomial approximations". J. Math. Phys., Vol. 20, No. 3 (1941)

4. Hildebrand, F.B.,"Advanced Calculus for Engineers" Prentice-Hall Inc., New York, 1953.

5. Hildebrand, F.B. "Methods of Applied Mathematics" Prentice-Hall, Inc. New York 1954.

6. Peters, J. "8-place table of trigonometric functions" Verlag des Reichsamts fur Lamogsaufnahme, Berlin, 1939.

7. Burington, R.S.'"Handbook of Mathematical Tables and Formulas" Handbook Publishers, Inc. Sandusky, Ohio, 1947.