February 25 Notes:

Definitions:

- A convex quadrilateral is a **parallelogram** if the opposite sides are parallel.
- A **rhombus** is a parallelogram having two adjacent sides congruent.
- A **square** is a rhombus having two adjacent sides perpendicular.

A cascade of theorems, all of which rely on cutting things into triangles and using congruence theorems and transversal theorems:

- A diagonal of a parallelogram divides it into congruent triangles.
- A convex quadrilateral is a parallelogram iff its opposite sides are congruent.
- Opposite angles of a parallelogram are congruent
- Adjacent angles of a parallelogram are supplementary
- A convex quadrilateral is a parallelogram iff its diagonals bisect each other.
- A parallelogram is a rhombus iff its diagonals are perpendicular.
- A parallelogram is a rectangle iff its diagonals are congruent.
- A parallelogram is a square iff its diagonals are both perpendicular and congruent.
- Etc. **These make good exercises. Try a few!!**

Definition: A **trapezoid** is a convex quadrilateral if a pair of opposite sides parallel. The parallel sides are called **bases** and the other two sides are called **legs**. The segment joining the midpoint of the legs is called the **median**.

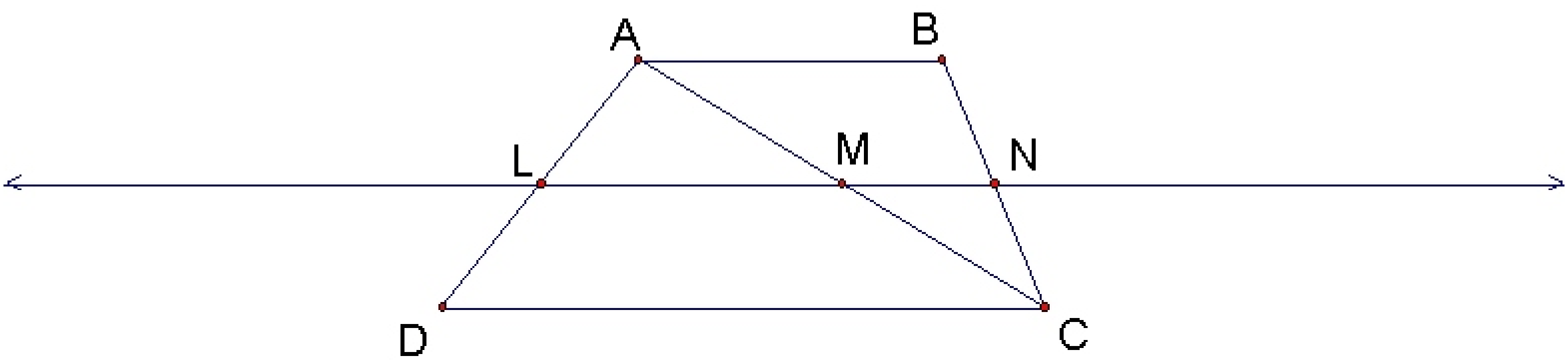
Our definition allows trapezoids to be parallelograms. Some books insist that trapezoids have exactly one pair of parallel sides.

If a trapezoid is not a parallelogram in disguise, then if its two legs are congruent, it is called an **isosceles trapezoid**.

Theorem (Midpoint Connector Theorem for Trapezoids): If a line segment bisects one leg of a trapezoid and is parallel to the base, then it is the median and its length is one-half the sum of the lengths of the two bases. Conversely, the median of a trapezoid is parallel to each of the two bases and has length equal to one-half the sum of the length of the bases.

Proof: Given trapezoid ◊ABCD with $\overline{AB} \parallel \overline{CD}$, let line *l* intersect leg $\overline{AD}$ at midpoint L. Draw diagonal $\overline{AC}$. Then *l* must intersect $\overline{AC}$ at a midpoint M by the midpoint connector theorem for triangles. Applying this again to △ABC, line *l* must intersect $\overline{BC}$ at a midpoint N. Segment $\overline{LN}$ is thus the median of the trapezoid. A straightforward argument establishes that L-M-N, so LN = LM + MN. Again by the midpoint connector theorem for triangles,

$$LN = LM + MN = \tfrac{1}{2}CD + \tfrac{1}{2}AB = \tfrac{1}{2}(CD + AB)$$



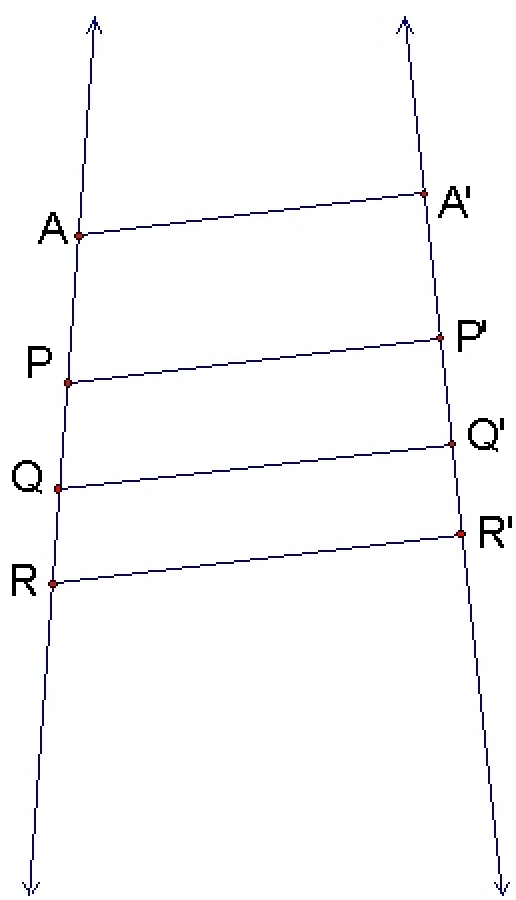
The converse is Problem 18 in Section 4.2, which you get to do as an exercise.

Parallel Projection: Given two lines *l* and *m*, locate points A and A′ on the two lines. We set up a correspondence P ↔ P′ between the points of *l* and *m* by requiring that $PP' \parallel AA'$, for all P on *l*. We claim that this mapping, called a **parallel projection**, 1) is one-to-one, 2) preserves betweeness, and 3) preserves ratios of segments.

1) follows almost immediately from the Parallel Postulate.

2) follows almost immediately from the fact that parallel lines cannot cross (given P-Q-R, if P′-R′-Q′, then Q is on one side of $\overleftrightarrow{RR'}$ and Q′ is on the other; segment $\overline{QQ'}$ must cross $\overleftrightarrow{RR'}$).

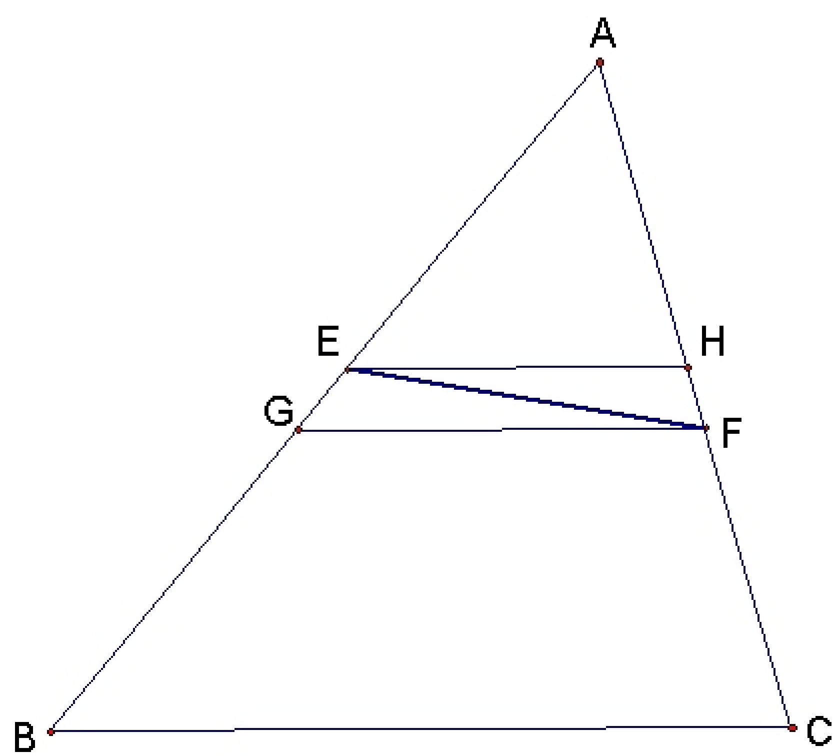
We spend the rest of our time proving 3.



We claim that $\frac{PQ}{QR} = \frac{P'Q'}{Q'R'}$. If the lines *l* and *m* are parallel, then the quadrilaterals become parallelograms, and opposite sides are equal, so PQ = P′Q′ and QR = Q′R′, and the result is trivial. So we assume that the lines meet at point A, forming a triangle △ABC. We prove the following:

Theorem: Given triangle △ABC, if E lies on $\overline{AB}$ and F on $\overline{AC}$ such that $\overline{EF} \parallel \overline{BC}$, then AE/AB = AF/AC.

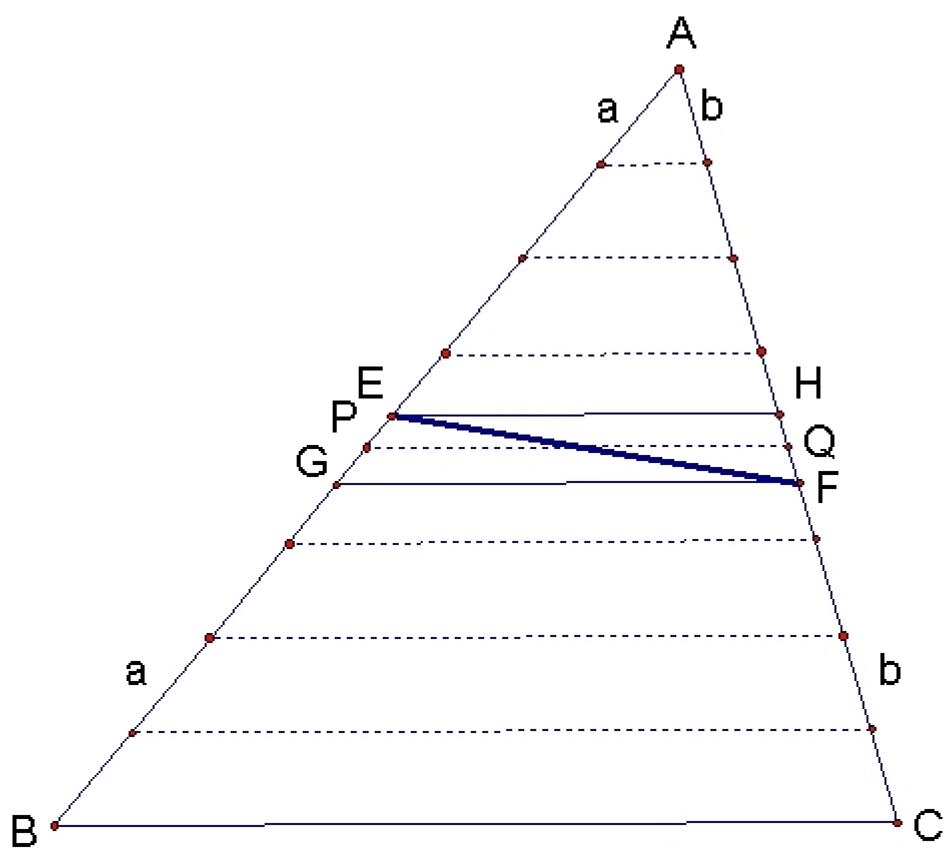
Proof: We assume that $\overline{EF}$ is **not** parallel to $\overline{BC}$, then show that AE/AB ≠ DF/DC. Begin by constructing the parallels $\overline{EH}$ and $\overline{GF}$ to $\overline{BC}$ through E and F, respectively. We have two cases, A-E-G or A-G-E. We will argue the case A-E-G, the other case being analogous. Since betweenness is preserved under parallel projection, A-H-F.



Now: Bisect segments $\overline{AB}$ and $\overline{AC}$, then bisect the segments determined by those midpoints, and so on, continuing the bisection process indefinitely. We claim that at some point, one of our bisecting points P will fall on $\overline{EG}$, and the corresponding midpoint Q will fall on $\overline{HF}$. This is true since for some n, $n \cdot \text{EG} > \text{AB}$ (the Archimedean Property of real numbers). So, it is also true that for some m, $2^m \cdot \text{EG} > \text{AB}$, so $\text{AB}/2^m < \text{EG}$. This is illustrated below.

Note that the segments joining midpoints (like $\overline{PQ}$) are parallel to $\overline{BC}$ by the midpoint connector theorems.

Moreover, the process of repeatedly finding midpoints partitions the segments $\overline{AB}$ and $\overline{AC}$ into n congruent segments of length a and b, respectively. There are also some number k parallel lines between A and $\overline{PQ}$, (counting $\overline{PQ}$). Then:



AP = ka AQ = kb
AB = na AC = nb.

By algebra, $\frac{AP}{AB} = \frac{ka}{na} = \frac{k}{a}$ and $\frac{AQ}{AC} = \frac{kb}{nb} = \frac{k}{n}$; therefore, $\frac{AP}{AB} = \frac{AQ}{AC}$.

However, because A-E-P-G and A-H-Q-F, AE < AP and AQ < AF. So,
$\frac{AE}{AB} < \frac{AP}{AB} = \frac{AQ}{AC} < \frac{AF}{AC}$.

Thus, $\frac{AE}{AB} \neq \frac{AF}{AC}$.

If it happens that A-G-E, an exactly analogous argument gives us $\frac{AE}{AB} > \frac{AF}{AC}$ so again $\frac{AE}{AB} \neq \frac{AF}{AC}$.

We have shown the contrapositive of the statement we wanted to prove. In summary,

If $\frac{AE}{AB} = \frac{AF}{AC}$, then $\overline{EF} \parallel \overline{BC}$.

Now we prove:

Theorem (The Side-Splitting Theorem): Parallel projection preserves ratios of line segments. Specifically, if a line $\overleftrightarrow{EF}$ parallel to the base $\overline{BC}$ of △ABC cuts the other two sides $\overline{AB}$ & $\overline{AC}$ at points E and F, respectively, then AE/AB = AF/AC, and AE/EB = AF/FC.

Proof: We locate F′ on $\overline{AC}$ such that AF′ = AC ·(AE/AB), so that AE/AB = AF′/AC, and construct line $\overleftrightarrow{EF'}$. By the preceding theorem, $\overleftrightarrow{EF'} \parallel \overline{BC}$. But $\overleftrightarrow{EF} \parallel \overline{BC}$ by hypothesis, so $\overleftrightarrow{EF'} = \overleftrightarrow{EF}$. Therefore, F′ = F, and AE/AB = DF/DC.

To get the other ratio, note that A-E-B and A-F-C so that AB = AE + EB and AC = AF + FC. Then

$$\frac{AE}{AE+EB} = \frac{AF}{AF+FC},$$ or taking reciprocals,

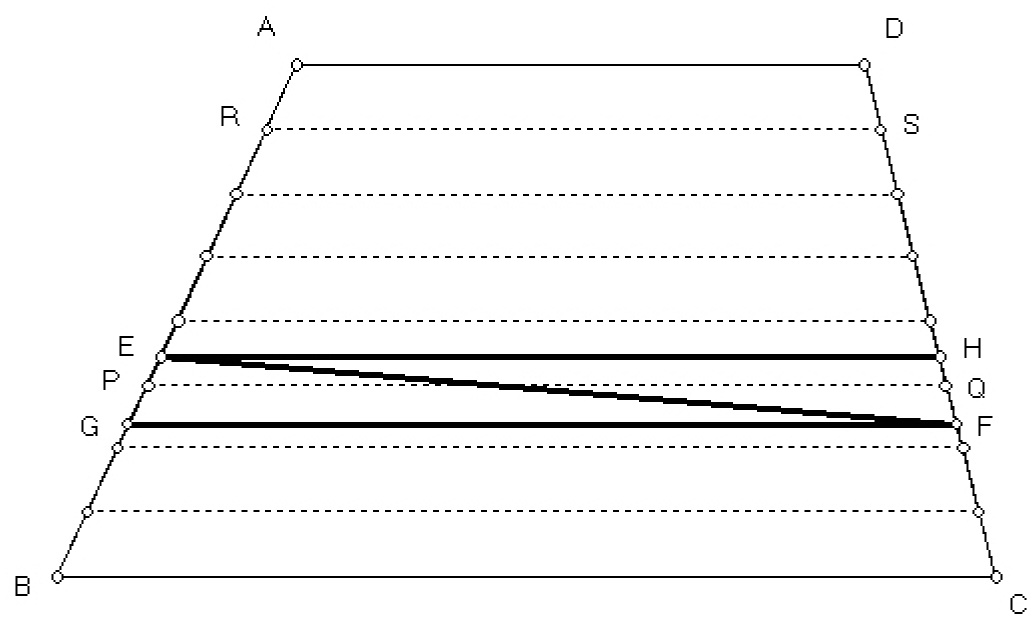
$$\frac{AE+EB}{AE} = \frac{AF+FC}{AF}.$$ Thus $1+\frac{EB}{AE} = 1+\frac{FC}{AF}$, or

$$\frac{EB}{AE} = \frac{FC}{AF}.$$ Taking reciprocals gives the desired result.

A final note: Everything we have done here could have been done on a trapezoid instead of a triangle. (Indeed, an earlier edition of the text did all of this on a trapezoid, and noted that it could have been done on a triangle.) In the case of a trapezoid, the point A at the top of the triangle is replaced by a segment $\overline{AD}$, and the major theorem is then stated as:

Given trapezoid ◇ABCD, if E lies on $\overline{AB}$ and F on $\overline{CD}$ such that $\overleftrightarrow{EF} \parallel \overline{BC}$, then AE/AB = DF/DC.

The picture is slightly different, but the main ideas are exactly the same.



All this forms the foundation for our study of similarity in the next section.