Chapter 1
First-Order Differential Equations

1.1 Definition of Differential Equations

A differential equation (DE) is a mathematical equation that relates some functions of one or more variables with their derivatives. A DE is used to describe changing quantities and it plays a major role in quantitative studies in many disciplines such as all fields of engineering, physical sciences, life sciences, and economics.

Examples

Are they DEs or not?

$ax^2 + bx + c = 0$ No

$ax^2 + bx' + c = 0$	Yes	Here $x' = \frac{dx}{dt}$
$ax^2 + bx' + cy' = 0$	Yes	Here $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$
$y'' = x^3$	Yes	Here $y' = \frac{dy}{dt}$

To solve a DE is to express the solution of the unknown function (the dependent variable or DV) in mathematical terms without the derivatives.

Examples

$$ax' + b = 0$$

$$x' = -\frac{b}{a} \qquad \text{is not a solution}$$

$$x = -\int \frac{b}{a} dt \qquad \text{is a solution}$$

In general, there are two common ways in solving DEs: analytically and numerically. Most DEs, difficult to solve by analytical methods, must be "solved" by using numerical methods, although many DEs are too stiff to solve by numerical techniques. Solving DEs by numerical methods is a different subject requiring basic knowledge of computer programming and numerical analysis; this book focuses on introducing analytical methods for solving very small families of DEs that are truly solvable. Although the DEs are quite simple, the solution methods are not and the essential solution steps and terminologies involved are fully applicable to much more complicated DEs.

Classification of DEs

Classification of DEs is itself another subject in studying DEs. We will introduce classifications and terminologies to make the contents of the book flow but one may still need to look up terms undefined here or abbreviations introduced at the end of the book. First, we introduce the terms of dependent variables (DVs) and independent variables (IVs) of functions. A DV represents the output or effect while the IV represents the input or the cause. Truly, a DE is an equation that relates these two variables. A DE may have more than one variable for each and the DE with one IV and one DV is called an ordinary differential equation or ODE. The ODE, or simply referred to as DE, is the object of our book. Continuing, you understand why this 1-to-1 relationship is called *ordinary*. A DE that has one DV and $N \geq 2$ IVs, 1-to-N relationship, is called a partial differential equation or PDE. Studying PDEs, out of the scope of this book, requires solid understanding of partial derivatives and, more desirably, full knowledge of multiple variable calculus. By now, you certainly want to see two other types of DEs. With $N \geq 2$ DVs and 1 IV, you may compose an N-to-1 *system* of ODEs. Similarly, with $N \geq 2$ DVs and $M \geq 2$ IVs, you may compose an N-to-M *system* of PDEs. We have several other classifications to categorize DEs.

Order of DEs

The order of a DE is determined by the highest order derivative of the DV (and sometimes we interchangeably call it unknown function). If you love to generalize things, algebraic equations (AE) may be classified as 0th order DEs as there are no derivatives for the unknowns in the AEs.

Examples

Determine the orders of the following DEs.

1st-order (1st.O) DE:

$$x' + ax = 0 \tag{1.1}$$

$$x' + ax^2 = 0 \tag{1.2}$$

2nd-order (2nd.O) DE:

$$x'' + bx = 0 \tag{1.3}$$

$$x'' + bx^5 = 0 \tag{1.4}$$

nth-order (nth.O) DE:

$$x^{(n)} + bx'' + cx = 0 \tag{1.5}$$

$$x^{(n)} + 5x^{(n-1)} + x = 0 \tag{1.6}$$

	Definition	Examples
ODEs vs. PDEs	ODEs contain only one IV. PDEs contain two or more IVs.	ODE: $x'' + \omega^2 x = f(t)$ PDE: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
Order of DEs	The highest order of the derivatives of the DVs determines the order of the DEs	First-order (1st.O): $y' = x + y$ Second-order (2nd.O): $y'' = x + y^2$
Linear vs. Nonlinear	A DE containing nonlinear term(s) for the DVs is a nonlinear DE regardless of the nature of the IVs	Linear: $y' = x^2 + y$ Nonlinear: $y'' = x + y^2$

Homogeneous vs. Inhomogeneous	A 1st-order (1st.O) DE $M(x,y)dx + N(x,y)dy = 0$ is Homo DE if both $M(x,y)$ and $N(x,y)$ are homogeneous functions. Otherwise, it is InHomo DE. In general, a linear DE of order n is a Homo DE if it is of the form $$\sum_{j=0}^{n} a_j(x)y^{(j)} = 0$$ Otherwise, it is InHomo DE.	1st.O Homo DE: $y^2y' = x^2 + xy$ 1st.O InHomo DE: $y' = x^2 + y^2$ Homo DE: $y'' + y^2 \cos x = 0$ InHomo DE: $y'' + y = \cos x$

Linearity of DEs

DEs can also be classified as linear or nonlinear according to the linearity of the DVs regardless of the nature of the IVs. A DE that contains only linear terms for the DV or its derivative(s) is a linear DE. Otherwise, a DE that contains at least one nonlinear term for the DV or its derivative(s) is a nonlinear DE.

Linear	Nonlinear
$y'' + y = 0$	$y'' + y^2 = 0$
$y^{(n)} + x + y = 0$	$y^{(n)} + xy + y^3 = 0$
$y^{(n)} + x^3 + y = 0$	$\left(y^{(n)}\right)^2 + x^3 + y = 0$

Solutions

A function that satisfies the DE is a solution to that DE. Seeking such functions is the main objective of the book while composing the DEs, which may excite engineering majors more, is the secondary objective of this book.

Solutions can also be classified into several categories, for example, general solutions (GS), particular solutions (PS), and singular solutions (SS). A GS, *i.e.*, a complete solution, is the set of all solutions to the DE and it can usually be expressed in a function with arbitrary constant(s). A PS is a subset of the GS whose arbitrary constant(s) are determined by the initial conditions (ICs) or the boundary conditions (BCs) or both. An SS is a solution that is singular. In a less convoluted definition, an SS is one for which the DE, or the initial value problem (IVP) or the Cauchy problem, fails to have a unique solution at some point in the solution. The set in which the solution is singular can be a single point or the entire real line.

Examples

Try to *guess* the solutions of the following DEs:

(1) $x' + \omega^2 x = 0$

(2) $x' + \omega^2 x = t$

(3) $x'' + \omega^2 x = 0$

(4) $x'' + \omega^2 x = \sin(\omega t)$

(5) $x'' + \omega^2 x = \sin(\omega_1 t)$ where, in general, $\omega \neq \omega_1$

As briefly mentioned before, there are several methods to find the solutions of DEs. This book covers the first of the two common methods, *i.e.*, analytical method and numerical method.

1. To obtain analytical (closed form) solutions.

   Only a small percentage of linear DEs and a few special nonlinear DEs are simple enough to allow findings of analytical solutions. There are two major types: the exact methods and the approximation methods. Examples of the exact methods are (1) method of undetermined coefficients (MUC); (2) integrating factor (IF) method; (3) method of separation of variables (SOV); and (4) variation of parameters (VOP). The examples of the approximation but convergent methods are (1) successive approximations; (2) series methods including power series methods and the generalized Fourier series methods; (3) multiple scale analysis; and (4) perturbation methods.

2. To obtain numerical solutions.

   Most DEs in science, engineering, and finance are too complicated to allow findings of analytical solutions and numerical methods are the only viable alternatives for *approximate* numerical solutions. Unfortunately, most DEs that are of vital importance for practical purposes belong to this type. Indeed, every rose has its thorn. To solve DEs numerically, one must acquire a different set of skills: numerical analysis and computer programming. The types of numerical methods for DEs are too numerous to name. For ODEs, the examples are the Euler method and the general linear methods such as Runge-Kutta methods and the linear multi-step method. For PDEs, the examples are finite difference methods and finite element methods.

This book does not cover any of the numerical methods.

Problems

Problem 1.1.1 Verify by substitution that each given function is a solution to the given DE. Throughout these problems, primes denote derivatives with respect to (wrt) x.

$$y'' + y = 3\cos 2x\,, \qquad y_1 = \cos x - \cos 2x\,, \qquad y_2 = \sin x - \cos 2x$$

Problem 1.1.2 Verify by substitution that each given function is a solution to the given DE. Throughout these problems, primes denote derivatives wrt x.

$$x^2y'' - xy' + 2y = 0, \qquad y_1 = x\cos(\ln x), \qquad y_2 = x\sin(\ln x)$$

Problem 1.1.3 A function $y = g(x)$ is described by some geometric properties of its graph. Write a DE of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions). The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.

Problem 1.1.4 Determine by inspection at least one solution to the DE $xy' + y = 3x^2$. That is, use your knowledge of derivatives to make an intelligent guess and, then, test your hypothesis.

Problem 1.1.5 Determine by inspection at least one solution to the DE $y'' + y = 0$. That is, use your knowledge of derivatives to make an intelligent guess and, then, test your hypothesis.

Problem 1.1.6 Verify by substitution that the given function is a solution to the given DE. Primes denote derivatives wrt x.

$$y' + 2xy^2 = 0, \qquad y = \frac{1}{1 + x^2}$$

Problem 1.1.7 Verify that $y(x)$ satisfies the given DE and determine a value of the constant C so that $y(x)$ satisfies the given IC.

$$y' + 3x^2y = 0, \qquad y(x) = C\exp(-x^3), \qquad y(0) = 7$$

Problem 1.1.8 Verify that $y(x)$ satisfies the given DE and determine a value of the constant C so that $y(x)$ satisfies the given IC.

$$y' + y\tan x = \cos x, \qquad y(x) = (x + C)\cos x, \qquad y(\pi) = 0$$

Problem 1.1.9 Verify that $y(x)$ satisfies the given DE and determine a value of the constant C so that $y(x)$ satisfies the given IC.

$$y' = 3x^2(y^2 + 1), \qquad y(x) = \tan(x^3 + C), \qquad y(0) = 1$$

Problem 1.1.10 Verify that $y(x)$ satisfies the given DE and determine a value of the constant C so that $y(x)$ satisfies the given IC.

$$xy' + 3y = 2x^5, \qquad y(x) = \frac{1}{4}x^5 + \frac{C}{x^3}, \qquad y(2) = 1$$

Problem 1.1.11 Verify by substitution that the given functions are solutions of the given DE. Primes denote derivatives wrt x.

$$y'' = 9y, \qquad y_1 = \exp(3x), \qquad y_2 = \exp(-3x)$$

Problem 1.1.12 Verify by substitution that the given functions are solutions of the given DE. Primes denote derivatives wrt x.

$$\exp(y)\, y' = 1, \qquad y(x) = \ln(x + C), \qquad y(0) = 0$$

Problem 1.1.13 Verify that $y(x)$ satisfies the given DE and determine a value of the constant C so that $y(x)$ satisfies the given IC. In the equation and its solution, n is a given constant.

$$y' + nx^{n-1}y = 0, \qquad y(x) = C\exp(-x^n), \qquad y(0) = 2014$$

1.2 Slope Fields and Solution Curves

Geometrical interpretation is an effective way to help understand the properties of the DEs and their solutions. A solution can be drawn as a curve, which is called a Solution Curve. A series of lines with the same slope (for each line), such as a family of curves $f(x, y) = s$ for DE $y' = f(x, y)$ (Figure 1.1) are called Isocline, meaning the same slope. A line in the xy-plane whose slope is $f(x, y)$ is called a Slope Curve and a collection of such slope curves forms a Slope Field (Figure 1.2), aka, the direction field.

Consider a classical example:

$$y' = 2xy \tag{1.7}$$

whose GS is

$$y = C \exp(x^2) \tag{1.8}$$

In the xy-plane, we can draw the isocline by selecting a few appropriate constant values s for the following:

$$2xy = s \tag{1.9}$$

and the solution curves with the given solution by selecting a few constant values C.

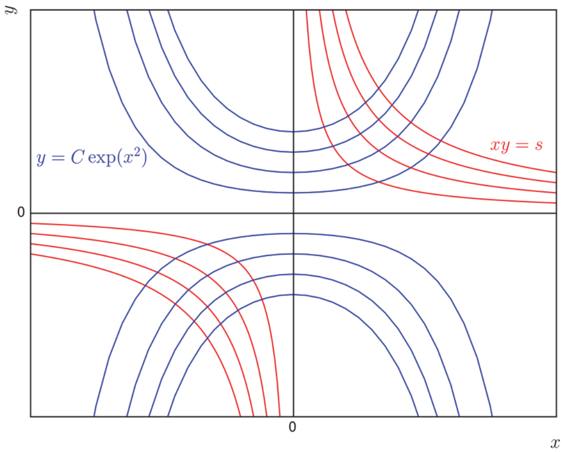

Figure 1.1 Examples of solution curves $y = C\exp(x^2)$ and isocline $xy = s$.

Example 1
Find the isoclines

$$y' = \sin(x - y) \tag{1.10}$$

Solution

$$x - y = \arcsin c$$
$$y = x - \arcsin c$$

Example 2
Find the isoclines

$$y' = x^2 + y^2 = C \tag{1.11}$$

Solution
The isocline equation

$$y' = f(x, y) = x^2 + y^2 = C$$

It is a circle.

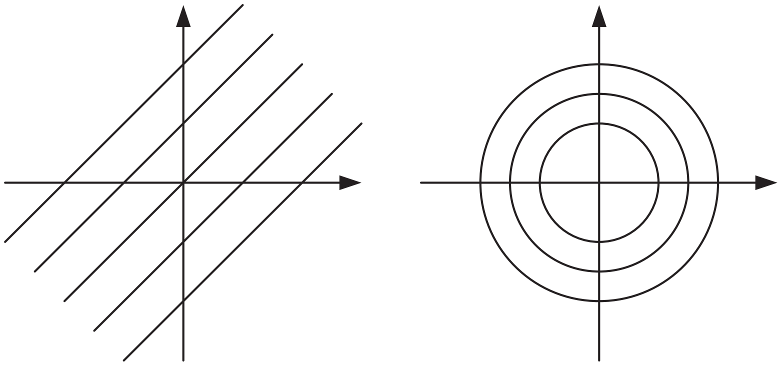

Figure 1.2 The isoclines for the two examples.

Problems

Problem 1.2.1 Plot the solution curves and slope field for the DE for the appropriate ranges for variables x and y

$$y' = x^2 - y$$

Problem 1.2.2 Plot the solution curves and slope field for the DE for the appropriate ranges for variables x and y

$$y' + y = x + 2$$

1.3 Separation of Variables

All 1st.O initial value problem (IVP), aka, the Cauchy problem, can be written as

$$\begin{cases} y' = F(x, y) \\ y(x = a) = b \end{cases} \tag{1.12}$$

If $F(x, y)$ can be written as

$$y' = \frac{g(x)}{f(y)} \tag{1.13}$$

Then, the 1st.O IVP is said to be separable.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{g(x)}{f(y)} \\ \int f(y)dy &= \int g(x)dx \\ F(y) &= G(x) + C \end{aligned} \tag{1.14}$$

where C is a constant determined by the IC $y(a) = b$.

Example 1A

Find the PS to the following IVP

$$\begin{cases} y' = -6xy \\ y(0) = 7 \end{cases} \tag{1.15}$$

Solution

$$\begin{aligned} \frac{dy}{y} &= -6xdx \\ \int \frac{dy}{y} &= \int (-6x)dx \\ \ln y &= -3x^2 + \ln C \\ y &= C\exp(-3x^2) \end{aligned}$$

is the GS to (1.15) as shown in Figure 1.3.

Applying the IC $y(0) = 7$, *i.e.*, $y(0) = C\exp(0) = 7$, we get $C = 7$ and the PS

$$y(x) = 7\exp(-3x^2)$$

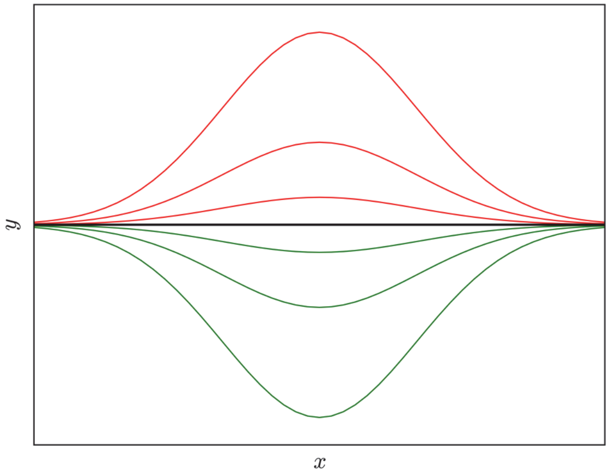

Figure 1.3 The GS $y = C\exp(-3x^2)$ and the SS $y = 0$.

Example 1B

Is $y = 0$ a solution to DE $y' = -6xy$?

Solution

If $y = 0$, for the given DE, we have

$$\text{LHS} = y' = 0$$

and

$$\text{RHS} = -6xy = -6x \cdot 0 = 0$$

Thus,

$$\text{LHS} = \text{RHS}$$

Therefore,

$$y = 0$$

is an SS to $y' = -6xy$.

Example 2

Find the PS to the following IVP

$$\begin{cases} y' = \dfrac{4-2x}{3y^2-5} \\ y(1) = 3 \end{cases} \tag{1.16}$$

Solution

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

$$\int (3y^2-5)dy = \int (4-2x)dx$$

Integrating both sides, we get the GS

$$y^3 - 5y = 4x - x^2 + C$$

Applying the IC $y(1) = 3$, *i.e.*, $3^3 - 15 = 4 - 1 + C$, we get $C = 9$ and the PS to the IVP is

$$y^3 - 5y = 4x - x^2 + 9$$

Example 3

Find the GS to the following DE

$$y^2 + x^2y' = 0 \tag{1.17}$$

Solution

$$y^2 + x^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$

$$\frac{1}{y^2}dy = -\frac{1}{x^2}dx$$

$$\int \frac{1}{y^2}dy = -\int \frac{1}{x^2}dx$$

$$-\frac{1}{y} = \frac{1}{x} - C$$

where we selected $-C$, instead of the more common $+C$, as the integration constant for minor convenience. Therefore,

$$\frac{1}{x} + \frac{1}{y} = C$$

is the GS to the given DE.

Singular Solutions

A singular solution is a solution for which the IVP fails to have a unique solution at some point on the solution. The domain on which a solution is singular is weird as it could be a single point or the entire real line. The SS usually appear(s) when one divides the DE by a term that might be zero while solving the DE.

Suppose we have a DE

$$y' = H(y)G(x) \tag{1.18}$$

To solve (1.18), we divide it by $H(y)$.

$$\frac{dy}{H(y)} = G(x)dx \tag{1.19}$$

That gives

$$\int \frac{dy}{H(y)} = \int G(x)dx \tag{1.20}$$

Such division is mathematically allowed if and only if $H(y) \neq 0$. However,

$$H(y) = 0 \tag{1.21}$$

is a solution to the DE because it indeed satisfies the original DE (1.18). This solution is peculiar and, thus, it is called SS.

Problems

Problem 1.3.1 Find the GS to the following DE

$$3x(y-2)dx + (x^2+1)dy = 0$$

Problem 1.3.2 Find the GS (implicit if necessary, explicit if convenient) of the following DE. Prime denotes derivatives wrt x.

$$y' = (64xy)^{\frac{1}{3}}$$

Problem 1.3.3 Find the GS (implicit if necessary, explicit if convenient) of the following DE. Prime denotes derivatives wrt x. (*Hint: Factorize the RHS.*)

$$y' = 1 + x + y + xy$$

Problem 1.3.4 Find the explicit PS to the following IVP

$$\begin{cases} \dfrac{dy}{dx} = x\exp(-x) \\ y(0) = 1 \end{cases}$$

Problem 1.3.5 Find the explicit PS to the following IVP

$$\begin{cases} y' = -2\cos 2x \\ y(0) = 2014 \end{cases}$$

Problem 1.3.6 Find the explicit PS to the following IVP

$$\begin{cases} 2\sqrt{x}y' = (\cos y)^2 \\ y(4) = \dfrac{\pi}{4} \end{cases}$$

Problem 1.3.7 Find the explicit PS to the following IVP

$$\begin{cases} y' = 2xy + 3x^2 y\exp(x^3) \\ y(0) = 5 \end{cases}$$

Problem 1.3.8 Find the GS (implicit if necessary, explicit if convenient) of the following DE. Primes denote derivatives wrt x.

$$y^3 y' = (y^4 + 1)\cos x$$

Problem 1.3.9 Find the GS to the following DE

$$4xy^2 + y' = 5x^4 y^2$$

Problem 1.3.10 Find the explicit PS to the following IVP

$$\begin{cases} y' = 2xy^2 + 3x^2 y^2 \\ y(1) = -1 \end{cases}$$

Problem 1.3.11 Find the GS (implicit if necessary, explicit if convenient) of the following DE. Primes denote derivatives wrt x.

$$y' = 2x\sec y$$

Problem 1.3.12 Find the explicit PS to the following IVP

$$\begin{cases} \tan x\, y' = y \\ y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \end{cases}$$

Problem 1.3.13 Find the GS (implicit if necessary, explicit if convenient) of the following DE. Primes denote derivatives wrt x.

$$(x^2+1)\tan(y)\, y' = x$$

Problem 1.3.14 Find the GS (implicit if necessary, explicit if convenient) of the following DE

$$(x+3)^3 y' = (y-2)^2$$

Problem 1.3.15 Find the PS to the following DE

$$\begin{cases} y' - xy = 3y \\ y(1) = 1 \end{cases}$$

Problem 1.3.16 Find the PS to the following DE

$$\begin{cases} (1+x)y' + y = \cos(x) \\ y(0) = 1 \end{cases}$$

Problem 1.3.17 Find the GS to the following DE

$$\frac{dy}{dt} = \beta y(\alpha - \ln y)$$

where α and β are parameters independent of y and t. If $y(t=0) = y_0$ and $y(t \to \infty) = y_\infty$, can you express α or β or both in terms of y_0 and y_∞? If so, do it.

1.4 First-Order Linear DEs

All 1st.O linear DEs can be written as

$$A(x)y' + B(x)y = C(x) \tag{1.22}$$

If $A(x) = 0$, the equation is no longer a DE. Thus, we set $A(x) \neq 0$.

If $B(x) = 0$, DE (1.22) becomes

$$A(x)y' = C(x) \tag{1.23}$$

which is a separable DE.

If $C(x) = 0$, DE (1.22) becomes

$$A(x)y' + B(x)y = 0 \tag{1.24}$$

and, after dividing both sides by y, we get

$$\frac{y'}{y} = -\frac{B(x)}{A(x)} \tag{1.25}$$

which is also a separable DE.

For a general 1st.O linear DE, we divide both sides (1.22) by $A(x)$,

$$y' + \frac{B(x)}{A(x)}y = \frac{C(x)}{A(x)} \tag{1.26}$$

Let

$$\frac{B(x)}{A(x)} = P(x), \quad \frac{C(x)}{A(x)} = Q(x)$$

Now, we have

$$y' + P(x)y = Q(x) \tag{1.27}$$

Let's now derive the method to solve the general 1st.O linear DE in the form of (1.27). The key idea is to absorb the $P(x)y$ term to a whole derivative by using an IF $\rho(x)$ whose precise composition will be introduced shortly.

$$\begin{aligned} \rho(x)(y' + P(x)y) &= Q(x)\rho(x) \\ \rho(x)y' + \rho(x)P(x)y &= Q(x)\rho(x) \end{aligned} \tag{1.28}$$

If we can make the LHS of (1.28)

$$\rho(x)y' + \rho(x)P(x)y = \frac{d}{dx}(\rho(x)y) \tag{1.29}$$

a whole derivative, (1.28) can be solved easily in terms of $\rho(x)y$. Equation (1.29) can be written as

$$\rho(x)y' + \rho(x)P(x)y = \rho(x)y' + \rho'(x)y \tag{1.30}$$

Cancelling $\rho(x)y'$ on both sides of DE (1.30), we have

$$\begin{aligned} \rho(x)P(x)y &= \rho'(x)y \\ \rho(x)P(x) &= \rho'(x) \end{aligned} \tag{1.31}$$

which is a separable DE with unknown function $\rho(x)$. To solve this DE, we have

$$\begin{aligned} \frac{d\rho}{\rho} &= P(x)dx \\ \int \frac{d\rho}{\rho} &= \int P(x)dx \\ \ln \rho &= \int P(x)dx \end{aligned} \tag{1.32}$$

Therefore, we have

$$\rho(x) = \exp\left(\int P(x)dx + C_1\right) = C\exp\left(\int P(x)dx\right) \tag{1.33}$$

Since this function $\rho(x)$ can be multiplied by any non-zero constant to serve as a factor, without loss of generality while gaining simplicity, we set $C = 1$. The resulting factor

$$\rho(x) = \exp\left(\int P(x)dx\right) \tag{1.34}$$

is called the integrating factor (IF) of DE (1.27) and this IF satisfies the condition (1.29). Plugging (1.29) into (1.28), we have

$$\frac{d}{dx}(\rho(x)y) = Q(x)\rho(x) \tag{1.35}$$

i.e.,

$$d(\rho(x)y) = Q(x)\rho(x)dx \tag{1.36}$$

Integrating both sides of DE (1.36), we get

$$\rho(x)y = \int Q(x)\rho(x)dx \tag{1.37}$$

Thus, the GS to the DE (1.27) is

$$y(x) = \frac{1}{\rho(x)}\left(\int Q(x)\rho(x)dx + C\right) \tag{1.38}$$

Replacing the IF $\rho(x)$ by the given function $P(x)$, we find the GS to the general 1st.O linear DE as

$$y(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx + C\right) \tag{1.39}$$

Now, we outline the steps to solve the 1st.O linear DEs without memorizing the formula (1.39).

Step 1: Convert the DE to the standard form $y' + P(x)y = Q(x)$
Step 2: Identify $P(x)$
Step 3: Compute the IF

$$\rho(x) = \exp\left(\int P(x)dx\right) \tag{1.40}$$

Step 4: Multiply both sides of the converted DE by the IF

$$\begin{aligned}\exp\left(\int P(x)dx\right)y' + \exp\left(\int P(x)dx\right)P(x)y \\ = Q(x)\exp\left(\int P(x)dx\right)\end{aligned} \tag{1.41}$$

Step 5: Write the LHS as a whole derivative using the product rule,

$$\begin{aligned}\exp\left(\int P(x)dx\right)y' + \exp\left(\int P(x)dx\right)P(x)y \\ = \frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right)\end{aligned} \tag{1.42}$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right) = Q(x)\exp\left(\int P(x)dx\right) \\ \exp\left(\int P(x)dx\right)y = \int Q(x)\exp\left(\int P(x)dx\right)dx\end{aligned} \tag{1.43}$$

Finally, the solution to the DE is

$$y(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right) \tag{1.44}$$

Example 1
Find the GS to the following DE

$$(x^2+1)y' + 3xy = 6x \tag{1.45}$$

Solution
Step 1: Converting the DE

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

Step 2: Identify

$$P(x) = \frac{3x}{x^2+1}$$

Step 3: Computing the IF

$$\begin{aligned}\rho(x) &= \exp\left(\int \frac{3x}{x^2+1}dx\right)\\ &= \exp\left(\frac{3}{2}\int \frac{d(x^2+1)}{x^2+1}\right)\\ &= \exp\left(\frac{3}{2}\ln(x^2+1)\right)\\ &= (x^2+1)^{\frac{3}{2}}\end{aligned}$$

Step 4: Multiplying the DE by the IF

$$y'(x^2+1)^{\frac{3}{2}} + \frac{3x}{x^2+1}y(x^2+1)^{\frac{3}{2}} = \frac{6x}{x^2+1}(x^2+1)^{\frac{3}{2}}$$

$$y'(x^2+1)^{\frac{3}{2}} + 3xy(x^2+1)^{\frac{1}{2}} = 6x(x^2+1)^{\frac{1}{2}}$$

Step 5: Completing the solution

$$\frac{d}{dx}\left(y(x^2+1)^{\frac{3}{2}}\right) = 6x(x^2+1)^{\frac{1}{2}}$$

$$y(x^2+1)^{\frac{3}{2}} = \int 6x(x^2+1)^{\frac{1}{2}}dx$$

$$\begin{aligned}y &= \frac{1}{(x^2+1)^{\frac{3}{2}}}\int 3(x^2+1)^{\frac{1}{2}}d(x^2+1)\\ &= (x^2+1)^{-\frac{3}{2}}\left(2(x^2+1)^{\frac{3}{2}} + C\right)\\ &= 2 + C(x^2+1)^{-\frac{3}{2}}\end{aligned}$$

This is the GS to the original DE.

Example 2

Find the PS to the following IVP

$$\begin{cases} x^2y' + xy = \sin x \\ y(1) = y_0 \end{cases} \tag{1.46}$$

Solution

Method 1: The 1st.O linear DE method

$$y' + \frac{y}{x} = \frac{\sin x}{x^2}$$

The IF

$$\rho(x) = \exp\left(\int \frac{1}{x}dx\right) = \exp(\ln x) = x$$

Multiplying the IF yields

$$(xy)' = \frac{\sin x}{x^2}\cdot x$$

$$xy = \int \frac{\sin x}{x} dx$$

$$y(x) = \frac{1}{x}\int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{C}{x}$$

This is the GS to the original DE where x_0 and C are constants.
Using the IC $y(1) = y_0$

$$y(1) = y_0 = \frac{1}{1}\int_{x_0}^{1} \frac{\sin t}{t} dt + \frac{C}{1}$$

$$C = y_0 - \int_{x_0}^{1} \frac{\sin t}{t} dt$$

Now plugging C back into $y(x)$, we have

$$y(x) = \frac{1}{x}\int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{y_0}{x} - \frac{1}{x}\int_{x_0}^{1} \frac{\sin t}{t} dt$$

$$= \frac{y_0}{x} + \frac{1}{x}\int_{1}^{x} \frac{\sin t}{t} dt$$

Method 2: Direct use of the formula (1.35)

$$y(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)$$

$$P(x) = \frac{1}{x}$$

$$Q(x) = \frac{\sin x}{x^2}$$

$$y(x) = \exp\left(-\int \frac{1}{x}dx\right)\left(\int \frac{\sin x}{x^2}\exp\left(\int \frac{1}{x}dx\right)dx + C\right)$$

$$= \frac{1}{x}\int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{C}{x}$$

Using the IC $y(1) = y_0$, we have

$$C = y_0 - \int_{x_0}^{1} \frac{\sin t}{t} dt$$

Therefore,

$$y(x) = \frac{y_0}{x} + \frac{1}{x}\int_{1}^{x} \frac{\sin t}{t} dt$$

Problems

Problem 1.4.1 Find the PS to the following IVP

$$\begin{cases}(x^2+1)y' + 3x^3y = 6x\exp\left(-\frac{3}{2}x^2\right)\\ y(0) = 1\end{cases}$$

Problem 1.4.2 (a) Show that

$$y_C(x) = C\exp\left(-\int P(x)dx\right)$$

is a GS to $y' + P(x)y = 0$.
(b) Show that

$$y_P(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)$$

is a PS to $y' + P(x)y = Q(x)$.
(c) Suppose that $y_C(x)$ is any GS to $y' + P(x)y = 0$ and that $y_P(x)$ is any PS to $y' + P(x)y = Q(x)$. Show that $y(x) = y_C(x) + y_P(x)$ is a GS to $y' + P(x)y = Q(x)$.

Problem 1.4.3 Find the GS to the following IVP

$$\begin{cases}xy' + 2y = 7x^2\\ y(2) = 5\end{cases}$$

Problem 1.4.4 Find the GS to the following DE

$$y' + y\cot x = \cos x$$

Problem 1.4.5 Find the GS to the following DE using two different methods

$$y' = 3(y+7)x^2$$

Problem 1.4.6 Find the GS to the following DE

$$xy' = 2y + x^3\cos x$$

Problem 1.4.7 Find the GS to the following DE

$$(2x+1)y' + y = (2x+1)^{\frac{3}{2}}$$

Problem 1.4.8 Find the GS to the following DE using two different methods

$$y' = \frac{2xy + 2x}{x^2+1}$$

Problem 1.4.9 Find the GS to the following DE

$$(1+2xy)y' = 1+y^2$$

(Hint: Regard x as DV and y as IV.)

Problem 1.4.10 Find the GS to the following DE

$$2xy' + y = 10\sqrt{x}$$

Problem 1.4.11 Find the GS to the following DE

$$(x + y\exp(y))y' = 1$$

(Hint: Regard x as DV and y as IV.)

Problem 1.4.12 Find the GS to the following DE

$$2y + (x+1)y' = 3(x+1)$$

Problem 1.4.13 Find the GS to the following IVP

$$\begin{cases} (1+x)y' + y = \sin x \\ y(0) = 1 \end{cases}$$

Problem 1.4.14 Find the GS to the following DE

$$y' = 2(xy' + y)y^3$$

(Hint: Regard x as DV and y as IV.)

Problem 1.4.15 Find the GS to the following 2nd.O DE

$$x^2y'' + 3xy' = 4x^4$$

(Hint: Use substitution $v = y'$.)

1.5 The Substitution Methods

Substitution methods (S-methods) are those introducing one or more new variables to represent the variables in the original DE. The procedure will convert the original DE to a separable or other more easily solvable DE, *e.g.*, one can change one variable x to u,

$$F_1(x,y) = 0 \xRightarrow{f(x)\to u} F_2(u,y) \tag{1.47}$$

One may also consider changing both variables.

$$G_1(x,y) = 0 \xRightarrow[h(x,y)\to v]{g(x,y)\to u} G_2(u,v) \tag{1.48}$$

Let's discover the power of the S-methods by solving several families of DEs.

1.5.1 Polynomial Substitution

Solve

$$y' = F(ax+by+c) \tag{1.49}$$

where a, b, and c are constants.

Step 1: Introducing a new variable v.

$$v = ax+by+c \tag{1.50}$$

Step 2: Transform the original DE (1.49) into a new DE of v.

$$\begin{aligned} v' &= a + by' \\ y' &= \frac{1}{b}v' - \frac{a}{b} \end{aligned} \tag{1.51}$$

After the substitution, the original DE (1.49) now becomes

$$y' = F(v) \tag{1.52}$$

Substituting y' with the function containing v', we have

$$\begin{aligned} \frac{1}{b}v' - \frac{a}{b} &= F(v) \\ \frac{1}{b}v' &= F(v) + \frac{a}{b} \end{aligned} \tag{1.53}$$

which is now separable.

Step 3: Solve DE (1.53).

$$\begin{aligned} \frac{1}{b}\frac{dv}{dx} &= F(v) + \frac{a}{b} \\ \frac{\frac{1}{b}dv}{F(v) + \frac{a}{b}} &= dx \\ \frac{dv}{bF(v) + a} &= dx \\ x &= \int \frac{dv}{bF(v) + a} \end{aligned} \tag{1.54}$$

Example

Find the GS to the following DE

$$y' = (x + y + 3)^2 \tag{1.55}$$

Solution

Let

$$\begin{aligned} v &= x + y + 3 \\ v' &= 1 + y' \\ y' &= v' - 1 \end{aligned}$$

After the substitution, the original DE becomes

$$\begin{aligned} v' - 1 &= v^2 \\ \frac{dv}{v^2 + 1} &= dx \\ \int \frac{dv}{v^2 + 1} &= \int dx \\ \tan^{-1} v &= x + C \\ v &= \tan(x + C) \end{aligned}$$

Substituting v back into the above DE, we obtain

$$x + y + 3 = \tan(x + C)$$

Thus,

$$y = \tan(x + C) - x - 3$$

1.5.2 Homogeneous DEs

For the 1st.O homogeneous DEs (Homo DEs)

$$y' = F\left(\frac{y}{x}\right) \tag{1.56}$$

After a different substitution $v = \frac{y}{x}$ with which $y = xv$ and $y' = xv' + v$, we convert the original DE (1.56) into

$$\begin{aligned} xv' + v &= F(v) \\ v' &= \frac{F(v) - v}{x} \end{aligned} \tag{1.57}$$

which is a separable DE. Solving it, we get

$$\begin{aligned} \int \frac{dv}{F(v) - v} &= \int \frac{dx}{x} \\ \int \frac{dv}{F(v) - v} &= \ln x - C \\ x &= \exp\left(\int \frac{dv}{F(v) - v} + C_1\right) \\ x &= C \exp\left(\int \frac{dv}{F(v) - v}\right) \end{aligned} \tag{1.58}$$

Example

Find the GS to the following DE

$$2xyy' = 4x^2 + 3y^2 \tag{1.59}$$

Solution

Dividing DE (1.59) by $2xy$

$$y' = \frac{2x}{y} + \frac{3y}{2x}$$

This is a Homo DE.
Let $v = \frac{y}{x}$. Then, $y = vx$ and $y' = v + xv'$.
Substituting this back into the DE, we have

$$v + xv' = \frac{2}{v} + \frac{3}{2}v$$
$$v' = \frac{1}{x}\left(\frac{2}{v} + \frac{v}{2}\right)$$
$$\frac{2v dv}{4 + v^2} = \frac{dx}{x}$$
$$\int \frac{d(4 + v^2)}{4 + v^2} = \int \frac{dx}{x}$$
$$\ln(4 + v^2) = \ln x + C$$
$$4 + v^2 = Cx$$

Plugging v back in, we have

$$4 + \left(\frac{y}{x}\right)^2 = Cx$$
$$y^2 + 4x^2 = Cx^3$$

1.5.3 Bernoulli DEs

Given the DE

$$y' + P(x)y = Q(x)y^n \tag{1.60}$$

We have a few cases depending on n.

Case 1 $(n < 0)$:

DE (1.60) is a general 1st.O nonlinear DE.

Case 2 $(n = 0)$:

DE (1.60) is the 1st.O linear DE $y' + P(x)y = Q(x)$ whose solution method was introduced before.

Case 3 $(n = \frac{1}{2})$:

DE (1.60) is a general 1st.O nonlinear DE.

Case 4 $(n = 1)$:

DE (1.60) is 1st.O linear DE $y' + \big(P(x) - Q(x)\big)y = 0$ which is separable.

Case 5 $(n = 2)$:

DE (1.60) is a special 1st.O nonlinear DE, *i.e.*, the Riccati DE.

Case 6 $(n > 2)$:

DE (1.60) is a general 1st.O nonlinear DE.

To sum up, for $n \neq 0, 1$, we know that DE (1.60) is a 1st.O nonlinear DE and we have not learned any methods to solve the DE for such cases. Now, let's use a substitution to solve the Bernoulli DEs for $n \neq 0, 1$.

Solution steps

Step 1: Select a proper substitution

$$v = y^{1-n} \tag{1.61}$$

Step 2: Find a relationship to replace one variable

$$v' = (1-n)y^{-n}y' \tag{1.62}$$

Step 3: Divide DE (1.60) by y^n

$$y^{-n}y' + P(x)y^{1-n} = Q(x) \tag{1.63}$$

Since $n \neq 1$, we have

$$\frac{1}{1-n}\big((1-n)y^{-n}y'\big) + P(x)y^{1-n} = Q(x) \tag{1.64}$$

Inserting v and v' in (1.61) and (1.62), respectively, into DE (1.64) yields

$$\left(\frac{1}{1-n}\right)v' + P(x)v = Q(x) \tag{1.65}$$

Or

$$v' + (1-n)P(x)v = (1-n)Q(x) \tag{1.66}$$

Step 4: Solve this DE (1.66) in terms of v. Since it is now a 1st.O linear DE, we can solve it by any of the methods that were introduced before.
Step 5: Solve DE (1.66) to find v.
Step 6: Back substituting variable v with y to express the solution in the original variables.

Example
Find the GS to the following DE

$$xy' + 6y = 3xy^{\frac{4}{3}} \tag{1.67}$$

Solution
Slight reformatting $y' + \frac{6}{x}y = 3y^{\frac{4}{3}}$ leads to an obvious Bernoulli DE for which we have

$$P(x) = \frac{6}{x}$$
$$Q(x) = 3$$
$$n = \frac{4}{3}$$

Let's follow the step-by-step procedure to solve this Bernoulli DE.
Step 1:

$$v = y^{1-n} = y^{-\frac{1}{3}}$$
$$y = v^{-3}$$

Step 2:

$$v' = -\frac{1}{3}y^{-\frac{4}{3}}y'$$
$$y' = -3v'(v^{-3})^{\frac{4}{3}}$$
$$= -3v'v^{-4}$$

Step 3:

$$x(-3v'v^{-4}) + 6v^{-3} = 3x(v^{-3})^{\frac{4}{3}}$$
$$-3xv'v^{-4} + 6v^{-3} = 3xv^{-4}$$
$$v'x - 2v = -x$$

$$v' - \frac{2v}{x} = -1$$

Step 4: Now, it becomes a 1st.O linear DE that can be solved using IF

$$\begin{aligned}\rho(x) &= \exp\left(\int P(x)dx\right)\\ &= \exp\left(-\int \frac{2}{x}dx\right)\\ &= \exp(-2\ln x)\\ &= x^{-2}\end{aligned}$$

where we used $P(x) = -\frac{2}{x}$.

Multiplying the IF on both sides of the DE

$$\begin{aligned}\frac{1}{x^2}v' - \frac{2v}{x^3} &= -\frac{1}{x^2}\\ \left(\frac{1}{x^2}v\right)' &= -\frac{1}{x^2}\\ \frac{1}{x^2}v &= -\int \frac{1}{x^2}dx\\ \frac{v}{x^2} &= \frac{1}{x} + C\\ v &= x + Cx^2\end{aligned}$$

Step 5: Plug y back in

$$\begin{aligned}y^{-\frac{1}{3}} &= x + Cx^2\\ y &= \frac{1}{(x + Cx^2)^3}\\ y &= \frac{1}{x^3(1 + Cx)^3}\end{aligned}$$

Problems

Problem 1.5.1 Find the GS to the following DE

$$x^2y' = xy + y^2$$

Problem 1.5.2 Find the GS to the following DE

$$xyy' = y^2 + x\sqrt{4y^2 + x^2}$$

Problem 1.5.3 Show that the substitution $v = \ln y$ transforms the DE $y' + P(x)y = Q(x)(y \ln y)$ into the linear DE $v' + P(x) = Q(x)v(x)$.

Problem 1.5.4 Find the GS to the following DE. Prime denotes derivatives wrt x.

$$5y^4 y' = x^2 y' + 2xy$$

Problem 1.5.5 Find the GS to the following DE

$$xy' - 4x^2 y + 2y \ln y = 0$$

Problem 1.5.6 Find the GS to the following DE

$$tx' - (m+1)t^m x + 2x \ln x = 0$$

Problem 1.5.7 Find the GS to the following DE

$$xy' = 6y + 12x^4 y^{\frac{2}{3}}$$

Problem 1.5.8 Find the GS to the following DE

$$yy'' = 3(y')^2$$

Problem 1.5.9 Find the GS to the following DE

$$y^3 y'' = 3$$

Problem 1.5.10 Find the GS to the following DE using two different methods.

$$y' = xy^3 - xy$$

Problem 1.5.11 Find the GS to the following DE using two different methods.

$$xy' - y = y^2 \sin x$$

Problem 1.5.12 Show that the solution curves of the following DE

$$y' = -\frac{y(2x^3 - y^3)}{x(2y^3 - x^3)}$$

are of the form $x^3 + y^3 = 3Cxy$.

(Hint: Use substitution $u = y/x$.)

Problem 1.5.13 Find the GS to the following DE

$$xy' = y + x\exp\left(\frac{y}{x}\right)$$

Problem 1.5.14 Find the GS to the following DE

$$(x + y)y' = 1$$

Problem 1.5.15 Find the GS to the following DE

$$(2x\sin y\cos y)y' = 4x^2 + \sin^2 y$$

Problem 1.5.16 Find the GS to the following DE

$$y'' = (x + y')^2$$

(Hint: Use substitution $v = y'$.)

Problem 1.5.17 Find the GS to the following DE

$$(x + \exp(y))y' = x\exp(-y) - 1$$

Problem 1.5.18 Find the GS to the following DE

$$(3xy)^2 + x^{\frac{3}{2}}y' = y^2$$

Problem 1.5.19 Find the GS to the following DE

$$y^2(xy' + 1)(1 + x^4)^{\frac{1}{2}} = x$$

Problem 1.5.20 Find the GS to the following DE

$$x\exp(y)\,y' = 2(\exp(y) + x^3\exp(2x))$$

Problem 1.5.21 Find the GS to the following DE

$$\frac{dy}{dx} - 4xy\ln y + 2\frac{y}{x}(\ln y)^n = 0$$

where $n \geq 2$ is a constant integer.

Problem 1.5.22 Find the GS to the following DE

$$yy'' + (y')^2 = yy'$$

Problem 1.5.23 Find the GS to the following DE

$$6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)y' = 0$$

Problem 1.5.24 Given a 1st.O nonlinear DE

$$y' + 7yx^{-1} - 3y^2 = 3x^{-2}$$

(1) Prove that after substitution, $y(x) = x^{-1} + u(x)$, one can transform this DE into a Bernoulli DE;

(2) Solve the resulting Bernoulli DE.

Problem 1.5.25 Find the GS to the following DE

$$x^2y'' + 3xy' = 4$$

(Hint: Use substitution $v = y'$.)

Problem 1.5.26 Find the GS to the following DE

$$tx' - 1000t^{1000}x + 2x^2 = 0$$

Problem 1.5.27 Find the GS to the following DE

$$(x^2 + xy)dx - (xy + y^2)dy = 0$$

Problem 1.5.28 Find the GS to the following DE

$$2x^2yy' + 2xy^2 + 1 = 0$$

Problem 1.5.29 Find the GS to the following DE using two different methods

$$(x^2 + 1)y' - 2xy - 2x = 0$$

Problem 1.5.30 Find the PS to the following IVP

$$\begin{cases} y' = \dfrac{y}{x} - b\left(1 + \left(\dfrac{y}{x}\right)^2\right)^{\frac{1}{2}} \\ y(a) = 0 \end{cases}$$

where a and b are constants and $x \in [0, a]$. Additionally, please

(1) sketch the solution for $b < 1$;

(2) sketch the solution for $b = 1$;

(3) sketch the solution for $b > 1$;

(4) identify a case for which $y(x = 0) = 0$ possible?

Problem 1.5.31 Find the GS to the following DE

$$x' = 3x^2 - \frac{8}{t}x + \frac{4}{t^2}$$

(*Hint: Use substitution* $x = \frac{1}{t} + u$.)

Problem 1.5.32 Find the GS to the following DE

$$y' = (K(x) + y + \beta)(y - \beta)$$

When the function $K(x)$ is given as $K(x) = x^{2011}$ and $\beta = 1$, find the specific form of the solution.

Problem 1.5.33 Find the GS to the following DE

$$tx' - x = \beta(x'x + t)$$

Problem 1.5.34 Find the GS to the following DE

$$y' = by^2 + cx^n$$

where $b, c \neq 0$ are constants and $n = 0, -2$. You need to consider both values of n.

1.6 Riccati DEs

DEs that can be expressed as follows are called Riccati DEs, named after the Italian mathematician J. F. Riccati (1676–1754),

$$y' = A_0(x) + A_1(x)y + A_2(x)y^2 \tag{1.68}$$

If $A_0(x) = 0$, DE (1.68) reduces to a Bernoulli DE.
If $A_2(x) = 0$, DE (1.68) reduces to the 1st.O linear DE.

By intuition or by *guessing*, one may propose a *pilot* solution $y_1(x)$ with which the following solution is proposed,

$$y(x) = y_1(x) + \frac{1}{Z(x)} \tag{1.69}$$

Thus,

$$\begin{aligned} y' &= y_1' - \frac{Z'}{Z^2} \\ y^2 &= y_1^2 + \frac{1}{Z^2} + \frac{2y_1}{Z} \end{aligned} \tag{1.70}$$

Plugging (1.69) and its associated terms (1.70) into the original Riccati DE (1.68), we have

$$y_1' - \frac{Z'}{Z^2} = A_0 + A_1\left(y_1 + \frac{1}{Z}\right) + A_2\left(y_1^2 + \frac{1}{Z^2} + \frac{2y_1}{Z}\right) \tag{1.71}$$

which, after reorganization of the terms, can be written as

$$\begin{aligned} &y_1' - A_0 - A_1 y_1 - A_2 y_1^2 \\ &\qquad = \frac{Z'}{Z^2} + A_1\left(\frac{1}{Z}\right) + A_2\left(\frac{1}{Z^2} + \frac{2y_1}{Z}\right) \end{aligned} \tag{1.72}$$

Because y_1 is a solution to the original Riccati DE, the LHS of (1.72) must be zero, *i.e.*,

$$y_1' - A_0 - A_1 y_1 - A_2 y_1^2 = 0 \tag{1.73}$$

Thus,

$$\frac{Z'}{Z^2} + A_1\left(\frac{1}{Z}\right) + A_2\left(\frac{1}{Z^2} + \frac{2y_1}{Z}\right) = 0 \tag{1.74}$$

Multiplying DE (1.74) by Z^2, we get

$$Z' + (A_1(x) + 2A_2(x)y_1(x))Z = -A_2(x) \tag{1.75}$$

which is a 1st.O linear DE that can be solved, conveniently.

If defining

$$P(x) = A_1(x) + 2A_2(x)y_1(x) \tag{1.76}$$

$$Q(x) = -A_2(x) \tag{1.77}$$

we get the standard form of a 1st.O linear DE

$$Z' + P(x)Z = Q(x) \tag{1.78}$$

Example 1

Find the GS to the following DE

$$y' + y^2 = \frac{2}{x^2} \tag{1.79}$$

Solution

This is a simple Riccati DE and, by inspection, we may assume a PS in the following form $y_1 = \frac{A}{x}$ where A is a constant to be determined. Since it is a PS, it must satisfy the original DE, *i.e.*,

$$-\frac{A}{x^2} + \frac{A^2}{x^2} = \frac{2}{x^2} \tag{1.80}$$

We can easily find two roots $A = -1, 2$ to satisfy (1.80) for arbitrary values of x, resulting in two PS's: $y_1 = -\frac{1}{x}$ or $y_1 = \frac{2}{x}$.

Next, we may select any one of the PS's to compose the GS as

$$y = -\frac{1}{x} + \frac{1}{Z}$$

where we selected the PS: $y_1 = -\frac{1}{x}$.

Thus,

$$y' = \frac{1}{x^2} - \frac{Z'}{Z^2}$$

$$y^2 = \frac{1}{x^2} - \frac{2}{xZ} + \frac{1}{Z^2}$$

Plugging the above two formulas into the original DE:

$$\left(\frac{1}{x^2}-\frac{Z'}{Z^2}\right)+\left(\frac{1}{x^2}-\frac{2}{xZ}+\frac{1}{Z^2}\right)=\frac{2}{x^2}$$

We get

$$Z'+\frac{2}{x}Z=1$$

whose solution is

$$Z=\frac{x}{3}+\frac{C}{x^2}$$

Finally, the GS for the DE is

$$y(x)=-\frac{1}{x}+\frac{3x^2}{x^3+C_1}$$

Example 2

Find the GS to the following DE

$$y'+2xy=1+x^2+y^2 \tag{1.81}$$

Solution

This is a simple Riccati DE and, by inspection, we found a PS $y_1(x)=x$. The original DE can be written as

$$\begin{aligned} y' &= (1+x^2)+(-2x)y+y^2 \\ &= A_0(x)+A_1(x)y+A_2(x)y^2 \end{aligned}$$

where $(x)=1+x^2$, $A_1(x)=-2x$, $A_2(x)=1$. Using substitution $y(x)=y_1(x)+Z(x)^{-1}$, we have

$$Z'+(A_1(x)+2A_2(x)y_1(x))Z=-A_2(x)$$

i.e.,

$$Z'=-1$$

whose GS is

$$Z(x)=-x+c$$

After back substitution, we get

$$y=x-\frac{1}{x-c}$$

Alternatively, we may use the S-method: $y'=1+(x-y)^2$ with substitution $u=x-y$, we get $y'=1-u'$ and

$$1-u'=1+u^2$$

Thus, $\frac{1}{u}=x+c$. After back substitution, we get

$$y = x - \frac{1}{x - c}$$

Problems

Problem 1.6.1 The DE

$$y' = A(x)y^2 + B(x)y + C(x)$$

is called a Riccati DE. Suppose that one PS y_1 of this DE is known, show that the substitution $y = y_1 + \frac{1}{v}$ can transform the Riccati DE into a linear DE $v' + (B + 2Ay_1)v = -A$.

Problem 1.6.2 Find the GS to the following DE

$$y' + y^2 = 1 + x^2$$

given that $y_1 = x$ is a solution.

Problem 1.6.3 Find the GS to the following DE

$$y' = 1 + \frac{1}{4}(x - y)^2$$

given that $y_1 = x$ is a solution.

Problem 1.6.4 Find the GS to the following DE

$$y' - 13(x^2 + y^2) + 26xy = 1$$

given that $y_1 = x$ is a solution.

Problem 1.6.5 Find the GS to the following DE

$$\frac{dy}{dx} = (f(x) + y + a)(y - a)$$

given that $y_1 = a$ is a solution.

Problem 1.6.6 Find the GS to the following DE

$$y' = -(a^2 + 4ax^3) + 4x^3y + y^2$$

given that $y_1 = a$ is a solution.

Problem 1.6.7 Find the GS to the following DE

$$y' = \frac{2\cos^2 x - \sin^2 x + y^2}{2\cos x}$$

given that $y_1 = \sin x$ is a solution.

Problem 1.6.8 Find the GS to the following DE

$$y' = y^2 + \alpha(x)(y - x^2) + 2x - x^4$$

and express it in terms of the given function $\alpha(x)$. We know that this DE has one solution $y = x^2$.

Problem 1.6.9 Find the GS to the following DE

$$x^3 y' + x^2 y - y^2 = 2x^4$$

given that $y_1 = x^2$ is a solution.

1.7 The Exact DEs

Consider a DE

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0 \tag{1.82}$$

Is it

- ✓ 1st-order?
- ✓ Nonlinear?
- ✓ Homo DE?
- ✓ InHomo DE?
- ✓ Inseparable?
- ✓ Non-Bernoulli?

New methods must be introduced in order to solve DEs of this kind.

The GS to all DEs can be written as

$$F(x, y) = C \tag{1.83}$$

With this claim, we get

$$d\big(F(x, y)\big) = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = dC = 0 \tag{1.84}$$

Thus,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \tag{1.85}$$

Therefore, we have

$$\left(\frac{\partial F}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y}\right) dy = 0 \tag{1.86}$$

Let

$$\begin{aligned} M(x,y) &= \frac{\partial F}{\partial x} \\ N(x,y) &= \frac{\partial F}{\partial y} \end{aligned} \tag{1.87}$$

This gives

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.88}$$

All 1st.O DEs can be written in the above manner.

The condition for the DEs to be exact is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \tag{1.89}$$

That is, if $N_x = M_y$, (1.88) is exact. In fact, one prove that the necessary and sufficient condition for (1.88) to be exact is $N_x = M_y$.

Example 1
Is DE

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0 \tag{1.90}$$

exact?

Solution
We have

$$\begin{aligned} M(x,y) &= 6xy - y^3 \\ N(x,y) &= 4y + 3x^2 - 3xy^2 \end{aligned}$$

Here

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2 \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2) = 6x - 3y^2 \end{aligned}$$

Since

$$M_y = N_x = 6x - 3y^2$$

The DE is exact.

Example 2
Is DE

$$ydx + 3xdy = 0 \tag{1.91}$$

exact?

Solution
From DE, we get

$$M(x, y) = y$$
$$N(x, y) = 3x$$

Here

$$M_y = \frac{\partial}{\partial y}(y) = 1$$
$$N_x = \frac{\partial}{\partial x}(3x) = 3$$

That means

$$M_y \neq N_x$$

The DE is not exact.

Example 3
Is DE

$$y^3 dx + 3xy^2 dy = 0 \tag{1.92}$$

exact?

Solution
We get

$$M(x, y) = y^3$$
$$N(x, y) = 3xy^2$$

Here

$$M_y = \frac{\partial}{\partial y}(y^3) = 3y^2$$
$$N_x = \frac{\partial}{\partial x}(3xy^2) = 3y^2$$

That means

$$N_x = M_y = 3y^2$$

The DE is exact.

Let's find its solution. There is a new set of methods for solving the exact DEs. As mentioned in the beginning of this section, the solution to any DE can be written as

$$F(x, y) = C \tag{1.93}$$

From the above example, we have

$$M(x,y) = \frac{\partial F}{\partial x} = y^3 \tag{1.94}$$

$$N(x,y) = \frac{\partial F}{\partial y} = 3xy^2 \tag{1.95}$$

So, from (1.94), we have

$$F(x,y) = \int y^3 dx = y^3 x + C$$

Here we have C as a constant wrt x. Consider the constant be a function of y: $C = g(y)$. Now we have

$$F(x,y) = y^3 x + g(y) \tag{1.96}$$

Plugging (1.96) back into (1.95), we get

$$\begin{aligned} \frac{\partial}{\partial y}\left(y^3 x + g(y)\right) &= 3xy^2 \\ 3xy^2 + g'(y) &= 3xy^2 \\ g'(y) &= 0 \\ g(y) &= C_1 \end{aligned}$$

So, we find that $g(y)$ is actually a constant.

Plugging $g(y) = C_1$ into (1.96), we get

$$F(x,y) = y^3 x + C_1 \tag{1.97}$$

Therefore, from (1.93), we have

$$y^3 x + C_1 = C_2 \tag{1.98}$$

Thus, the solution is

$$y^3 x = C \tag{1.99}$$

Steps for solving exact DEs:

Step 1: Write the DE in the following form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.100}$$

and get the formulas for

$$M(x,y) = \frac{\partial F}{\partial x} \tag{1.101}$$

$$N(x,y) = \frac{\partial F}{\partial y} \tag{1.102}$$

Step 2: Find

$$F(x,y) = \int M(x,y)dx + g(y) \tag{1.103}$$

Step 3: Plugging $F(x,y)$ into (1.102) obtains $g(y)$.
Step 4: The GS is

$$\int M(x,y)dx + g(y) = C \tag{1.104}$$

Remarks

In the above steps, we start from (1.101), integrate it and, then, plug it back into (1.102) to get the solution. Similarly, we can start from (1.102), integrate it and plug back into (1.101) to get the *same* solution, different by a constant.

Let's now clarify a few points:

(1) Not all DEs are exact. Some non-exact DEs can be converted to exact DEs while others cannot. So, the question is, under what condition(s) a non-exact DE can be converted to an exact DE.

(2) If a non-exact DE can be converted to an exact DE, is the conversion unique? In other words, is there more than one method to convert a non-exact DE into an exact DE?

Let's now convert a non-exact DE to an exact DE. Suppose we have a DE

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.105}$$

If we have the IF $I(x)$ such that

$$I(x)M(x,y)dx + I(x)N(x,y)dy = 0 \tag{1.106}$$

is exact, we should have

$$\frac{\partial}{\partial y}\big(I(x)M(x,y)\big) = \frac{\partial}{\partial x}\big(I(x)N(x,y)\big) \tag{1.107}$$

That means

$$\begin{aligned} I(x)\frac{\partial}{\partial y}M(x,y) &= N(x,y)\frac{\partial}{\partial x}I(x) + I(x)\frac{\partial}{\partial x}N(x,y) \\ I \cdot M_y &= N \cdot I' + I \cdot N_x \\ I\big(M_y - N_x\big) &= I' \cdot N \\ \frac{I'}{I} &= \frac{M_y - N_x}{N} \\ I(x) &= \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) \end{aligned} \tag{1.108}$$

Here the integrand

$$\frac{M_y - N_x}{N} \tag{1.109}$$

must be a function with a single variable x.

Exact DE Theorem

For DE

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.110}$$

1) If

$$\frac{M_y - N_x}{N} = f(x) \tag{1.111}$$

is a function of one variable x, we have the IF

$$\rho(x) = \exp\left(\int f(x)dx\right)$$

2) If

$$\frac{M_y - N_x}{M} = g(y) \tag{1.112}$$

is a function of one variable y, we have the IF

$$\rho(y) = \exp\left(-\int g(y)dy\right) \tag{1.113}$$

Remarks

(1) For one DE, $\rho(x)$ and $\rho(y)$ may both exist. In this case, use the most convenient form.
(2) It is possible that neither $\rho(x)$ nor $\rho(y)$ exists.
(3) You must have noticed that $\rho(x)$ has no negative sign in the exponential term while $\rho(y)$ has a negative sign in the exponential term. So, given below is the proof of why we need the negative sign for $\rho(y)$ but not for $\rho(x)$.

Proof of the negative sign in $\rho(y)$ in (1.113).

$M(x,y)dx + N(x,y)dy = 0$ is the given DE. The IF is chosen as $\rho(y)$

$$\rho(y)M(x,y)dx + \rho(y)N(x,y)dy = 0 \tag{1.114}$$

Let

$$\bar{M}(x,y) = \rho(y)M(x,y) \tag{1.115}$$

$$\bar{N}(x,y) = \rho(y)N(x,y) \tag{1.116}$$

We get the DE: $\bar{M}(x,y)dx + \bar{N}(x,y)dy = 0$. We want it to be exact and, thus, we set

$$\bar{M}_y = \bar{N}_x \tag{1.117}$$

i.e.,

$$\frac{\partial}{\partial y}\big(\rho(y)M(x,y)\big) = \frac{\partial}{\partial x}\big(\rho(y)N(x,y)\big)$$

$$M(x,y)\frac{\partial}{\partial y}\rho(y) + \rho(y)\frac{\partial}{\partial y}M(x,y) = \rho(y)\frac{\partial}{\partial x}N(x,y) \tag{1.118}$$

$$M\rho' + \rho M_y = \rho N_x$$

$$\frac{\rho'}{\rho} = \frac{N_x - M_y}{M}$$

That is

$$\frac{\rho'}{\rho} = -g(y) \tag{1.119}$$

where

$$g(y) = \frac{M_y - N_x}{M} \tag{1.120}$$

as noted above.

Solving (1.119), we get

$$\rho(y) = \exp\left(-\int g(y)dy\right) \tag{1.121}$$

We sum up the above discussion in the following theorems:

Theorem 1

For a given 1st.O DE $M(x,y)dx + N(x,y)dy = 0$, if

$$\frac{M_y - N_x}{N} = f(x) \tag{1.122}$$

is a function of purely x, the DE can be converted into an exact DE by multiplying the original DE with

$$\rho(x) = \exp\left(\int f(x)dx\right) \tag{1.123}$$

Theorem 2

For a given 1st.O DE $M(x,y)dx + N(x,y)dy = 0$, if

$$\frac{M_y - N_x}{M} = g(y) \tag{1.124}$$

is a function of purely y, the DE can be converted into an exact DE by multiplying the original DE with

$$\rho(y) = \exp\left(-\int g(y)dy\right) \tag{1.125}$$

Next, let's work on one example to enhance the understanding of the theorems expressed by the IF's (1.123) and (1.125). We also make a few remarks afterwards.

Example 4

Determine whether the DE

$$ydx + 3xdy = 0 \tag{1.126}$$

is exact or not. If not, convert it to an exact DE and solve it.

Solution

From the given DE, we get

$$M(x,y) = y$$
$$N(x,y) = 3x$$

Here, we have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y) = 1$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3x) = 3$$

Since $M_y \neq N_x$, we know that the given DE is not exact. For solving this DE, there are two methods of doing this.

Method 1: Using the IF $\rho(x)$ to covert the DE.

$$\frac{M_y - N_x}{N} = \frac{1-3}{3x} = -\frac{2}{3x} = f(x)$$

It is clear that $f(x)$ is a function of purely x.

$$\begin{aligned}\rho(x) &= \exp\left(\int f(x)dx\right) \\ &= \exp\left(-\int \frac{2}{3x}dx\right) \\ &= \exp\left(-\frac{2}{3}\ln x\right) \\ &= x^{-\frac{2}{3}}\end{aligned}$$

Multiplying $\rho(x)$ on both sides of the original DE, we have

$$x^{-\frac{2}{3}}ydx + 3x \cdot x^{-\frac{2}{3}}dy = 0$$

This is now our new DE. Here we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = x^{-\frac{2}{3}}y \tag{1.127}$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3x^{\frac{1}{3}} \tag{1.128}$$

Thus, we have

$$\bar{M}_y = \frac{\partial}{\partial y}\left(x^{-\frac{2}{3}}y\right) = x^{-\frac{2}{3}}$$

$$\bar{N}_x = \frac{\partial}{\partial x}\left(3x^{\frac{1}{3}}\right) = x^{-\frac{2}{3}}$$

Since $\bar{M}_y = \bar{N}_x$, the new DE is exact and can be solved by using either (1.127) or (1.128) to find $F(x,y)$. It appears the function in (1.128) is easier to integrate and we opt for (1.128) to find $F(x,y)$. We have

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3x^{\frac{1}{3}}$$

$$F(x,y) = \int 3x^{\frac{1}{3}}dy + g(x) = 3x^{\frac{1}{3}}y + g(x) \tag{1.129}$$

Substituting (1.129) back into (1.127), we have

$$\frac{\partial}{\partial x}\left(3x^{\frac{1}{3}}y + g(x)\right) = x^{-\frac{2}{3}}y + g'(x) = x^{-\frac{2}{3}}y$$

This solves $g'(x) = 0$. That means $g(x) = C_1$. Substituting this back into (1.129), we have

$$F(x,y) = 3x^{\frac{1}{3}}y + C_1$$

That gives the solution to the DE

$$F(x,y) = C_2$$

That is

$$x^{\frac{1}{3}}y = C$$

One may even write the GS as

$$xy^3 = C_3$$

where C_3 is another constant.

Method 2: Using the IF $\rho(y)$ to covert the DE.

We now discuss the second method of converting a non-exact DE to an exact DE.

$$\frac{M_y - N_x}{M} = \frac{1-3}{y} = -\frac{2}{y} = g(y)$$

The IF

$$\begin{aligned}\rho(y) &= \exp\left(-\int g(y)dy\right)\\ &= \exp\left(\int \frac{2}{y}dy\right)\\ &= y^2\end{aligned}$$

Multiplying $\rho(y)$ on both sides of the original DE, we have

$$y^3dx + 3xy^2dy = 0$$

as our new DE. Here we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = y^3 \tag{1.130}$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3xy^2 \tag{1.131}$$

and

$$\bar{M}_y = \frac{\partial}{\partial y}(y^3) = 3y^2$$

$$\bar{N}_x = \frac{\partial}{\partial x}(3xy^2) = 3y^2$$

Since $\bar{M}_y = \bar{N}_x$, the new DE formed is now exact.

We can now use either (1.130) or (1.131) to find $F(x,y)$. Since the function in (1.130) is easier to integrate, we choose (1.130) to find $F(x,y)$. We have

$$F(x,y) = \int y^3dx + g(y) = xy^3 + g(y) \tag{1.132}$$

Substituting (1.132) back into (1.130), we have

$$\frac{\partial}{\partial y}\left(xy^3 + g(y)\right) = 3xy^2 + g'(y) = 3xy^2$$

It gives

$$g'(y) = 0$$

which means

$$g(y) = C_1$$

Substituting this back into (1.132), we have

$$F(x, y) = xy^3 + C_1$$

Thus, the GS to the DE is

$$xy^3 + C_1 = C_2$$

That is

$$xy^3 = C$$

This is the same answer as we found in Method 1.

Summary

We have four ways for solving a non-exact DE, which is summarized as follows.

Steps for solving a non-exact DE by converting it into an exact DE

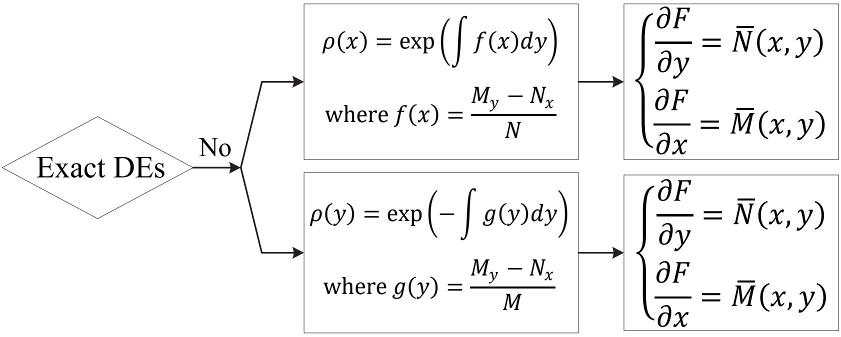

Figure 1.4 Steps for converting a non-exact DE to an exact DE.

Step 1 Determine $M(x, y)$ and $N(x, y)$.

Check if $M_y = N_x$? If so, go to Step 5

Step 2 Check if $\frac{M_y - N_x}{N} = f(x)$ is a function of purely x or if $\frac{M_y - N_x}{M} = g(y)$ is a function of purely y.

Step 3 Find the easiest IF

$$\rho(x) = \exp\left(\int f(x)dx\right) \tag{1.133}$$

Or

$$\rho(y) = \exp\left(-\int g(y)dy\right) \tag{1.134}$$

Step 4 Compute the new $\bar{M}$ and $\bar{N}$ using either $\rho(x)$ or $\rho(y)$

$$\bar{M}(x,y) = \rho\, M(x,y) \tag{1.135}$$

$$\bar{N}(x,y) = \rho\, N(x,y) \tag{1.136}$$

Step 5 Construct the partial DEs

$$\begin{cases} \dfrac{\partial F}{\partial x} = \bar{M}(x,y) \\ \dfrac{\partial F}{\partial y} = \bar{N}(x,y) \end{cases} \tag{1.137}$$

Step 6 Use either of the DEs (1.137) to find $F(x,y)$

$$F(x,y) = \int \bar{M}(x,y)dx + g(y) \tag{1.138}$$

or equivalently,

$$F(x,y) = \int \bar{N}(x,y)dy + g(x) \tag{1.139}$$

Step 7 Substituting the $F(x,y)$ obtained as (1.138) or (1.139) into one of the DEs (1.137), we have either

$$\frac{\partial}{\partial y}\left(\int \bar{M}(x,y)dx + g(y)\right) = \bar{N}(x,y) \tag{1.140}$$

Or

$$\frac{\partial}{\partial x}\left(\int \bar{N}(x,y)dy + g(x)\right) = \bar{M}(x,y) \tag{1.141}$$

Step 8 Solving DE (1.140) or (1.141) produces $g(y)$ or $g(x)$. Inserting $g(y)$ or $g(x)$ into (1.138) or (1.139) forms the final GS to the DE $F(x,y)$ =C.

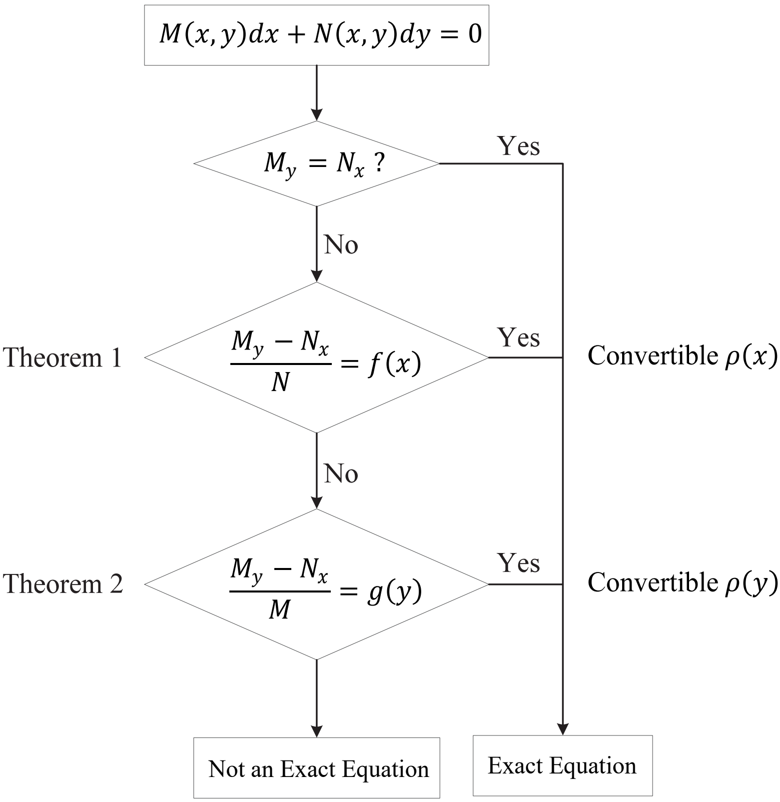

Figure 1.5 Steps for solving a non-exact DE.

Example 5

Find the GS to the following DE

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.142}$$

Solution

We can compose two simple partial DEs with F as the DV and x and y as the IVs

$$\frac{\partial F}{\partial x} = M(x,y) \tag{1.143}$$

$$\frac{\partial F}{\partial y} = N(x,y) \tag{1.144}$$

Solving (1.143), we get

$$F(x,y) = \int M(x,y)dx + g(y)$$

Plugging the above into (1.144), we get

$$\frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + g'(y) = N(x,y)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right)$$

Solving this, we have

$$g(y) = \int N(x,y)dy - \int \left(\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)\right)dy$$

Thus, we get the GS to the DE as

$$\int M(x,y)dx + \int N(x,y)dy - \int \left(\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)\right)dy = C$$

Example 6

Is the following DE exact? Find the GS.

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0 \tag{1.145}$$

Solution

From the given DE, we have

$$\frac{\partial F}{\partial x} = M(x,y) = 6xy - y^3$$

$$\frac{\partial F}{\partial y} = N(x,y) = 4y + 3x^2 - 3xy^2$$

Now we evaluate

$$M_y = \frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2$$

$$N_x = \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2) = 6x - 3y^2$$

Since $M_y = N_x$, the given DE is exact. Now, we find $F(x, y)$

$$\frac{\partial F}{\partial x} = M(x, y) = 6xy - y^3$$

$$F(x, y) = \int (6xy - y^3)dx + g(y)$$

$$= 3x^2y - y^3x + g(y)$$

Plugging it into

$$\frac{\partial F}{\partial y} = N(x, y)$$

we have

$$\frac{\partial}{\partial y}\left(3x^2y - y^3x + g(y)\right) = 4y + 3x^2 - 3xy^2$$

$$3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

$$g'(y) = 4y$$

$$g(y) = 2y^2$$

Therefore, the GS to the DE is

$$3x^2y - xy^3 + 2y^2 = C$$

Let's examine the following to show that not all DEs can be solved this way.

Example 7

Find the GS to the following DE

$$(3y^2 + 5x^2y)dx + (3xy + 2x^3)dy = 0 \qquad (1.146)$$

Solution

From the given DE, we have

$$M(x, y) = 3y^2 + 5x^2y$$

$$N(x, y) = 3xy + 2x^3$$

$$M_y = \frac{\partial}{\partial y}(3y^2 + 5x^2y) = 6y + 5x^2$$

$$N_x = \frac{\partial}{\partial x}(3xy + 2x^3) = 3y + 6x^2$$

Since $M_y \neq N_x$, the given DE is not exact. Let's now check if we can convert it to an exact DE.

$$\frac{M_y - N_x}{N} = \frac{3y - x^2}{3xy + 2x^3}$$

is not a function of purely x, and

$$\frac{M_y - N_x}{M} = \frac{3y - x^2}{3y^2 + 5x^2 y}$$

is not a function of purely y. Therefore, the given DE cannot be converted to an exact DE.

Finally, we show the relationship between the exact DE method and the linear DE we previously studied.

Converting the 1st.O linear DE to an exact DE

Now the question arises whether all linear DEs are exact. If not, are they convertible? Let's now answer these queries. The standard form of the 1st.O linear DE is

$$y' + P(x)y = Q(x) \tag{1.147}$$

That is

$$\begin{aligned} \frac{dy}{dx} + P(x)y - Q(x) &= 0 \\ \big(P(x)y - Q(x)\big)dx + dy &= 0 \end{aligned} \tag{1.148}$$

Now we have

$$\begin{aligned} M(x,y) &= P(x)y - Q(x) \\ N(x,y) &= 1 \\ M_y &= P(x) \\ N_x &= 0 \end{aligned} \tag{1.149}$$

Since in general $M_y \neq N_x$, (1.147) is not an exact DE.

If a DE is not exact in its original form, we may convert it to an exact DE.

Compare the following two formulas.

$$\frac{M_y - N_x}{N} = P(x) \tag{1.150}$$

$$\frac{M_y - N_x}{M} = \frac{P(x)}{P(x)y - Q(x)}$$

Apparently, working with the first one is much easier than working with the second one. Thus, we have our IF

$$\rho(x) = \exp\left(\int P(x)dx\right) \tag{1.151}$$

Multiplying $\rho(x)$ on both sides of (1.148), we have

$$\begin{aligned}\exp\left(\int P(x)dx\right)\left(P(x)y - Q(x)\right)dx \\ + \exp\left(\int P(x)dx\right)dy = 0\end{aligned} \tag{1.152}$$

which is now the new DE. Thus, we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = \exp\left(\int P(x)dx\right)\left(P(x)y - Q(x)\right) \tag{1.153}$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = \exp\left(\int P(x)dx\right) \tag{1.154}$$

$$\begin{aligned}\bar{M}_y &= \frac{\partial}{\partial y}\left(\exp\left(\int P(x)dx\right)\left(P(x)y - Q(x)\right)\right) \\ &= \exp\left(\int P(x)dx\right)\frac{\partial}{\partial y}\left(P(x)y - Q(x)\right) \\ &= P(x)\exp\left(\int P(x)dx\right)\end{aligned} \tag{1.155}$$

$$\begin{aligned}\bar{N}_x &= \frac{\partial}{\partial x}\exp\left(\int P(x)dx\right) \\ &= P(x)\exp\left(\int P(x)dx\right)\end{aligned} \tag{1.156}$$

Since $\overline{M}_y = \overline{N}_x$, the new DE is exact.

Now we can use either (1.153) or (1.154) to find $F(x, y)$. Since the function in (1.154) is easier to integrate, we use it to find:

$$\begin{aligned} F(x,y) &= \int \exp\left(\int P(x)dx\right)dy \\ &= y\exp\left(\int P(x)dx\right) + g(x) \end{aligned} \tag{1.157}$$

Substituting this back to (1.153), we have

$$\begin{aligned} &\frac{\partial}{\partial x}\left(y\exp\left(\int P(x)dx\right) + g(x)\right) \\ &\quad = \exp\left(\int P(x)dx\right)\left(P(x)y - Q(x)\right) \\ &\qquad + y\,P(x)\exp(\textstyle\int P(x)dx) + g'(x) \\ &\quad = y\,P(x)\exp\left(\int P(x)dx\right) - Q(x)\exp\left(\int P(x)dx\right) \end{aligned} \tag{1.158}$$

$$\begin{aligned} g'(x) &= -Q(x)\exp\left(\int P(x)dx\right) \\ g(x) &= -\int Q(x)\exp\left(\int P(x)dx\right)dx \end{aligned} \tag{1.159}$$

Substituting this back to $F(x, y)$, we have

$$F(x,y) = y\exp\left(\int P(x)dx\right) - \int Q(x)\exp\left(\int P(x)dx\right)dx \tag{1.160}$$

Therefore, the GS to the DE is

$$y \exp\left(\int P(x)dx\right) - \int Q(x) \exp\left(\int P(x)dx\right) dx = C \quad (1.161)$$

That is

$$y = \exp\left(-\int P(x)dx\right)\left(\int Q(x) \exp\left(\int P(x)dx\right) dx + C\right) \quad (1.162)$$

Problems

Problem 1.7.1 Find the GS to the following DE

$$(1 + \ln(xy))dx + \left(\frac{x}{y}\right) dy = 0$$

Problem 1.7.2 Given a 1st.O DE

$$A(x,y)dx + B(x,y)dy = 0$$

which is not an exact DE in general, but the function $A(x,y)$ and $B(x,y)$ satisfy the following relationship

$$\left(\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right) / B(x,y) = P(x)$$

where $P(x)$ is a function of one variable x. Prove that, after multiplying the original DE by an IF $\rho(x) = \exp(\int P(x)\, dx)$, one can transform the original non-exact DE into an exact DE.

Problem 1.7.3 Given 1st.O linear DE

$$\alpha(x)\frac{dy}{dx} + \beta(x)y + \gamma(x) = 0$$

where $\alpha(x) \neq 0$.

(1) Use the exact DE method to solve the DE and express the GS in terms of $\alpha(x), \beta(x)$, and $\gamma(x)$.

(2) Use the 1st.O linear DE method to solve the DE and express the GS in terms of $\alpha(x), \beta(x)$, and $\gamma(x)$.

Problem 1.7.4 The 1st.O linear DE can be expressed alternatively as

$$(P(x)y - Q(x))dx + dy = 0$$

where functions $P(x) \neq 0$ and $Q(x) \neq 0$ are given. Please

(1) Check if this DE is exact.

(2) If not, convert it to an exact DE.

(3) Solve the DE using the exact DE method; your solution may be expressed in terms of the functions $P(x)$ and $Q(x)$.

(4) If $P(x) = \frac{1}{x}$ and $Q(x) = \frac{\cos x}{x}$, get the specific solution.

Problem 1.7.5 Find the GS to the following DE

$$(2x - y^2)dx + xydy = 0$$

Problem 1.7.6 Find the GS to the following DE using two different methods.

$$y' = -\frac{3x^2 + 2y^2}{4xy}$$

Problem 1.7.7 Find the GS to the following DE using two different methods.

$$y' - \frac{x}{x^2 + 1}y = 2x(x^2 + 1)$$

Problem 1.7.8 Find the GS to the following DE using two different methods.

$$y' = \frac{x + 3y}{y - 3x}$$

Problem 1.7.9 Find the GS to the following DE

$$ydx + (2x + y^4)dy = 0$$

Problem 1.7.10 Find the GS to the following DE

$$y'(2xy + 1) = 2x - y^2$$

Problem 1.7.11 Find the GS to the following DE

$$y^3 + xy^2y' - y' = 0$$

Problem 1.7.12 Check whether the following DE is exact

$$(\cos x + \ln y)dx + \left(\frac{x}{y} + \exp(y)\right) dy = 0$$

If it is exact according to your verification above, solve it using the exact DE method. Otherwise, use a different method to solve it.

Problem 1.7.13 Find the GS to the following DE using two different methods.

$$\exp(y) + y\cos x + (x\exp(y) + \sin x)y' = 0$$

1.8 Summary

Most of the DEs we discuss can be summarized in the following Figure 1.6.

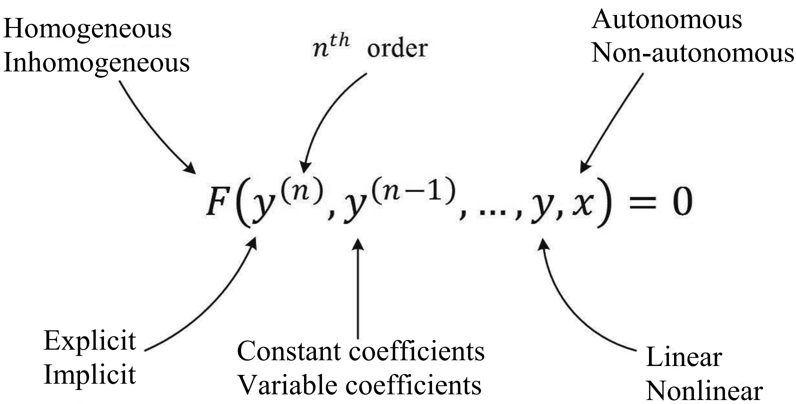

Figure 1.6 This diagram illustrates the classification of DEs by various categories.

Classifying DEs according to homogeneous (Homo) and inhomogeneous (InHomo) will produce two types. Of each type, further classification according to autonomous and non-autonomous will produce two sub-types. Thus, we have four types using these two classifications. Adding linear and nonlinear, explicit and implicit, constant coefficients (c-coeff) and variable coefficients (v-coeff), we have a total of $2^5 = 32$ types. Further, if adding the orders of the DEs, we have 32 types of 1st.O DEs and 32 types of 2nd.O DEs, *etc*.

Based on their intrinsic properties and solution methods, 1st.O DEs (like many other types of DEs) may be classified into several overlapping groups: separable, 1st.O linear, substitutable, the Bernoulli, the Riccati, and the exact DEs.

Commonly, one single DE may belong to multiple groups and, thus, can be solved by multiple solution methods although the algebraic complexity of the methods may differ dramatically. Therefore, when given a DE to solve, the most important first step is to identify the group(s) the DE belongs to and, then, select the most effective solution method(s) to solve it. The following Venn diagram (Figure 1.7) illustrates the logical relationship of all 1st.O DEs and their solution methods.

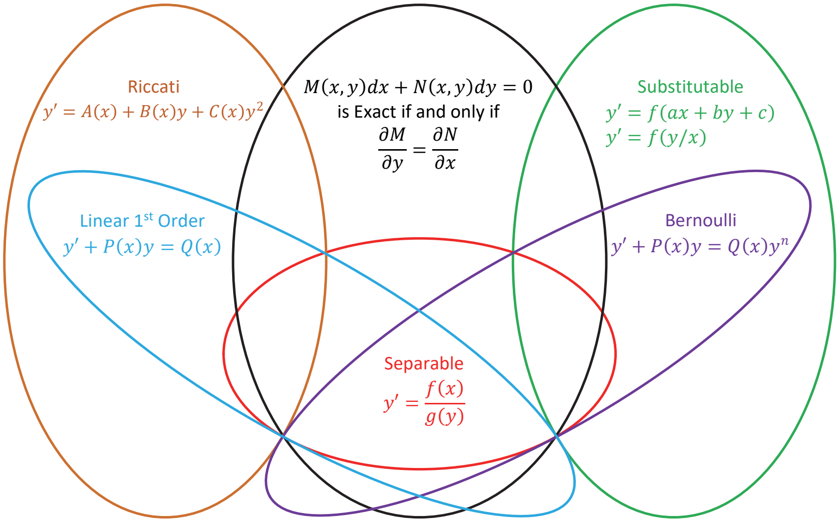

Figure 1.7 This Venn diagram illustrates the logical relationships of 1st.O DEs and their solution methods.

Problems

Problem 1.8.1 Find the GS to the following DE using at least two methods

$$xy' - y = y^2 \sin x$$

Problem 1.8.2 Find the GS to the following DE using at least two methods:

$$y' + 30xy = 1 + 15(x^2 + y^2)$$