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Positive periodic solutions of Hill's equations with singular nonlinear perturbations

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Abstract

We study the existence and multiplicity of positive periodic solutions of Hill's equations with singular nonlinear perturbations. The new results are applicable to the case of a strong singularity as well as the case of a weak singularity. The proof relies on a nonlinear alternative principle of Leray–Schauder and a fixed point theorem in cones. Some recent results in the literature are generalized and improved.

MSC: 34B15; 34B16; 34D20

Keywords: Positive periodic solution; Hill equation; Strong singularity; Weak singularity; Leray–Schauder alternative principle; Fixed point theorem in cones

1. Introduction

In this paper, we study the existence and multiplicity of positive periodic solutions of the perturbation of Hill's equation

$$x'' + a(t)x = f(t, x) + e(t), \tag{1.1}$$

here $a(t)$, $e(t)$ are continuous, T-periodic functions. The nonlinearity $f(t, x)$ is continuous in (t, x) and T-periodic in t and has a singularity at $x = 0$. Such singular equations appear in many problems of applications such as the Brillouin focusing system [15] and nonlinear elasticity [4,5]. Another typical example are the so-called Ermakov–Pinney equations [21].

Beginning with the paper of Lazer and Solimini [12], the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \tag{1.2}$$

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with $a, b, c \in \mathbb{C}[0, T]$ and $\lambda > 0$, has attracted the attention of many researchers during the last two decades [4, 9]. Some strong force conditions introduced by Gordon [8] are standard in the related works [5,19]. This condition corresponds to the case that $\lambda > 1$ in Eq. (1.2). With a strong singularity, the energy near $x = 0$ becomes infinite and this fact is very useful for obtaining a priori bounds needed for the classical applications of the degree theory [13].

Compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities is more recent and the number of references is much smaller [7,10,14,16]. Here we mention the following results. From now on, let us denote by p^* and p_* the essential supremum and infimum of a given function $p \in L^1[0, 1]$, if they exist. In [14], it is proved that for $0 < k^2 \leq \mu_1 = (\frac{\pi}{T})^2$ and $\lambda, b > 0$, the equation

$$x'' + k^2 x = \frac{b}{x^\lambda} + e(t) \tag{1.3}$$

has a positive periodic solution if the following inequality holds:

$$e_* > -(\lambda + 1) b \left(\frac{\pi^2 - T^2 k^2}{T^2 \lambda b} \right)^{\frac{\lambda}{\lambda+1}}. \tag{1.4}$$

In particular, (1.4) is fulfilled when $e_* \geq 0$. In [16], Torres gave another existence condition: $e_* < 0$ and

$$e^* \leq \frac{e_*}{\cos^\lambda \left(\frac{kT}{2}\right)} + \frac{k}{T} \sin kT \left(\frac{b}{|e_*|} \right)^{\frac{1}{\lambda}}. \tag{1.5}$$

Both conditions (1.4) and (1.5) describe the dependence of the range of $e(t)$ upon the parameter $\lambda > 0$. Condition (1.5) has been slightly improved in [2,6] by the following condition

$$e^* \leq \frac{e_*}{\cos^\lambda \left(\frac{kT}{2}\right)} + \left(\frac{k}{T}\right)^2 \left(\frac{b}{|e_*|} \right)^{\frac{1}{\lambda}} \cos\left(\frac{kT}{2}\right). \tag{1.6}$$

Some classical tools have been used to study singular equations in the literature. These classical tools include the Poincaré–Birkhoff Theorem or Moser Twist Theorem [5], the coincidence degree theory of Mawhin [13,18,21], the method of upper and lower solutions [1,3], and some fixed point theorems in cones for completely continuous operators [10,16]. In this paper, we will apply a nonlinear alternative principle of Leray–Schauder and a fixed point theorem in cones to prove the main results.

Our main motivation is to partially solve an open problem posed by Torres in [17] and obtain new existence results for positive periodic solutions of the equation

$$x'' + a(t)x = \frac{b(t)}{x^\alpha} + \mu c(t) x^\beta + e(t), \tag{1.7}$$

with $a, b, c, e \in \mathbb{C}[0, 1]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a given parameter. Here we emphasize that the new results are applicable to the case of a strong singularity as well as the case of a weak singularity, and that e does not need to be positive. Therefore we generalize and improve some results contained in [10,14,16,17]. Some differences between our results and those obtained in [17] will given in Remark 3.3 in Section 3.

Finally we fix some notation to be used. Given $\varphi \in L^1[0, T]$, we write $\varphi \succ 0$ if $\varphi \geq 0$ for a.e. $t \in [0, T]$ and it is positive in a set of positive measure. $\mathbb{R}^+$ denotes the set of positive real numbers. The usual L^p-norm is denoted by $\|\cdot\|_p$. The conjugate exponent of p is denoted by $\tilde{p}$: $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$.

2. Preliminaries

Throughout this paper, we assume that Hill's equation

$$x'' + a(t)x = 0 \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T) \tag{2.2}$$