A TWO-LEVEL LES CONSTRUCTED WITH UNFILTERED EQUATIONS AND SGS MODELS OF PHYSICAL VARIABLES

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CURRENT OFTEN-STUDIED FORMS OF LES

- Smagorinsky—it's easy and efficient, relatively at least
- Combinations of LES and RANS—*e.g.*, DES
- Dynamic Models—produce backscatter; hence able to transition
- Implicit LES, *e.g.*, MILES—do not filter equations of motion
- "Synthetic-Velocity" Models in General, *e.g.*, estimation models, LEM, ODT—all utilize subgrid-scale (SGS) models based on physical variables

A NOT-TOO-DIFFERENT ALTERNATIVE

- Basically a Synthetic-Velocity Model Constructed Using the Following Ideas
  - Filter solutions rather than equations—just implementation of *mollification* from pure PDE theory
  - Model sub-grid physical variables instead of their statistics—one approach is use "*symbols*" of the differential operators
  - Directly use SGS results to enhance large-scale (deliberately under-resolved) part—analogous to a *multi-level* formalism
- Most of These Already Being Used in One or More of Typical Approaches, But Usually Not All Applied Simultaneously
- Some Details on Each Follow

FILTERING

- **Why Filter Anything?—LES resolved-scale solutions under resolved and will alias**
  - **Filtering equations produces terms whose models provide dissipation**
  - **Filtering solutions is easier, and more easily analyzed—just mollification in pure-math sense; eliminate high-frequency/high-wavenumber components leading to nonsmoothness**

- **How?— define C^∞ function $u_\epsilon(x,t)$ as** $u_\epsilon(x,t) = \int_{-\epsilon}^{\epsilon} u(\xi,t)\, \delta_\epsilon(x-\xi)\, d\xi$,

  where δ_ϵ is C_0^∞ with support in $[-\epsilon,\epsilon]$
  - **Error easily determined for most practical filters**
  - **Essentially same form of filter usually applied to equations of motion**

- **Where, When?—at all points of resolved-scale grid, and for every time step**

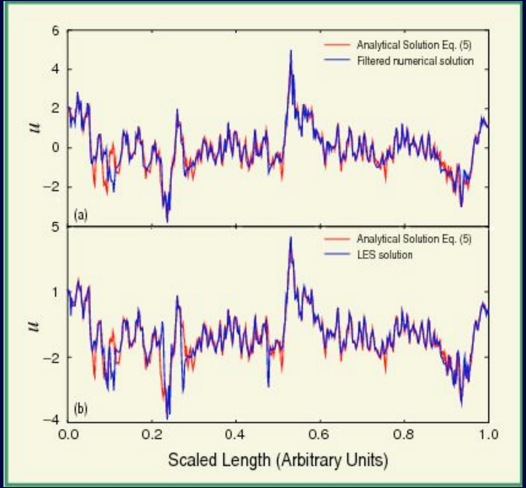
FILTERING (Cont.)

- **Present Approach Utilizes *Shuman Filter* Formally Derived From Mollifier Shown Above (**McDonough & Yang**, in preparation)**
- **Basic Discrete Form:**

$$\tilde{u}_{\boldsymbol{i}}(\boldsymbol{x},t) = \frac{\beta\, u_{\boldsymbol{i}} + \sum_{j \in S_i} u_{\boldsymbol{j}}}{\beta + 2^d},$$

where $\beta \in (-2^d, \infty)$ **is filter parameter;** $\boldsymbol{i}, \boldsymbol{j}$ **are multi-indices of dimension** d**;** $S_{\boldsymbol{i}}$ **is index set of nearest neighbors of** $\boldsymbol{i}$

- **Can be shown to provide needed dissipation as required for mollification**
- **Solution shown at right requires minimum 4096 points for stable DNS, but can be computed using only 512 points with filtering**
- **Leads to global (in time) first-order accuracy with overall results comparable to those obtained using Smagorinsky model of classical LES**



SUBGRID-SCALE MODELS

- Typical Models Are Statistical; *i.e.*, they model velocity correlations
- Seems Difficult (and expensive) to Model Dependent Variables on Sub-Grid Scales
- But There Are (at least) Several Alternatives
  - Use only high-wavenumber terms of local Fourier representation—one of many approaches to low-dimensional models, but generally still too expsive in 3D
  - Use 1-D equations on sub-grid scales—but raises numerous fundamental questions; examples include LEM and ODT of Kerstein and coworkers
  - Employ local (in each large-scale grid cell) formulations based on discrete dynamical systems (DDSs)—the approach to be considered here
    - not too different from shell models—hence, very low dimensional locally, but still contains considerable global information (unlike a shell model)
    - can be constructed directly from governing equations

ALTERNATIVE FORM OF SGS MODEL

- **Model Takes Form**

$$q_i^* = A_i\, M_i$$

  - q_i^* **— i^{th} component of small-scale solution vector**
  - A_i **— amplitude determined from “extended” Kolmogorov theory**
  - M_i **— i^{th} component of discrete dynamical system (DDS) consisting of equations for velocity components and any associated scalars**

Text

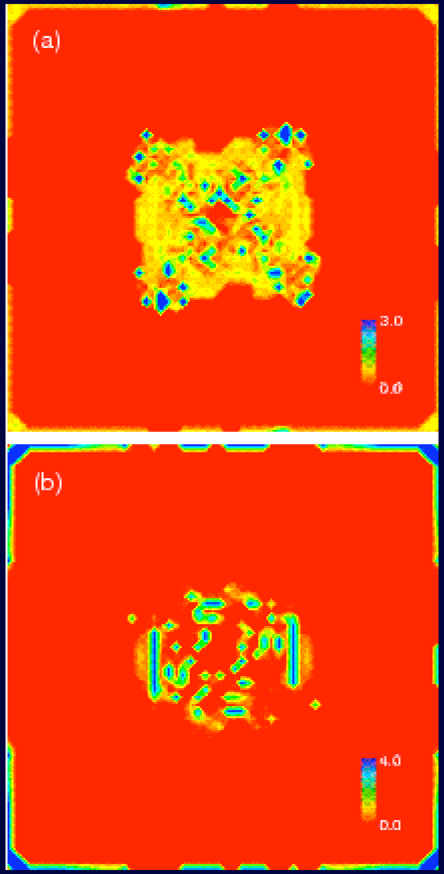
- **Details of Computing A_i s**
  - **Based on <u>local</u> in space and time “on-the-fly” analysis of 2nd-order structure functions for each small-scale variable:**

$$S_{2,i}(r) \equiv \langle [\check{q}_i(\boldsymbol{x}+\boldsymbol{r}) - \check{q}_i(\boldsymbol{x})]^2 \rangle$$

    - $\check{q}_i$ **— high-pass filtered large-scale q_i**
    - $\langle \cdot \rangle$ **— average over all neighboring grid points of x at distance r $(=|\boldsymbol{r}|)$**

CALCULATION OF AMPLITUDES (Cont.)

- Extension of these formulas to case of anisotropic turbulence achieved by replacing 2/3 exponent with arbitrary (but computed) one, and similarly for the Kolmogorov constant
  - results in general power laws: $S_{2,i}(r) = C_{2,i}(\langle \varepsilon \rangle r)^{\beta}$
  - exponent and constant computed locally in space and time
  - figures show snapshot (in time) of one horizontal plane of LES of fire whirl: (a) exponent, (b) constant
  - note that exponent would be ~2/3 (yellow) in regions of flow associated with the inertial subrange



- Finally, observe that amplitude A_i is just $\| q_i^* \| \Rightarrow$ by Parseval's identity

$$A_i = \Big(\sum_k E(k) \Big)^{1/2}$$

THE DISCRETE DYNAMICAL SYSTEMS

- **Details of Computing M_i s**
  - M_i s provide small-scale temporal fluctuations
  - Obtained starting with "projection-method form" of N.–S. equations:

$$U_t + U \cdot \nabla U = \nu \Delta U$$

where, in 2-D, $U = (u,v)^T$

  - Assume Fourier representations for velocity components (and also for all scalars), *e.g.*,

$$u^*(\boldsymbol{x},t) = \sum_{k=-\infty}^{\infty} a_k(t)\, \varphi_k(\boldsymbol{x}) \qquad v^*(\boldsymbol{x},t) = \sum_{k=-\infty}^{\infty} a_k(t)\, \varphi_k(\boldsymbol{x})$$

where $\{\varphi_k\}$ is orthonormal with properties analogous to complex exponentials *wrt* differentiation

DISCRETE DYNAMICAL SYSTEMS (Cont.)

- Apply Galerkin procedure, and decimate result to single wavevector in each equation

$$\dot{a}_k + A^{(1)}_{k\ell m} a_k^2 + A^{(2)}_{k\ell m} a_k b_k = -\frac{|k|^2}{Re} a_k$$

$$\dot{b}_k + B^{(1)}_{k\ell m} b_k^2 + B^{(2)}_{k\ell m} a_k b_k = -\frac{|k|^2}{Re} b_k$$

  - $A^{(i)}_{k\ell m}$ *, etc.* contain Galerkin triple products and wavevector components associated with Fourier representation of velocity gradients
  - wavevector k selected to correspond to Taylor microscale

- Use simple forward-Euler time integration with time step τ, and suppress subscripts

$$a^{(n+1)} = a^{(n)} - \tau \left[\frac{|k|^2}{Re} a^{(n)} + A^{(1)} \left(a^{(n)}\right)^2 + A^{(2)} a^{(n)} b^{(n)} \right]$$

$$b^{(n+1)} = b^{(n)} - \tau \left[\frac{|k|^2}{Re} b^{(n)} + B^{(1)} \left(b^{(n)}\right)^2 + B^{(2)} a^{(n)} b^{(n)} \right]$$

DISCRETE DYNAMICAL SYSTEMS (Cont.)

- Motivated by Frisch's epithet "poor man's Navier–Stokes" equation when referring to a generic quadratic map, and the fact that any such map can be transformed to the well-understood logistic map, we transform each of the above by setting

$$\tau A^{(1)} = 1 - \tau \frac{|k|^2}{Re}$$

and defining

$$\beta_1 \equiv 4\left(1 - \tau \frac{|k|^2}{Re}\right) \quad \textbf{and} \quad \gamma_1 \equiv \tau A^{(2)}$$

- With analogous definitions associated with the second of the above equations, we obtain the **2-D poor man's N.–S. (PMNS) equations:**

$$a^{(n+1)} = \beta_1 a^{(n)}\left(1 - a^{(n)}\right) - \gamma_1 a^{(n)} b^{(n)}$$

$$b^{(n+1)} = \beta_2 b^{(n)}\left(1 - b^{(n)}\right) - \gamma_2 a^{(n)} b^{(n)}$$

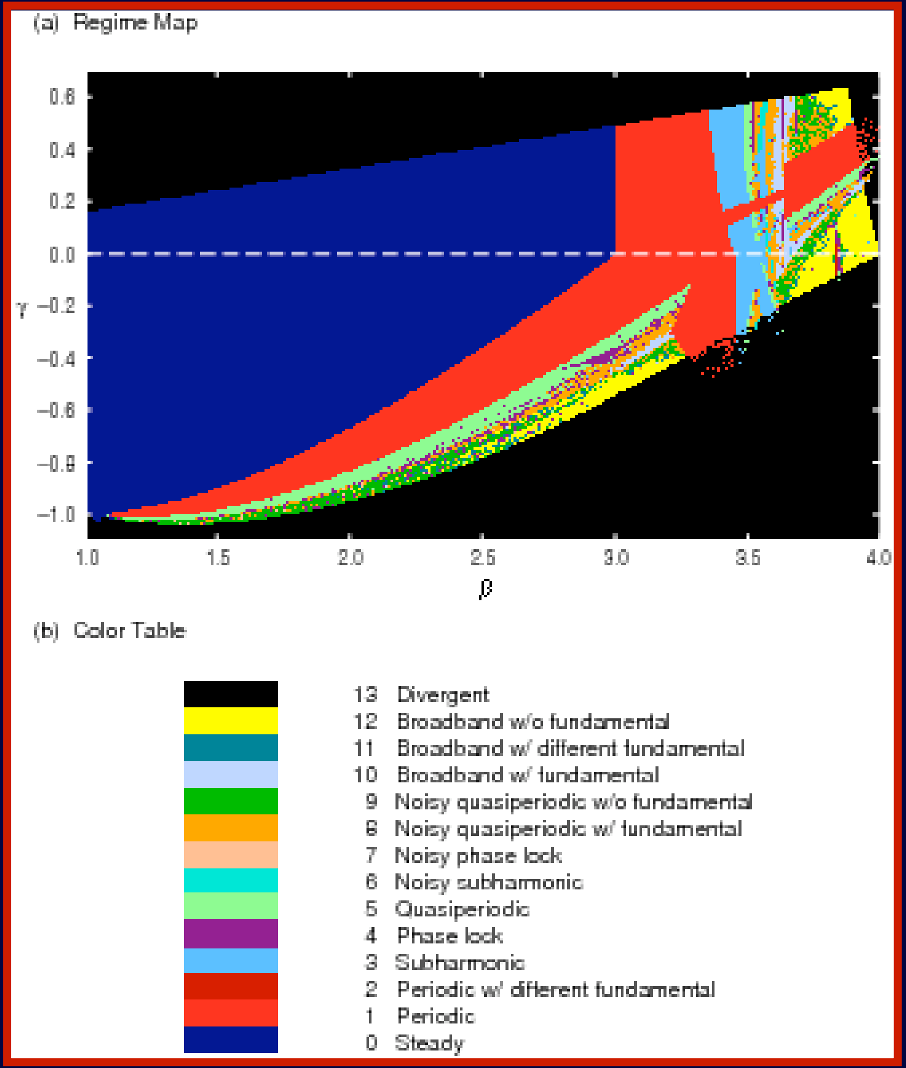
PMNS EQUATIONS

- **2-D PMNS equations studied extensively**
  - McDonough & Huang, *Int. J. Numer. Meth. Fluids* (2004)
  - McDonough *et al.*, *J. Turbulence* (2003)
  - Bible & McDonough, *Int. J. Bifurcation and Chaos* (2004)
- **14 possible states identified, as indicated in figure at right**
- **Figure also displays potential of PMNS eqs. undergoing many different bifurcation sequences including the well-known ones**

Steady → Periodic → Quasiperiodic → Chaotic

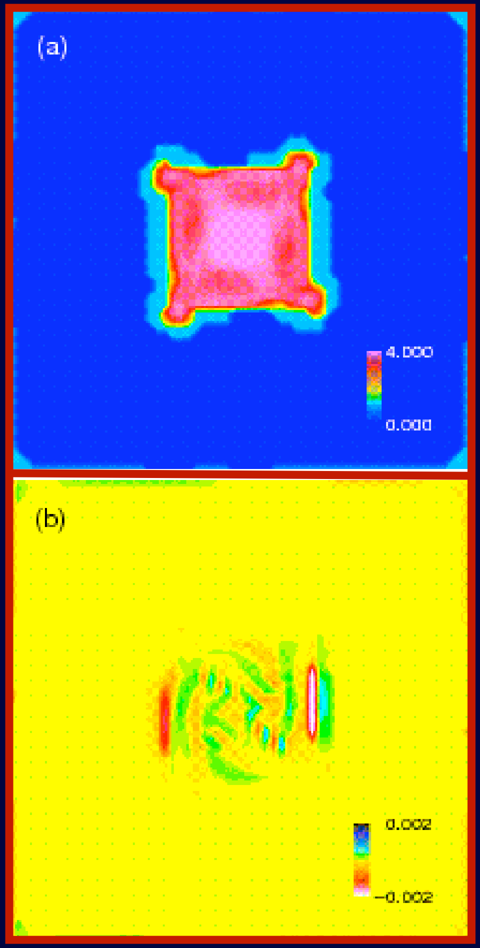
Steady → Periodic → Subharmonic → Chaotic

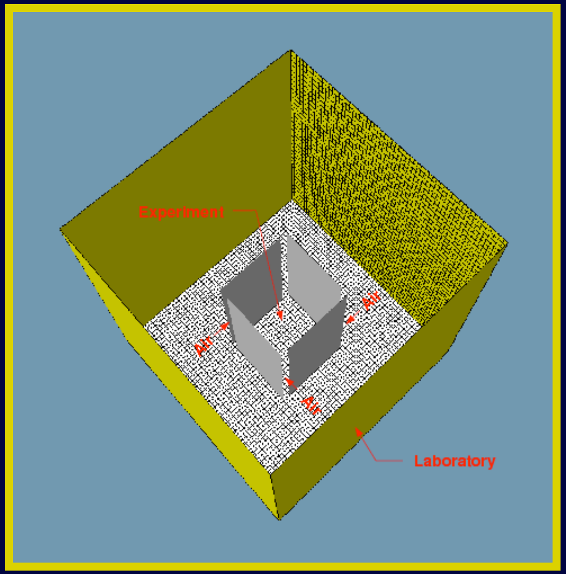
Steady → Periodic → Intermittent → Chaotic



PMNS EQUATIONS (Cont.)

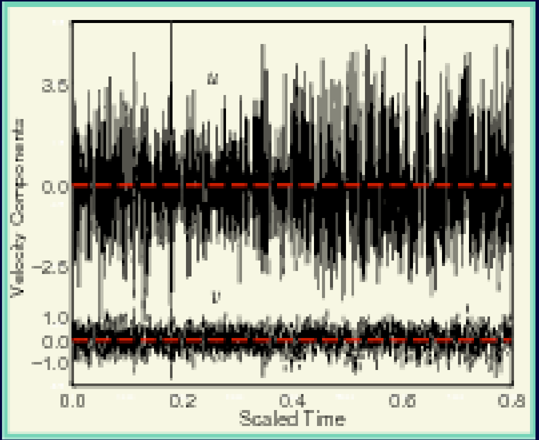
- observe that β_i s are directly related to Re, a wave-vector and a time scale—all of which are readily computed
- figures to right display contours of β (part a) and γ (part b) in a single horizontal plane of a 3-D fire-whirl simulation
- schematic of computational domain shown below

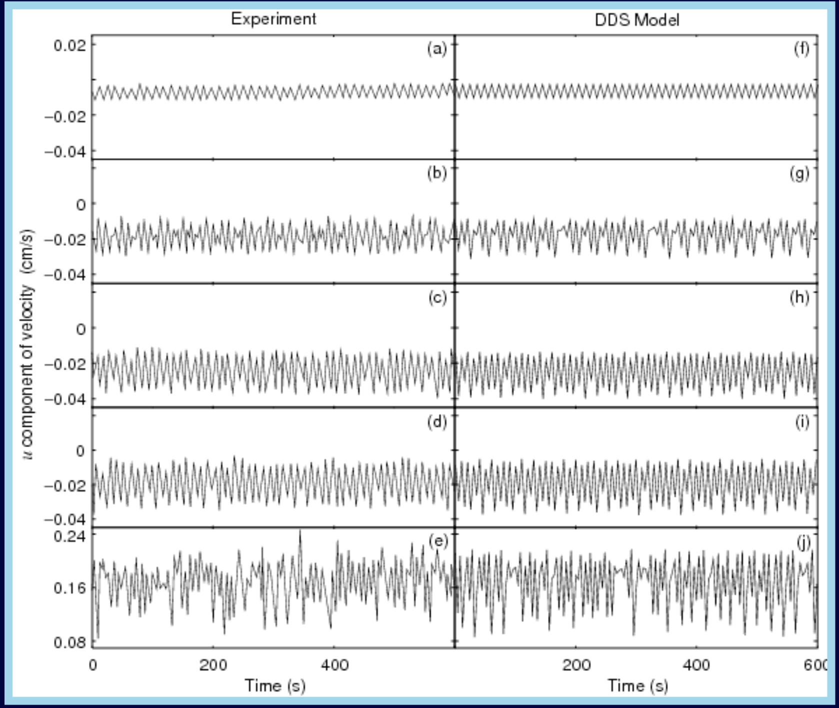




PMNS EQUATIONS (Cont.)

- **PMNS equations also have been fit to experimental data by** Yang *et al., AIAA J.* (2003) **using fractal least-squares algorithm first introduced by** McDonough *et al., Appl. Math. Comput.* (1998)
  - **corresponds to flow behind backward-facing step**





- **Treatment extended to thermal convection (both free and forced) by** McDonough & Joyce (2002) **and by** McDonough *et al.* (2005) **in preparation**
  - **experiments due to** Gollub & Benson, *JFM* (1980)
  - **DDS results obtained with only <u>single</u> realization of PMNS + thermal energy equations**

MULTI-LEVEL FORMALISM

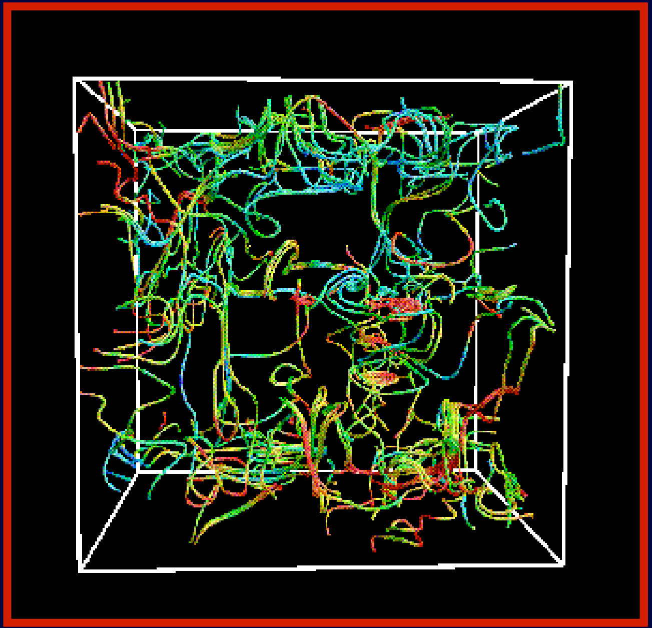
- LES Intrinsically Multilevel—but not usually implemented to reflect this
- Consider LES Decomposition of Dependent Variables:

$$Q(x,t) = \overline{q}(x,t) + q'(x,t)$$

  - Viewed as consisting of “large-” and “small-scale” parts
  - But usually only effects of small scale are modeled—not the small-scale variables themselves
- Synthetic-Velocity Models Are the Exception
  - In implementing these we may—or, may not—be faced with now recognized mathematical difficulties of correctly combining large- and small-scale parts
  - Approach reported here imposes requirement of numerical consistency of complete algorithm with differential equations being solved

SAMPLE CALCULATIONS

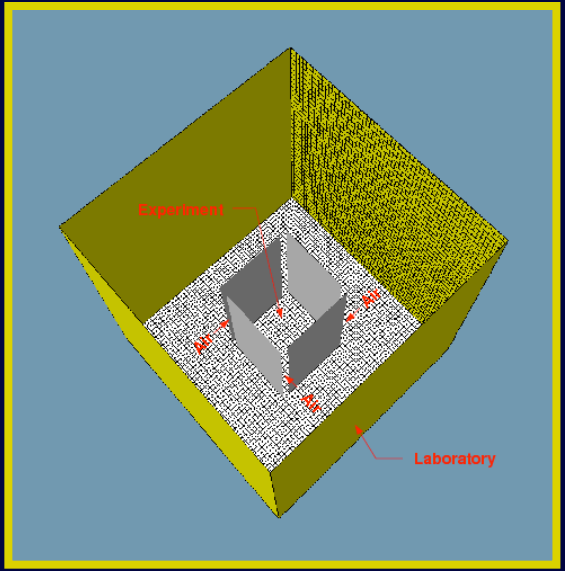
- **Several Standard Turbulence Test Problems Have Been Run (but have not been completely analyzed)**
  - **Decay of isotropic turbulence in $2\pi^3$ periodic box**
    - initial Taylor microscale Reynolds number $Re_\lambda \sim 150$
    - result computed on 16^3 grid
    - required only minutes to obtain energy spectrum close to that of Huang & Leonard (1997) computed on 256^3 grid
    - colors represent small-scale fluctuating vorticity (red ~ high positive, blue ~ high negative)
    - vortices smaller than grid scale created by SGS model



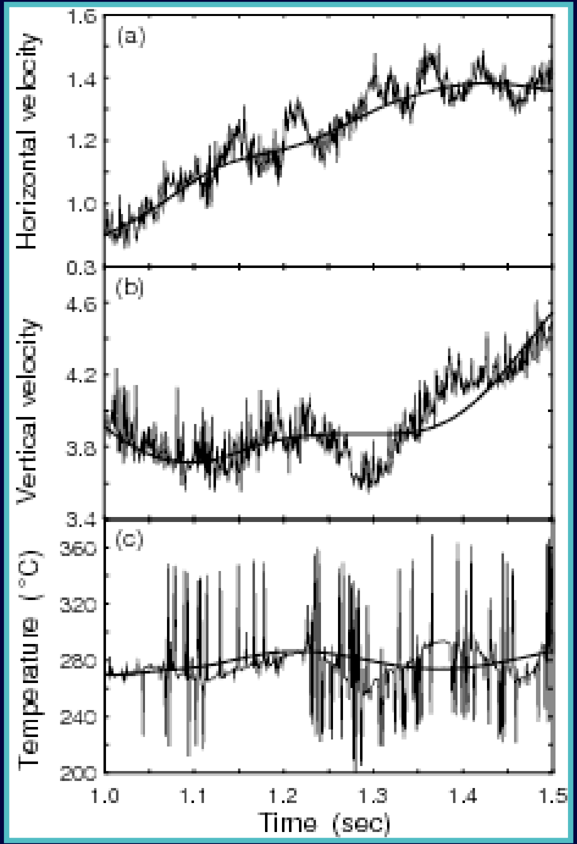
  - **3-D channel flow with four solid walls, periodicity in streamwise direction**
  - **3-D backward-facing step flow with no periodic directions**

SAMPLE CALCULATIONS (Cont.)

- Two Physical Problems Simulated to Date—both involving natural convection in addition to flow field
  - Simulation of laboratory fire whirl experiment
    - computational domain shown earlier consists of laboratory 3 m on a side and 3 m high with 1 $m^2 \times$ 1.6 m high fire-whirl generating apparatus having 0.1 m slots in each corner

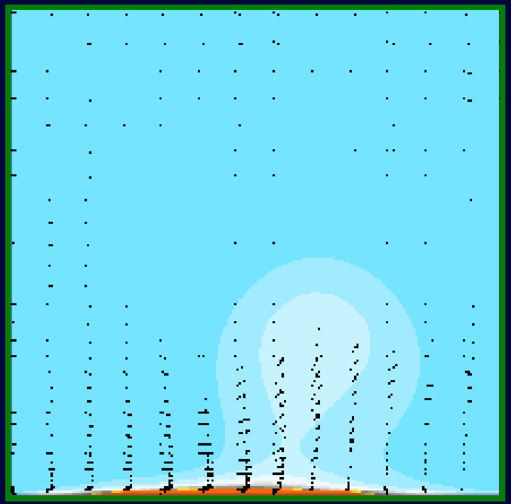


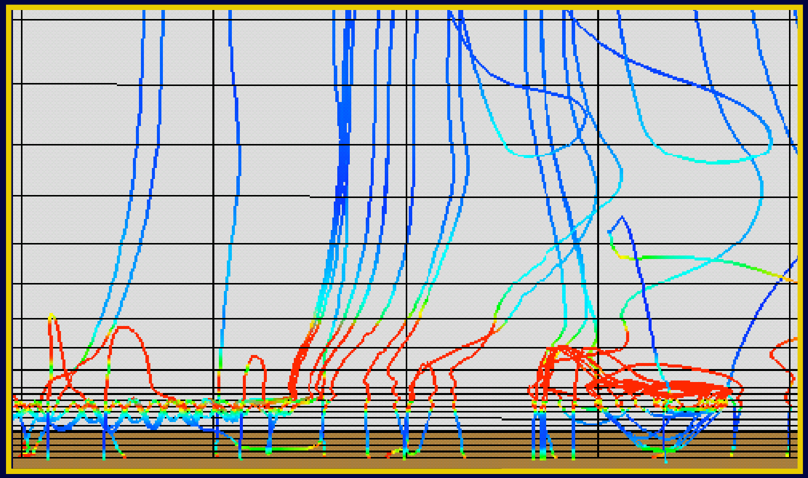
- grid employed was rather coarse 61 × 61 × 41
- figure at right displays resolved-scale and fluctuating time series of two velocity components and temperature



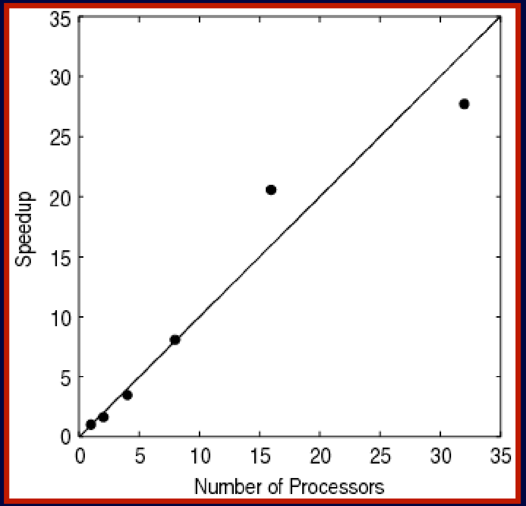
SAMPLE CALCULATIONS (Cont.)

- Simulated **Forest Fire Spread**
  - 7 km x 1.5 km x 2 km (vertical) with approximately 50 m x 50 m horizontal grid spacing
  - Side view of fire after 10 min ➡





  - Streamlines of fluctuating velocity field colored with small-scale vorticity magnitude



  - Parallel speedup ➡

FOREST FIRE SIMULATION (Cont.)

- Zoomed View of Fire Spread Showing More Details of Temperature Fluctuations and Fire Whirls

SUMMARY

- Slightly Different LES Alternative Presented
  - Based on filtering solutions, modeling physical variables and explicitly combining large- and small-scale parts
  - Each part of process has sound mathematical basis—but connecting all the pieces is nontrivial
- Preliminary *a priori* tests suggest SGS model generally works well
  - Has successfully identified turbulent and non-turbulent flow regions in quite complicated flow fields including buoyancy effects
  - Able to generate small-scale structures below grid scale—and do this very efficiently with no adjustable parameters except those used in filter
- Open Questions Remain Concerning Coupling of Large- and Small-Scale Parts
- Results for Fundamental Problems Have Yet to Be Thoroughly Analyzed—so they are, in a sense, still anecdotal