SUPPLEMENTARY INFORMATION

S-A

As mentioned in the paper, the transition from a state of phase locking to state of phase drifting happens through intermittent phase locking or imperfect phase locking with arbitrary phase slips (Figs. 3b-I, 4b-I, and 4d-I in the main article). These phase slips happen due to the phase jump observed in one of the signals. The instantaneous phase of p' and $\dot{q}'$ are shown along with the relative phase in Fig. S1. A linear fit curve is subtracted from the unwrapped instantaneous phase (in Fig. S1a and S1c) so as to view the fluctuations in relative phase in bounded region. Figs. S1a and S1b correspond to the flame location (x_f) 15.1 cm (corresponding to the results shown in Fig. 4a in the main article) and Figs. S1c and S1d correspond to the flame location (x_f) 15.3 cm (corresponding to the results shown in Fig. 4b in the main article).

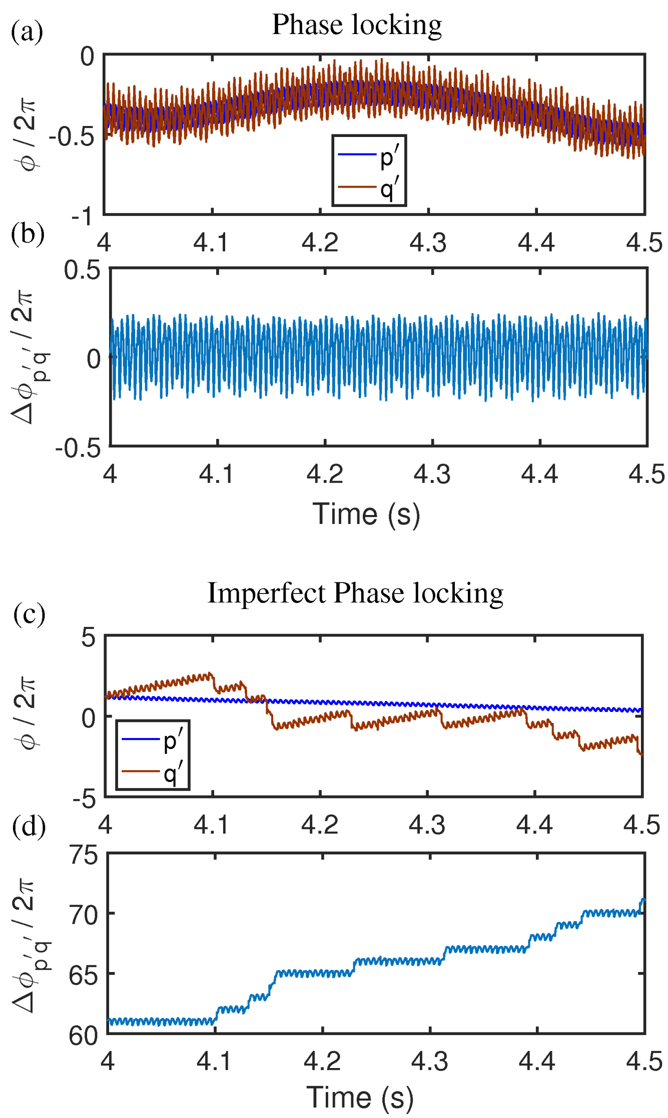

FIG. S1. (a) Instantaneous phase of p' and $\dot{q}'$ and (b) relative phase between them during phase locking (x_f = 15.1 cm). (c) Instantaneous phase of p' and $\dot{q}'$ and (d) relative phase between them during imperfect phase locking (x_f = 15.3 cm). Occasional phase jump in $\dot{q}'$ (c) causes to the slips in relative phase, making the relative phase unbounded.

During phase locking, instantaneous phases of p' and $\dot{q}'$ do not show any departure from each other which, in turn, results in bounded relative phase. On the other hand, during imperfect phase locking, occasional phase jumps are observed in any one of the signals resulting in slips in relative phase. At $x_f = 15.3$ cm, jumps are observed in $\dot{q}'$ (Figs. S1c) whereas, at $x_f = 16.1$ cm, jumps are observed in p' (not shown).

S-B

During quasiperiodic oscillations, PLV shows lower value in phase locking region compared to that in periodic oscillations (Fig. 5a in main article). PLV also shows a decrease in the phase locking region (Fig. 5a-I in the main article). Difference in PLV in the state of phase locking (precisely, phase trapping) is due to the variation in amplitude of the bounded oscillations of the relative phase. Time series of relative phase between p' and $\dot{q}'$ along with corresponding phase locking values are shown for different x_f during phase locking state in Fig. S2. From Fig. S2, it can be concluded that PLV decreases as the amplitude of relative phase oscillations increases.

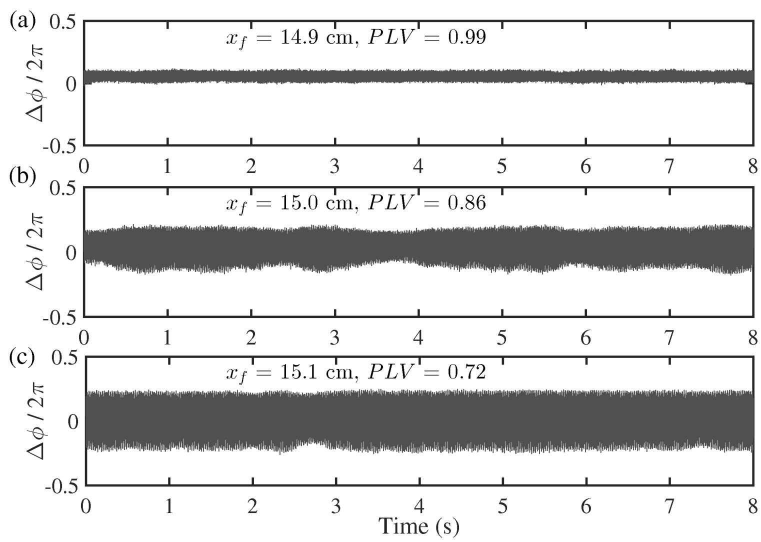

FIG. S2. Time series of relative phase at different flame locations (x_f) exhibiting phase locking between p' and $\dot{q}'$. (a) During periodic oscillations. (b) and (c) during quasiperiodic oscillations. Corresponding phase locking values (PLV) show that PLV decreases as the amplitude of relative phase oscillations increases.

S-C

In the analysis of synchronization, frequency locking and phase locking appear hand in hand. For signals with single frequency, frequency locking straightaway indicates phase locking. However, for more complex signals with many frequencies, although locking in mean frequency ($\langle d\phi/dt\rangle$) indicates phase locking, locking in dominant frequency does not.

Here, we are interested in quasiperiodic signals having two dominant frequencies. We observe different phase dynamics between the pressure oscillations (p') and the heat release rate oscillations ($\dot{q}'$) even when two dominant frequencies are completely locked (Fig. 4-I in the main article). The FFT plots for individual signals along with the relative phase are shown in Fig. S6.

Now, the question which arises from the experimental observations is: does locking in both the frequencies of such quasiperiodic signals always mean phase locking? In other words, if both frequencies of two quasiperiodic signals are same, will the dynamics of relative phase between them be always bounded?

As a probable cause, we first check the relative phase of two time lagged quasiperiodic signals. As, in the course of experimentation we change the flame locations in the acoustic field keeping the location of pressure measurement fixed, there might be a constant time lag between p' and $\dot{q}'$ signals. Intuitively, a time lag should lead to a constant shift in relative phase with a complete phase locking. To check this with the quasiperiodic signals, we generate signals with frequency ratio being golden mean ratio ($\omega = (\sqrt{5}-1)/2$) as

$$y(t) = \cos(2\pi f t) + a * \cos(2\pi f \omega t) \tag{1}$$

We check the dynamics of the relative phase between $y_1(t) = y(t)$ and $y_2(t) = y(t+\tau)$, where τ is, in effect, a time lag between y_1 and y_2. Two overlapped signals are shown in Fig. S3c and amplitude spectrum of $y_1(t)$ and $y_2(t)$ are in Fig. S3a and S3b. Surprisingly, the time lag does not lead to any constant phase lag, however the relative phase becomes oscillatory (Figure S3d).

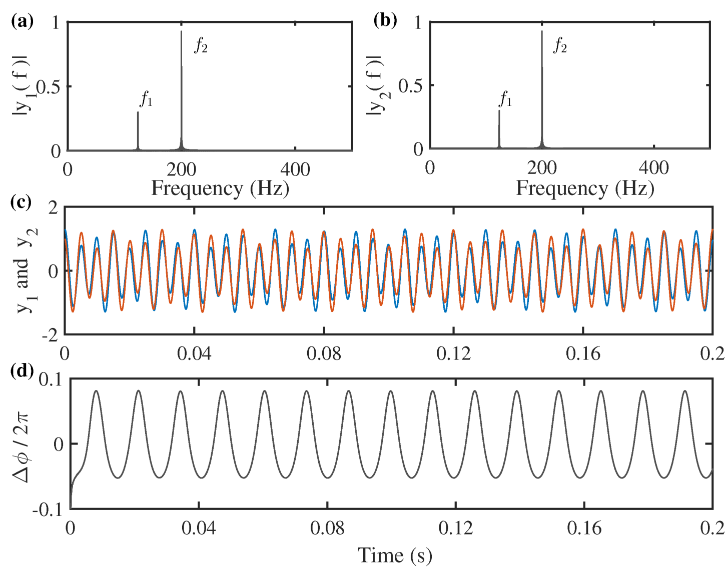

FIG. S3. (a) and (b) FFT plots of two time delayed quasiperiodic signals ($f_1 = 123.6$ Hz and $f_2 = 200$ Hz). (c) Overlapped time series of two such quasiperiodic signals and (d) relative phase between them which is oscillatory and bounded, characteristics of phase trapping.

Relative phase ($\Delta\phi$) being bounded shows a periodic oscillations, known as phase trapping [1]. Therefore, an initial phase lag can lead to phase trapping for quasi-periodic signals when the spectral contents (height of the FFT peaks) of two different frequencies are not same (implemented setting $a = 0.3$). For two quasiperiodic signals with frequencies of equal spectral content ($a = 1$), it is trivial to show analytically that $\Delta\phi$ is a constant.

Let us say, $\phi_1(t)$ is the phase of a general function $u_1(t)$. Now, if we define $u_2(t)$ as $u_1(t-\tau_0)$, then $\phi_2(t)$ will be equal to $\phi_1(t-\tau_0)$. Therefore, the relative phase ($\Delta\phi = \phi_1(t) - \phi_2(t)$) will be constant iff ϕ is a linear function of t (for $\tau_0 \neq 0$).

For a simple quasiperiodic signal such as $y(t) = \cos(2\pi f t) + \cos(2\pi f \omega t)$, phase, ϕ becomes $(1+\omega)t/2$ which is a linear function of t. Therefore, constant phase shift is expected from two time-delayed signals. In short, even a time lag leads to oscillatory relative phase, it remains unbounded. The question remains: what is the reason for the relative phase to be unbounded even when the frequencies are locked?

We examined the spectral content of two incommensurate frequencies. To that end, we generate two quasiperiodic signal having similar frequencies with different amplitudes as

$$\begin{aligned} y_1(t) &= \cos(2\pi f t) + a * \cos(2\pi f \omega t) \\ y_2(t) &= \cos(2\pi f t) + b * \cos(2\pi f \omega t) \end{aligned} \tag{2}$$

We set the values of a and b as 0.3 and 0.7 respectively so as to keep the dominant frequencies for both the signals similar (Fig. S4a and b). FFT plots, time series and relative phase are shown in Fig. S4. Relative phase shows phase trapping (Fig. S4d) even when the spectral contents are different for two signals.

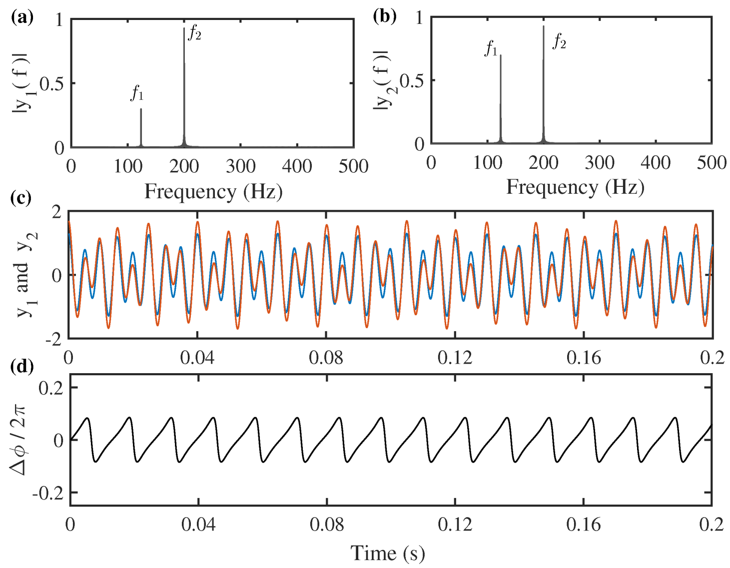

FIG. S4. (a) and (b) FFT plots of two quasiperiodic signals ($f_1 = 123.6$ Hz and $f_2 = 200$ Hz) with different spectral contents (height of the peaks). (c) Overlapped time series of two such quasiperiodic signals and (d) relative phase between them which shows phase trapping.

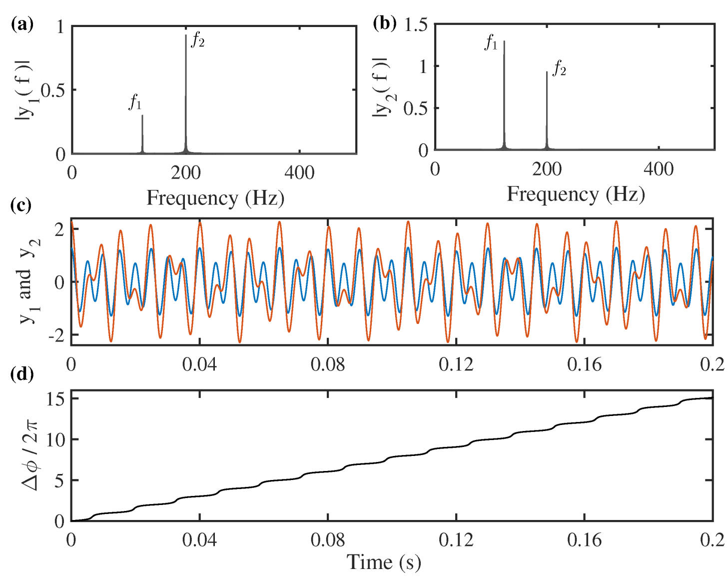

FIG. S5. (a) and (b) FFT plots of two quasiperiodic signals ($f_1 = 123.6$ Hz and $f_2 = 200$ Hz) where dominant frequency is different. (c) Overlapped time series of two such quasiperiodic signals and (d) relative phase between them which shows phase drfting.

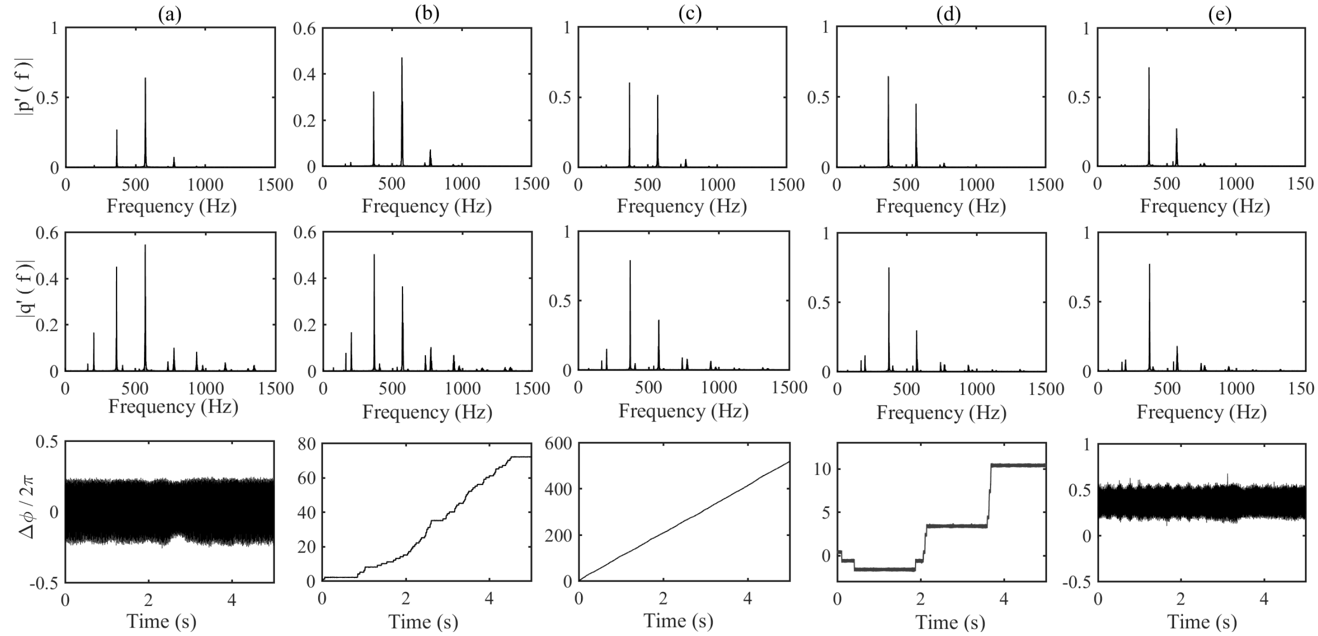

FIG. S6. FFT plots of p' (1st roq) and $\dot{q}'$ (2nd row) and relative phase (3rd row) between them are shown for flmae location (x_f) 15.1 cm (a), 15.3 cm (b), 15.6 cm (c), 16.1 cm (d), 16.3 cm (e)

Next, we swap the dominant frequency keeping the frequencies same by properly choosing the coefficients of sinusoids in eq. 2. To that end, we set the values of a (0.3) and b (1.3) in such a way that dominant frequency is not the same anymore for the signals chosen (Fig. S5a and b). For such signals, FFT plots, time series and relative phase are shown in Fig. S5. The relative phase of such signals becomes unbounded (Fig. S5d). The fact that the frequency with higher spectral content repeats more in time causes to differ the mean frequencies of the signals. Therefore, two frequency locked quasiperiodic signals with different dominant frequency can lead to a phase drift.

To summarize, having different dominant frequency can be one of the causes for two quasiperiodic signals which have equal independent frequencies to be phase drifted. However, in experimental data, there are more than two frequency components having comparably small spectral content. FFT plots and relative phase between p' and $\dot{q}'$ signals are shown in Fig. S6 (corresponding to the results shown in Fig. 4 in the main article). In experimental data, change in dominant frequency is observed at flame location (x_f) 15.3 cm (Fig. S6b) where the relative phase is observed to be unbounded. However, at other flame locations (Fig. S6c), phase drifting is observed even though the signals have same dominant frequency. There are a lot more frequencies in both signals with low spectral content which might effectively change the dynamics in relative phase. We conjecture here that this could be one of the many causes that lead to unbounded phase for two frequency locked quasiperiodic signals.

[1] J. Thévenin, M. Romanelli, M. Vallet, M. Brunel, and T. Erneux, Physical review letters **107**, 104101 (2011).