A First Course in Elementary Differential Equations: Problems and Solutions

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24   The Method of Variation of Parameters

Problem 24.1
Solve $y'' + y = \sec t$ by variation of parameters.

Solution.
The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and

$$y_h(t) = c_1 \cos t + c_2 \sin t$$

Also, $y_1(t) = \cos t$ and $y_2(t) = \sin t$ so that $W(t) = \cos^2 t + \sin^2 t = 1$. Now,

$$u_1 = -\int \sin t \sec t \, dt = \int \frac{d(\cos t)}{\cos t} = \ln|\cos t|$$

and

$$u_2 = \int \cos t \sec t \, dt = \int dt = t$$

Hence, the particular solution is given by

$$y_p(t) = \ln|\cos t| \cos t + t \sin t$$

and the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + \ln|\cos t| \cos t + t \sin t \ \blacksquare$$

Problem 24.2
Solve $y'' - y = e^t$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution.
The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots $r = \pm 1$. The homogeneous solution is

$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

Undetermined Coefficients Summary. The basic trial solution method gives initial trial solution $y_p(t) = d_1 t e^t$ since 1 is a root of the characteristic equation. Substitution into $y'' - y = e^t$ gives $2d_1 e^t + d_1 t e^t - d_1 t e^t = e^t$. Cancel e^t and equate coefficients of like powers of t to find $d_1 = 1/2$. Then $y_p = \frac{t e^t}{2}$.
Variation of Parameters Summary. The homogeneous solution $y_h(t) =$

$c_1 e^t + c_2 e^{-t}$ found above implies $y_1 = e^t$, $y_2 = e^{-t}$ is a suitable independent pair of solutions. Their Wronskian is $W = -2$. The variation of parameters formula applies:

$$y_p(t) = e^t \int \frac{e^{-t}}{2} e^t dt - e^{-t} \int \frac{e^t}{2} e^t dt$$

Integration, followed by setting all constants of integration to zero, gives $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$.

Differences. The two methods give respectively $y_p(t) = \frac{te^t}{2}$ and $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$. The solutions $y_p(t) = \frac{te^t}{2}$ and $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$ differ by the homogeneous solution $-\frac{e^t}{4}$. In both cases, the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2} te^t$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2 ∎

Problem 24.3
Solve the following 2nd order equation using the variation of parameters method:

$$y'' + 4y = t^2 + 8 \cos 2t.$$

Solution.
The characterisitc equation $r^2 + 4 = 0$ has roots $r = \pm 2i$ so that $y_h(t) = c_1 \cos 2t + c_2 \sin 2t$. Hence, $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$, and $W(t) = 2$. Thus,

$$\begin{aligned}
y_p &= -\cos 2t \int \frac{\sin 2t(t^2 + 8\cos 2t)}{2} dt + \sin 2t \int \frac{\cos 2t(t^2 + 8\cos 2t)}{2} dt \\
&= -\cos 2t(\frac{1}{4}t\sin 2t + \frac{1}{8}\cos 2t - \frac{1}{4}t^2 \cos 2t - \cos^2 2t) \\
&\quad + \sin 2t(\frac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t + \frac{1}{4}t^2 \sin 2t + 2t + \frac{1}{2}\sin 4t) \\
&= -\frac{1}{8} + \frac{1}{4}t^2 + \cos^2 2t \cos 2t + 2t\sin 2t + \frac{1}{2}\sin 4t \sin 2t
\end{aligned}$$

The general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + 2t\sin 2t \quad ∎$$

Problem 24.4

Find a particular solution by the variation of parameters to the equation

$$y'' + 2y' + y = e^{-t} \ln t.$$

Solution.

The characteristic equation

$$r^2 + 2r + 1 = 0$$

has roots $r_1 = r_2 = -1$, so the fundamental solutions of the reduced equation are

$$y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}$$

Compute the Wronskian.

$$W(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}$$

$$= e^{-t}(e^{-t} - te^{-t}) + e^{-t} \cdot te^{-t}$$

$$= e^{-2t} - te^{-2t} + te^{-2t}$$

$$= e^{-2t}$$

Compute $u_1(t)$.

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)} dt$$

$$= -\int \frac{te^{-t} \cdot e^{-t} \ln t}{e^{-2t}} dt$$

$$= -\int t \ln t \, dt = -\frac{t^2}{2} \ln t + \int \frac{t^2}{2} \cdot \frac{1}{t} dt$$

$$= -\frac{t^2}{2} \ln t + \frac{t^2}{4}$$

Compute $u_2(t)$.

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} dt$$

$$= \int \frac{e^{-t} \cdot e^{-t} \ln t}{e^{-2t}} dt$$

$$= \int \ln t \, dt = t \ln t - \int t \cdot \frac{1}{t} dt$$

$$= t \ln t - t$$

Note. We used integration by parts to compute the integrals $\int t \ln t \, dt$ and $\int \ln t \, dt$.

The particular solution to our complete equation is

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$= \left(-\frac{t^2}{2} \ln t + \frac{t^2}{4} \right) e^{-t} + (t \ln t - t)te^{-t}$$

$$= \frac{t^2}{2} \ln te^{-t} - \frac{3t^2}{4}e^{-t}$$

$$= (\frac{t^2}{2} \ln t - \frac{3t^2}{4})e^{-t} \quad \blacksquare$$

Problem 24.5

Solve the following initial value problem by using variation of parameters:

$$y'' + 2y' - 3y = te^t, \quad y(0) = -\frac{1}{64}, \quad y'(0) = \frac{59}{64}.$$

Solution.

From the characteristic equation, we obtain $y_1(t) = e^t, y_2(t) = e^{-3t}$ and $W(t) = -4e^{-2t}$. Integration then yields

$$u_1(t) = -\int \frac{e^{-3t}te^t}{-4e^{-2t}} dt = \frac{t^2}{8}$$

$$u_2(t) = \int \frac{e^t te^t}{-4e^{-2t}} dt = -\frac{1}{16}te^{4t} + \frac{e^{4t}}{64}$$

Thus. $y_p(t) = \frac{e^t}{64}(8t^2 - 4t + 1)$ and the general solution is

$$y(t) = c_1 e^t + c_2 e^{-3t} + \frac{t^2}{8}e^t - \frac{1}{16}te^t$$

Initial conditions:

$$y(0) = c_1 + c_2 = -\frac{1}{64}$$

$$y'(0) = c_1 - 3c_2 - \frac{4}{64} = \frac{59}{64}$$

These are satisfied by $c_1 = \frac{15}{64}$ and $c_2 = -\frac{1}{4}$. Finally the solution to the initial value problem is

$$y = \frac{e^t}{64}(8t^2 - 4t + 15) - \frac{1}{4}e^{-3t} \quad \blacksquare$$

Problem 24.6
(a) Verify that $\{e^{\sqrt{t}}, e^{-\sqrt{t}}\}$ is a fundamental set for the equation

$$4ty'' + 2y' - y = 0$$

on the interval $(0, \infty)$. You may assume that the given functions are solutions to the equation.
(b) Use the method of variation of parameters to find one solution to the equation

$$4ty'' + 2y' - y = 4\sqrt{t}e^{\sqrt{t}}.$$

Solution.
(a) Usually the first thing to do would be to check that $y_1(t) = e^{\sqrt{t}}$ and $y_2(t) = e^{-\sqrt{t}}$ really are solutions to the equation. However, the question says that this can be assumed and so we move on to the next step, which is to check that the Wronskian of the two solutions is non-zero on $(0, \infty)$. We have

$$y_1' = \frac{1}{2\sqrt{t}}e^{\sqrt{t}} \text{ and } y_2' = -\frac{1}{2\sqrt{t}}e^{-\sqrt{t}}$$

and so

$$W(t) = y_1y_2' - y_1'y_2 = -\frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{t}} = -\frac{1}{\sqrt{t}}$$

This is indeed non-zero and so $\{e^{\sqrt{t}}, e^{-\sqrt{t}}\}$ is a fundamental set for the homogeneous equation.
(b) The variation of parameters formula says that

$$y = -y_1 \int \frac{y_2 g}{W(t)}dt + y_2 \int \frac{y_1 g}{W(t)}dt$$

is a solution to the nonhomogeneous equation in the form $y'' + py' + qy = g$. To get the right g, we have to divide the equation through by $4t$ and so $g = \frac{1}{\sqrt{t}}e^{\sqrt{t}}$. Thus

$$y = - e^{\sqrt{t}} \int \frac{e^{-\sqrt{t}}(\frac{1}{\sqrt{t}})e^{\sqrt{t}}}{-1/\sqrt{t}}dt + e^{-\sqrt{t}} \int \frac{e^{\sqrt{t}}(\frac{1}{\sqrt{t}})e^{\sqrt{t}}}{-1/\sqrt{t}}dt$$

$$= e^{\sqrt{t}} \int dt - e^{-\sqrt{t}} \int e^{2\sqrt{t}}dt$$

$$= te^{\sqrt{t}} - e^{-\sqrt{t}} \int e^{2\sqrt{t}}dt$$

To evaluate the integral, we substitute $u = 2\sqrt{t}$ so that $dt = \frac{1}{2}u du$. We get

$$\int e^{2\sqrt{t}} dt = \frac{1}{2}\int u e^u du = \frac{1}{2}(u-1)e^u = (\sqrt{t} - 1/2)e^{2\sqrt{t}}.$$

Thus

$$y = (t - \sqrt{t} + 1/2)e^{\sqrt{t}}$$

is one solution to the equation. You might notice that the $1/2$ can be dropped (because $(1/2)e^{\sqrt{t}}$ is a solution to the homogeneous equation) so that

$$y = (t - \sqrt{t})e^{\sqrt{t}}$$

would also work ∎

Problem 24.7
Use the method of variation of parameters to find the general solution to the equation

$$y'' + y = \sin t.$$

Solution.
The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ so that the solution to the homogeneous equation is $y_h(t) = c_1 \cos t + c_2 \sin t$. The Wronskian is $W(\cos t, \sin t) = 1$. Now $u_1'(t) = -\sin^2 t = \frac{\cos(2t)-1}{2}$. Hence $u_1(t) = \frac{1}{2}(\frac{1}{2}\sin(2t) - t)$. Similarly, $u_2'(t) = \sin t \cos t$. Hence $u_2(t) = \frac{1}{2}\sin^2 t$. So $y_p(t) = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$. The general solution is given by

$$y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2}t\cos t \ \blacksquare$$

Problem 24.8
Consider the differential equation

$$t^2 y'' + 3ty' - 3y = 0, \ t > 0.$$

(a) Determine r so that $y = t^r$ is a solution.
(b) Use (a) to find a fundamental set of solutions.
(c) Use the method of variation of parameters for finding a particular solution to

$$t^2 y'' + 3ty' - 3y = \frac{1}{t^3}, \ t > 0.$$

Solution.
(a) Inserting y, y', and y'' into the equation we find $r^2 + 2r - 3 = 0$. Solving for r to obtain $r_1 = 1$ and $r_2 = -3$.
(b) Let $y_1(t) = t$ and $y_2(t) = t^{-3}$. Since

$$W(t) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

$\{y_1, y_2\}$ is a fundamental set of solutions for $t > 0$.
(c) Recall that the variation of parameters formula states that if y_1 and y_2 form a fundamental solution set for $y'' + p(t)y' + q(t)y = 0$, then $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ is a particular solution to the equation $y'' + p(t)y' + q(t)y = g(t)$, where

$$u_1(t) = -\int \frac{t^{-3}t^{-5}}{-4t^{-3}} dt = -\frac{1}{16}t^{-4}$$

$$u_2(t) = \int \frac{t \cdot t^{-5}}{-4t^{-3}} dt = -\frac{1}{4}\ln t$$

Thus,

$$y_p(t) = -\frac{1}{16}t^{-3} - \frac{1}{4}t^{-3}\ln t \blacksquare$$

Problem 24.9
Use the method of variation of parameters to find the general solution to the differential equations

$$y'' + y = \sin^2 t.$$

Solution.
The characterisitc equation $r^2 + 1 = 0$ has roots $r = \pm i$ so that $y_1(t) = \cos t$, $y_2(t) = \sin t$, and $W(t) = 1$. Hence,

$$u_1(t) = -\int \sin t \sin^2 t\, dt = \int (1 - \cos^2 t)d(\cos t) = \cos t - \frac{1}{3}\cos^3 t$$

$$u_2(t) = \int \cos t \sin^2 t\, dt = \frac{1}{3}\sin^3 t$$

Thus,

$$y_p(t) = \cos^2 t - \frac{1}{3}\cos^4 t + \frac{1}{3}\sin^4 t$$

and

$$y(t) = c_1 \cos t + c_2 \sin t + \cos^2 t - \frac{1}{3}\cos^4 t + \frac{1}{3}\sin^4 t \blacksquare$$