Chapter 6

Stability Analysis

6.1 Introduction

In 1964, Chandrasekhar [73] first developed a method to study the stability of a star against radial oscillations. Bardeen *et al* [153] presented a variety of methods to analyze the stability of a star against radial oscillations.

In our earlier chapters we have shown that the solution of Mukherjee *et al* [67] can be used to generate the EOS for strange matter as well as quark-diquark mixed state in connection with pulsars SAX J1808.4-3658 and Her X-1 respectively. The crucial issue that needs to be studied in this connection is the stability of these solutions. Some of the earlier workers who investigated radial oscillations in strange stars include Glendenning *et al* [34], Kettner *et al* [120] and Cervillera *et al* [154]. Gleiser [139] investigated the dynamical stability of boson stars against radial pulsations. Negi and Durgapal [155] proved the stability of Tolman's type VII solution using the same technique. Using the method developed by Chandrasekhar [73], Knutsen [156] and Tikekar and Thomas [157] proved that Vaidya-Tikekar model [63] is stable for some specific values of the curvature parameter, e.g., $\lambda = 2$ & 7. The general solution of Einstein's field equations [67] helps us to do stability calculations for any value of the parameter λ. This allows us to extend the stability calculations to realistic stars like SAX J1808.4-3658 or Her X-1. Since we have already shown that for some specific values of λ the model can describe SAX and Her X-1, all we need to do to prove their stability is to show whether the solutions for the relevant values of λ are stable or not. Earlier, Bhowmick *et al* [134] analyzed the stability of SAX J1808.4-3658 and claimed that SAX is a stable star. The calculation presented here permits us to check this claim by using a totally different technique. In this chapter we shall show that the solutions presented in chapter 4 for SAX J1808.4-3658 and also the configuration obtained for Her X-1 in chapter 5, are stable with respect to small radial oscillations.

6.2 Methodology

Let us consider a perturbation of the metric for a spherically symmetric star,

$$ds^2 = -e^{2\gamma(r,t)}dt^2 + e^{2\mu(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.1}$$

where

$$\gamma(r,t) = \gamma_0(r) + \delta\gamma(r,t)$$
$$\mu(r,t) = \mu_0(r) + \delta\mu(r,t).$$

Here γ_0 and μ_0 denote equilibrium configuration, and $\delta\nu(r,t)$ and $\delta\mu(r,t)$ are small perturbations from this configuration. We also perturb the energy density ρ, pressure p and number density n as

$$\rho(r,t) = \rho_0(r) + \delta\rho(r,t)$$
$$p(r,t) = p_0(r) + \delta p(r,t)$$
$$n(r,t) = n_0(r) + \delta n(r,t).$$

The small perturbation in radial parameter r is assumed to be

$$\delta r(r,t) = u_n(r)e^{\gamma_0(r)}e^{-i\omega_n t}/r^2. \tag{6.2}$$

Utilizing energy conservation, baryon number conservation and Einstein's field equations, we get the dynamical equation governing the stellar pulsation in its n-th normal mode ($n = 0$ gives the fundamental mode) which has the Sturm-Liouville's form (the calculation can be found in any standard text book, e.g., [158])

$$P(r)\frac{d^2u_n(r)}{dr^2} + \frac{dP}{dr}\frac{du_n}{dr} + \left[Q(r) + \omega_n^2 W(r)\right] u_n(r) = 0, \tag{6.3}$$

where $u_n(r)$ and ω_n are the amplitude and frequency of the n-th normal mode, respectively. Functions $P(r)$, $Q(r)$ and $W(r)$ are expressed in terms of the equilibrium configuration of the star given by

$$P(r) = \Gamma p_0 r^{-2} e^{\mu_0 + 3\gamma_0} \tag{6.4}$$

$$Q(r) = e^{\mu_0+3\gamma_0}\left[\frac{(p_0')^2}{r^2(\rho_0+p_0)} - \frac{4p_0'}{r^3} - \frac{8\pi}{r^2}(\rho_0+p_0)p_0 e^{2\mu_0}\right] \tag{6.5}$$

$$W = \frac{(\rho_0+p_0)}{r^2} e^{3\mu_0+\gamma_0}. \tag{6.6}$$

For the fundamental mode ($n = 0$), the above equation can be written as

$$\omega_0^2 \int_0^b e^{3\mu_0+\gamma_0}(\rho_0+p_0)\tfrac{u_0^2}{r^2}dr = \int_0^b e^{\mu_0+3\gamma_0}\tfrac{(\rho_0+p_0)}{r^2}\times \left[\left(-\tfrac{4\gamma_0'}{r} - (\gamma_0')^2 + 8\pi p_0 e^{2\mu_0}\right)u_0^2 + \tfrac{dp_0}{d\rho_0}\left(\tfrac{du_0}{dr}\right)^2\right] dr, \tag{6.7}$$

where we substituted the varying adiabatic index Γ, given by

$$\Gamma = \tfrac{(\rho_0+p_0)}{p_0}\tfrac{dp_0}{d\rho_0}, \tag{6.8}$$

where ρ_0 and p_0 denote energy-density and pressure at equilibrium state and are given by equations (2.29) and (2.30) obtained in chapter 2.

The model will be stable if ω is real. Since integration of the left hand side of equation (6.7) is always positive definite, all we need to do is to show that the right hand side of this equation is positive. To integrate the right hand side of equation (6.7) we employ the method 2-D given by Bardeen *et al* [153]. We take a trial solution for u_0 as a polynomial in r, which can be written conveniently as

$$u_0 = R^3(1-kz^2)^{3/2}\left[1 + a_1(1-kz^2) + b_1(1-kz^2)^2\right], \tag{6.9}$$

where, a_1 and b_1 are arbitrary constants, $k = (1+\lambda)/\lambda$, $z = \sqrt{(1 - r^2/R^2)/k}$ and R is the parameter discussed in chapter 2.

At the surface of the star, we impose the condition that the Lagrangian change in pressure vanishes, i.e., $\Delta p \to 0$ as $r \to b$, ($\Delta p \approx \delta p + p_0' \delta r(r,t) = -\frac{1}{r^2}e^{\gamma_0}\Gamma p_0 \frac{du_0}{dr}$), which gives

$$\frac{du_0}{dr} = 0$$

at $r = b$. For the trial solution (6.9) this boundary condition gives

$$3 + 5a_1 y + 7b_1 y^2 = 0, \tag{6.10}$$

where $y = \frac{b^2}{R^2}$. In the next section we shall integrate the right hand side of equation (6.7) for different choices of the parameter λ together with the boundary condition (6.10).

6.3 Results

We begin with SAX J1808.4-3658 and consider the configuration SS1. This corresponeds to $\lambda = 53.34$ and gives $b = 7.07$ km and $R = 43.245$ km and hence $y = \frac{b^2}{R^2} = 0.026728$. Substituting these values we integrate the right hand side of equation (6.7) for different values of a_1 and b_1, large or small, positive or negative, but satisfying equation (6.10). In each case the result is found to be positive, which clearly indicates that the model is stable against small radial pulsations. The results are given in Table 6.1.

If we consider the configuration for the EOS SS2, the relevant parameters are $\lambda = 230.58$, $b = 6.55$ km, $R = 82.35$ km and $y = \frac{b^2}{R^2} = 0.00633$. Employing the same technique we can draw the conclusion this configuration is also stable. The values of the integral of the right hand side of (6.7) for different choices of the parameters a_1 and b_1 are shown in Table 6.2.

Let us now consider the case of Her X-1. In chapter 5 we have already shown that if we take the mass and radius of Her X-1 as $M = 0.88\ M_{\odot}$ and $b = 7.7\ km$, respectively, the corresponding EOS for $\lambda = 100$ agrees well with the EOS obtained by Horvath and Pacheco [72] for a quark-diquark mixture. To prove that our configuration is stable in this case too, we integrate the right hand side of equation (6.7) once again with $\lambda = 100$, $b = 7.7\ km$ and $R = 108.779\ km$. The results are shown in Table 6.3. The results, clearly suggest that the configuration is stable under radial oscillations.

6.4 Discussions

We note that equation (6.7) can also be integrated for different values of the parameter $y = \frac{b^2}{R^2}$. Changing the value of y ($y < 0.5$), Knutsen [156] verified stability of the Vaidya-Tikekar [63] model for $\lambda = 2$ case and showed that the model is stable. Since the solution presented by Mukherjee *et al* [67] can admit any value of $\lambda > \frac{3}{17}$, we can do the same type of calculation for any value of λ subject to the constraint on $y(= \frac{b^2}{R^2})$ given in equation (2.41), which follows from physical consideration. In Table 6.4., the upper bound on the possible values of y for different λ are given. However, in this chapter, we have shown explicitly how the simple analytic solution [67] helps in studying the stability of realistic compact objects, provided an equation of state is available.

b_1	a_1	Integral
1	-22.4858	0.0005
10	-22.8225	0.0005
10^2	-26.1903	0.0004
10^3	-59.8676	0.0004
10^4	-396.641	0.1494
10^5	-3764.37	16.5377
10^6	-37441.7	1670.24
10^{10}	-3.74×10^8	1.67×10^{11}
0	-22.4483	0.0006
-10	-22.0741	0.0006
-10^2	-18.7064	0.0008
-10^3	14.9709	0.0041
-10^4	351.744	0.1861
-10^5	3719.48	16.9049
-10^6	37396.8	1673.91
-10^{10}	3.74×10^8	1.67×10^{11}
10^{-10}	-22.4483	0.0005

Table 6.1: Value of the integral (equation (6.7)) for $\lambda = 53.34$, $y = \frac{b^2}{R^2} = 0.026728$

b_1	a_1	Integral
1	-94.8498	0.00025
10	-94.9295	0.00025
10^2	-95.7266	0.00025
10^3	-103.698	0.00021
10^4	-183.41	0.00002
10^5	-980.534	0.0204
10^6	-8951.78	2.44813
10^{10}	-8.857×10^7	2.495×10^8
0	-94.8409	0.00025
-10	-94.7524	0.00025
-10^2	-93.9552	0.00026
-10^3	-85.984	0.0003
-10^4	-6.2716	0.00098
-10^5	790.852	0.0300
-10^6	8762.09	2.54395
-10^{10}	8.857×10^7	2.496×10^8
10^{-10}	-94.8409	0.00025

Table 6.2: Value of the integral (equation (6.7)) for $\lambda = 230.58$, $y = \frac{b^2}{R^2} = 0.00633$

b_1	a_1	Integral
1	-119.753	0.00174
10	-119.816	0.00174
10^2	-120.447	0.00173
10^3	-126.761	0.00167
10^4	-189.894	0.00113
10^5	-821.232	0.00585
10^6	-7134.61	1.05879
10^{10}	-7.61488×10^7	1.13×10^8
0	-119.746	0.00174
-10	-119.676	0.00174
-10^2	-119.044	0.00175
-10^3	-112.73	0.00181
-10^4	-29.597	0.00257
-10^5	581.741	0.02021
-10^6	6895.12	1.20234
-10^{10}	7.01×10^7	1.13×10^8
10^{-10}	-119.746	0.00174

Table 6.3: Value of the integral (equation (6.7)) for $\lambda = 100$, $y = \frac{b^2}{R^2} = 0.00501062$

λ	$y_{max.}$
1	0.5166
2	0.5000
3	0.4574
4	0.4166
5	0.3810
10	0.2631
20	0.1611
50	0.0742
100	0.0390
200	0.0200

Table 6.4: Upper bound on the possible values of $y = (\frac{b^2}{R^2})$ as a function of λ, as given by equation (2.41).