Report 6

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Chapter 5
De la Vallee Poussin's Theorem

This chapter is about estimate the rate at which the relative error of $\pi(x) \sim Li(x)$ appraches zero as x approaches infinity. De la Vallee Poussin proved that there exist a constant $c > 0$ such that the relative error approaches zero at least as fast as $exp[-(clogx)^{1/2}]$ does, i.e

$$\left| \frac{\pi(x) - Li(x)}{Li(x)} < e^{-\sqrt{clogx}} \right|$$

for all sufficiently large x.
Therefore this chapter proves this theorem and some application of it.

1. An improvement of $Re\ \rho < 1$

   Theorem: There exist constant $c > 0, K > 1$, such that

   $$\beta < 1 - \frac{c}{log\gamma}$$

   for all roots $\rho = \beta + i\gamma$, in the range $\gamma > K$. This inequality is stronger than $\beta < 1$, however this doesn't not preclude the possibility that there are roots ρ arbitrarily near to the line $Re\ s = 1$.

   De la Vallee Poussin's proof of this is based on the elementary inequality

   $$4 \geq 2(1 - cos\theta)$$
   $$4(1 + cos\theta) \geq 2(1 - cos^2\theta) = 1 - cos2\theta$$
   $$3 + 4cos\theta + cos2\theta \geq 0$$

   This is true for all θ. Combine this with the formula $-\zeta'(s)/\zeta(s) = \int_0^\infty x^{-s} d\psi(x)$ gives

   $$Re\left\{-3\frac{\zeta'(s)}{\zeta(s)} - 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right\}$$
   $$= \int_0^\infty x^{-\sigma}[3 + 4cos(tlogx) + cos(2tlogx)]d\psi(x) \geq 0$$

   Hence
   $$Re\left\{3\frac{\zeta'(s)}{\zeta(s)} + 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right\} \leq 0$$
   for all $\sigma > 1$ andforall real t.
   This can be used to prove $\beta < 1$.
   Then De la Vallee Poussin used this inequality to prove $\beta < 1 - \frac{c}{log\gamma}$.[1]

[1] In the proof,he uses a formula $\Pi'(x)/\Pi(x) \sim logx$, This formula will be in the next chapter

2. De la Vallee Poussin's estimate of the error

The main step in the proof of the prime number theorem is to use the estimate $\beta < 1$ to prove $\sum x^{\rho-1}/\rho(\rho+1)$ approaches 0 as x approaches infinity. Since we have found a better estimate $\beta < 1 - c(log\gamma)^{-1}$, now it is natural to use it.

De la Vallee Poussin accomplished it by first considering

$$\left|\sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho+1)}\right| \leq \sum_{\rho} \frac{x^{\beta}}{\gamma^2} = \sum_{|\rho|<K} \frac{x^{\beta}}{\gamma^2} + 2\sum_{\rho \geq K} \frac{x^{\beta}}{\gamma^{1-\delta}} \cdot \frac{1}{\gamma^{1+\delta}}$$

The first term on the right is the sum of finite number of terms, each of them is a constant times a negative power of x ($x^{\beta-1}$), hence there are constant C, ϵ such that this termis less than $c^{-\epsilon}$ for all $x > 1$. Now if δ is any postive constant, then $2\sum \gamma^{-1-\delta}$ converges, so the second term on the right is less than a constant times the maximum of $x^{\beta-1}/\gamma^{1-\delta}$.[2]

By setting $\delta = \frac{3}{4}$ and let C_1 denote $2\sum \gamma^{-1-\delta}$, Then the above estimates gives

$$\left|\sum \frac{x^{\rho-1}}{\rho(\rho+1)}\right| < Cx^{-\epsilon} + C_1 exp[-(c\ logx)^{1/2}]$$

for all sufficiently large x. Finally, since x^{ϵ} converges to zero much faster than $exp[-(c\ logx)^{1/2}]$, since the constant C, C_1 can be absorbed by decreasing c slightly, and since $2\sum x^{\rho-1}/\rho(\rho+1)$ is the relatve error in the approximation $\int_0^x \psi(t)dt \sim x^2/2$, this proves that there is a constant $c > 0$ such that the relative error is lessthan $exp[-(c\ logx)^{1/2}]$ for all sufficiently large x.

Last step is using the same technique as in last chapter, where $\pi(x) \sim Li(x)$ from $\int_0^x \psi(t)dt \sim x^2/2$ was deduced.[3] This will leads to

$$\pi(y) \geq const + Li(y) - \frac{y}{logy}\left[\epsilon(y) + \frac{const\ logy}{y^{1/2}} + \frac{4\epsilon(y^{1/2})}{logy}\right]$$

Where $epsilon(x) = exp[-(c\ logx)^{1/2}]$, and c is as above.
Since the quantity in square brackets is less than $\epsilon(y^{1/2})$for all sufficinetly large y, it will suffice to prove that $y/logy$ divided by $Li(y)$ is bounded as y goes to infinity.

[2] up to this point, consider the continuous variable $\frac{clogx}{(log\gamma)^2}\frac{1}{\gamma} - \frac{1-\delta}{\gamma}$ which can be negative, zero or positive. Thus for sufficient large x that $clogx > (1-\delta)(logK)^2$, this will lead to the inequality

$$\frac{x^{\beta-1}}{\gamma^{1-\delta}} \leq \frac{x^{-c/(log\gamma)}}{\gamma^{1-\delta}} \leq \frac{exp\{-[c(1-\delta)logx]^{1/2}\}}{exp\{[c(1-\delta)logx]^{1/2}\}} = exp[-(clogx)^{1/2}]$$

[3] by consider the least and most value

3. Other formulas for $\pi(x)$

Ledendre came up with a different approximation formula for $\pi(x)$

$$\pi(x) \sim \frac{x}{logx - A}$$

where A is a constant, its value were given by Legendre as 1.08366 (on empirical grounds).
The prime number theorem shows if this approximation is true if and only if $Li(x) \sim x/(logx - A)$ for some and hence all values of A. However by integraion by parts

$$\begin{aligned} Li(x) &= Li(2) + \int_2^x \frac{dt}{logt} \\ &= Li(2) + \frac{x}{logx} - \frac{2}{log2} + \int_2^x \frac{dt}{(logt)^2} \\ Li(x) - \frac{x}{logx} &= const + \int_2^x \frac{dt}{(logt)^2} \end{aligned}$$

Now it is suffice to show tha thte integral on the right divided by $x/logx$ approaches zero as x goes to infinity in order to conclude that the approximation is true with $A = 0$.

Chebyshev was able to show that if any value of A is any better than any other then its value must be $A = 1$. This is needed for the fact that the approximation

$$Li(x) \sim \frac{x}{logx - A}$$

is best when $A = 1$. [4]

Another application of De la Vallee Poussin estimate is that, First consider integrating $Li(x)$ by parts shows that it can be generalised to the form

$$Li(x) \sim \frac{x}{logx} + \frac{x}{(logx)^2} + 2\frac{x}{(logx)^3} + 6\frac{x}{(logx)^4} + \cdots + (n-1)!\frac{x}{(logx)^n}$$

where the error grows much less rapidly than the last term $x(logx)^{-n}$ as x goes to infinity. De la Vallee Poussin estimate shows that hte error in $\pi(x) \sim Li(x)$ also grows less rapidly than $x(logx)^{-n}$ and hence proves that the approximation

$$\pi(x) \sim \frac{x}{logx} + \frac{x}{(logx)^2} + \cdots + (n-1)!\frac{x}{(logx)^n}$$

is valid.

[4]The prove of this uses the de la Vallee Poussin estimate fo the error

4. Error estimates and the Riemman Hypothesis

So far we can see that there is a strong relationship between de la Vallee Poussin's estimate of the error in the PNT and his estimate $\beta < 1 - c(log\gamma)^{-1}$ of $\beta = Re\ \rho$. It is not surprising that the Riemman Hypothesis $Re\ \rho = 1/2$ should imply much stronger estimates of the error. The best error estimate so far is proved by von Koch, which says if the Riemman Hypothesis is true then the relative errors for al lsufficiently large x. This estimate implies that the relative error are eventually less than a constant times $(logx)^2 x^{-1/2}$ for all sufficiently large x.

On the other hand, if the Riemman hypothesis is false, then there is a root ρ with $Re\ \rho > \frac{1}{2}$ and hence a "'periodic"' termin Riemann's formula for $\pi(x)$ which grows more rapidly in magnitude than $x^{1/2}$, so it is reasonable to assume the error in PNT would not in the case grow less rapidly less than $x^{(1/2)+\epsilon}$. [5]

5. A postscript of De la Vallee Poussins Proof

In this section it talked about Sum of Mobius function, and it begins with Euler's product formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{s} - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{\mu(n)}{n^s}$$

Where $\mu(n)$ is Mobius function. Since $zeta(s)$ has a pole at s=1, $[\zeta(s)]^{-1}$ has a zero at $s = 1$; so if Euler's product formula for $\zeta(s)$ is valid for $s = 1$, it would say

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} + \cdots$$

de la Vallee Poussin proved that

$$\left| \sum_{n<x} \frac{\mu(n)}{n} \right| < \frac{K}{logx}$$

for all sufficient large x. As x goes to infinity, this will imply the Euler's product formula with $s = 1$.

[5] Last part in this section gives a theorem which states that, The Riemann Hypothesis is equivalent to the statement that for every $\epsilon > 0$ the relative error in the prime number theorem $\pi(x) \sim Li(x)$ is less than $x^{-(1/2)+\epsilon}$ for all sufficiently large x.