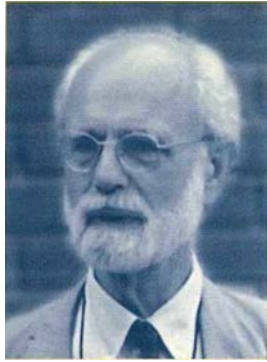


Classics in Mathematics

Tonny A. Springer    Jordan Algebras and Algebraic Groups



Born on February 13, 1926 at the Hague, Holland, Tonny A. Springer studied mathematics at the University of Leiden, obtaining his Ph. D. in 1951. He has been at the University of Utrecht since 1955, from 1959–1991 as a full professor, and since 1991 as an emeritus professor.

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Throughout his career T. A. Springer has been involved in research on various aspects of the theory of linear algebraic groups (conjugacy classes, Galois cohomology, Weyl groups).

Tonny A. Springer

# Jordan Algebras and Algebraic Groups

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## Preface

The aim of this book is to give an exposition of a part of the theory of Jordan algebras, using linear algebraic groups. That such groups play a role in Jordan algebra theory is a well-established fact, pointed out, for example, by H. Braun and M. Koecher in their book “Jordan Algebren”. We have tried to exploit that fact as much as possible. In particular, the classification of simple Jordan algebras is derived here from the Cartan-Chevalley theory of semi-simple linear algebraic groups and their irreducible representations.

The first part of the book (until §11) is, in the main, of an elementary character. It contains part of the basic theory of finite dimensional Jordan algebras with identity. But these appear in disguise: instead of Jordan algebras we use the “J-structures”, introduced in §1. The notion of a J-structure contains an axiomatization of the notion of inverse. The algebraic group which is central in the theory, the so-called structure group (introduced by Koecher), enters already in the definition of J-structures.

If the characteristic is not 2, a J-structure is essentially the same thing as a Jordan algebra (as is established in §6). One of the advantages of J-structures is that the characteristic 2 case needs no special care, at least in the elementary theory. This is not so in Jordan algebra theory, where in characteristic 2 the so-called quadratic Jordan algebras come in. The relations between these and J-structures are discussed in §7. Examples of J-structures are discussed in §2 and §5. The “quadratic map”, which is familiar from Jordan algebra theory, is introduced in §3. It plays an important role. In particular, we use it in §10 for a group-theoretical version of the Peirce decomposition with respect to an idempotent element.

In §11 a classification problem for semisimple groups is solved. This solution leads quickly to the classification of simple J-structures over algebraically closed fields of characteristic not 2 in §12. The more troublesome characteristic 2 case is dealt with in §13. In §14 we then discuss the explicit determination of the structure group of the various simple J-structures, as well as the related Lie algebras (which are in-

troduced in §4). §15 deals with the classification of J-structures over fields which are not algebraically closed. §0 contains some preliminary material, for example about polynomial and rational maps of a vector space.

The notes at the end of the sections contain various remarks and references to the literature. In the latter we have not attempted to achieve completeness. Nor does the bibliography at the end of the book claim to be comprehensive (more complete references are given in the books of Braun-Koecher and Jacobson, quoted in the bibliography).

References to the bibliography are given in square brackets. Formulas are numbered consecutively in the sections, § $x$ , ( $y$ ) means formula ( $y$ ) of § $x$ .

I am grateful to F. D. Veldkamp for a number of critical remarks and to Miss J. van der Mars for the preparation of the manuscript.

Utrecht, December 1972

T. A. Springer

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