# §8.3 The Unit Circle

Outline

- Background Trig Function Information
- Unit circle
- Relationship between unit circle and background information
- 6 Trigonometric Functions
- Values of 6 Trig Functions

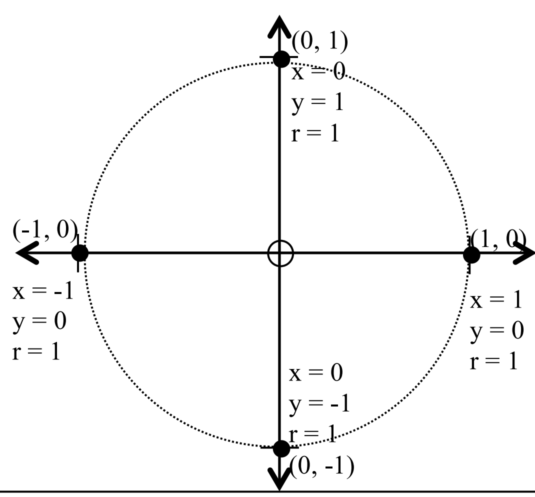
## The Unit Circle and Points on Circle

The function $\mathbf{x^2 + y^2 = 1}$, is the algebraic function that describes a circle with radius = 1.

I will call, ***s*** , the arc transcribed by rotating a ray located in the initial position (defined by some texts as the x-axis between QI and QIV) to a terminal position (rotation is always assumed to be counter clock-wise). The ordered pair **(x, y)** describes the position on the unit circle after said rotation, it is called the **terminal point**.

The arc ***s*** and the angle $\boldsymbol{\theta}$ are identical except in definition and position when the radius of the circle is 1 (since $\theta = {}^{s}/_{1}$). The arc lies along the unit circle and the angle is the angle made between the x-axis (between QI & QIV) and the ray. The angle, $\boldsymbol{\theta}$, will be described using radian or degree measure in this supplemental text. A radian is a portion of the total $2\pi$ arc that can be transcribed in a complete revolution of the initial side.

$s = 0, {}^{\pi}/_{2}, \pi, {}^{3\pi}/_{2}$ & $2\pi$ and their multiples $s + n \cdot 2\pi$




$s = 0^\circ, 90^\circ, 180^\circ, 270^\circ$ & $360^\circ$ and their multiples $s + n \cdot 360^\circ$

## Terminal Points

The position along the unit circle, resulting from the rotation ***s***, can be described by an ordered pair – the terminal point. This ordered pair is dependent upon the equation defining the unit circle:

$$x^2 + y^2 = 1$$

The above picture shows the terminal points for $\boldsymbol{s} = 0, {}^{\pi}/_{2}, \pi, {}^{3\pi}/_{2}$ & $2\pi$ or 0, 90, 180, 270, 360º.

**Example:** For a rotation $\boldsymbol{s} = {}^{\pi}/_{2}$ or 90º the terminal point is (0, 1)
For a rotation $\boldsymbol{s} = {}^{3\pi}/_{2}$ or 270º the terminal point is (0, -1)

**Example:** a) For a rotation $\boldsymbol{s} = \pi$ or 180º, what is the terminal point?

b) For a rotation $\boldsymbol{s} = 2\pi$ or 360º, what is the terminal point?

It can be shown, using the fact that the unit circle is symmetric about the line y = x, that the terminal point for $\boldsymbol{s} = {}^{\pi}/_{4}$ is $({}^{\sqrt{2}}/_{2}, {}^{\sqrt{2}}/_{2})$.

| | Statements | | Reasons |
|---|---|---|---|
| 1. | The unit circle is symmetric to y = x | 1. | Given |
| 2. | $s = {}^{\pi}/_{4}$ is equidistant from (1, 0) & (0, 1) on the unit circle | 2. | Midpoint of 0 & ${}^{\pi}/_{2}$ |
| 3. | Terminal point P for $s = {}^{\pi}/_{4}$ lies at the intersection of $y = x$ and $x^2 + y^2 = 1$ | 3. | Symmetry |
| 4. | $x^2 + x^2 = 1$ | 4. | Follows from 3 & substitution |
| 5. | $x^2 = {}^{1}/_{2}$ | 5. | Multiplication Prop. |
| 6. | $x = \pm{}^{1}/_{\sqrt{2}}$ | 6. | Square Root Prop. |
| 7. | $x = {}^{1}/_{\sqrt{2}}$ | 7. | Quadrant I o.p.'s |
| 8. | $y = {}^{1}/_{\sqrt{2}}$ | 8. | Since y = x, Given |
| 9. | $x = y = {}^{\sqrt{2}}/_{2}$ | 9. | Rationalization |
| 10. | Terminal point for $t = {}^{\pi}/_{4}$ lies at $P({}^{\sqrt{2}}/_{2}, {}^{\sqrt{2}}/_{2})$ in the 1st quadrant | 10. | Follows from 1-9 |

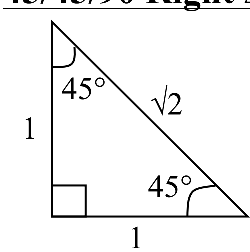
It can also be shown, through a little more complicated argument that $\boldsymbol{s} = {}^{\pi}/_{6}$, has the terminal point $({}^{\sqrt{3}}/_{2}, {}^{1}/_{2})$.

| | Statements | | Reasons |
|---|---|---|---|
| 1. | Points R(0,1), P(x, y), Q(x, -y) lie on the unit circle | 1. | Given |
| 2. | The arc determined by the terminal point P(x, y) is $s = {}^{\pi}/_{6}$ | 2. | Given |
| 3. | $\overline{PR} = \overline{PQ}$ | 3. | Both subtend arcs of length ${}^{\pi}/_{3}$ because R subtends a arc of ${}^{\pi}/_{2}$ and P subtends ${}^{\pi}/_{6}$ |
| 4. | $d(P, Q) = \sqrt{(x-x)^2 + (y - -y)^2} = 2y$ | 4. | Distance formula & algebra |
| 5. | $d(P, R) = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{2-2y}$ | 5. | Distance formula, $x^2 + y^2 = 1$ & algebra |
| 6. | $2y = \sqrt{2-2y}$ | 6. | Follows from 3-5 |
| 7. | $4y^2 = 2 - 2y \Rightarrow 4y^2 + 2y - 2 = 0 \Rightarrow 2(2y-1)(y+1) = 0 \Rightarrow y = -1$ or $y = {}^{1}/_{2}$ | 7. | Algebra |
| 8. | P is in QI $\therefore y = {}^{1}/_{2}$ | 9. | Given & Quadrant op's |
| 9. | $x^2 + ({}^{1}/_{2})^2 = 1 \Rightarrow x^2 = {}^{3}/_{4} \Rightarrow x = {}^{\sqrt{3}}/_{2}$ | 10. | Since P(x, y) is on unit circle, substitution & algebra & QI |
| 10. | $P(x, y) = ({}^{\sqrt{3}}/_{2}, {}^{1}/_{2})$ | 11. | Follows from 1-9 |

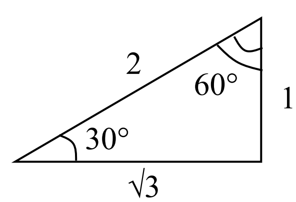
And finally, that that $\boldsymbol{s} = {}^{\pi}/_{3}$, which is nothing more than the reflection of $\boldsymbol{s} = {}^{\pi}/_{6}$ across the line of symmetry $y = x$, has the terminal point $({}^{1}/_{2}, {}^{\sqrt{3}}/_{2})$. Your #45 on p.541 exercises relates to this!

With all of this said, the bottom line that you need to know is what I was taught when I first took trigonometry. My teacher made us memorize two special triangles and this helped me throughout my studies. **MEMORIZE** these triangles. I will refer to them often.

**<u>45/45/90 Right Δ</u>**




**<u>30/60/90 Right Δ</u>**



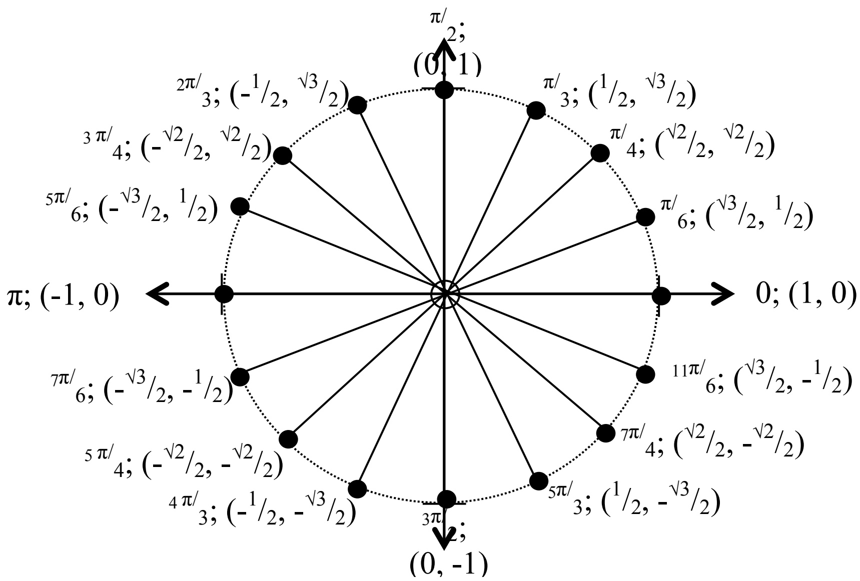

***Note:*** *Saying "ONE TO ONE TO THE SQUARE ROOT OF TWO" and "ONE TO TWO TO THE SQUARE ROOT OF THREE" helped me to memorize them. Recall that the sides of a triangle are in proportion to the angles, so the larger the angle the larger the side length must be (that helped me to place the correct side lengths).*

You need to memorize the following table for the unit circle

| ***s; θ*** | Terminal Pt.; P(x, y) |
|---|---|
| 0; 0 | (1, 0) |
| $^{\pi}/_{6}$ ; 30º | $(^{\sqrt{3}}/_{2}, {}^{1}/_{2})$ |
| $^{\pi}/_{4}$ ; 45º | $(^{\sqrt{2}}/_{2}, {}^{\sqrt{2}}/_{2})$ |
| $^{\pi}/_{3}$ ; 60º | $(^{1}/_{2}, {}^{\sqrt{3}}/_{2})$ |
| $^{\pi}/_{2}$ ; 90º | (0, 1) |

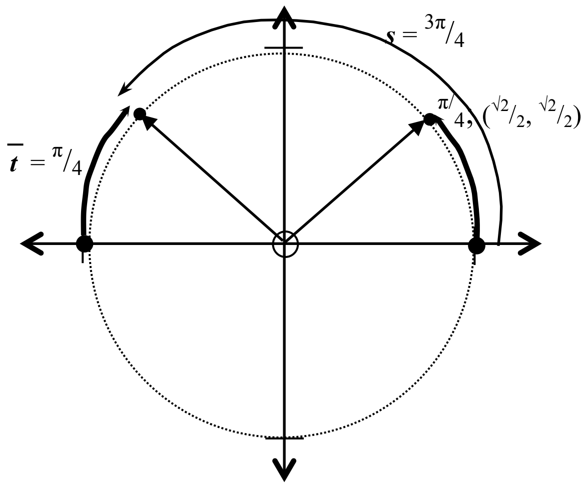
Draw a picture of the unit circle with the arc length & terminal points to the left labeled in QI

Converting between Radian and Degree Measure is extremely important and having a copy of the unit circle broken into equivalent measure although nice in my opinion is not necessary. I will not let you have a copy of the unit circle on tests. I instead will expect you to recreate these values via methods elaborated upon in class such as the above triangles, reference angles, 3 basic trig functions and fundamental identities and signs of trig functions in each quadrant.



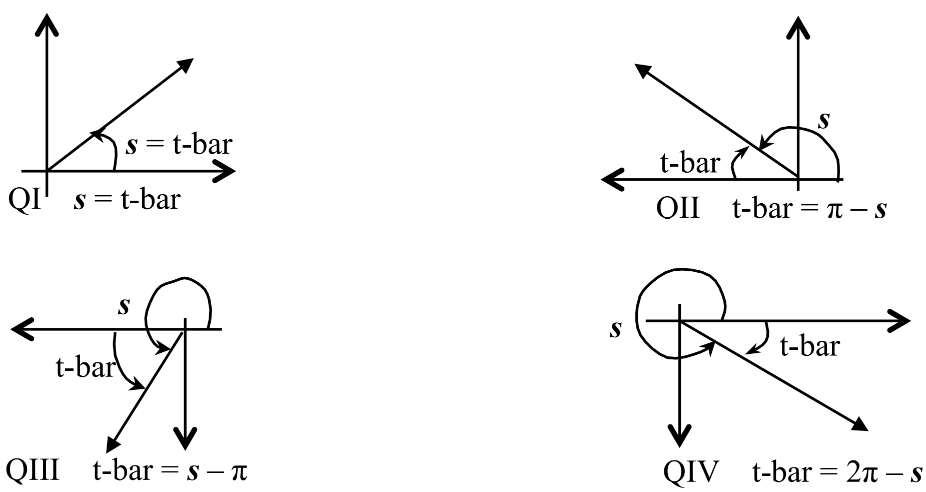

***Note 1:*** *Make all 45° marks by 4ths and make all 30º/60º marks by 6ths and count your way around.*
***Note 2:*** *Think in terms of x-axis and* $^{\pi}/_{6}$*=30°,* $^{\pi}/_{4}$*=45°, and* $^{\pi}/_{3}$*=60° adding and subtracting your way around the circle 2π.*

Instead of memorizing the entire unit circle, we can instead use the QI information and the idea of a reference number/angle. To do this we will introduce a concept known as a **reference number** (angle). Lets refers to it as $\bar{t}$ (I'll call that s-bar sometimes for easy typing). A reference number is the shortest distance between the x-axis and the terminal point.




Now that we have a visual of what a reference number is, we need to be able to find one without the visual. Here is the process:

**Reference Number** – An arc length, t-bar, is a positive arc length less than ${}^{\pi}/_2$ made by the terminal side and the x-axis.




For $s > 2\pi$ ($s > 360^\circ$) or for $s < 0$, first apply the methods developed in §6.1 appendix for finding the least positive co-terminal angle between 0 and $2\pi$ (0 & 360º). You may then have to apply the above methodologies of finding t-bar.

Lastly, we need to find the terminal point on the unit circle for a reference number, t-bar. This is done quite simply by using the reference number, quadrant information and having memorized the terminal points for the first quadrant as shown in the table above. The following is the process:

**Finding the Terminal Point for a Reference Number**

1. Determine the quadrant for which ***s*** lies
    a) Know that QI (+, +), QII (–, +), QIII (–, –) and QIV (+, –)
2. Use the Reference Number t-bar to determine the terminal point's coordinates (see table above or unit circle or **use 45/45/90 or 30/60/90 triangles**)
3. Give appropriate signs to the terminal point's coordinates according to the quadrant – see step #1

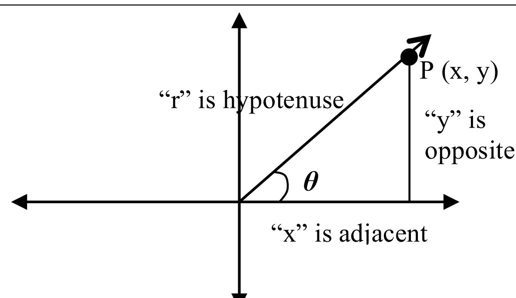
**Example:** Find the reference number for the following & then give the terminal point for each angle.(#34 p. 407 Stewart)

a) $s = {}^{5\pi}/_{6}$ b) $s = {}^{7\pi}/_{6}$

c) $s = {}^{11\pi}/_{3}$ d) $s = {}^{-7\pi}/_{4}$

Next we will **define the trigonometric functions**, review some basic geometry and make the connection to the special triangles that I asked you to memorize in the last section.

**Standard Position of $\theta$**




$r = \sqrt{x^2 + y^2}$ (Pythagorean Theorem)
r > 0 since it is a distance
(an undirected vector meaning it has no direction)

**Note*: P(x, y) can be denoted as P(cos **θ** , sin **θ** ) when the terminal points are located on the unit circle. This leads us to the connection between my triangles and the unit circle.*

Based on the $\boldsymbol{\theta}$ in Standard Position the 6 trigonometric functions can be defined. The names of the 6 functions are sine, cosine, tangent, cotangent, secant and cosecant. Because there are many relationships that exist between the 6 trig f(n) you should get in a habit of thinking about them in a specific order. I've gotten used to the following order and I'll show you some of the important links.

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r}$ ***Note:*** When r = 1, $\sin\theta = y$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r}$ ***Note:*** When r = 1, $\cos\theta = x$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sin s}{\cos s} = \dfrac{y}{x} \qquad x \neq 0$

$\cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{1}{\tan s} = \dfrac{\cos s}{\sin s} = \dfrac{x}{y} \qquad y \neq 0$

$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{1}{\cos s} = \dfrac{r}{x} \qquad x \neq 0$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{1}{\sin s} = \dfrac{r}{y} \qquad y \neq 0$

***Note 1:*** *These are the **exact** values for the 6 trig f(n). A calculator will yield only the approximate values of the functions.*

***Note 2:*** *This is both the functions and their reciprocal identities and relations that tie to the coordinate system (an ∠ in standard position). The definitions given in terms of opposite, adjacent and hypotenuse will help later.*

At this point every text gives the following table to fill in and "memorize" for ease of finding the values of the trig functions. However, with the MEMORIZATION of the **above triangles** and the **definitions of the trig functions** you won't have to "memorize" the table, it will write itself.

**Example:** Fill in the following table using the definitions of the trig functions and the above triangles. I want you to write "undefined" for division by zero!

## Values of the 6 Trig F(n) for *t*

| | sin t | cos t | tan t | cot t | sec t | csc t |
|---|---|---|---|---|---|---|
| **0** | | | | | | |
| $^{\pi}/_{6}$ | | | | | | |
| $^{\pi}/_{4}$ | | | | | | |
| $^{\pi}/_{3}$ | | | | | | |
| $^{\pi}/_{2}$ | | | | | | |

***Note:*** *Another trick for the sin t and the cos t for these special angles is to do* $^{\sqrt{\square}}/_{2}$ *and fill in 0, 1, 2, 3 & 4 for the sin t and reverse the order 4, 3, 2, 1, 0 for cos t.*

At this point you should also know the domains of the six trig functions. The importance of this should be obvious once you have completed the above table and see which functions are undefined at what points. (That is showing you what is not in their domain.)

**<u>Domains of 6 Trig Functions</u>**

Sine and Cosine $\{s \mid s \in \text{Real Number}\}$

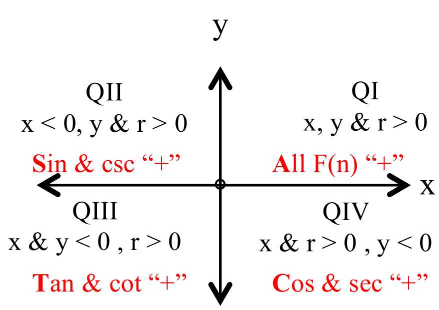
Tangent and Secant $\{s \mid s \neq {}^{(2n+1)\pi}/_2,\ n \in \ I\}$ (odd multiples of $^{\pi}/_2$ )

Cotangent and Cosecant $\{s \mid s \neq n\pi,\ n \in I\}$ (even multiples of $^{\pi}/_2$)

In order to find the values of the 6 trig functions for values of ***s*** that are not between 0 and $^{\pi}/_2$, the following information is helpful:

## <u>Signs & Ranges of Function Values</u>

You don't have to memorize this, but you at least have to be able to develop it, which is dependent upon knowing quadrant information and standard position.




| **<u>This Saying Will Help Remember the Positive F(n)</u>** | |
|---|---|
| **A**ll | **A**ll f(n) **"+"** |
| **S**tudents | **s**in & csc **"+"** |
| **T**ake | **t**an & cot **"+"** |
| **C**alculus | **c**os & sec **"+"** |

Let's go through the QII information using the definitions of the 6 trig f(n) to see how this works:

In QII, x is negative (x < 0) while y & r are positive (y, r > 0)

So,

$$\sin = \frac{y}{r} = \frac{+}{+} = + \qquad \csc = \frac{r}{y} = \frac{+}{+} = +$$

$$\cos = \frac{x}{r} = \frac{-}{+} = - \qquad \sec = \frac{r}{x} = \frac{+}{-} = -$$

$$\tan = \frac{y}{x} = \frac{+}{-} = + \qquad \cot = \frac{x}{y} = \frac{-}{+} = +$$

You can always develop the table below using sign information, memorize it or use "**A**ll **S**tudents **T**ake **C**alculus" to help know the signs of the trig functions in each of the quadrants.

| **θ in Quad** | **sin θ** | **cos θ** | **tan θ** | **cot θ** | **sec θ** | **csc θ** |
|---|---|---|---|---|---|---|
| **I** | + | + | + | + | + | + |
| **II** | + | - | - | - | - | + |
| **III** | - | - | + | + | - | - |
| **IV** | - | + | - | - | + | - |

Now, we can take all our newfound knowledge and put it together with our earlier knowledge.

**<u>How to Find the Exact Values of the 6 Trig F(n)</u>**

**Step 1:** Draw the ***s*** in the coordinate system creating a Δ w/ the terminal side and the x-axis. In other words, find the reference number for ***s***.

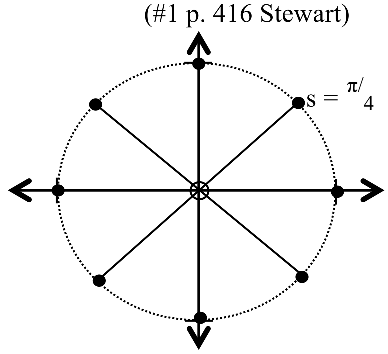
**Step 2:** Place ***s***, x & y and find r (using $r = \sqrt{x^2 + y^2}$) if you don't already know a basic Δ

**Step 3:** Use x, y & r (opp, adj & hyp) w/ definitions to write the exact values of 6 trig f(n)

***Note:*** *Once you've got sin, cos & tan you've got the others due to the reciprocal identities.*

**Step 4:** Simplify (you'll need to review your radicals)

**Example:** Find the sin s and cos s for the points on the unit circle and use them to give the coordinates of the terminal points.
(#1 p. 416 Stewart)




**Example:** For ***s*** = $^{5\pi}/_6$
(essentially #4 p. 416 Stewart)

a) What quadrant would ***s*** be in?

b) Find the reference number

c) Find sin s

d) Find cos s

e) Find tan s

f) Find csc s

***Note:*** *Your book puts a lot of focus on angles other than 30º, 45º, 60º, 90º and their reference & co-terminal angles. While this does not appear to be what I've taught in this lesson, the focus is still the same – relationships between x & y coordinates of terminal point, those points' relationships to the unit circle, those points relationships to the trig functions and angles and arc lengths. My focus mainly will be on the be on those angles I've focused on here.*