In this lecture we will find the eigenvalues and eigenvectors of 3×3 matrices.

EXAMPLE. An example of three distinct eigenvalues.

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix}.$$

SOLUTION: Recall,

Steps to find eigenvalues and eigenvectors:

1. Form the characteristic equation
$$\det(\lambda I - A) = 0.$$
2. To find all the eigenvalues of A, solve the characteristic equation.
3. For each eigenvalue λ, to find the corresponding set of eigenvectors, solve the linear system of equations
$$(\lambda I - A)\vec{x} = 0$$

Step 1. **Form the Characteristic Equation.**
The characteristic equation is:

$$\begin{aligned} \det\,(\lambda I - A) &= 0 \\ \begin{vmatrix} \lambda - 4 & 0 & -1 \\ 1 & \lambda + 6 & 2 \\ -5 & 0 & \lambda \end{vmatrix} &= 0 \\ (\lambda - 4)((\lambda + 6)(\lambda) - 0) - 1(0 - (-5)(\lambda + 6)) &= 0 \\ \lambda^3 + 2\lambda^2 - 29\lambda - 30 &= 0 \end{aligned}$$

Step 2. **Find the eigenvalues.**
We need to solve the characteristic equation. i.e. we need to factorize the characteristic polynomial. We can factorize it by either using long division or by directly trying to spot a common factor.

Method 1: Long Division.
We want to factorize this cubic polynomial. In general it is quite difficult to guess what the factors may be. We try $\lambda = \pm 1, \pm 2, \pm 3, etc.$ and hope to quickly find one factor. Let us try $\lambda = -1$. We divide the polynomial $\lambda^3 + 2\lambda^2 - 29\lambda - 30$ by $\lambda + 1$, to get,

$$\begin{array}{rrrrr}
 & \lambda^2 & +\lambda & -30 \\ \cline{2-5}
\lambda+1 \,| \lambda^3 & +2\lambda^2 & -29\lambda & -30; \\
-\lambda^3 & +\lambda^2 & & \\ \cline{1-2}
 & \lambda^2 & -29\lambda & -30 \\
- & \lambda^2 & +\lambda & \\ \cline{2-4}
 & & -30\lambda & -30 \\
 & - & -30\lambda & -30 \\ \cline{3-4}
 & & & 0
\end{array}$$

The quotient is $\lambda^2 + \lambda - 30$.
The remainder is 0.
Therefore $\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + 1)(\lambda^2 + \lambda - 30) + 0$.

$$\begin{aligned}
\lambda^3 + 2\lambda^2 - 29\lambda - 30 &= 0 \\
(\lambda + 1)(\lambda^2 + \lambda - 30) &= 0 \\
(\lambda + 1)(\lambda + 6)(\lambda - 5) &= 0
\end{aligned}$$

Therefore the eigenvalues are: $\{-1, -6, 5\}$.
Method 2: Direct factorization by spotting common factor.

$$\begin{aligned}
\begin{vmatrix} \lambda - 4 & 0 & 0 \\ 1 & \lambda + 6 & 2 \\ -5 & 0 & \lambda \end{vmatrix} &= 0 \\
(\lambda - 4)((\lambda + 6)(\lambda) - 0) - 1(0 - (-5)(\lambda + 6)) &= 0 \\
(\lambda - 4)((\lambda + 6)(\lambda)) - 5(\lambda + 6) &= 0 \\
(\lambda + 6)(\lambda(\lambda - 4) - 5) &= 0 \\
(\lambda + 6)(\lambda^2 - 4\lambda - 5) &= 0 \\
(\lambda + 6)(\lambda - 5)(\lambda + 1) &= 0
\end{aligned}$$

Therefore the eigenvalues of A are: $\lambda_1 = -1$, $\lambda_2 = -6$ and $\lambda_3 = 5$.

Step 3. **Find Eigenvectors corresponding to each Eigenvalue:**
We now need to find eigenvectors corresponding to each eigenvalue.

case(i) $\lambda_1 = -1$.
The eigenvectors are the solution space of the following system:

$$\begin{pmatrix} -5 & 0 & -1 \\ 1 & 5 & 2 \\ -5 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -5x_1 - x_3 &= 0; \quad x_1 = \frac{-1}{5}x_3 \\ x_1 + 5x_2 + 2x_3 &= 0; \quad x_2 = \frac{-9}{25}x_3 \end{aligned}$$

The set of eigenvectors corresponding to $\lambda_1 = -1$ is,

$$\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} | x_1 = \frac{-1}{5}x_3, x_2 = \frac{-9}{25}x_3 \}$$

$$\{ \begin{pmatrix} \frac{-1}{5}x_3 \\ \frac{-9}{25}x_3 \\ x_3 \end{pmatrix} | x_3 \text{ is a non-zero real number} \}$$

$$\{ x_3 \begin{pmatrix} \frac{-1}{5} \\ \frac{-9}{25} \\ 1 \end{pmatrix} | x_3 \text{ is a nn-zero real number} \}$$

An eigenvector corresponding to $\lambda_1 = -1$ is $\begin{pmatrix} \frac{-1}{5} \\ \frac{-9}{25} \\ 1 \end{pmatrix}$.

case(ii) $\lambda_2 = -6$.
The eigenvectors are the solution space of the following system:

$$\begin{pmatrix} -10 & 0 & -1 \\ 1 & 0 & 2 \\ -5 & 0 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -10x_1 - x_3 &= 0; \quad x_3 = -10x_1 \\ x_1 + 2x_3 &= 0; \quad x_1 = -2x_3 = -2(-10)x_1 \\ x_1 &= 0 = x_3 \end{aligned}$$

The set of eigenvectors corresponding to $\lambda_2 = -6$ is,

$$\{ \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} |x_1 = x_3 = 0\}$$

$$\{ \quad \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} |x_2 \text{ is a non-zero real number}\}$$

$$\{ \quad x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} |x_2 \text{ is a non-zero real number}\}$$

An eigenvector corresponding to $\lambda_2 = -6$ is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

case(iii) $\lambda_3 = 5$.

The eigenvectors are the solution space of the following system:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 11 & 2 \\ -5 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_3$$

$$x_2 = \frac{-3}{11} x_3$$

The set of eigenvectors corresponding to $\lambda_3 = 5$ is,

$$\{ \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} |x_1 = x_3, x_2 = \frac{-3}{11} x_3\}$$

$$\{ \quad \begin{pmatrix} x_3 \\ \frac{-3}{11} x_3 \\ x_3 \end{pmatrix} |x_3 \text{ is a non-zero real number}\}$$

$$\{ \quad x_3 \begin{pmatrix} 1 \\ \frac{-3}{11} \\ 1 \end{pmatrix} |x_3 \text{ is a non-zero real number}\}$$

An eigenvector corresponding to $\lambda_3 = 5$ is $\begin{pmatrix} 1 \\ \frac{-3}{11} \\ 1 \end{pmatrix}$.

Example. An example of repeated eigenvalue having only two eigenvectors.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Solution: Recall,

Steps to find eigenvalues and eigenvectors:

1. Form the characteristic equation

$$\det(\lambda I - A) = 0.$$

2. To find all the eigenvalues of A, solve the characteristic equation.

3. For each eigenvalue λ, to find the corresponding set of eigenvectors, solve the linear system of equations

$$(\lambda I - A)\vec{x} = 0$$

Step 1. **Form the Characteristic Equation.**
The characteristic equation is:

$$\begin{aligned} \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} &= 0 \\ \lambda(\lambda^2 - (-1)(-1)) - (-1)((-1)\lambda - (-1)(-1)) - 1((-1)\lambda - (-1)(-1)) &= 0 \\ \lambda^3 - 3\lambda^2 - 2 &= 0 \end{aligned}$$

Step 2. **Find the eigenvalues.**
We need to solve the characteristic equation. i.e. we need to factorize the characteristic polynomial. We can factorize it by either using long division or by directly trying to spot a common factor.
Method 1: Long Division.
We want to factorize this cubic polynomial. In general it is quite difficult to guess what the factors may be. We try $\lambda = \pm 1, \pm 2, \pm 3, etc.$

and hope to quickly find one factor. Let us try $\lambda = -1$. We divide the polynomial $\lambda^3 + 2\lambda^2 - 29\lambda - 30$ by $\lambda + 1$, to get,

$$
\begin{array}{rrrrr}
 & \lambda^2 & -\lambda & -2 & \\
\cline{2-5}
\lambda+1 \,| & \lambda^3 & & -3\lambda & -2; \\
 -\lambda^3 & +\lambda^2 & & & \\
\cline{2-3}
 & -\lambda^2 & -3\lambda & -2 & \\
 - & -\lambda^2 & -\lambda & & \\
\cline{2-3}
 & & -2\lambda & -2 & \\
 & - & -2\lambda & -2 & \\
\cline{3-4}
 & & & 0 &
\end{array}
$$

The quotient is $\lambda^2 - \lambda - 2$.
The remainder is 0.
Therefore $\lambda^3 - 3\lambda - 2 = (\lambda + 1)(\lambda^2 - \lambda - 2) + 0$.

$$
\begin{aligned}
\lambda^3 - 3\lambda - 2 &= 0 \\
(\lambda + 1)(\lambda^2 - \lambda - 2) &= 0 \\
(\lambda + 1)(\lambda + 1)(\lambda - 2) &= 0
\end{aligned}
$$

Therefore the eigenvalues are: $\{-1, 2\}$.

Method 2: Direct factorization by spotting common factor.

$$
\begin{aligned}
\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} &= 0 \\
\lambda(\lambda^2 - (-1)(-1)) - (-1)((-1)\lambda - (-1)(-1)) - 1((-1)\lambda - (-1)(-1)) &= 0 \\
\lambda(\lambda^2 - 1) + 1(-\lambda - 1) - 1(1 + \lambda) &= 0 \\
\lambda(\lambda - 1)(\lambda + 1) - 1(\lambda + 1) - 1(1 + \lambda) &= 0 \\
(\lambda + 1)(\lambda(\lambda - 1) - 1 - 1) &= 0 \\
(\lambda + 1)(\lambda^2 - \lambda - 2) &= 0 \\
(\lambda + 1)^2(\lambda - 2) &= 0
\end{aligned}
$$

Therefore the eigenvalues of A are: $\{-1, 2\}$.