

Trapezoid Convex Proof

Definition: A quadrilateral $\square ABCD$ is a trapezoid if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

Statement: A trapezoid is convex.

Proof: [WLOG, assume that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. We need to prove that each point is in the interior of the angle formed by the other three vertices. Since the proof is similar for each vertex, we will prove that D is in the interior of $\angle ABC$, with the understanding that the same proof can be used for the other three vertices.]

[To show that D is in the interior of $\angle ABC$, we have to show that C and D on the same side of \overleftrightarrow{AB} and A and D on the same side of \overleftrightarrow{BC} .]

C and D on the same side of \overleftrightarrow{AB} :

1. Suppose C and D are on opposite sides of \overleftrightarrow{AB} .
2. Then \overleftrightarrow{CD} intersects \overleftrightarrow{AB} .
3. This is a contradiction.

A and D on the same side of \overleftrightarrow{BC} :

4. Suppose that A, D are on opposite sides of \overleftrightarrow{BC} .
5. Let E be the point of intersection between \overleftrightarrow{AD} and \overleftrightarrow{BC} .
6. \overleftrightarrow{AD} and \overleftrightarrow{BC} cannot intersect.
7. $E * B * C$ or $B * C * E$.

*Case 1: Suppose that $E * B * C$. [See second figure.]*

8. Note that because B is between E and C, then B is an interior point of $\angle CAE$.
9. But D is collinear with A and E, which implies that B is an interior point of $\angle CAD$.
10. \overleftrightarrow{AB} intersects \overleftrightarrow{CD} .
11. This is a contradiction.

*Case 2: Suppose that $B * C * E$. [See third figure.]*

12. Then $\triangle AEB$ is a triangle.
13. \overleftrightarrow{CD} intersects the interior of \overleftrightarrow{EB} .
14. \overleftrightarrow{CD} must intersect \overleftrightarrow{AE} or \overleftrightarrow{AB} .
15. But \overleftrightarrow{CD} cannot intersect \overleftrightarrow{AE} .
16. So \overleftrightarrow{CD} must intersect \overleftrightarrow{AB} .
17. This is a contradiction.

