

Solution Manual for

**Structural Dynamics:
Theory and Computation**

Sixth Edition

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1.1

If the weight w is displaced by amount, y , the beam and the springs will exert a total force on the mass of

$$P = \left(\frac{3EI}{L^3} + 2k \right) u$$

The beam and springs act in parallel. The equivalent stiffness is:

$$k_e = \frac{P}{u} = \left(\frac{3EI}{L^3} + 2k \right)$$

Natural frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{W} \left(\frac{3EI}{L^3} + 2k \right)}$$

Natural period:

$$T = \frac{2\pi}{\omega} = 2\pi L \sqrt{\frac{W}{g} \left(\frac{L}{3EI + 2kL^3} \right)}$$

1.2

Stiffness:

$$k_e = \left(\frac{3EI}{L^3} + 2k \right) = \frac{3 \times 10^8}{100^3} + 2 \times 1000 = 4,300 \text{ lb/in.}$$

Natural frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4300 \times 386}{3000}} = 23.52 \text{ rad/sec}$$

Free vibration response of undamped oscillator:

$$u(t) = u_0 \cos \omega t + \frac{\dot{u}_0}{\omega} \sin \omega t$$

$$\dot{u}(t) = -u_0 \omega \sin \omega t + \dot{u}_0 \cos \omega t$$

Displacement and velocity at $t = 1\text{ sec}$ with the initial values

$$u_0 = 1 \text{ in.}, \dot{u}_0 = 20 \text{ in./sec.}$$

$$u(1) = 1 \cdot \cos(23.5 \cdot 1) + \frac{20}{23.5} \sin(23.5 \cdot 1) = -0.89 \text{ in.}$$

$$\dot{u}(1) = -1 \cdot 23.5 \sin(23.5 \cdot 1) + 20 \cos(23.5 \cdot 1) = 22.66 \text{ in./sec}$$

1.3

The stiffness of the beam is

$$k = \left(\frac{12EI_1}{L^3} + \frac{3E(2I_2)}{L^3} \right) = \frac{12 \cdot (30 \cdot 10^6) \cdot 170.9}{144^3} + \frac{3 \cdot (30 \cdot 10^6) \cdot 82.5}{144^3} = 25,577 \text{ lb/in.}$$

Natural frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{25,577 \times 386}{50,000}} = 2.24 \text{ cps}$$

1.4

a) Infinitely rigid horizontal member

Stiffness:

$$k = 2 \left(\frac{12EI}{L^3} \right) = \frac{12 \cdot (30 \cdot 10^6) \cdot 171}{(12 \cdot 15)^3} = 21,100 \text{ lb/in.}$$

Natural frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{21,100 \times 386}{25,000}} = 18.05 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = 2.87 \text{ cps}$$

b) Flexible horizontal member consisting of W18X30

Compute the stiffness by moment distribution method. Displace the frame horizontally by one inch and determine the stiffness of the frame as the sum of the shear forces in both columns. Take advantage of the symmetry by modifying the stiffness of horizontal member by factor 3/2.

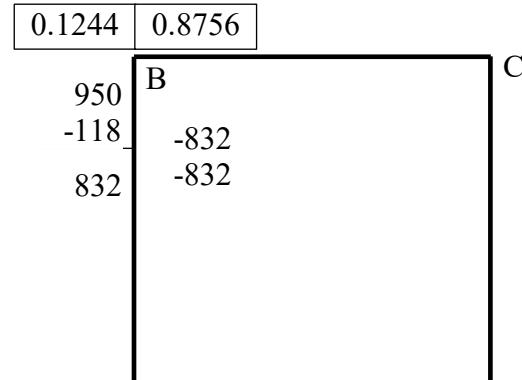
Distribution factors

$$k_{BA} = \frac{4EI}{L} = \frac{4E}{L} \cdot 171 \rightarrow 171 \rightarrow 0.1244$$

$$k_{BC} = \frac{4EI}{L} \frac{3}{2} = \frac{4E}{L} \cdot \frac{3}{2} \cdot 802 \rightarrow 1203 \rightarrow 0.8756$$

Fixed end moments:

$$M_{BA} = M_{AB} = \frac{6EI}{L^2} = \frac{6 \cdot (30 \cdot 10^6) \cdot 170.9}{(12 \cdot 15)^2} = 950 \text{ (k - in.)}$$



Shear force:

$$k = \frac{832 + 891}{180} \cdot 2 = 19.14 \text{ kip/in.}$$

Natural frequency:

$$\begin{array}{r} 950 \\ -59 \\ \hline 891 \end{array}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{19,140 \times 386}{25,000}} = 2.74 \text{ cps}$$

Note:

- Assuming a flexible girder decreases the natural frequency by only

$$1 - \frac{2.74}{2.87} = 0.05 = 5\%$$

2. Adding the weight of the girder and half of the weight of the two columns decrease the natural frequency by:

$$W = 25,000 + 50 \cdot 15 + 2 \cdot 33 \cdot \frac{15}{2} = 26,2456 \text{ lb}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{19,140 \times 386}{26,245}} = 2.67 \text{ cps}$$

$$1 - \frac{2.67}{2.74} = 0.05 = 2.5\%$$

1.5

Stiffness coefficient:

$$k = 2 \left(\frac{12EI}{(2/L)^3} \right) = \frac{192 \cdot EI}{(L)^3}$$

Natural frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{192 \cdot EIg}{(L)^3 W}}$$

$$f = \frac{\omega}{2\pi} = \frac{4}{\pi} \sqrt{\frac{3 \cdot EIg}{(L)^3 W}}$$

1.6

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{192 \cdot 10^9 \cdot 386}{(120)^3 \cdot 5,000}} = 92.61 \text{ rad/sec}$$

$$u(t = 2) = u_0 \cos \omega t + \frac{\dot{u}_0}{\omega} \sin \omega t = 0.5 \cos(92.61 \cdot 2) + \frac{15}{92.61} \sin(92.61 \cdot 2)$$

$$u(t = 2) = -0.474 \text{ in.}$$

$$\dot{u}(t) = -u_0 \omega \sin \omega t + \dot{u}_0 \cos \omega t = -92.61 \cdot 0.5 \cdot \sin(92.61 \cdot 2) + 15 \cos(92.61 \cdot 2)$$

$$\dot{u}(t = 2) = -21.05 \text{ in./sec}$$

$$\ddot{u}(t) = -\omega^2 u_0 \cos \omega t - \dot{u}_0 \omega \sin \omega t = -\omega^2 u(t)$$

$$\ddot{u}(t = 2) = 4,665 \text{ in./sec}^2$$

1.7

$$\sum M_o = I \cdot \alpha$$

$$\alpha = \text{avg. acc.}$$

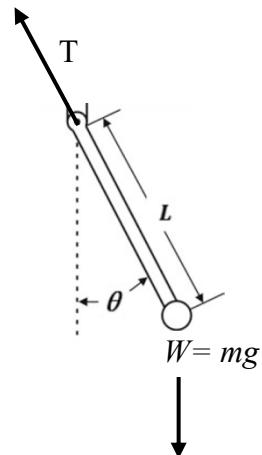
$$-m \cdot g \cdot L \cdot \sin \theta = m \cdot L^2 \cdot \ddot{\theta}$$

For θ small, $\sin \theta \approx \theta$

$$0 = \ddot{\theta} + \frac{g}{L} \cdot \theta$$

$$\theta = \theta_0 \cos \omega t + \frac{\dot{\theta}_0}{\omega} \sin \omega t$$

Where $\omega = \sqrt{\frac{g}{L}}$ is the natural frequency.



1.8

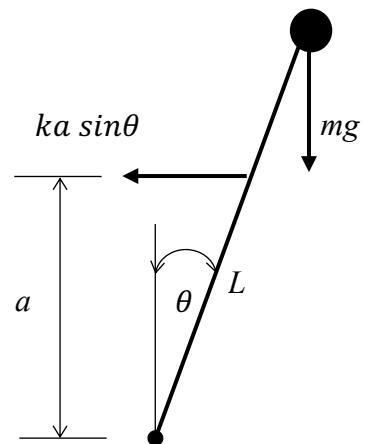
$$\sum M_o = I \cdot \alpha$$

$$-k \cdot a \cdot \tan\theta \cdot a + m \cdot g \cdot L \cdot \sin\theta = m \cdot L^2 \cdot \ddot{\theta}$$

For θ small, $\sin\theta \approx \theta$ $\tan\theta \approx \theta$

$$m \cdot L^2 \cdot \ddot{\theta} + (k \cdot a^2 - m \cdot g \cdot L)\theta = 0$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k \cdot a^2 - m \cdot g \cdot L}{m \cdot L^2}}$$



Note: If $m \cdot g \cdot L > k \cdot a^2$, the inverted pendulum is unstable.

1.13

The deflection curve may be assumed to have the shape shown in Fig. 1.13 (a), which is bend due to a force at the top of the pole.

In this case the slope θ at the tip is given as

$$\theta = \frac{PL^2}{2EI}$$

and the displacement δ as

$$\delta = \frac{PL^3}{3EI}$$

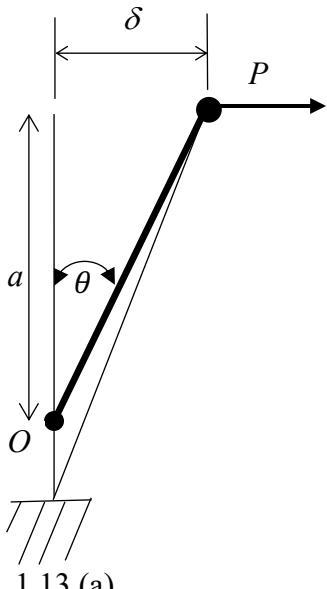


Fig. 1.13 (a)

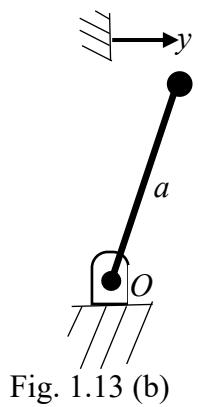


Fig. 1.13 (b)

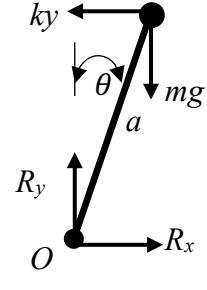


Fig. 1.13 (c)

The distance a is

$$a = \frac{\delta}{\theta} = \frac{2}{3}L$$

and is independent of θ . Go a first approximation: therefore, it may be assumed that the system is essentially equivalent to that of Fig. 1.13 (b) in which the stiffness is given by

$$k = \frac{P}{\delta} = \frac{3EI}{L^3}$$

From Fig. 1.13 (c) taking moments about O,

$$\sum M_o = I \cdot \alpha$$

$$-k \cdot a \cdot \cos\theta \cdot a + m \cdot g \cdot L \cdot \sin\theta = m \cdot a^2 \cdot \ddot{\theta}$$

For small θ $\cos\theta \approx 1$ $\sin\theta \approx \theta$ $y = a\theta$

Then

$$m \cdot a^2 \cdot \ddot{\theta} + (k \cdot a^2 - m \cdot g \cdot a)\theta = 0$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{g}{a}} = \frac{1}{2\pi} \sqrt{\frac{3EI}{mL^3} - \frac{3g}{2L}}$$

1.15

Case a)

The spring constant k_b for the beam is

$$k_b = \frac{3EI}{L^3}$$

Springs are in series, therefore the equivalent stiffness is

$$\frac{1}{k_e} = \frac{1}{k} + \frac{1}{k_b} = \frac{1}{k} + \frac{L^3}{3EI} = \frac{3EI + kL^3}{3EI \times k}$$

$$k_e = \frac{3EI \times k}{3EI + kL^3}$$

Natural frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_e}{m}} = \frac{1}{2\pi} \sqrt{\frac{3EI \cdot k \cdot g}{(3EI + kL^3)W}}$$

Case b)

The spring constant of the beam is

$$k_b = \frac{48EI}{L^3}$$

Springs are in series, therefore the equivalent stiffness is

$$\frac{1}{k_e} = \frac{1}{k} + \frac{1}{k_b} = \frac{1}{k} + \frac{L^3}{48 \cdot EI} = \frac{48EI + kL^3}{48EI \cdot k}$$

Natural frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_e}{m}} = \frac{1}{2\pi} \sqrt{\frac{48EI \cdot k \cdot g}{(48EI + k \cdot L^3)W}}$$

Case c)

Deflection of simply supported beam with load P is

$$\delta = \frac{P \cdot a^2 \cdot b^2}{3EI \cdot L}$$

So

$$k = \frac{P}{\delta} = \frac{3EI}{a^2 b^2}$$

and

$$f = \frac{1}{2\pi} \sqrt{\frac{3EI \cdot L \cdot g}{a^2 b^2 \cdot W}}$$

Case d)

Stiffness of the beam, from case c)

$$k_b = \frac{3EIL}{a^2 L^2}$$

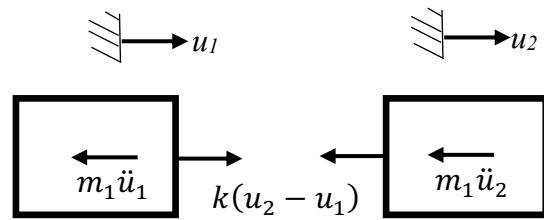
Spring in series:

$$\frac{1}{k_e} = \frac{1}{k} + \frac{1}{k_b} = \frac{1}{k} + \frac{a^2 b^2}{3EI \cdot L} = \frac{3EI + k \cdot a^2 b^2}{3EI \cdot k \cdot L}$$

Natural frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_e}{m}} = \frac{1}{2\pi} \sqrt{\frac{3EI \cdot k \cdot g}{(3EI \cdot L + k \cdot a^2 b^2)W}}$$

1.16



Newton's Law gives: $\sum F = 0$

$$m_1 \ddot{u}_1 - k(u_2 - u_1) = 0 \quad \text{Eq.1}$$

$$m_2 \ddot{u}_2 + k(u_2 - u_1) = 0 \quad \text{Eq.2}$$

Multiplying Eq. 1 by m_2 and Eq. 2 by m_1 , and subtract Eq. 1 from Eq. 2

$$m_1 m_2 (\ddot{u}_1 - \ddot{u}_2) - k(m_1 + m_2)(u_2 - u_1) = 0$$

Let $y = u_2 - u_1$

$$\ddot{y} - k \left(\frac{1}{m_1} + \frac{1}{m_2} \right) y = 0$$

Natural frequency:

$$f = \frac{1}{2\pi} \sqrt{k \left(\frac{1}{m_1} + \frac{1}{m_2} \right)}$$

2.1

The following numerical values are given:

$$\begin{aligned} L &= 100 \text{ in.} & EI &= 10^8 \text{ lb.}m^2 & W &= 3,000 \text{ lb} \\ k &= 2,000 \text{ lb/in.} & u_0 &= 1.0 \text{ in.} & \dot{u}_0 &= 20 \text{ in./sec} & \xi &= 0.15 \end{aligned}$$

Stiffness:

$$k_e = \frac{3EI}{L^3} + 2k = \frac{3 \cdot 10^8}{100^3} + 2 \cdot 2,000 = 4,300 \text{ lb/in.}$$

Natural frequency:

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{4,300 \cdot 386}{3,000}} = 23.52 \text{ rad/sec} \\ \omega_D &= \omega \sqrt{1 - \xi^2} = 23.52 \sqrt{1 - 0.15^2} = 23.25 \text{ rad/sec} \end{aligned}$$

Displacement and velocity after $t = 1 \text{ sec.}$

$$\begin{aligned} u(t) &= Ce^{-\xi\omega t} \cos(\omega_D t - \alpha) \\ \dot{u}(t) &= -Ce^{-\xi\omega t} [\xi\omega \cos(\omega_D t - \alpha) + \omega_D \sin(\omega_D t - \alpha)] \end{aligned}$$

where

$$C = \sqrt{u_0^2 + \frac{(\dot{u}_0 + u_0\xi\omega)^2}{\omega_D^2}} = \sqrt{1^2 + \frac{(20 + 0.15 \cdot 23.52)^2}{23.52^2}} = 1.423 \text{ in.}$$

2.2

From problem 1.6

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{192 \cdot 10^9 \cdot 386}{(120)^3 \cdot 5,000}} = 92.61 \text{ rad/sec}$$

Natural frequency:

$$\omega_D = \omega \sqrt{1 - \xi^2} = 92.61 \sqrt{1 - 0.10^2} = 92.15 \text{ rad/sec}$$

$$\begin{aligned} u(t) &= Ce^{-\xi\omega t} \cos(\omega_D t - \alpha) \\ \dot{u}(t) &= -Ce^{-\xi\omega t} [\xi\omega \cos(\omega_D t - \alpha) + \omega_D \sin(\omega_D t - \alpha)] \\ \ddot{u}(t) &= Ce^{-\xi\omega t} [2\xi\omega \omega_D \sin(\omega_D t - \alpha) + (\xi^2\omega^2 - \omega_D^2)\cos(\omega_D t - \alpha)] \end{aligned}$$

For initial conditions ($t = 0 \text{ sec}$)

$$u_0 = 0.5 \text{ in.} \quad \dot{u}_0 = 15 \text{ in./sec}$$

$$\begin{aligned} C &= \sqrt{u_0^2 + \frac{(\dot{u}_0 + u_0 \xi \omega)^2}{\omega_D^2}} = \sqrt{0.5^2 + \frac{(15 + 0.5 \cdot 0.1 \cdot 92.61)^2}{92.15^2}} = 0.5435 \text{ in.} \\ \tan \alpha &= \frac{\dot{u}_0 + u_0 \xi \omega}{\omega_D u_0} = \frac{15 + 0.5 \cdot 0.1 \cdot 92.61}{0.5 \cdot 92.15} = 0.4261 \\ \alpha &= \arctan(0.4261) = 23.08 \end{aligned}$$

Final conditions ($t = 2 \text{ sec}$)

$$\begin{aligned} e^{-\xi\omega t} &= e^{-0.1 \cdot 92.61 \cdot 2} = 9.0364 \cdot 10^{-9} \\ \cos(\omega_D t - \alpha) &= \cos(92.15 \cdot 2 - 23.077) = -0.9468 \\ \sin(\omega_D t - \alpha) &= \cos(92.15 \cdot 2 - 23.077) = 0.3219 \end{aligned}$$

$$\begin{aligned} u(t = 2) &= 0.5435 \cdot 9.0364 \cdot 10^{-9} \cdot -0.9468 \\ \dot{u}(t = 2) &= -0.5435 \cdot 9.0364 \cdot 10^{-9} [-0.1 \cdot 92.61 \cdot 0.9468 + 92.15 \cdot 0.3219] \\ &= -4.084 \cdot 10^{-8} \text{ in./sec} \\ \ddot{u}(t = 2) &= 0.5435 \cdot 9.0364 \\ &\cdot 10^{-9} [2 \cdot 0.1 \cdot 92.61 \cdot 92.15 \cdot 0.3219 - (0.1^2 \cdot 92.61^2 - 92.15^2) \cdot 0.9468] \\ &= 4.18 \cdot 10^{-5} \text{ in./sec}^2 \end{aligned}$$

2.3

The following numerical values are given:

$$k = 200 \text{ lb/in.}$$
$$m = 10 \text{ lb} \cdot \text{sec}^2/\text{in.}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{200 \cdot 10} = 89.443$$

$$u_1 = 1.0 \text{ in.}$$
$$u_2 = 0.95 \text{ in.}$$

$$\delta = \ln \frac{u_1}{u_2} = \ln \frac{1}{0.95} = 0.0513$$

$$\xi = \frac{\delta}{2\pi} = \frac{0.0513}{2\pi} = 0.00816$$

$$c = \xi \cdot c_{cr} = 0.00816 \cdot 89.443 \text{ lb} \cdot \text{sec/in.}$$

2.4

Ratio between first amplitude u_0 and amplitude after k cycles:

$$\ln \frac{u_0}{u_k} = k\delta$$
$$\delta = \frac{1}{k} \ln \frac{u_0}{u_k} = \frac{1}{10} \ln \frac{1}{0.4} = 0.09163$$

Damping:

$$\xi = \frac{\delta}{2\pi} = \frac{0.09163}{2\pi} = 0.01458$$

$$\xi = 1.5\%$$

2.5

a) for $\xi = 1$

$$\begin{aligned} u &= (C_1 + C_2 t)e^{-\frac{c_{cr}}{2m}t} \\ c_{cr} &= 2m\omega \\ u &= (C_1 + C_2 t)e^{-\omega t} \\ \dot{u} &= -\omega(C_1 + C_2 t)e^{-\omega t} + C_2 e^{-\omega t} \end{aligned}$$

At $t = 0$

$$u_0 = C_1 \quad \dot{u}_0 = -\omega C_1 + C_2$$

$$u = [u_0(1 + \omega t) + \dot{u}_0 t]e^{-\omega t}$$

b) for $\xi > 1$

$$\begin{aligned} u &= C_1 e^{P_1 t} + C_2 e^{P_2 t} \\ P_1 &= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{aligned}$$

$$\text{Using } \xi = \frac{c}{c_{cr}} \quad c_{cr} = 2\sqrt{km} \quad \omega = \sqrt{\frac{k}{m}}$$

$$\frac{P_1}{P_2} = -\xi\omega \pm \omega'_D \quad \omega'_D = \omega\sqrt{\xi^2 - 1}$$

$$\begin{aligned} u &= e^{-\xi\omega t}[C_1 e^{\omega'_D t} + C_2 e^{-\omega'_D t}] \\ \dot{u} &= -e^{-\xi\omega t}\xi\omega[C_1 e^{\omega'_D t} + C_2 e^{-\omega'_D t}] + e^{-\xi\omega t}[\omega'_D C_1 e^{\omega'_D t} - C_2 \omega'_D e^{-\omega'_D t}] \end{aligned}$$

at $t = 0$:

$$\begin{aligned} u_0 &= C_1 + C_2 \\ \dot{u}_0 &= -\xi\omega(C_1 + C_2) + \omega'_D(C_1 - C_2) \end{aligned}$$

Solving for C_1 and C_2

$$C_1 = \frac{u_0}{2} + \frac{\dot{u}_0 + \xi\omega u_0}{2\omega'_D}; \quad C_2 = \frac{u_0}{2} - \frac{\dot{u}_0 + \xi\omega u_0}{2\omega'_D}$$

$$\begin{aligned} u &= e^{-\xi\omega t} \left[\left(\frac{u_0}{2} + \frac{\dot{u}_0 + \xi\omega u_0}{2\omega'_D} \right) (\cosh(\omega'_D t) + \sinh(\omega'_D t)) + \left(\frac{u_0}{2} - \frac{\dot{u}_0 + \xi\omega u_0}{2\omega'_D} \right) (\cosh(\omega'_D t) - \sinh(\omega'_D t)) \right] \\ &= e^{-\xi\omega t} \left[u_0 \cosh(\omega'_D t) + \frac{\dot{u}_0 + \xi\omega u_0}{2\omega'_D} \sinh(\omega'_D t) \right] \end{aligned}$$

2.6

The following values are given:

$$k = 30,000 \frac{lb}{in.} \quad \omega = 25 \text{ rad/sec}$$

Damping force:

$$\begin{aligned} F_D &= c\dot{u} \\ c &= \frac{F_D}{\dot{u}} = \frac{1000}{1.0} = 1000 \frac{lb \cdot sec}{in.} \\ \omega^2 &= \frac{k}{m} \quad m = \frac{k}{\omega^2} = \frac{30,000}{25^2} = 48.0 \frac{lb \cdot sec^2}{in.} \\ c_{cr} &= 2\sqrt{km} = 2\sqrt{30,000 \cdot 48.0} = 2,400 \frac{lb \cdot sec^2}{in.} \end{aligned}$$

a)

$$\xi = \frac{c}{c_{cr}} = \frac{1000}{2400} = 0.4167$$

b)

$$T_D = \frac{2\pi}{\omega\sqrt{1-\xi^2}} = \frac{2\pi}{25\sqrt{1-0.4167^2}} = 0.2765 \text{ sec.}$$

c)

$$\delta = \xi\omega T_D = 0.4167 \cdot 25 \cdot 0.2767 = 2.8801$$

d)

$$\ln \frac{u_1}{u_2} = \delta$$

$$\frac{u_1}{u_2} = e^\delta = e^{2.8801} = 17.8161$$

2.7

The peaks occur whenever $\dot{u} = 0$.

$$u(t) = Ce^{-\xi\omega t} \cos(\omega_D t - \alpha)$$

$$\dot{u}(t) = -Ce^{-\xi\omega t} [\xi\omega \cos(\omega_D t - \alpha) + \omega_D \sin(\omega_D t - \alpha)]$$

$$\dot{u}_0 = 0 \quad \text{say time } t_0$$

$$0 = [\xi\omega \cos(\omega_D t_0 - \alpha) + \omega_D \sin(\omega_D t_0 - \alpha)]$$

Then at time $t = t_0 + T_D$

$$\dot{u}(t_0 + T_D) = -Ce^{-\xi\omega(t_0+T_D)} [\xi\omega \cos(\omega_D t_0 + \omega_D T_D - \alpha) + \omega_D \sin(\omega_D t_0 + \omega_D T_D - \alpha)]$$

$$\omega_D T_D = 2\pi$$

Therefore, the bracket in $\dot{u}(t_0 + T_D)$ to equal to zero and there is a peak at this time. So peaks occurs at 2π interval in $\omega_D t$

2.8

The ratio $\frac{u_i}{u_{i+k}}$ may be written as

$$\frac{u_i}{u_{i+k}} = \frac{u_i}{u_{i+1}} \cdot \frac{u_{i+1}}{u_{i+2}} \cdot \frac{u_{i+2}}{u_{i+3}} \cdots \frac{u_{i+k-1}}{u_{i+k}}$$

Taken log on both sides

$$\begin{aligned} \ln\left(\frac{u_i}{u_{i+k}}\right) &= \ln\left[\frac{u_i}{u_{i+1}} \cdot \frac{u_{i+1}}{u_{i+2}} \cdot \frac{u_{i+2}}{u_{i+3}} \cdots \frac{u_{i+k-1}}{u_{i+k}}\right] \\ &= \ln\left(\frac{u_i}{u_{i+1}}\right) + \ln\left(\frac{u_{i+1}}{u_{i+2}}\right) + \ln\left(\frac{u_{i+2}}{u_{i+3}}\right) + \cdots + \ln\left(\frac{u_{i+k-1}}{u_{i+k}}\right) \\ \ln\left(\frac{u_i}{u_{i+k}}\right) &= k\delta \end{aligned}$$

2.10

a) The damping force is

$$c = \frac{\dot{u}}{F_D} = \frac{100}{12} = 8.33 \frac{lb \cdot sec}{in.}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{3,000 \cdot 1} = 109.54 \frac{lb \cdot sec^2}{in.}$$

$$\xi = \frac{c}{c_{cr}} = \frac{8.33}{109.54} = 0.076$$

b) The damping frequency is

$$\omega_D = \omega\sqrt{1 - \xi^2} = 57.77\sqrt{1 - 0.076^2} = 54.61 rad/sec$$

$$\omega = \sqrt{\frac{3000}{1}} = 57.77 \frac{rad}{sec}$$

$$f_D = \frac{\omega_D}{2\pi} = \frac{54.61}{2\pi} = 8.69 cps$$

c) δ is

$$\delta = \frac{2\pi\xi}{\sqrt{1 - \xi^2}} = \frac{2\pi \cdot 0.076}{\sqrt{1 - 0.076^2}} = 0.48$$

d) $\frac{u_1}{u_2}$ is

$$\ln\left(\frac{u_1}{u_2}\right) = \delta = 0.48$$

$$\frac{u_1}{u_2} = e^\delta = e^{0.48} = 1.61$$

2.11

From Problem 2.10

a)

a) The damping force is

$$c_{cr} = 2\sqrt{km} = 2\sqrt{3,000 \cdot 1} = 109.54 \frac{lb \cdot sec^2}{in.}$$

$$\xi = \frac{c}{c_{cr}} = \frac{2}{109.54} = 0.018$$

b) The damping frequency is

$$\omega_D = \omega\sqrt{1 - \xi^2} = 57.77\sqrt{1 - 0.018^2} = 57.76 \text{ rad/sec}$$

$$f_D = \frac{\omega_D}{2\pi} = \frac{57.76}{2\pi} = 9.19 \text{ cps}$$

c) δ is

$$\delta = \frac{2\pi\xi}{\sqrt{1 - \xi^2}} = \frac{2\pi \cdot 0.018}{\sqrt{1 - 0.018^2}} = 0.113$$

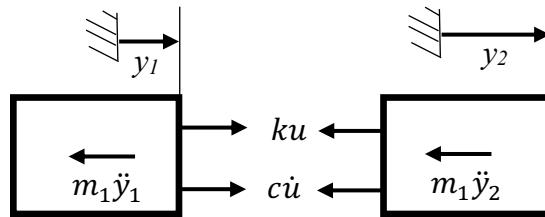
d) $\frac{u_1}{u_2}$ is

$$\ln\left(\frac{u_1}{u_2}\right) = \delta = 0.113$$

$$\frac{u_1}{u_2} = e^\delta = e^{0.113} = 1.12$$

2.14

Let $u = y_1 - y_2$



Newton's law $\sum F = 0$

$$m_2\ddot{y}_2 + c\dot{u} + ku = 0 \quad (Eq. 1)$$

$$m_1\ddot{y}_1 - c\dot{u} - ku = 0 \quad (Eq. 2)$$

Multiply eq. (1) by m_1 , eq. (2) by m_2 and subtract

$$m_1m_2\ddot{u} + (m_1 + m_2)c\dot{u} + (m_1 + m_2)ku = 0$$

where $\ddot{u} = \ddot{y}_1 - \ddot{y}_2$

2.15

Let in Problem 2.14 divide by $(m_1 + m_2)$ and let

$$M = \frac{m_1m_2}{(m_1 + m_2)}$$

Then, $M\ddot{u} + c\dot{u} + ku = 0$

Also,

$$\begin{aligned}\omega &= \sqrt{\frac{k}{M}} \\ \omega_D &= \omega\sqrt{1 - \xi^2} \\ \xi &= \frac{c}{c_{cr}}\end{aligned}$$

Substituting,

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2u = 0$$

3.2

Stiffness:

$$k_b = \frac{48EI}{L^3} = \frac{48 \cdot 30 \cdot 10^6 \cdot 110}{(15 \cdot 12)^3} = 27,160 \text{ lb/in.}$$

Natural frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{27,160 \cdot 386}{1000}} = 102.4 \text{ rad/sec}$$

Force frequency:

$$\bar{\omega} = \frac{900 \cdot 2\pi}{60} = 94.25 \text{ rad/sec}$$

Frequency ratio:

$$r = \frac{\bar{\omega}}{\omega} = \frac{94.25}{102.4} = 0.9204$$

Amplitude of force:

$$F_o = m'e_o\bar{\omega} = \frac{1}{386} 94.25^2 = 23.01 \text{ lb}$$

Amplitude of vertical motion:

$$Y = \frac{F_o/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = \frac{23.01/27,160}{\sqrt{(1-0.92^2)^2 + (2 \cdot 0.1 \cdot 0.92)^2}} = 0.0037 \text{ in.}$$

Note the high dynamic amplification factor $\frac{Y}{y_{st}} = 4.375$; the frequency ratio is close to the point of resonance, that is $r = 1$.

3.3

From Problem 3.2:

$$\begin{aligned}F_o &= 23.01 \text{ lb} \\k &= 27,160 \text{ lb/in.} \\\xi &= 0.1 \\r &= 0.9204\end{aligned}$$

Transmissibility

$$T_r = \frac{A_T}{F_o} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}} = \sqrt{\frac{1 + (2 \cdot 0.1 \cdot 0.92)^2}{(1 - 0.92^2)^2 + (2 \cdot 0.1 \cdot 0.92)^2}} = 4.45$$

Force transmitted to the supports (per support):

$$A_T = \frac{1}{2} F_o T_r = \frac{1}{2} 23.01 \cdot 4.45 = 51.2 \text{ lb}$$

In addition, each support carries one-half of the total motor weight of $W = 1000 \text{ lbs.}$

3.4

Stiffness:

$$k = 2 \frac{12EI}{L^3} = 2 \frac{23 \cdot 30 \cdot 10^6 \cdot 170}{(15 \cdot 12)^3} = 20,990 \text{ lb/in.}$$

Natural frequency:

$$\begin{aligned}\omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{29,990 \cdot 386}{2000 \cdot 20}} = 14.23 \text{ rad/sec} \\r &= \frac{\bar{\omega}}{\omega} = \frac{12}{14.23} = 0.843\end{aligned}$$

Dynamic magnification factor:

$$D = \frac{1}{1 - r^2} = \frac{1}{1 - 0.843^2} = 3.456$$

Steady-state amplitude:

$$Y = y_{st}D = \frac{F_o}{k}D = \frac{5,000}{20,990} \cdot 3.456 = 0.823 \text{ in.}$$

3.5

$$\xi = 0.08$$

From problem 3.4:

$$\begin{aligned} F_o &= 5,000 \text{ lb} \\ k &= 20,990 \text{ lb/in.} \\ r &= 0.843 \end{aligned}$$

Dynamic amplification factor:

$$D = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{1}{\sqrt{(1 - 0.843^2)^2 + (2 \cdot 0.08 \cdot 0.843)^2}} = 3.132$$

Steady-state amplitude:

$$Y = y_{st}D = \frac{F_o}{k}D = \frac{5,000}{20,990} \cdot 3.132 = 0.746 \text{ in.}$$

Note: Structural damping of 8 percent of critical damping reduced in this case. The dynamic magnification factor from 3.456 to 3.132.

3.6

a)

$$A_T = F_o \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}} = 5,000 \sqrt{\frac{1 + (2 \cdot 0.1 \cdot 0.843)^2}{(1 - 0.843^2)^2 + (2 \cdot 0.1 \cdot 0.843)^2}} = 15,803 \text{ lb}$$

b)

$$T_r = \frac{A_T}{F_o} = 3.16$$

3.7

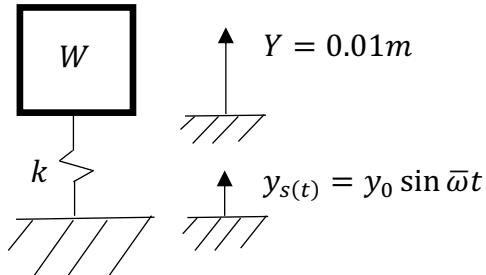
Harmonic motion

$$y_0 = 0.1$$

$$\bar{\omega} = 10 \cdot 2\pi = 62.83 \text{ rad/sec}$$

From equation for transmissibility:

$$T_r = \frac{Y}{y_o} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$



$$\xi = 0$$

$$\pm(1 - r^2) = \frac{y_o}{Y} = \frac{0.1}{0.01} = 10$$

Retain negative root

$$r^2 = 11$$

$$r = 3.317 = \frac{\bar{\omega}}{\omega} = \frac{62.83}{\sqrt{\frac{386 \cdot k}{100}}}$$

$$k = 93 \text{ lb/in.}$$

3.8

Effective force acting on the tower:

$$F_{eff} = -m\ddot{y}_s = -\frac{W}{g} 0.1 \cdot g \cdot \sin \bar{\omega} t$$

$$F_{eff} = -10 \sin(10 \cdot 2\pi \cdot t)$$

$$F_0 = 10 \text{ kip}$$

$$y_{st} = \frac{F_0}{k} = \frac{10}{3000/12} = 0.04$$

Natural Frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000/12}{100/386}} = 31.06 \text{ rad/sec}$$

$$\omega_D = \omega \sqrt{1 - \xi^2} = 30.9 \text{ rad/sec}$$

$$\bar{\omega} = 10 \cdot 2\pi = 62.83 \text{ rad/sec}$$

$$r = \frac{\bar{\omega}}{\omega} = \frac{62.83}{31.06} = 2.023$$

Tower relative to foundation motion by the following equation:

$$U = \frac{y_{st}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{0.04}{\sqrt{(1 - 2.023^2)^2 + (2 \cdot 2.023 \cdot 0.1)^2}} = 0.013 \text{ in.}$$

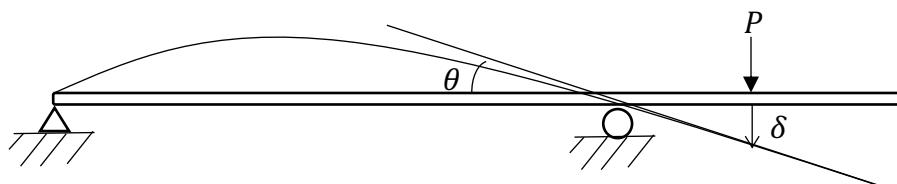
3.9

Transmissibility is given by

$$T_r = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}} = \sqrt{\frac{1 + (2 \cdot 2.023 \cdot 0.1)^2}{(1 - 2.023^2)^2 + (2 \cdot 2.023 \cdot 0.1)^2}} = 0.34$$

3.10

First find the equivalent spring constant, that is the force to produce a unit deflection on the beam at the location of the motor.



$$\theta = \frac{M_0 L^2}{3EI} = \frac{\frac{PL}{4}L^2}{3EI} = \frac{PL^3}{12EI}$$

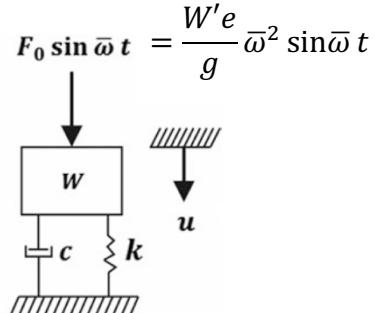
$$\delta = \frac{PL^3}{12EI} \cdot \frac{L}{4} + \frac{P \left(\frac{L}{4}\right)^3}{3EI} = \frac{5PL^3}{192EI}$$

$$k = \frac{P}{\delta} = \frac{192EI}{5PL^3} = \frac{192 \cdot 30 \cdot 10^8}{5 \cdot 120^3} = 66,670 \text{ lb/in.}$$

$$m = \frac{W}{g} = \frac{3330}{386} = 8.63 \frac{lb \cdot sec^2}{in.}$$

$$\omega = \sqrt{\frac{k}{m}} = 87.89 \frac{rad}{sec} = 839 RPM$$

Mathematical model is the damped oscillator:



Amplitude of motion:

$$Y = \frac{\frac{W'e}{g} \bar{\omega}/k}{\sqrt{(1 - r^2)^2 + (2\xi)^2}}$$

$$\bar{\omega}_1 = \frac{800 \cdot 2\pi}{60} = 83.77 rad/sec$$

$$\bar{\omega}_2 = \frac{1000 \cdot 2\pi}{60} = 104.72 rad/sec$$

$$\bar{\omega}_3 = \frac{1200 \cdot 2\pi}{60} = 125.66 rad/sec$$

$$r_1 = \frac{\bar{\omega}_1}{\omega} = 0.95; r_2 = \frac{\bar{\omega}_2}{\omega} = 1.19; r_3 = \frac{\bar{\omega}_3}{\omega} = 1.43$$

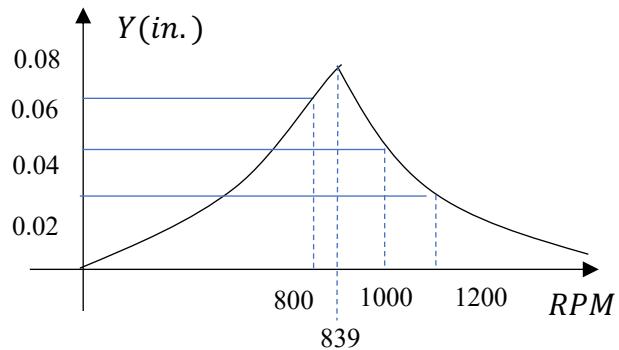
$$y_{st} = \frac{W' \cdot e \cdot \bar{\omega}^2}{g \cdot \frac{k}{m} \cdot m} = \frac{W' \cdot e \cdot r^2}{g \cdot \frac{W}{g}} = \frac{W' \cdot e \cdot r^2}{W}$$

$$Y(r=1) = \frac{y_{st}}{2\xi} = \frac{W'e}{W} = \frac{50}{3300 \cdot 2 \cdot 0.1} = 0.076 in.$$

$$Y_1 = 0.064 in.$$

$$Y_2 = 0.0446 in.$$

$$Y_3 = 0.0302 in.$$



3.14

Let the stiffness of the floor be k . Then the resonant frequency is

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m + m_s}}$$

And the natural frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Dividing

$$\frac{f}{f_n} = \sqrt{\frac{m + m_s}{m}}$$

Then

$$f = f_n \sqrt{1 + \frac{m_s}{m}}$$

3.15

$$D = \frac{Y}{y_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

The peak will occurs when $\frac{dD}{dr} = 0$

$$\frac{dD}{dr} = \frac{2r_p(1 - r_p^2) - 4\xi^2 r_p}{[(1 - r_p^2)^2 + (2\xi r_p)^2]^{3/2}} = 0$$

$$r_p^2 = 1 - 2\xi^2 = \frac{\omega_p^2}{\omega^2}$$

ω_p is the peak frequency.

$$\omega_p^2 = \omega \sqrt{1 - 2\xi^2} \text{ for } \xi < \frac{1}{\sqrt{2}}$$

And

$$Y_p = \frac{y_{st}}{2\xi \sqrt{1 - \xi^2}}$$

The corresponding phase angel is given by the following equation.

$$\theta_p = \tan^{-1} \frac{2\xi r_p}{1 - r_p^2} = \tan^{-1} \frac{2\xi \sqrt{1 - 2\xi^2}}{2\xi^2} = \tan^{-1} \frac{\sqrt{1 - 2\xi^2}}{\xi}$$

Note: These results are, of course, only valid for $2\xi^2 < 1$ or $\xi < \frac{1}{\sqrt{2}} = 0.707$.

Since the amount of damping in structures $\bar{\omega}$ usually small ($\xi < 0.1$) the peak frequency, ω_p may be considered to be the same as the natural frequency ω , that is, the resonant frequency.

3.16

a)

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{89,000 \cdot 386}{2,520}} = 18.58 \text{ cps}$$

b)

$$\xi = \frac{c}{c_{cr}} = \frac{112}{2\sqrt{89,000 \cdot 2520/386}} = 0.0735$$

c)

From solution of Problem 3.15

$$Y_p = \frac{y_{st}}{2\xi\sqrt{1-\xi^2}}$$

$$y_{st} = \frac{F_0}{k}$$

$$F_0 = 2\xi Y_p k \sqrt{1-\xi^2} = 2 \cdot 0.0735 \cdot 0.37 \cdot 89,000 \sqrt{1 - 0.0735^2} = 4825 \text{ lb}$$

d)

At resonance

$$\frac{Y}{y_{st}} = \frac{1}{2\xi}$$

$$F_0 = 2\xi Y k = 2 \cdot 0.0735 \cdot 0.37 \cdot 89,000 = 4840 \text{ lb}$$

3.17

From Eq. in illustrative Example 3.1

$$m\ddot{u} + c\dot{u} + ku = m'e_0\bar{\omega}^2 \sin\bar{\omega}t$$

Using eq. 3.20

$$Y = \frac{\frac{m'e}{k}\bar{\omega}^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = \frac{\frac{m'e}{m}r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

At resonance $r = 1$:

$$Y_r = \frac{m'e}{2m\xi}$$

or

$$2\xi Y_r = \frac{m'e}{m}$$

at

$$r = r_1$$

$$Y_1 = \frac{\frac{m'e}{m}r_1^2}{\sqrt{(1-r_1^2)^2 + (2\xi r_1)^2}}$$

Solve for ξ

$$\begin{aligned} 2\xi Y_r r_1^2 &= Y_1 \sqrt{(1-r_1^2)^2 + (2\xi r_1)^2} \\ 4\xi^2 Y_r^2 r_1^4 &= Y_1^2 [(1-r_1^2)^2 + (2\xi r_1)^2] \end{aligned}$$

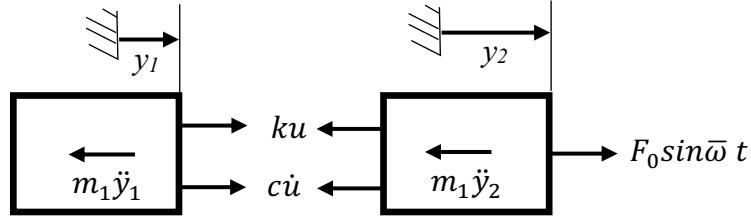
$$\xi = \frac{Y_1(1-r_1^2)}{2r_1\sqrt{Y_r^2 r_1^2 - Y_1^2}}$$

Force at resonance:

$$\begin{aligned} F_r &= m'e\bar{\omega}_r^2 = m'e\frac{k}{m} \\ F_r &= 2\xi Y_r k \\ F_r &= \frac{Y_1 Y_r (1-r_1^2) k}{r_1 \sqrt{Y_r^2 r_1^2 - Y_1^2}} \end{aligned}$$

3.18

Let $u = y_1 - y_2$



$$m_2 \ddot{y}_2 + c\dot{u} + ku = F_0 \sin \bar{\omega} t \quad (\text{Eq. 1})$$

$$m_1 \ddot{y}_1 - c\dot{u} - ku = 0 \quad (\text{Eq. 2})$$

Multiply eq. (1) by m_1 , eq. (2) by m_2 and subtract

$$m_1 m_2 \ddot{u} + (m_1 + m_2) c\dot{u} + (m_1 + m_2) k u = m_1 F_0 \sin \bar{\omega} t$$

divide by $(m_1 + m_2)$ and let

$$M = \frac{m_1 m_2}{(m_1 + m_2)}$$

a)

$$M \ddot{u} + c\dot{u} + k u = \frac{m_1 F_0}{m_1 + m_2} \sin \bar{\omega} t$$

Steady-state solution:

b)

$$u = \frac{\frac{m_1 F_0}{k(m_1 + m_2)} \sin(\bar{\omega} t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\tan \theta = \frac{2\xi r}{1 - r^2}$$

where

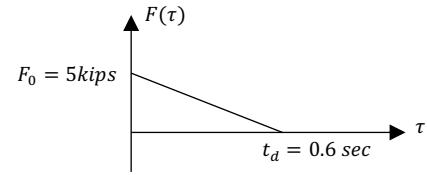
$$r = \frac{\bar{\omega}}{\omega}; \quad \omega = \sqrt{\frac{k}{m}}$$

$$c_{cr} = 2\sqrt{km}$$

$$\xi = \frac{c}{c_{cr}}$$

4.3

Columns are massless and girder are rigid. Neglect damping



$$k_1 + k_2 = \frac{12EI}{L_1^3} + \frac{3EI}{L_2^3}$$

$$k = \frac{12 \cdot 30 \cdot 10^6 \cdot 82.8}{(15 \cdot 12)^3} + \frac{3 \cdot 30 \cdot 10^6 \cdot 82.8}{(20 \cdot 12)^3} = 5111.1 + 539.1 = 5650.2 \frac{\text{lb}}{\text{in}}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5650.2 \cdot 386}{20,000}} = 10.44 \text{ rad/sec}$$

Then, for $t < t_d$:

(a)

$$u(t) = \frac{F_0}{k}(1 - \cos\omega t) + \frac{F_0}{kt_d} \left(\frac{\sin\omega t}{\omega} - t \right)$$

at $t = 0.5 \text{ sec.}$

$$u(t) = \frac{5,000}{5,650.2} (1 - \cos(10.44 \cdot 0.5)) + \frac{5,000}{5,650.2 \cdot 0.6} \left(\frac{\sin(10.44 \cdot 0.5)}{10.44} - 0.5 \right)$$

$$= -0.4072 \text{ in.}$$

(b) Maximum Horizontal Deflection using Chart Fig. 4.5

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10.44} = 0.602 \text{ sec.}$$

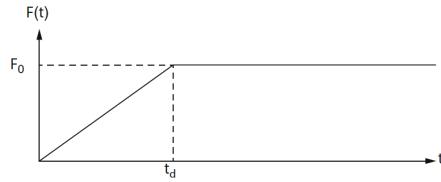
$$\frac{t_d}{T} = \frac{0.6}{0.602} = 0.9972$$

$$\text{for } \frac{t_d}{T} = 0.997 \rightarrow (DLF)_{max} = 1.55$$

$$\therefore \frac{u_{max}}{u_{st}} = 1.55$$

$$u_{max} = 1.55 \cdot u_{st} = 1.55 \cdot \frac{5,000}{5,650.2} = 1.37 \text{ in.}$$

4.5



Determine DLF.

First interval for $t < t_d$:

$$F(\tau) = F_0 \frac{\tau}{t_d}; u_0 = 0; \dot{u}_0 = 0$$

Duhamel's integral

$$u(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t-\tau) d\tau$$

$$u(t) = \frac{1}{m\omega} \int_0^t F_0 \frac{\tau}{t_d} \sin \omega(t-\tau) d\tau$$

where

$$\begin{aligned} \int_0^t \tau \sin \omega(t-\tau) d\tau &= \frac{\tau}{\omega} \cos \omega(t-\tau) - \frac{1}{\omega} \int_0^t \cos \omega(t-\tau) d\tau \\ &= \frac{\tau}{\omega} \cos \omega(t-\tau) + \frac{1}{\omega^2} \sin \omega(t-\tau) \end{aligned}$$

Then:

$$\begin{aligned} u(t) &= \frac{F_0}{m\omega^2 t_d} \left[\tau \cdot \cos \omega(t-\tau) + \frac{1}{\omega} \sin \omega(t-\tau) \right]_0^t \\ &= \frac{F_0}{k t_d} \left(t - \frac{\sin \omega t}{\omega} \right) \quad Eq. 1; \\ \frac{F_0}{k} &= u_{st} \end{aligned}$$

$$DLF = \frac{u(t)}{u_{st}} = \frac{1}{t_d} \left(t - \frac{\sin \omega t}{\omega} \right)$$

First interval for $t \geq t_d$:

$$F(\tau) = F_0; \quad u_0 = u_d; \quad \dot{u}_0 = \dot{u}_d$$

From Eq.1 at $t = t_d$

$$u_d = \frac{F_0}{k t_d} \left(t_d - \frac{\sin \omega t_d}{\omega} \right); \quad u_d = \frac{F_0}{k} \left(1 - \frac{\sin \omega t_d}{\omega t_d} \right)$$

Derivate Eq. 1

$$\dot{u}(t) = \frac{F_0}{kt_d}(1 - \cos \omega t); \quad \dot{u}_d = \frac{F_0}{kt_d}(1 - \cos \omega t_d)$$

The governing equation for the second interval is Eq. 4.4

$$\begin{aligned} u(t) &= u_d \cos \omega t + \frac{\dot{u}_d}{\omega} \sin \omega t + \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau) d\tau \\ u(t) &= u_d \cos \omega t + \frac{\dot{u}_d}{\omega} \sin \omega t + \frac{F_0}{m\omega} \int_0^t \sin \omega(t - \tau) d\tau \\ u(t) &= u_d \cos \omega t + \frac{\dot{u}_d}{\omega} \sin \omega t + \frac{F_0}{m\omega^2} (1 - \cos \omega t) \end{aligned}$$

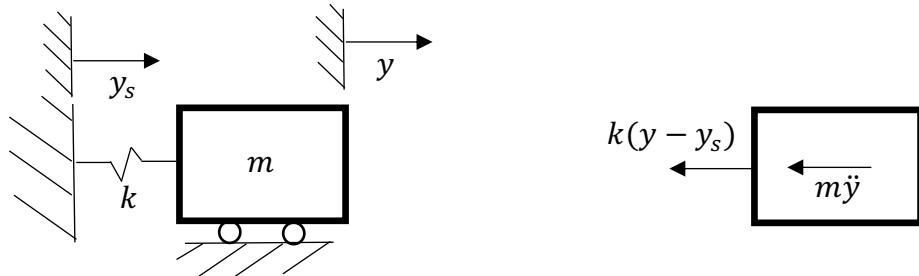
Plugging values of u_d and \dot{u}_d :

$$\begin{aligned} u(t) &= \frac{F_0}{k} \left(1 - \frac{\sin \omega t_d}{\omega t_d} \right) \cos \omega t + \frac{F_0}{\omega k t_d} (1 - \cos \omega t_d) \sin \omega t + \frac{F_0}{k} (1 - \cos \omega t) \\ u(t) &= u_{st} \left(1 - \frac{\sin \omega t_d}{\omega t_d} \right) \cos \omega t + \frac{u_{st}}{\omega} \left(\frac{1}{t_d} - \frac{\cos \omega t_d}{t_d} \right) \sin \omega t + u_{st} (1 - \cos \omega t) \end{aligned}$$

$$\begin{aligned} DLF &= \frac{u(t)}{u_{st}} = \left(1 - \frac{\sin \omega t_d}{\omega t_d} \right) \cos \omega t + \frac{1}{\omega} \left(\frac{1}{t_d} - \frac{\cos \omega t_d}{t_d} \right) \sin \omega t + (1 - \cos \omega t) \\ &= 1 + \frac{1}{\omega t_d} (-\sin \omega t_d \cos \omega t + \sin \omega t - \cos \omega t_d \sin \omega t) \\ DLF &= 1 + \frac{1}{\omega t_d} (\sin \omega t - \sin \omega(t + t_d)) \end{aligned}$$

4.6

The model of the system is the free body diagram.



where

$$u = y - y_s$$

$$\ddot{y} = \ddot{u} + \ddot{y}_s$$

$$m(\ddot{u} + \ddot{y}_s) + ku = 0$$

$$m\ddot{u} + ku = -m\ddot{y}_s$$

$$\ddot{u} + \omega u = -\ddot{y}_s$$

In this case, the acceleration is a constant, $\ddot{y}_s = 0.5g$

$$\ddot{u} + \omega u = -0.5g$$

The solution of this equation is:

$$u = A \cos \omega t + B \sin \omega t + u_p$$

The particular solution is:

$$u_p = C;$$

$$0 + \omega^2 C = -0.5g;$$

$$C = \frac{-0.5g}{\omega^2};$$

$$u_p = \frac{-0.5g}{\omega^2}$$

Then,

$$u = A \cos \omega t + B \sin \omega t - \frac{0.5g}{\omega^2}$$

From initial conditions at $t = 0$; $u_0 = 0$; $\dot{u}_0 = 0$

$$u_0 = A - \frac{0.5g}{\omega^2} = 0, \quad A = \frac{0.5g}{\omega^2}$$

$$\dot{u} = -\omega A \sin \omega t + B \omega \cos \omega t$$

$$\dot{u}_0 = B\omega = 0, \quad B = 0$$

Finally

$$u = \frac{0.5g}{\omega^2} \cos \omega t - \frac{0.5g}{\omega^2}$$

Find u_{max} :

$$\begin{aligned}\dot{u} &= \frac{0.5g}{\omega^2} \sin \omega t = 0 \\ t &= 0; \quad u_0 = 0, \\ t &= \frac{\pi}{\omega}; \quad u = u_{max}\end{aligned}$$

$$u_{max} = \frac{0.5g}{\omega^2} \cos \omega \frac{\pi}{\omega} - \frac{0.5g}{\omega^2} = -\frac{0.5g}{\omega^2} - \frac{0.5g}{\omega^2} = -\frac{g}{\omega^2} = -\frac{386}{109.05} = -3.54 \text{ in.}$$

Maximum Shear forces in columns:

$$\text{Left column } V_{max} = k_1 u_{max} = 5,111.1 (3.54) = 18,093 \text{ lb}$$

$$\text{Right column } V_{max} = k_2 u_{max} = 539.1 (3.54) = 1,908 \text{ lb}$$

4.7

Repeat problem 4.6 for 10% of critical damping.

$$\xi = 0.1; \quad \omega = 10.44 \text{ rad/sec}; \quad k_e = 5650 \text{ lb/sec}; \quad F_{eff} = 10,000 \text{ lb}$$

$$\omega_D = \omega\sqrt{1 - \xi^2} = 10.44\sqrt{1 - 0.1^2} = 10.39 \text{ rad/sec}$$

$$u(t) = \frac{1}{m\omega_D} \int_0^t F(\tau) e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau$$

$$= \frac{10,000}{m\omega_D} \left[\frac{e^{-\xi\omega(t-\tau)}}{(\xi\omega)^2 + \omega_D^2} (\xi\omega \sin \omega_D(t-\tau) + \omega_D \cos \omega_D(t-\tau)) \right]_0^t$$

$$= \frac{10,000}{m\omega_D} \left\{ \frac{1}{(\xi\omega)^2 + \omega_D^2} (0 + \omega_D) - \frac{e^{-\xi\omega t}}{(\xi\omega)^2 + \omega_D^2} [\xi\omega \sin \omega_D t + \omega_D \cos \omega_D t] \right\} =$$

$$= \frac{10,000 \cdot 386}{20,000 \cdot 10.39} \left\{ \frac{10.39}{1.044^2 + 10.39^2} - \frac{e^{-1.044t}}{1.044^2 + 10.39^2} [1.044 \sin 10.39t + 10.39 \cos 10.39t] \right\}$$

$$= 0.17035 \{10.39 - e^{-1.044t} [1.044 \sin 10.39t + 10.39 \cos 10.39t]\}$$

For $u_{max} = 0$;

$$\frac{du}{dt} = e^{-1.044t} [1.044 \cdot (1.044 \sin 10.39t + 10.39 \cos 10.39t) - (10.39 \cdot 1.044 \cos 10.39t - 10.39^2 \sin 10.39t)] = 0$$

$$\sin(10.39t) = 0; \quad 10.39t = k\pi; \quad k = 0, 1, 2, \dots$$

$$t_m = 0.302 \text{ sec}$$

$$u_{max} = 0.17035 [10.39 - e^{-0.3157} (1.044 \sin \pi + 10.39 \cos \pi)] = 0.17035 \cdot 10.39 \cdot 1.729$$

$$= 3.06 \text{ in.}$$

$$\text{Left column } V_{max} = \frac{12EI}{L^3} u_{max} = \frac{12 \cdot 30 \cdot 10^6 \cdot 825}{(15 \cdot 12)^3} 3.06 = 15,640 \text{ lb}$$

$$\text{Right column } V_{max} = k_2 u_{max} = 1,648 \text{ lb}$$

4.19

The stiffness of the frame is

$$k = \frac{3E(2I)}{L^3} = \frac{12 \cdot 30 \cdot 10^6 \cdot 2 \cdot 69.2}{120^3} = 7208.75 \text{ lb/in.}$$

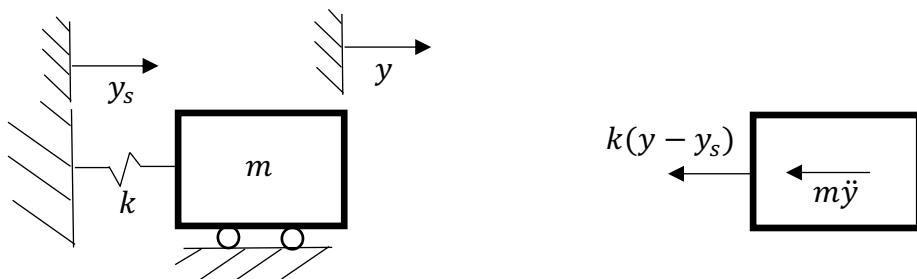
$$m\ddot{u} + k(u - u_s) = 0$$

$$m\ddot{u} + ku = ku_s = F(t)$$

$$m = \frac{20,000}{386} = 51.814 \frac{\text{lb.sec}^2}{\text{in.}}$$

$$F(t) = 7208.75 \cdot u_s(t)$$

t	$F(t)$
0	0
0.25	7208.75
0.5	0
1.0	0



4.20

From Problem 4.19

$$k = 7208.75 \text{ lb/in.}$$

$$m = 51.81 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

$$c_{cr} = 2\sqrt{km} = 1222.21 \frac{\text{lb} \cdot \text{sec}}{\text{in.}}$$

$$c = \xi c_{cr} = 0.1 \cdot 1222.21 = 122.23$$

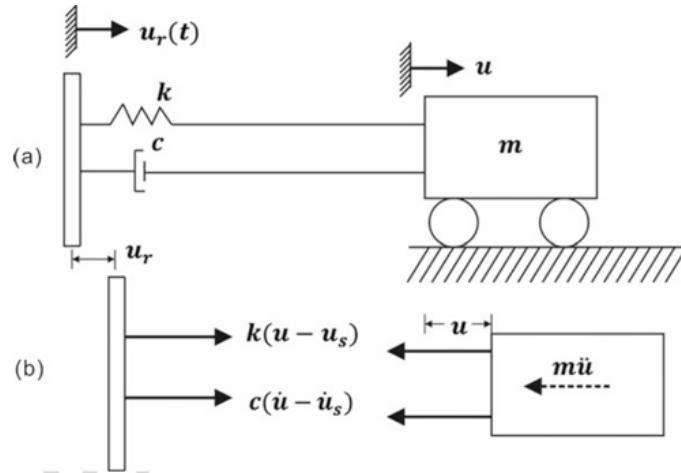
Equation of Motion:

$$m\ddot{u} + c(\dot{u} - \dot{u}_s) + k(u - u_s) = 0$$

$$m\ddot{u} + c\dot{u} + ku = c\dot{u}_s + ku_s = F(t)$$

$$\dot{u}_s = \frac{1}{0.25} = 4 \text{ in.} \rightarrow \text{Assuming uniform speed of support in motion}$$

t	$F(t)$
0	489.0
0.25	7697.5
0.5	489.0
1.0	489.0



4.25

$$m = \frac{1000}{386} = 233.44 \frac{lb \cdot sec^2}{in.}$$

$$k = \frac{3EI}{L_1^3} + \frac{12EI}{L_1^3} = \frac{3 \cdot 30 \cdot 10^6 \cdot 1892.5}{(20 \cdot 12)^3} + \frac{12 \cdot 30 \cdot 10^6 \cdot 1892.5}{(30 \cdot 12)^3} = 26,924 \text{ (lb/in.)}$$

$$c = \xi c_{cr} = 2\xi \sqrt{km} = 490.55 \frac{lb \cdot sec}{in.}$$

$$T = 2\pi \sqrt{\frac{k}{m}} = 0.5724 \text{ sec}$$

$$\bar{\omega} = \frac{10}{2\pi} = 1.5724 \text{ rad/sec}$$

$$\Delta T = \frac{T}{10} \approx 0.05 \text{ sec}$$

$$t_{max} = 2.5 \text{ sec}$$

$$F(t) = 0.1 \sin\left(\frac{10t}{2\pi}\right)$$

4.26

Maximum shear forces:

$$V_{1max} = \frac{3EI}{L_1^3} u_{max} = \frac{3 \cdot 30 \cdot 10^6 \cdot 1892.5}{(20 \cdot 12)^3} 0.342 = 4214 \text{ lb}$$

$$V_{2max} = \frac{12EI}{L_2^3} u_{max} = \frac{12 \cdot 30 \cdot 10^6 \cdot 1892.5}{(30 \cdot 12)^3} 0.342 = 4994 \text{ lb}$$

The bending moments:

$$M_{1max} = V_{1max} \cdot L_1 = 4214 \cdot 240 = 1,011,360 \text{ lb} \cdot \text{in.}$$

$$M_{2max} = V_{2max} \cdot L_2 = 4994 \cdot 360 = 1,797,840 \text{ lb} \cdot \text{in.}$$

And the max. stress:

$$\sigma_{1max} = \frac{M_{1max}}{S} = \frac{1,011,360}{263.2} = 3842 \text{ psi}$$

$$\sigma_{2max} = \frac{M_{2max}}{S} = \frac{1,797,840}{263.2} = 6831 \text{ psi}$$

where $S = 263.2 \text{ in.}^3$ is the section modulus.

To check the calculations using the response in terms of the maximum absolute acceleration, we consider the differential equation for relative motion,

$$m\ddot{y} + c\dot{u} + ku = 0 \quad \text{Eq. (a)}$$

where

$$u = y - y_s \quad \text{Eq. (b)}$$

y_s = support displacement

Solving Eq.(a) for the total force in the column $V = ku$ yields

$$V = -m\ddot{y} - y_s$$

and replacing values from the response obtained in Problem 3.15 at time $t = 1 \text{ sec.}$ which is the time for maximum displacement:

$$\ddot{y}_{max} = 41.308 \text{ in./sec}^2 \quad \dot{u} = -0.036 \text{ in./sec}$$

gives

$$V_{max} = 9212 \text{ lb}$$

This last value checks closely with the total shear force calculated using the relative displacements in the columns, that is,

$$V_{max} = V_{1max} + V_{2max} = 4214 + 4994 = 9208 \text{ lb}$$

$$V_{1max} = \frac{3EI}{L_1^3} u_{max} = \frac{3 \cdot 30 \cdot 10^6 \cdot 1892.5}{(20 \cdot 12)^3} 3.845 = 47,376 \text{ lb}$$

$$V_{2max} = \frac{12EI}{L_2^3} u_{max} = \frac{12 \cdot 30 \cdot 10^6 \cdot 1892.5}{(30 \cdot 12)^3} 3.845 = 56,146 \text{ lb}$$

The bending moments:

$$M_{1max} = V_{1max} \cdot L_1 = 47,376 \cdot 240 = 11,370,240 \text{ lb} \cdot \text{in.}$$

$$M_{2max} = V_{2max} \cdot L_2 = 56,146 \cdot 360 = 20,212,560 \text{ lb} \cdot \text{in.}$$

And the max. stress:

$$\sigma_{1max} = \frac{M_{1max}}{S} = \frac{11,370,240}{263.2} = 43,200 \text{ psi}$$

$$\sigma_{2max} = \frac{M_{2max}}{S} = \frac{20,212,560}{263.2} = 76,795 \text{ psi}$$

where $S = 263.2 \text{ in.}^3$ is the section modulus.

4.27

The deflection δ produced by a force P applied at the center of a fixed beam is given by

$$\delta = \frac{PL^3}{192EI}$$

Therefore, the equivalent spring constant K_S for the beam is

$$K_S = \frac{P}{\delta} = \frac{192EI}{L^3} = \frac{192 \cdot 30 \cdot 10^6 \cdot 348}{(20 \cdot 12)^3} = 145,000 \text{ lb/in.}$$

Since the coil spring and the beam are in series, the combined spring constant K_e for the system is:

$$\frac{1}{K_e} = \frac{1}{K_c} + \frac{1}{K_S} = \frac{1}{18000} + \frac{1}{145000}$$

or

$$K_e = 16,012 \text{ lb/in.}$$

Problem Data:

$$m = \frac{3000}{386} = 7.772 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

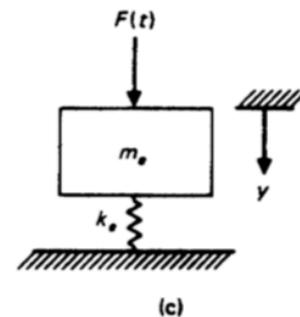
$$k = 16,012 \text{ (lb/in.)}$$

$$c = 0$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 7.224 \text{ cps}$$

$$T = 0.138$$

$$\Delta T = \frac{T}{10} \approx 0.01 \text{ sec}$$



4.28

$$P_{max} = k_e u_{max}$$

$$k_e = 16,012 \text{ lb}$$

$$u_{max} = 0.71 \\ P_{max} = k_e u_{max} = 16,012 \cdot 0.71 = 11,368 \text{ lb}$$

The maximum moment on fixed beam resulting from a concentrated force P_{max} applied at its center is given by,

$$M_{max} = \frac{P_{max}L}{8} = \frac{11,368.5 \cdot 240}{8} = 341,055 \text{ lb} \cdot \text{in.}$$

And the maximum dynamic stress σ_{max} by

$$\sigma_{max} = \pm \frac{M_{max}c}{I} = \pm \frac{341,055 \cdot 5}{348} = \pm 4900 \text{ psi}$$

The static moment M_{st} and the static stress σ_{st} resulting from the weight of the motor are then calculated as

$$M_{st} = \frac{1}{8}WL = \frac{1}{8} \cdot 3000 \cdot 240 = 90,000 \text{ lb} \cdot \text{in.}$$

and

$$\sigma_{st} = \frac{M_{st}c}{I} = \frac{90,000 \cdot 5}{348} = 1293 \text{ psi}$$

The total maximum stresses are then

$$\sigma_{tens} = 4900 + 1293 = 6193 \text{ psi} \\ \sigma_{compr} = 4900 + 1293 = 6193 \text{ psi}$$

Maximum displacement of the beam:

$$u_{B,max} = \frac{P_{max}}{K_S} = \frac{11,368.5}{145,000} = 0.0784 \text{ in.}$$

Maximum stretch of the coil spring:

$$u_{c,max} = 0.71 - 0.0784 = 0.632 \text{ in.}$$

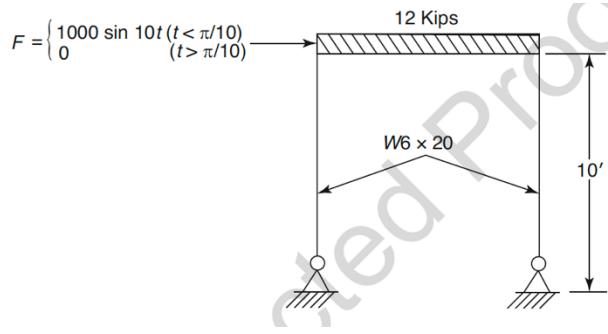
Maximum force $F_{c,max}$ in the coil spring:

$$F_{c,max} = K_c u_{c,max} = 18000 \cdot 0.632 = 11,376 \text{ lb.}$$

5.1

$$F = \begin{cases} 1000 \sin 10t & (t \leq \frac{\pi}{10}) \\ 0 & (t > \frac{\pi}{10}) \end{cases}$$

$$\bar{\omega} = 10 \text{ cps}$$



$$k_1 = \frac{3E(2I)}{L^3} = \frac{3 \cdot 30 \cdot 10^3 \cdot 2 \cdot 82.8}{120^3} = 4.3125 \text{ k/in.}$$

$$\omega = \sqrt{\frac{4.3125 \cdot 386}{12}} = 11.78 \frac{\text{rad}}{\text{sec}}$$

$$f = \frac{\omega}{2\pi} = \frac{11.78}{2\pi} = 1.875 \text{ cps}$$

$$T = \frac{1}{f} = \frac{1}{1.875} = 0.5333 \text{ sec}$$

$$t_d = \frac{\pi}{10} = 0.31416$$

$$u_{st} = \frac{F_o}{k} = \frac{1}{4.3125} = 0.232$$

Using Fig. 5.3

$$\frac{t_d}{T} = 0.589 \rightarrow \frac{u}{u_{st}} \approx 1.6$$

$$u_{max} = 1.6u_{st} = 1.6(0.232) = 0.374 \text{ in.}$$

5.2

From Problem 5.1,

$$u_{max} = 0.374 \text{ in.}$$

$$k = 4.3125 \text{ k/in.}$$

$$E = 30 \cdot 10^3 \text{ ksi}$$

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{V_{max}Lc}{I} = \frac{3EIy_{max}Lc}{L^3 I} = \frac{3EIy_{max}c}{L^2} = \frac{3 \cdot 30 \cdot 10^3 \cdot 0.374 \cdot 3.1}{120^2} = 7.246$$

$$\sigma_{max} = 7.246 \text{ ksi}$$

5.3

Mathematical model:

Effective force

$$F_{eff} = m a(t) = \frac{5}{386} \cdot 200 \sin 10t = 2.5907 \sin 10t$$

$$\bar{\omega} = 10 \text{ rad/sec}$$

$$\bar{\omega} t_d = \pi$$

$$t_d = \frac{\pi}{\bar{\omega}} = \frac{\pi}{10} = 0.1\pi$$

$$k = \frac{12E(2I_1)}{L^3} + \frac{3EI_2}{L^3} = \frac{12 \cdot 30 \cdot 10^3 \cdot 2 \cdot 82.8}{216^3} + \frac{3 \cdot 30 \cdot 10^3 \cdot 2 \cdot 170}{216^3} = 7.4338 \text{ k/in.}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{7.4338 \cdot 386}{5}} = 38.5 \text{ cps}$$

$$T = \frac{1}{f} = \frac{1}{38.5} = 0.260 \text{ sec}$$

$$\frac{t_d}{T} = \frac{0.1\pi}{0.260} = 1.2$$

Using Fig. 5.3

$$\frac{t_d}{T} = 1.2 \rightarrow \frac{u}{u_{st}} \approx 1.6$$

$$u_{st} = \frac{F_o}{k} = \frac{2.5907}{7.4338} = 0.348 \text{ in.}$$

$$u_{max} = 1.6u_{st} = 1.6(0.348) = 0.557 \text{ in.}$$

5.4

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{V_{max}Lc}{I}$$

$$u_{max} = 0.557 \text{ in.}$$

$$V_{max} = \frac{12EIu_{max}}{L^3} \text{ (External column)}$$

$$V_{max} = \frac{3EIu_{max}}{L^3} \text{ (Internal column)}$$

External columns:

$$\sigma_{max} = \frac{12EIy_{max}Lc}{L^3I} = \frac{12 \cdot 30 \cdot 10^3 \cdot 0.557 \cdot 3.965}{216^2} = 17.0409 \text{ ksi}$$

Internal columns:

$$\sigma_{max} = \frac{3EIy_{max}Lc}{L^3I} = \frac{3 \cdot 30 \cdot 10^3 \cdot 0.557 \cdot 4.875}{216^2} = 5.2380 \text{ ksi}$$

5.5

From problem 5.1:

$$f = 1.875 \text{ cps}$$

From Fig. 5.8 with $\xi = 0.1$

$$S_D = 1.9 \text{ in.}$$

$$S_V = 22.4 \text{ in./sec}$$

$$S_a = 0.68g$$

5.6

From Fig. 5.9 with

$$f = 1.575 \text{ cps}$$

And for $\xi = 0.1$

$$S_D = 4.0 \cdot 0.32 = 1.28 \text{ in.}$$

$$S_V = 48 \cdot 0.32 = 15.36 \text{ in./sec}$$

$$S_a = 1.5 \cdot 0.32 = 0.48g$$

5.7

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8 \cdot 386}{400}} = 2.778 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = \frac{2.778}{2\pi} = 0.442 \text{ cps}$$

From Fig. 5.8

$$S_D = 11.0 \text{ in.}$$

$$(F_S)_{max} = K \cdot S_D = 8 \cdot 11.0 = 88.0 \text{ kips}$$

5.8

From Fig. 5.8, with $f = 0.442 \text{ cps}$ and $\xi = 0.1$

$$S_D = 4.5 \text{ in.}$$

$$(F_S)_{max} = K \cdot S_D = 8 \cdot 4.5 = 36.0 \text{ kips}$$

5.9

Force transmits to Foundation:

$$m\ddot{y} + c\dot{u} + ku = 0$$

$$F_T = c\dot{u} + ku = m\ddot{y}$$

$$(F_T)_{max} = m\ddot{y}_{max} = mS_a$$

From Fig. 5.8 for $f = 0.442 \text{ cps}$ and $\xi = 0.1$:

$$S_a = 0.119$$

$$(F_T)_{max} = 400 \cdot 0.119 = 44.0 \text{ kips}$$

5.10

From Problem 5.7: Max. Force in spring $(F_S)_{max} = 88.0 \text{ kips}$

$$R_T = \frac{88.0}{2} = 44 \text{ kips}$$

$$R_C = -44 \text{ kips}$$

From Fig. 5.14 with $\mu = 2.0$,

$$T = \frac{1}{f} = \frac{1}{0.442} = 2.26 \text{ sec:}$$

$$S_D = 6.0 \text{ in.}$$

yield point:

$$y_t = \frac{R_T}{k} = \frac{44}{8} = 5.5$$

$$\mu = \frac{6.0}{5.5} = 1.1$$

From Fig. 5.14 with $\mu = 1.25$, $T = 2.26 \text{ sec}$,

$$S_D = 10 \text{ in.}$$

$$\mu = \frac{10}{5.5} = 1.8$$

From Fig. 5.14 with $\mu = 1.5$, $T = 2.26 \text{ sec}$,

$$S_D = 8 \text{ in.}$$

$$\mu = \frac{8}{5.5} = 1.45$$

5.11

From Problem 5.10 $T = 2.26 \text{ sec}$

From Fig. 5.15 with $T = 2.26 \text{ sec}$

$$\mu = 2.0, \quad S_D = 3 \text{ in.}$$

From Problem 5.8 $(F_S)_{max} = 36.00$

$$R_T = \frac{36}{2} = 18 \text{ kips}$$

$$y_{yield} = \frac{R_T}{k} = \frac{18}{8} = 2.25$$

$$\mu = \frac{S_D}{y_{yield}} = \frac{3.0}{2.25} = 1.33$$

From Fig. 5.15 with $T = 2.26 \text{ sec}$

$$\mu = 1.50 \quad S_D = 4.5 \text{ in.}$$

$$\mu = \frac{S_D}{y_{yield}} = \frac{4.0}{2.25} = 1.8$$

5.12

a) From Fig. 5.8 with

$$f = \frac{1}{T} = \frac{1}{0.5} = 2.0 \text{ cps}, \quad \omega = 12.566 \text{ rad/sec}$$

$$S_D = 4.0 \text{ in.}$$

$$S_V = 50.3 \text{ in./sec}$$

$$S_a = 1.63g$$

a) From Fig. 5.9 with

$$f = \frac{1}{T} = \frac{1}{0.5} = 2.0 \text{ cps}, \quad \xi = 0$$

$$S_D = 16.0 \cdot 0.3 = 4.8 \text{ in.}$$

$$S_V = 200 \cdot 0.3 = 60.0 \text{ in./sec}$$

$$S_a = 6.54g \cdot 0.3 = 1.96g$$

5.13

a) $T = 0.5 \text{ sec}$

From Fig. 5.12 for $\mu = 4$

$$S_D = 0.48 \text{ in.} \times 4 = 1.92 \text{ in.}$$

$$S_V = 6.0 \text{ in./sec}$$

$$S_a = 0.2g$$

b) From Fig. 5.18

$$f = 2.0 \text{ cps}, \quad \mu = 4$$

$$S_D = 5 \cdot 4 \cdot 0.3 = 6.0 \text{ in.}$$

$$S_V = 60 \cdot 0.3 = 18.0 \text{ in./sec}$$

$$S_a = 2g \cdot 0.3 = 0.6g$$

6.7

Ductility factor μ is defined as the ratio of maximum displacement to the yield displacement

From Problem 6.2

Max. displacements:

$$u_{max} = 2.56 \text{ in.}$$

Yield displacements:

$$u_y = \frac{R}{k} = \frac{30k}{20k/in.} = 1.5 \text{ in.}$$

Ductility ratio:

$$\mu = \frac{u_{max}}{u_y} = \frac{2.56}{1.5} = 1.7$$

7.1

$$k_1 = \frac{12EI}{L_1^3} = \frac{12 \cdot 5 \cdot 10^8}{(12 \cdot 15)^3} = 1028.81 \text{ lb/in.}$$

$$k_2 = \frac{12EI}{L_2^3} = \frac{12 \cdot 2.5 \cdot 10^8}{(12 \cdot 12)^3} = 1004.69 \text{ lb/in.}$$

$$m_1 = \frac{3860}{386} = 10 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

$$m_2 = \frac{1930}{386} = 5 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

For free vibration:

$$\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Non-trivial solution:

$$\begin{vmatrix} 2033.50 - 10\omega^2 & -1004.69 \\ -1004.69 & 1004.69 - 5\omega^2 \end{vmatrix} = 0$$

$$50\omega^4 - 20214.4\omega^2 + 1033.63 = 0$$

$$\omega_1^2 = 60.05, \quad \omega_1 = 7.75 \text{ rad/sec}$$

$$\omega_2^2 = 344.23, \quad \omega_2 = 18.55 \text{ rad/sec}$$

Set $\omega_1^2 = 60.05$

$$1433a_{11} - 1004.69a_{21} = 0$$

$$a_{11} = 1.000$$

$$a_{21} = 1.426$$

Set $\omega_2^2 = 344.24$

$$1433a_{12} - 1004.69a_{22} = 0$$

$$a_{12} = 1.000$$

$$a_{22} = -1.402$$

Normalize first modes by factor:

$$\sqrt{\sum m_i a_{i1}^2} = 4.491$$

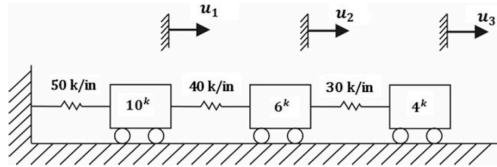
$$\sqrt{\sum m_i a_{i2}^2} = 4.453$$

$$\phi_{11} = \frac{1.000}{4.491} = 0.2227, \quad \phi_{12} = \frac{1.000}{4.453} = 0.2246$$

$$\phi_{21} = \frac{1.426}{4.491} = 0.3175, \quad \phi_{22} = \frac{-1.402}{4.453} = -0.3148$$

$$[\Phi] = \begin{bmatrix} 0.2227 & 0.2246 \\ 0.3175 & -0.3148 \end{bmatrix}$$

7.2



$$k_1 = 50,000 \text{ lb/in.}$$

$$k_2 = 40,000 \text{ lb/in.}$$

$$k_3 = 30,000 \text{ lb/in.}$$

$$W_1 = 10k, \quad W_2 = 6k, \quad W_3 = 4k$$

$$M_1 = 25.91 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}, \quad M_2 = 15.54 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}, \quad M_3 = 10.36 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}$$

$$[K] = \begin{bmatrix} 90000 - 25.91\omega^2 & -40000 & 0 \\ -40000 & 70000 - 15.54\omega^2 & -30000 \\ 0 & -30000 & 30000 - 10.36\omega^2 \end{bmatrix}$$

$$\begin{aligned} |K| &= (90000 - 25.91\omega^2)[(70000 - 15.54\omega^2)(30000 - 10.36\omega^2) - (-30000)(-30000)] \\ &\quad + 40000[-40000(30000 - 10.36\omega^2)] = 0 \\ &\quad -4169.641\omega^6 + 45346360\omega^4 - 1.2173 \cdot 10^4\omega^2 + 6.0 \cdot 10^{13} = 0 \end{aligned}$$

$$\begin{aligned} \omega_1^2 &= 633.78, \quad \omega_1 = 25.17 \text{ rad/sec} \\ \omega_2^2 &= 3244, \quad \omega_2 = 56.96 \text{ rad/sec} \\ \omega_3^2 &= 6996, \quad \omega_3 = 83.64 \text{ rad/sec} \end{aligned}$$

$$(90000 - 25.91\omega^2)a_{11} - 40000a_{21} = 0$$

$$\begin{aligned} a_{11} &= 1, \quad a_{21} = 1.8396, \\ -30000a_{21} + (30000 - 10.36\omega_1^2)a_{31} &= 0 \\ a_{31} &= 2.3551 \\ [a] &= \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.8396 & 0.1490 & -2.2804 \\ 2.3551 & -1.2362 & 1.6102 \end{bmatrix} \end{aligned}$$

$$j_1 = \sqrt{(25.91)(1)^2 + (15.52)(1.8396)^2 + (10.36)(2.3551)^2} = 11.6602$$

$$j_2 = \sqrt{(25.91)(1)^2 + (15.52)(0.1490)^2 + (10.36)(1.2362)^2} = 6.4875$$

$$j_3 = \sqrt{(25.91)(1)^2 + (15.52)(2.2804)^2 + (10.36)(1.6102)^2} = 11.5604$$

$$\phi_{ij} = \frac{a_{ij}}{j_i}$$

$$[\Phi] = \begin{bmatrix} 0.0858 & 0.1541 & 0.0865 \\ 0.1578 & 0.0230 & -0.1973 \\ 0.2020 & -0.1906 & 0.1393 \end{bmatrix}$$

7.3

$$k_1 = \frac{3E(2I)}{L_1^3} + \frac{12EI}{L_1^3} = \frac{18 \cdot 1 \cdot 10^6}{(144)^3} = 6.028 \text{ k/in.}$$

$$k_2 = \frac{12E(2I)}{L_2^3} = \frac{24 \cdot 10^6}{(120)^3} = 13.889 \text{ k/in.}$$

$$M_1 = \frac{2 \cdot 40}{386} = 0.2073 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}, \quad M_2 = \frac{3 \cdot 40}{386} = 0.1154 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}$$

For free vibration:

$$\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Non-trivial solution:

$$\begin{vmatrix} 19.917 - 0.2073\omega^2 & -13.889 \\ -13.889 & 13.889 - 0.1554\omega^2 \end{vmatrix} = 0$$

$$\begin{aligned} 0.03221\omega^4 - 5.9743\omega^2 + 83.7229 &= 0 \\ \omega_1^2 &= 15.27, \quad \omega_1 = 3.91 \text{ rad/sec} \\ \omega_2^2 &= 170.21, \quad \omega_2 = 13.046 \text{ rad/sec} \end{aligned}$$

Set $\omega_1^2 = 15.27$

$$\begin{aligned} 16.7515a_{11} - 13.889a_{21} &= 0 \\ a_{11} &= 1.000 \\ a_{21} &= 1.206 \end{aligned}$$

Set $\omega_2^2 = 170.31$

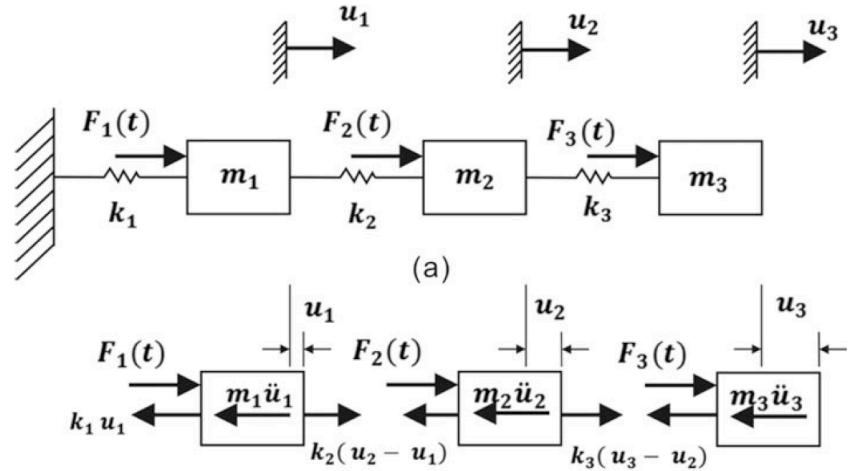
$$\begin{aligned} -15.3675a_{12} - 13.889a_{22} &= 0 \\ a_{12} &= 1.000 \\ a_{22} &= -1.1065 \end{aligned}$$

Normalize first modes by factor:

$$\begin{aligned} \sqrt{\sum m_i a_{i1}^2} &= 0.6583 \\ \sqrt{\sum m_i a_{i2}^2} &= 0.6306 \end{aligned}$$

$$[\Phi] = \begin{bmatrix} 1.5191 & 1.5858 \\ 1.8321 & -1.7547 \end{bmatrix}$$

7.5



$$m\ddot{u}_1 + 2ku_1 - ku_2 = 0$$

$$m\ddot{u}_2 - 2ku_1 + 2ku_2 - ku_3 = 0$$

$$m\ddot{u}_3 - ku_2 + ku_3 = 0$$

In general using matrices:

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

Where

$$[M] = m[I]_{N \times N}$$

$$[K] = \begin{bmatrix} 2 & -1 & & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ & -1 & 2 & -1 & 0 & 0 \\ & & -1 & 2 & -1 & 0 & 0 \\ & & & & \dots & & \\ & & & & & \dots & \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \\ & & & & & 0 & -1 \end{bmatrix}$$

7.6

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 50000 & -20000 & 0 \\ -20000 & 30000 & -10000 \\ 0 & -10000 & 10000 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

$$\omega_1^2 = 59.824, \quad \omega_2^2 = 260.84, \quad \omega_3^2 = 512.67,$$

$$[\Phi] = \begin{bmatrix} 0.03164 & 0.05429 & 0.05213 \\ 0.06490 & 0.02952 & -0.07012 \\ 0.09259 & -0.09703 & 0.4485 \end{bmatrix}$$

$$\begin{aligned} \omega_1 &= 7.73 \text{ rad/sec} \\ \omega_2 &= 16.15 \text{ rad/sec} \\ \omega_3 &= 22.64 \text{ rad/sec} \end{aligned}$$

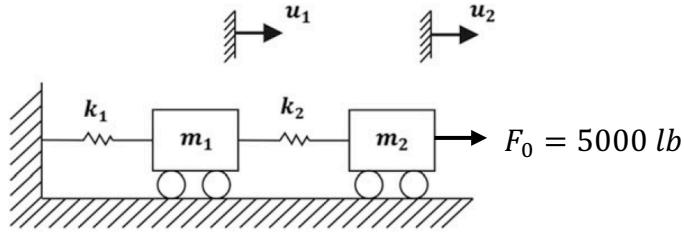
$$u_1 = C_1 \phi_{11} \sin \omega_1 t + C_2 \phi_{11} \cos \omega_1 t + C_3 \phi_{12} \sin \omega_2 t + C_4 \phi_{12} \cos \omega_2 t + C_5 \phi_{13} \sin \omega_3 t + C_6 \phi_{13} \cos \omega_3 t$$

$$u_2 = C_1 \phi_{21} \sin \omega_1 t + C_2 \phi_{21} \cos \omega_1 t + C_3 \phi_{22} \sin \omega_2 t + C_4 \phi_{22} \cos \omega_2 t + C_5 \phi_{23} \sin \omega_3 t + C_6 \phi_{23} \cos \omega_3 t$$

$$u_3 = C_1 \phi_{31} \sin \omega_1 t + C_2 \phi_{31} \cos \omega_1 t + C_3 \phi_{32} \sin \omega_2 t + C_4 \phi_{32} \cos \omega_2 t + C_5 \phi_{33} \sin \omega_3 t + C_6 \phi_{33} \cos \omega_3 t$$

8.1

Mathematical model:



$$k_1 = \frac{2 \cdot 12EI}{L_1^3} = \frac{2 \cdot 12 \cdot 30 \cdot 10^6 \cdot 248.6}{(12 \cdot 15)^3} = 30700 \text{ lb/in.}$$

$$k_2 = \frac{2 \cdot 12EI}{L_2^3} = \frac{2 \cdot 12 \cdot 30 \cdot 10^6 \cdot 106.3}{(12 \cdot 10)^3} = 44300 \text{ lb/in.}$$

$$m_1 = \frac{52500}{386} = 136 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

$$m_2 = \frac{25500}{386} = 66 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

From illustrative example 7.1 and 7.2

$$\begin{aligned}\omega_1 &= 11.83 \text{ rad/sec} \\ \omega_2 &= 32.9 \text{ rad/sec}\end{aligned}$$

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

Modal equations:

$$\begin{aligned}\ddot{q}_1 + 140q_1 &= 0.0813 \cdot 5000 = 406.5 \\ \ddot{q}_2 + 1082q_2 &= -0.0924 \cdot 5000 = -462.0\end{aligned}$$

Solution for constant force with zero initial conditions is:

$$\begin{aligned}q_1 &= q_{1st}(1 - \cos 11.83t), & q_{1st} &= \frac{406.5}{140} = 2.90 \\ q_2 &= q_{2st}(1 - \cos 32.9t), & q_{2st} &= \frac{-462.0}{1082} = -0.427\end{aligned}$$

Transforming coordinate, $\{u\} = [\Phi]\{q\}$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} \begin{Bmatrix} 2.90(1 - \cos 11.83t) \\ -0.427(1 - \cos 32.9t) \end{Bmatrix}$$

$$\begin{aligned}u_1 &= 0.2109 - 0.1867 \cos 11.83t - 0.0242 \cos 32.9t \\ u_2 &= 0.1250 - 0.1645 \cos 11.83t + 0.0394 \cos 32.9t\end{aligned}$$

8.2

Use data from Problem 8.1:

$$\begin{aligned} P_1 &= -m_1 \ddot{u}_s = -136 \cdot 0.5 \cdot 386 = -26248 \\ P_2 &= -m_2 \ddot{u}_s = -66 \cdot 0.5 \cdot 386 = -12738 \end{aligned}$$

Modal equations:

$$\begin{aligned} \ddot{q}_1 + 140q_1 &= \phi_{11}P_1 + \phi_{21}P_2 = -0.6437 \cdot 26248 - 0.0813 \cdot 12738 \\ \ddot{q}_1 + 140q_1 &= 17931 \end{aligned}$$

$$\begin{aligned} \ddot{q}_1 + 1082q_1 &= \phi_{12}P_1 + \phi_{22}P_2 = -0.0567 \cdot 26248 + 0.0924 \cdot 12738 \\ \ddot{q}_1 + 1082q_1 &= -311.3 \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{17931}{140}(1 - \cos 11.83t) = 128.1(1 - \cos 11.83t) \\ q_1 &= \frac{-311.3}{1082}(1 - \cos 32.9t) = -0.2874(1 - \cos 32.9t) \end{aligned}$$

Transforming coordinates

$$\{u\} = [\Phi]\{q\}$$

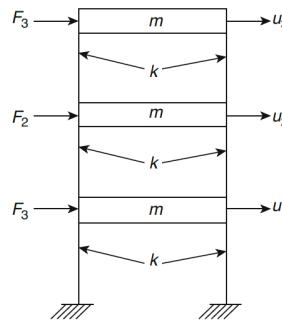
where

$$\begin{aligned} u_{r1} &= u_1 - u_s \\ u_{r2} &= u_2 - u_s \end{aligned}$$

$$\begin{Bmatrix} u_{r1} \\ u_{r2} \end{Bmatrix} = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} \begin{Bmatrix} 128.1(1 - \cos 11.83t) \\ -0.2874(1 - \cos 32.9t) \end{Bmatrix}$$

$$\begin{aligned} u_{r1} &= 8.23 - 8.25 \cos 11.83t - 0.0163 \cos 32.9t \\ u_{r2} &= 10.44 - 10.41 \cos 11.83t - 0.0266 \cos 32.9t \end{aligned}$$

8.3



Stiffness matrix:

$$[K] = \begin{bmatrix} 3000 & -1500 & 0 \\ -1500 & 3000 & -1500 \\ 0 & -1500 & 1500 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 0.3886 & 0 & 0 \\ 0 & 0.3886 & 0 \\ 0 & 0 & 0.3886 \end{bmatrix}$$

Solve Eigenproblem

$$\omega_1^2 = 764.52 \text{ rad/sec}$$

$$\omega_2^2 = 6002.2 \text{ rad/sec}$$

$$\omega_3^2 = 12533 \text{ rad/sec}$$

$$[\Phi] = \begin{bmatrix} 0.52614 & 1.1822 & 0.94808 \\ 0.94808 & 0.52614 & -1.1822 \\ 1.1822 & -0.94808 & 0.52614 \end{bmatrix}$$

Modal Equations:

$$F_1 = 1000 \left(1 - \frac{t}{0.2} \right) \quad (\text{lb})$$

$$F_2 = 2000 \left(1 - \frac{t}{0.2} \right) \quad (\text{lb})$$

$$F_3 = 3000 \left(1 - \frac{t}{0.2} \right) \quad (\text{lb})$$

$$\ddot{q}_1 + \omega_1^2 q_1 = \phi_{11} F_1 + \phi_{21} F_2 + \phi_{31} F_3$$

$$\ddot{q}_2 + \omega_2^2 q_2 = \phi_{12} F_1 + \phi_{22} F_2 + \phi_{32} F_3$$

$$\ddot{q}_3 + \omega_3^2 q_3 = \phi_{13} F_1 + \phi_{23} F_2 + \phi_{33} F_3$$

or

$$\ddot{q}_1 + 764.52 q_1 = 5969 \left(1 - \frac{t}{0.2} \right)$$

$$\ddot{q}_2 + 6002.2 q_2 = -610 \left(1 - \frac{t}{0.2} \right)$$

$$\ddot{q}_3 + 12533q_3 = 162 \left(1 - \frac{t}{0.2}\right)$$

For maximum values use spectral chart Fig. 4.5

$$t_d = 0.2 \text{ sec.}$$

$$\begin{aligned}\omega_1 &= \sqrt{764.52} = 27.65 \text{ rad/sec} \\ \omega_2 &= \sqrt{6002.2} = 77.46 \text{ rad/sec} \\ \omega_3 &= \sqrt{12533} = 111.95 \text{ rad/sec}\end{aligned}$$

$$\begin{aligned}T_1 &= 0.2272 \\ T_2 &= 0.0811 \\ T_3 &= 0.0561\end{aligned}$$

$$\begin{aligned}t_d/T_1 &= 0.88 \\ t_d/T_2 &= 2.466 \\ t_d/T_3 &= 3.565\end{aligned}$$

$$\begin{aligned}(DLF)_{1max} &= 1.5 \\ (DLF)_{2max} &= 1.8 \\ (DLF)_{3max} &= 1.85\end{aligned}$$

$$\begin{aligned}q_{1st} &= \frac{5969}{764.2} = 7.811 \\ q_{2st} &= \frac{-610}{6002.2} = -0.1016 \\ q_{3st} &= \frac{162}{12533} = 0.01292\end{aligned}$$

$$\begin{aligned}q_{1max} &= 7.811 \cdot 1.5 = 11.7165 \\ q_{2max} &= -0.1016 \cdot 1.8 = -0.1829 \\ q_{3max} &= 0.01292 \cdot 1.85 = 0.0239\end{aligned}$$

Estimate maximum response using square root sum of the squares of modal contributions:

$$u_{imax} = \sqrt{\sum_j (\phi_{ij} \cdot q_{imax})^2} \quad i = 1, 2, 3$$

$$\begin{aligned}u_{1max} &= \sqrt{(0.52614 \cdot 11.71)^2 + (1.1822 \cdot 0.1829)^2 + (0.94808 \cdot 0.0239)^2} = 6.16 \text{ in.} \\ u_{2max} &= \sqrt{(0.9408 \cdot 11.71)^2 + (0.52614 \cdot 0.1829)^2 + (1.1822 \cdot 0.0239)^2} = 11.02 \text{ in.} \\ u_{3max} &= \sqrt{(1.1822 \cdot 11.71)^2 + (0.94808 \cdot 0.1829)^2 + (0.52614 \cdot 0.0239)^2} = 13.84 \text{ in.}\end{aligned}$$

8.4

The maximum shear force V_{ij} at story i corresponding to mode j is given by

$$V_{ij} = z_{imax}(\phi_{ij} - \phi_{i-1,j})k_i$$

From 8.3

$$q_{1max} = 7.811 \cdot 1.5 = 11.7165$$

$$q_{2max} = -0.1829$$

$$q_{3max} = 0.0239$$

$$[\Phi] = \begin{bmatrix} 0.52614 & 1.1822 & 0.94808 \\ 0.94808 & 0.52614 & -1.1822 \\ 1.1822 & -0.94808 & 0.52614 \end{bmatrix}$$

$$V_{21} = 11.7165(0.94808 - 0.52614)1500 = 7415.5 \text{ lb}$$

$$V_{22} = -0.1839(0.52614 - 1.1822)1500 = 180.0 \text{ lb}$$

$$V_{23} = 0.0239(-1.1822 - 0.94808)1500 = -76.4 \text{ lb}$$

$$V_2 = \sqrt{7415.5^2 + 180^2 + 76.4^2} = 7418 \text{ lb}$$

8.8

$$k_1 = \frac{12EI}{L_1^3} = \frac{12 \cdot 10 \cdot 10^8}{(12 \cdot 15)^3} = 2058 \text{ lb/in.}$$

$$k_2 = \frac{12EI}{L_2^3} = \frac{12 \cdot 5 \cdot 10^8}{(12 \cdot 12)^3} = 2009 \frac{\text{lb}}{\text{in.}}$$

$$k_3 = \frac{12EI}{L_2^3} = \frac{12 \cdot 5 \cdot 10^8}{(10 \cdot 12)^3} = 3472 \text{ lb/in.}$$

$$m_1 = \frac{3860}{386} = 10 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

$$m_2 = \frac{1930}{386} = 5 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

$$m_3 = \frac{1930}{386} = 5 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}}$$

(a)

$$u_1 = 2.218 \sin t$$

$$u_2 = 3.981 \sin t$$

$$u_3 = 4.419 \sin t$$

(b)

$$u_1 = 4.436 \cos t$$

$$u_2 = 7.963 \cos t$$

$$u_3 = 9.128 \cos t$$

(c)

$$u_1 = 4.436 \cos t + 2.218 \sin t$$

$$u_2 = 7.963 \cos t + 3.981 \sin t$$

$$u_3 = 9.128 \cos t + 4.419 \sin t$$

9.1

(a)

$$[K] = \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix}, \quad [M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Apply Gauss-Jordan elimination to two-first rows:

$$\begin{aligned} & \begin{bmatrix} 1 & -0.2 & -0.1 & 0 \\ 0 & 5.6 & -3.2 & -2 \\ 0 & -3.2 & 11.9 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix} \\ & \begin{bmatrix} 1 & -0.2 & -0.1 & 0 \\ 0 & 5.6 & -0.5714 & -0.3591 \\ 0 & -3.2 & 11.9 & -1 \\ 0 & -2 & -2.1428 & 7.2858 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & -0.2193 & -0.07142 \\ 0 & 1 & -0.5714 & -0.3591 \\ 0 & 0 & 10.0715 & -2.1428 \\ 0 & 0 & -2.1428 & 7.2858 \end{bmatrix} \end{aligned}$$

By Eq. 9.11

$$[\bar{T}] = \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \end{bmatrix} \quad \text{so} \quad [T] = \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Check $[\bar{K}] = [\bar{T}]^T [K] [\bar{T}]$

$$[\bar{K}] = \begin{bmatrix} 0.2193 & 0.5714 & 1 & 0 \\ 0.07142 & 0.3571 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 10.0715 & -2.1425 \\ -2.1428 & 7.2858 \end{bmatrix}$$

Check $[\bar{M}] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

(b)

$$[\bar{K} - \bar{M}\omega^2]\{a\} = \{0\} \quad (\text{Eq. 1})$$

Set determinant equal to zero:

$$\begin{vmatrix} 10.0715 - 3\omega^2 & -2.1425 \\ -2.1428 & 7.2858 - 3\omega^2 \end{vmatrix} = 0$$

$$\begin{aligned} & \omega^4 - 7\omega^2 + 11.4645 = 0 \\ & \omega_1^2 = 2.6137, \quad \omega_1 = 1.6167 \text{ rad/sec} \\ & \omega_2^2 = 4.386, \quad \omega_2 = 2.094 \text{ rad/sec} \end{aligned}$$

From Eq. (1)

For $\omega_1^2 = 2.6137$

$$[10.0715 - 3(2.6137)]a_{11} - 2.1428a_{12} = 0$$

Set

$$\begin{aligned} a_{31} &= 1.0 \\ a_{41} &= 1.0409 \end{aligned}$$

or

$$\begin{aligned} \phi_{31} &= \frac{1}{\sqrt{3 \cdot 1.0^2 + 2 \cdot 1.0409^2}} = 0.4399 \\ \phi_{41} &= \frac{1.0409}{\sqrt{3 \cdot 1.0^2 + 2 \cdot 1.0409^2}} = 0.4579 \end{aligned}$$

For $\omega_2^2 = 4.386$

$$[10.0715 - 3(4.386)]a_{32} - 2.1428a_{42} = 0$$

Set

$$\begin{aligned} a_{31} &= 1.0 \\ a_{41} &= -1.4409 \end{aligned}$$

or

$$\begin{aligned} \phi_{32} &= \frac{1}{\sqrt{3 \cdot 1.0^2 + 2 \cdot 1.4409^2}} = 0.3739 \\ \phi_{42} &= \frac{-1.4409}{\sqrt{3 \cdot 1.0^2 + 2 \cdot 1.4409^2}} = -0.5388 \\ \{\phi\}_{P1} &= \begin{bmatrix} \phi_{31} \\ \phi_{41} \end{bmatrix} = \begin{bmatrix} 0.4399 \\ 0.4579 \end{bmatrix} \\ \{\phi\}_{P2} &= \begin{bmatrix} \phi_{32} \\ \phi_{42} \end{bmatrix} = \begin{bmatrix} 0.3739 \\ -0.5388 \end{bmatrix} \\ [T] &= \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

For the first mode:

$$\{\phi\}_1 = [T]\{\phi\}_{P1} = \begin{bmatrix} 0.1270 \\ 0.4148 \\ 0.4399 \\ 0.4579 \end{bmatrix}, \quad \{\phi\}_2 = [T]\{\phi\}_{P2} = \begin{bmatrix} 0.0416 \\ 0.0212 \\ 0.3739 \\ -0.5388 \end{bmatrix}$$

9.2

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

From Problem 9.1

$$[\bar{T}] = \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Check $[\bar{M}] = [\bar{T}]^T [M] [\bar{T}]$

$$[\bar{M}] = \begin{bmatrix} 0.2193 & 0.5714 & 1 & 0 \\ 0.07142 & 0.3571 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\bar{M}] = \begin{bmatrix} 3.372 & -0.2193 \\ -0.2193 & 2.133 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 10.0715 & -2.1425 \\ -2.1428 & 7.2858 \end{bmatrix}$$

$$[\bar{K} - \bar{M}\omega^2]\{a\} = \{0\} \quad (\text{Eq. 1})$$

Set determinant equal to zero:

$$\begin{vmatrix} 10.0715 - 3.372\omega^2 & -2.1425 - 0.2193\omega^2 \\ -2.1428 - 0.2193\omega^2 & 7.2858 - 2.133\omega^2 \end{vmatrix} = 0$$

$$\begin{aligned} \omega^4 - 6.5767\omega^2 + 9.6281 &= 0 \\ \omega_1^2 = 2.1997, \quad \omega_1 &= 1.4831 \text{ rad/sec} \\ \omega_2^2 = 4.377, \quad \omega_2 &= 2.0921 \text{ rad/sec} \end{aligned}$$

From Eq. (1)

For $\omega_1^2 = 2.1997$

$$[10.0715 - 3.372(2.1997)]a_{31} - [2.1428 + 0.2193(2.1997)]a_{41} = 0$$

Set

$$\begin{aligned} a_{31} &= 1.0 \\ a_{41} &= 1.011 \end{aligned}$$

or

$$\begin{aligned} \phi_{31} &= \frac{1}{2.45} = 0.4082 \\ \phi_{41} &= \frac{1.011}{2.45} = 0.4126 \end{aligned}$$

For $\omega_1^2 = 4.377$

$$[10.0715 - 3.372(2.1997)]a_{31} - [2.1428 + 0.2193(2.1997)]a_{41} = 0$$

Set

$$\begin{aligned} a_{32} &= 1.0 \\ a_{42} &= -1.5126 \end{aligned}$$

or

$$\begin{aligned} \phi_{31} &= \frac{1}{2.755} = 0.3630 \\ \phi_{41} &= \frac{1.011}{2.755} = 0.5490 \end{aligned}$$

To normalize compute:

$$\begin{aligned} \{a\}_1^T [\bar{M}] \{a\}_1 &= [1.0 \quad 1.011] \begin{bmatrix} 3.372 & 0.2193 \\ 0.2193 & 2.133 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.011 \end{bmatrix} = 2.45 \\ \{a\}_2^T [\bar{M}] \{a\}_2 &= [1.0 \quad -1.5126] \begin{bmatrix} 3.372 & 0.2193 \\ 0.2193 & 2.133 \end{bmatrix} \begin{bmatrix} 1.0 \\ -1.5126 \end{bmatrix} = 2.755 \end{aligned}$$

$$\{a\} = [T] \{a\}_p$$

From Problem 9.1:

$$[T] = \begin{bmatrix} 0.2193 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\{\phi\}_{P1} = \begin{bmatrix} \phi_{31} \\ \phi_{41} \end{bmatrix} = \begin{bmatrix} 0.4082 \\ 0.4126 \end{bmatrix}$$

$$\{\phi\}_{P2} = \begin{bmatrix} \phi_{32} \\ \phi_{42} \end{bmatrix} = \begin{bmatrix} 0.3630 \\ -0.5490 \end{bmatrix}$$

$$\{a\}_1 = \begin{bmatrix} 0.2143 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4082 \\ 0.4126 \end{bmatrix} = \begin{bmatrix} 0.1170 \\ 0.3808 \\ 0.4082 \\ 0.4126 \end{bmatrix} \rightarrow \{\phi\}_1 = \begin{bmatrix} 0.1170 \\ 0.3808 \\ 0.4082 \\ 0.4126 \end{bmatrix}$$

$$\{a\}_2 = \begin{bmatrix} 0.2143 & 0.07142 \\ 0.5714 & 0.3571 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3630 \\ -0.5490 \end{bmatrix} = \begin{bmatrix} 0.0386 \\ 0.0113 \\ 0.3630 \\ -0.5490 \end{bmatrix} \rightarrow \{\phi\}_2 = \begin{bmatrix} 0.0386 \\ 0.0113 \\ 0.3630 \\ -0.5490 \end{bmatrix}$$

9.5

From Problem 9.4: Natural frequencies

$$\begin{aligned}\omega_1 &= \sqrt{164} = 12.81 \\ \omega_2 &= \sqrt{1748} = 41.81\end{aligned}$$

From Fig. 5.10: Spectral displacements:

$$\begin{aligned}S_{D1} &= 6.274 \\ S_{D2} &= 0.6119\end{aligned}$$

Participation factors:

$$\Gamma_i = \sum m_j \phi_{ji}$$

$$\Gamma_1 = 1(0.1691) + 1(0.3397) + 1(0.5697) + 1(0.5697) = 2.103$$

Also,

$$\Gamma_2 = 0.606$$

Max. displacement

$$\begin{aligned}u_{i\max} &= \sqrt{\sum (\Gamma_i S_{Dj} \phi_{ji})^2} \\ u_{1\max} &= 2.246 \text{ in.} \\ u_{2\max} &= 4.492 \text{ in.} \\ u_{3\max} &= 6.001 \text{ in.} \\ u_{4\max} &= 7.520 \text{ in.} \\ u_{5\max} &= 7.520 \text{ in.}\end{aligned}$$

9.6

The modal shear force V_{ij} at story i due to mode j is:

$$V_{ij} = \Gamma_i S_{Dj} \Delta \phi_{ij} k_i$$

Where $\Delta \phi_{ij} = \phi_{ij} - \phi_{i-1,j}$ with $\phi_{0j} = 0$

$$\text{And } V_i = \sqrt{\sum_i V_{ij}^2}$$

$$V_1 = \sqrt{(2.104 \cdot 6.274 \cdot 0.1699)^2 + (0.6063 \cdot 0.6119 \cdot 0.3755)^2} \cdot 1949$$

$$\begin{aligned}V_1 &= 4371 \text{ lb} \\ V_2 &= 4002 \text{ lb} \\ V_3 &= 3327 \text{ lb} \\ V_4 &= 2345 \text{ lb} \\ V_5 &= 1410 \text{ lb}\end{aligned}$$

9.7

From Problem 9.5:

$$\begin{aligned}\omega_1^2 &= 2.3131 \\ \omega_2^2 &= 4.1622\end{aligned}$$

From Problem 9.1 and 9.4:

$$\begin{aligned}[K] &= \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix}, \quad [M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \\ [K - M\omega_1^2] &= \begin{bmatrix} 7.6868 & -2 & -1 & 0 \\ -2 & 3.6868 & -3 & -2 \\ -1 & -3 & 5.0604 & -1 \\ 0 & -2 & -1 & 3.3736 \end{bmatrix}\end{aligned}$$

Apply Gauss-Jordan elimination to two-first rows:

$$\begin{bmatrix} 1 & 0 & -0.3980 & -0.1643 \\ 0 & 1 & -1.0297 & -0.6316 \\ 0 & 0 & 1.5129 & -3.0593 \\ 0 & 0 & 3.0593 & 2.1104 \end{bmatrix}$$

$$[T_E] = \begin{bmatrix} T_E \\ I \end{bmatrix} = \begin{bmatrix} 0.3980 & 0.1643 \\ 1.0297 & 0.6316 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The impored modal shapes are:

$$\{a\} = [T]\{a\}_p$$

$$\{\phi\}_{P1} = \begin{bmatrix} \phi_{31} \\ \phi_{41} \end{bmatrix} = \begin{bmatrix} 0.1858 \\ 0.1592 \end{bmatrix}$$

$$\{\phi\}_{P2} = \begin{bmatrix} \phi_{32} \\ \phi_{42} \end{bmatrix} = \begin{bmatrix} 0.1565 \\ -0.2029 \end{bmatrix}$$

$$\{a\}_1 = \begin{bmatrix} 0.3980 & 0.1643 \\ 1.0297 & 0.6316 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1858 \\ 0.1592 \end{bmatrix} = \begin{bmatrix} 0.1001 \\ 0.2910 \\ 0.1858 \\ 0.1592 \end{bmatrix} \rightarrow \{\phi\}_1 = \begin{bmatrix} 0.2004 \\ 0.5844 \\ 0.3720 \\ 0.3187 \end{bmatrix}$$

$$\{a\}_2 = \begin{bmatrix} 0.3980 & 0.1643 \\ 1.0297 & 0.6316 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1565 \\ -0.2029 \end{bmatrix} = \begin{bmatrix} -0.0085 \\ -0.0470 \\ 0.1565 \\ -0.2029 \end{bmatrix} \rightarrow \{\phi\}_2 = \begin{bmatrix} -0.0085 \\ -0.0470 \\ 0.1565 \\ -0.2029 \end{bmatrix}$$

Normalizing factor = 0.6142

9.9

From Problem 9.4

a)

$$\begin{aligned}\omega_1^2 &= 164.56 \\ \omega_2^2 &= 1473 \\ \{\phi\}_1 &= \begin{Bmatrix} 0.1697 \\ 0.3394 \\ 0.5286 \\ 0.6231 \end{Bmatrix}, \quad \{\phi\}_2 = \begin{Bmatrix} 0.3466 \\ 0.6932 \\ -0.1360 \\ -0.5506 \end{Bmatrix}\end{aligned}$$

Modal response:

Participation factors:

$$\Gamma_i = \sum m_j \phi_{ji}$$

$$\Gamma_1 = -[1(0.1697) + 1(0.3394) + 1(0.4340) + 1(0.5256) + 1(0.6331)] = -2.0945$$

Also,

$$\Gamma_2 = 0.6318$$

Spectral response

$$T_1 = \frac{2\pi}{\omega_1} = 0.500, \quad T_2 = \frac{2\pi}{\omega_2} = 0.1637$$

$$\frac{t_d}{T_1} = \frac{0.25}{0.50} = 0.125, \quad \frac{t_d}{T_2} = \frac{0.25}{0.1637} = 1.527$$

$$\frac{z_{1max}}{z_{1st}} = 0.39, \quad \frac{z_{2max}}{z_{2st}} = 1.7$$

$$\begin{aligned}z_{1st} &= \frac{P_0 \Gamma_1}{\omega_1^2} = \frac{400 \cdot 2.0948}{164.56} = 5.09 \\ z_{2st} &= \frac{P_0 \Gamma_1}{\omega_1^2} = \frac{400 \cdot 0.6318}{1473.4} = 0.17\end{aligned}$$

$$z_{1max} = 0.39 \cdot 5.09 = 1.99$$

$$z_{2max} = 1.7 \cdot 0.17 = 0.29$$

b)

u_{ij} = max. displacement at floor "i" and "j" mode relative to base displacement

$$u_{ij} = \sqrt{\sum_j (\phi_{ij} z_{jmax})^2}$$

$$\begin{aligned}u_1 &= \sqrt{(0.1697 \cdot 1.99)^2 + (0.3466 \cdot 0.29)^2} = 0.3523 \text{ in.} \\ u_2 &= \sqrt{(0.3394 \cdot 1.99)^2 + (0.6932 \cdot 0.29)^2} = 0.7172 \text{ in.} \\ u_3 &= \sqrt{(0.4340 \cdot 1.99)^2 + (0.2786 \cdot 0.29)^2} = 0.8674 \text{ in.} \\ u_4 &= \sqrt{(0.5280 \cdot 1.99)^2 + (0.1360 \cdot 0.29)^2} = 1.0526 \text{ in.} \\ u_5 &= \sqrt{(0.6231 \cdot 1.99)^2 + (0.5506 \cdot 0.29)^2} = 1.2502 \text{ in.}\end{aligned}$$

c)

F_{ij} = max. inertial force at floor i in mode j :

$$F_{ij} = m_i \phi_{ij} \omega_j^2 z_{jmax}$$

$$m_i = 1 \frac{lb \cdot sec^2}{in.}$$

$$F_{11} = 1 \cdot 0.1697 \cdot 164.56 \cdot 1.99 = 55.57$$

$$F_{21} = 1 \cdot 0.3394 \cdot 164.56 \cdot 1.99 = 111.14$$

$$F_{31} = 142.12$$

$$F_{41} = 173.10$$

$$F_{51} = 204.05$$

$$F_{12} = 1 \cdot 0.3466 \cdot 1473.4 \cdot 0.29 = 148.10$$

$$F_{22} = 269.19$$

$$F_{32} = 119.04$$

$$F_{42} = -58.11$$

$$F_{52} = -235.26$$

V_{ij} = Max. shear force at story i in mode j

$$V_{ij} = \sum_{k=i}^N F_{kj}$$

$$V_{51} = 204.05$$

$$V_{41} = 377.15$$

$$V_{31} = 519.27$$

$$V_{21} = 630.41$$

$$V_{11} = 685.98$$

$$V_{52} = -235.26$$

$$V_{42} = -293.37$$

$$V_{32} = -174.33$$

$$V_{22} = 97.83$$

$$V_{12} = 245.93$$

V_i = Max. shear force at story i

$$V_i = \sqrt{\sum_{i=1}^2 V_{ij}^2}$$

$$V_1 = 728.73 \text{ lb}, \quad V_2 = 637.96 \text{ lb}, \quad V_3 = 547.50 \text{ lb}, \quad V_4 = 477.81 \text{ lb}, \quad V_5 = 311.42 \text{ lb}$$

10.2

$$m_4 = 0.01 \cdot 100 = 1 \left(lb \frac{\sec^2}{in.} \right)$$

$$m_5 = 1$$

$$[\bar{M}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10.8

From Eq. 10.45

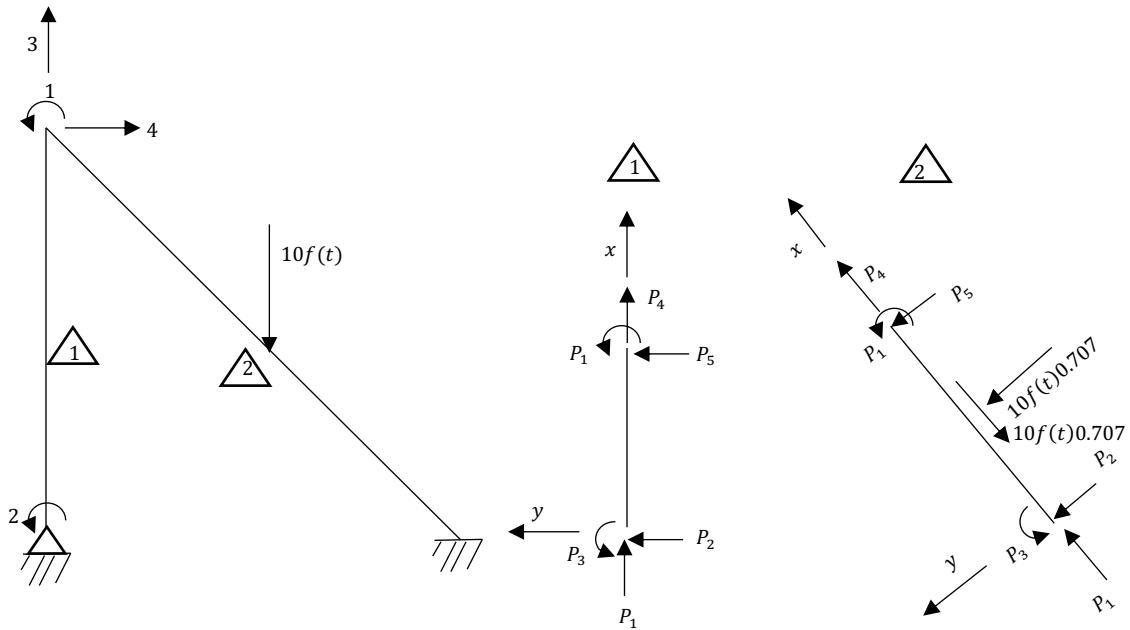
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{N}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

$$[\bar{K}_G] = \frac{10000}{30 \cdot 100} \begin{bmatrix} 36 & 300 & -36 & 300 \\ 300 & 40000 & -300 & -10000 \\ -36 & -300 & 36 & -300 \\ 300 & -10000 & -300 & 40000 \end{bmatrix}$$

System geometric matrix:

$$[\bar{K}_G] = \frac{10}{3} \begin{bmatrix} 80000 & -10000 & 0 & 0 & -300 \\ -10000 & 80000 & -10000 & 300 & 0 \\ 0 & -10000 & 0 & 0 & 300 \\ 0 & 300 & 0 & 72 & -36 \\ -300 & 0 & 300 & -36 & 72 \end{bmatrix}$$

11.8



For member 1

$$P_3 = \int_0^L -P(t)\psi_2(x)dx = -P(t) \int_0^L x \left(1 - \frac{x}{L}\right)^2 dx = -\frac{P(t)L^2}{12} = -633.3P(t)$$

$$P_6 = \frac{P(t)L^2}{12} = 833.3P(t)$$

$$P_5 = -\frac{P(t)L^2}{2} = -50P(t)$$

$$P_1 = P_2 = P_4 = 0$$

For member 2

$$\begin{aligned} P_4 &= P_1 = -3.533f(t) \\ P_2 &= P_5 = 3.533f(t) \end{aligned}$$

$$P_3 = 3.533f(t) \left| x \left(1 - \frac{x}{L}\right)^2 \right|_{x=\frac{L}{2}}$$

$$P_3 = 3.533f(t) \frac{L}{8} = 44.18f(t)$$

$$P_6 = -3.533f(t) \frac{L}{8} = -44.18f(t)$$

11.9

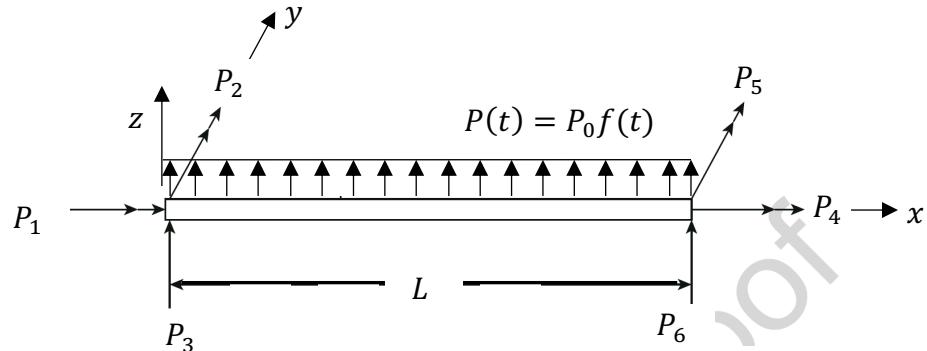
$$F_1 = 833.3P(t) - 44.18f(t)$$

$$F_2 = -833.3P(t)$$

$$F_3 = -5f(t)$$

$$F_4 = -50P(t)$$

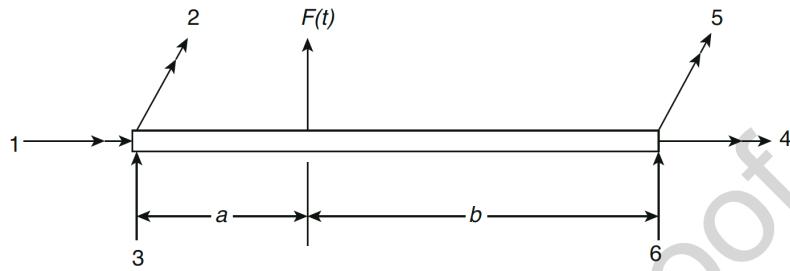
12.10



Equivalent forces = fixed-end reaction with opposite sign.

$$\begin{aligned} P_1 &= 0 \\ P_2 &= -\frac{1}{12}P_0f(t)L^2 \\ P_3 &= \frac{P_0f(t)L}{2} \\ P_4 &= 0 \\ P_5 &= \frac{1}{12}P_0f(t)L^2 \\ P_6 &= \frac{P_0f(t)L}{2} \end{aligned}$$

12.11



$$P_1 = 0$$

$$P_2 = -\frac{F(t)ab^2}{L^2}$$

$$P_3 = \frac{F(t)b^2}{L^2}(3a + b)$$

$$P_4 = 0$$

$$P_5 = \frac{F(t)a^2b}{L^2}$$

$$P_6 = \frac{F(t)a^2}{L^2}(3a + b)$$

i14.2

$$|[K] - [M]\omega^2| = 0$$

Using results from Problem 14.1

$$\begin{vmatrix} 1563 - 7.2\omega^2 & 902 \\ 902 & 4688 - 7.2\omega^2 \end{vmatrix} = 0$$

$$51.84\omega^4 - 45,007\omega^2 + 6,513,740 = 0$$

$$\omega_1^2 = 183.5, \quad f_1 = 2.16 \text{ cps}$$

$$\omega_2^2 = 684.4, \quad f_2 = 4.16 \text{ cps}$$

Modal shapes:

1st mode

$$[1563 - 7.2(183.5)]a_{11} - 902a_{21} = 0$$

Set

$$\begin{aligned} a_{11} &= 1.0 \\ a_{21} &= -0.268 \end{aligned}$$

or

$$\begin{aligned} \phi_{11} &= \frac{1}{2.778} = 0.35996 \\ \phi_{21} &= -\frac{1.011}{2.778} = -0.09647 \end{aligned}$$

2nd mode

$$[1563 - 7.2(684.4)]a_{12} - 902a_{22} = 0$$

Set

$$\begin{aligned} a_{12} &= 1.0 \\ a_{22} &= 3.73 \end{aligned}$$

or

$$\begin{aligned} \phi_{12} &= \frac{1}{10.3627} = 0.09649 \\ \phi_{22} &= \frac{3.73}{10.3627} = 0.35994 \end{aligned}$$

14.5

From Problem 14.2

Natural frequencies:

$$\begin{aligned}\omega_1^2 &= 183.5, & f_1 &= 2.16 \text{ cps} \\ \omega_2^2 &= 684.4, & f_2 &= 4.16 \text{ cps}\end{aligned}$$

$$\begin{aligned}\phi_{11} &= 0.35996 \\ \phi_{21} &= -0.09647 \\ \phi_{12} &= 0.09649 \\ \phi_{22} &= 0.35994\end{aligned}$$

$$T_1 = \frac{1}{2.16} = 0.463 \text{ sec}, \quad T_2 = 0.240, \quad t_d = 0.1 \text{ sec}$$

$$\begin{aligned}\frac{t_d}{T_d} &= 0.216, \quad DL F_{1\max} = \frac{z_{1\max}}{z_{1stat}} = 1.2 \\ \frac{t_d}{T_d} &= 0.417, \quad DL F_{2\max} = \frac{z_{2\max}}{z_{2stat}} = 1.9\end{aligned}$$

$$\begin{aligned}\ddot{q}_1 + \omega_1^2 q_1 &= \phi_{11} F_1 + \phi_{21} F_2 \\ \ddot{q}_2 + \omega_2^2 q_2 &= \phi_{12} F_1 + \phi_{22} F_2\end{aligned}$$

or

$$\begin{aligned}\ddot{q}_1 + 183.4 q_1 &= 10(0.35998) = 3.6, \quad t \leq 0.1 \text{ sec} \\ \ddot{q}_2 + 6002.2 q_2 &= 10(0.09646) = 0.9646, \quad t \leq 0.1 \text{ sec}\end{aligned}$$

$$z_{1,st} = \frac{F_0}{k} = \frac{3.6}{18.34} = 0.196, \quad z_{2,st} = \frac{F_0}{k} = \frac{0.9646}{684.6} = 0.0014$$

$$z_{1,max} = 1.2(0.196) = 0.235, \quad z_{2,max} = 1.9(0.0014) = 0.0026$$

Use SSS rule

$$\begin{aligned}y_{1,max} &= \sqrt{(z_{1,max}\phi_{11})^2 + (z_{2,max}\phi_{12})^2} = \sqrt{(0.235 \cdot 0.36)^2 + (0.0026 \cdot 0.09646)^2} = 0.0846 \text{ in.} \\ y_{2,max} &= \sqrt{(z_{1,max}\phi_{21})^2 + (z_{2,max}\phi_{22})^2} = \sqrt{(-0.235 \cdot 0.09646)^2 + (0.0026 \cdot 0.36)^2} = 0.0227 \text{ in.}\end{aligned}$$

17.1

$$EI = 3.5 \cdot 10^9 \text{ lb} \cdot \text{in.}^2$$

$$W = 150 \frac{\text{lb}}{\text{ft}^3}$$

$$\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{\bar{m}L^4}}$$

$$W = \frac{150}{12^3} = 0.0868 \frac{\text{lb}}{\text{in.}^3}$$

$$\bar{m} = \frac{0.0868 \cdot 24 \cdot 10}{386} = 0.054 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}^3}$$

$$\omega_1 = 13.4638, \quad f_1 = \frac{\omega_1}{2\pi} = 2.14 \text{ cps}$$

$$\omega_2 = 53.8552, \quad f_2 = \frac{\omega_2}{2\pi} = 8.57 \text{ cps}$$

$$\omega_3 = 121.1742, \quad f_3 = \frac{\omega_3}{2\pi} = 19.28 \text{ cps}$$

17.2

From Table 17.3

$$\omega_n = C_n \sqrt{\frac{EI}{\bar{m}L^4}}$$

$$C_1 = 22.3733, \quad C_2 = 61.6828, \quad C_3 = 120.9034$$

From Problem 17.1

$$\omega_n = C_n \frac{13.4638}{\pi^2} = 1.3642 C_n$$

$$\omega_1 = 30.52, \quad f_1 = 4.85 \text{ cps}$$

$$\omega_2 = 84.13, \quad f_2 = 13.39 \text{ cps}$$

$$\omega_3 = 164.93, \quad f_3 = 26.25 \text{ cps}$$

17.3

From Table 17.5

$$\omega_n = C_n \sqrt{\frac{EI}{\bar{m}L^4}}$$

$$\omega_n = C_n \frac{13.4638}{\pi^2} = 1.3642 C_n$$

$$C_1 = 15.4188, \quad C_2 = 49.9648, \quad C_3 = 140.2477$$

From Problem 17.1

$$\omega_n = C_n \frac{13.4638}{\pi^2} = 1.3642 C_n$$

$$\begin{aligned}\omega_1 &= 21.02, & f_1 &= 3.35 \text{ cps} \\ \omega_2 &= 68.16, & f_2 &= 10.85 \text{ cps} \\ \omega_3 &= 142.21, & f_3 &= 22.63 \text{ cps}\end{aligned}$$

17.4

$$y(x, t) = \frac{2P_0}{\bar{m}L} \sum \frac{1}{\omega_n^2} \sin \frac{n\pi x}{L} (1 - \cos \omega_n t) \sin \frac{n\pi x_1}{L}$$

Load

$$P_0 = -2000, \quad \text{at } x_1 = 9(12) = 108 \text{ in.}$$

Deflection at centers

$$x = \frac{(36)(12)}{2} = 216 \text{ in.}$$

For max. deflection

$$(1 - \cos \omega_n t) = 2, \quad \bar{m} = 0.054 \frac{\text{lb} \cdot \text{sec}^2}{\text{in.}^3}, \text{ from problem 17.1}$$

$$y_{max_center} = \frac{-2(2000)}{0.054(36)(12)} \sum \frac{2}{\omega_n^2} \sin \frac{n\pi(108)}{(36 \cdot 12)} \sin \frac{n\pi}{2} = -342.93 \sum \frac{1}{\omega_n^2} \sin(0.7854n) \sin \frac{n\pi}{2}$$

First 3 modes: $\omega_1 = 13.46 \text{ rad/sec}$, From Problem 17.1

$$y_{max_center} = -342.93 \left[\frac{1}{13.46^2} 0.707 + \frac{1}{53.85^2} (0) + \frac{0.707}{121.17^2} (-1) \dots \right]$$

$$= -342.93 [0.0039 - 0.000048]$$

$$= -1.321 \text{ in.}$$

17.5

$$\bar{m} = 0.3 \frac{lb \cdot sec^2}{in.^3}$$

$$EI = 10^6 lb \cdot in.^2$$

$$L = 150 in.$$

Modal differential equation:

$$M_n \ddot{z}_n(t) + \omega_n^2 M_n z_n(t) = F_n(t)$$

where

$$M_n = \int_0^L \bar{m} \phi_n^2(x) dx$$

$$F_n = \int_0^L \phi_n^2(x) P(x, t) dx$$

or

$$\ddot{z}_n + \omega_n^2 z_n = \frac{F(t)}{M_n} = \frac{P_0 \int \phi_n(x) dx}{\int_0^L \bar{m} \phi_n^2(x) dx}$$

$$I_n = \frac{\int \phi_n(x) dx}{\int_0^L \bar{m} \phi_n^2(x) dx}$$

Modal response:

$$z_n(t) = \frac{P_0 I_n}{\omega_n^2} (1 - \cos \omega_n t)$$

Total response:

$$u(t, x) = \sum_n \Phi_n(x) z_n(t), \quad \Phi_n = \sin \frac{n\pi x}{L} \text{ modal shape}$$

$$= \sum_n \frac{P_0 I_n}{\omega_n^2} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L}$$

First mode

$$\omega_1 = C_1 \sqrt{\frac{EI}{\bar{m} L^4}} = \pi^2 \sqrt{\frac{10^6}{0.3 (150)^4}} = 0.8$$

$$f_1 = \frac{\omega_1}{2\pi} = 0.13 cps$$

$$I_n = \frac{4}{\pi}$$

$$y_1(t, x) = \frac{4P_0}{\pi 0.13^2} (1 - \cos 0.13t) \sin \frac{\pi x}{150} = 75.34 P_0 (1 - \cos 0.13t) \sin \frac{x}{47.75}$$

17.6

$$\bar{m} = 0.3 \frac{lb \cdot sec^2}{in.^3}$$

$$EI = 10^6 lb \cdot in.^2$$

$$L = 150 \text{ in.}$$

$$\begin{aligned}\omega_n &= C_n \sqrt{\frac{EI}{\bar{m}L^4}} = C_n \sqrt{\frac{10^6}{0.3(150)^4}} = C_n 0.081 \\ \omega_1 &= \pi^2(0.081) = 0.8 \text{ rad/sec} \\ \omega_2 &= 4\pi^2(0.081) = 3.2 \text{ rad/sec}\end{aligned}$$

Modal differential equation:

$$M_n \ddot{z}_n(t) + \omega_n^2 M_n z_{n(t)} = F_{n(t)}$$

where

$$F_n = \int_0^L \phi_n^2(x) P(x, t) dx = \sin \frac{n\pi(\frac{L}{2})}{L} 1000 \sin 500t$$

$$\text{For } n = 1, \quad F_1(t) = 1000 \sin 500t$$

$$\text{For } n = 2, \quad F_2(t) = 0$$

$$\begin{aligned}M_n &= \int_0^L \bar{m} \phi_n^2(x) dx \\ M_1 &= \int_0^L 0.3 \sin^2 \frac{\pi x}{L} dx = \frac{0.3}{2} \int_0^L \left(1 - \cos \frac{2\pi x}{L}\right) dx = 0.15L = 0.15(150) = 22.5\end{aligned}$$

$$\ddot{z}_n + \omega_n^2 z_n = \frac{1000}{22.5} \sin 500t = 44.44 \sin 500t$$

$$z_1 = \frac{44.44}{0.64 - 500^2} \sin(500t) = 0.00018 \sin(500t)$$

$$y(t) = \phi_1(x)z_1(t) + \phi_2(x)z_2(t) + \dots$$

$$z_2(t) = 0, \quad \text{So consider only first mode}$$

$$y(t) = \frac{\sin \pi x}{150} 0.00018 \sin(500t) = 0.0000038 \sin(500t)$$

17.7

$$\ddot{z}_n(t) + 2\xi_n \omega_n \dot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n}$$

$$M_n = \int_0^L \bar{m} \phi_n^2(x) dx$$

$$M_1 = 22.5$$

$$F_n = \int_0^L \bar{m} \phi(x) P(x, t) dx$$

First mode

$$\begin{aligned} F_1(t) &= 1000 \sin 500t \\ \omega_1 &= 0.8 \text{ rad/sec} \end{aligned}$$

$$\ddot{z}_1(t) + 2 \cdot 0.1 \cdot 0.8 \dot{z}_1(t) + (0.8)^2 z_1(t) = \frac{F_1(t)}{M_1}$$

Solution

$$z(t) = \frac{z_{st} \sin(\bar{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\xi = 0.1, \quad r = \frac{500}{0.8} = 625$$

$$\tan \theta = \frac{2\xi r}{1 - r^2} = \frac{2(0.1)(625)}{1 - 625^2} = -0.00032$$

$$\theta = -0.018^\circ, \quad I_1 = \frac{4}{\pi}$$

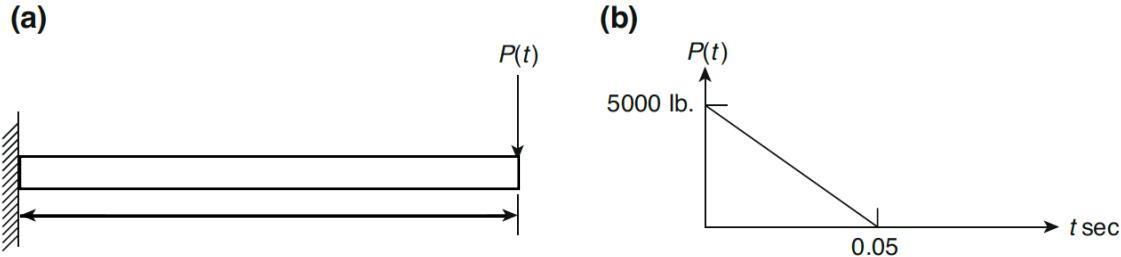
$$z_{st} = \frac{1000}{(0.8)^2} = 1562.5$$

$$z_1(t) = \frac{1562.5}{390624} \sin(500t + 0.0003) = 0.004 \sin(500t + 0.0003)$$

$$y(t) = \phi_1(x)z_1(t) + \phi_2(x)z_2(t) + \dots$$

$$y(t) = 0.004 \frac{\sin \pi x}{L} \sin(500t - 0.003)$$

17.8



$$\bar{m} = 0.5 \frac{lb \cdot sec^2}{in.^3}$$

$$EI = 30 \cdot 10^6 \text{ lb} \cdot in.^2$$

$$L = 100 \text{ in.}$$

$$I = 120 \text{ in.}^4$$

$$\omega_n = 3.516 \sqrt{\frac{EI}{\bar{m}L^4}} = 3.516 \sqrt{\frac{30 \cdot 10^6 (120)}{0.5 (100)^4}} = 24.83 \text{ rad/sec}$$

$$\ddot{z}_1(t) + \omega_1^2 z_1(t) = \frac{F_1(t)}{M_1}$$

$$F_1(t) = \int_0^L \phi_1(x) P(x, t) dx$$

$$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n (\sinh a_n x + \sin a_n x)$$

where

$$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}$$

$$(a_n L)^2 = C_n, \quad \text{Eq.(17.51)and Table 17.4}$$

$$a_n = \frac{\sqrt{3.5160}}{100} = 0.019$$

$$\phi_1(L) = \cosh \sqrt{3.516} - \cos \sqrt{3.516} - 0.734(\sinh \sqrt{3.516} - \sin \sqrt{3.516})$$

$$\sigma_1 = \frac{\cosh \sqrt{3.516} + \cos \sqrt{3.516}}{\sinh \sqrt{3.516} + \sin \sqrt{3.516}} = 0.734$$

$$\begin{aligned}\phi_1(L) &= 0.0248 \\ F_1(t) &= 0.0248 P(t)\end{aligned}$$

$$M_1 = \int_0^L \bar{m} \phi_1^2(x) dx$$

$$I_1 = \frac{\int_0^L \phi_1(x) dx}{\int_0^L \phi_1^2(x) dx} = 0.7830$$

$$\begin{aligned}\int_0^L \phi_1(x) dx &= \int_0^L [\cosh a_1 x - \cos a_1 x - \sigma_1 (\sinh a_1 x - \sin a_1 x)] dx \\ \sigma_1 &= \frac{\cosh a_1 L + \cos a_1 L}{\sinh a_1 L + \sin a_1 L} = 0.507\end{aligned}$$

$$(a_1 L)^2 = C_1 = 3.5160$$

$$a_1 L = \sqrt{3.5160} = 1.875$$

$$\begin{aligned}\int_0^L \phi_1(x) dx &= \frac{1}{a_1} \{ |\sinh a_1 x - \sin a_1 x|_0^{100} - 0.507 |\cosh a_1 x + \cos a_1 x|_0^{100} \} \\ a_1 &= \frac{1.875}{100} = 0.01875 \\ \int_0^L \phi_1(x) dx &= 53.3 \{(3.1837 - 0.9541) - 0.507(3.3371 - 0.2995 - 1 - 1)\} = 90.80\end{aligned}$$

$$\int_0^L \phi_1^2(x) dx = \frac{\int_0^L \phi_1(x) dx}{I_1} = \frac{90.80}{0.783} = 115.96$$

$$F_1(t) = \phi_1(L)P(t) = 0.0248P(t)$$

$$\ddot{z}_1(t) + 29.83^2 z_1(t) = \frac{0.0248P(t)}{115.96} = P_0 f(t), \quad t \leq 0.05$$

$$\begin{aligned}P_0 &= \frac{0.0249(5000)}{115.96} = 1.073f(t) \\ \frac{t_d}{T_1} &= \frac{0.05}{0.21} = 0.24, \quad (DLF)_{max} = 0.7 \quad (\text{From Fig. 4.5})\end{aligned}$$

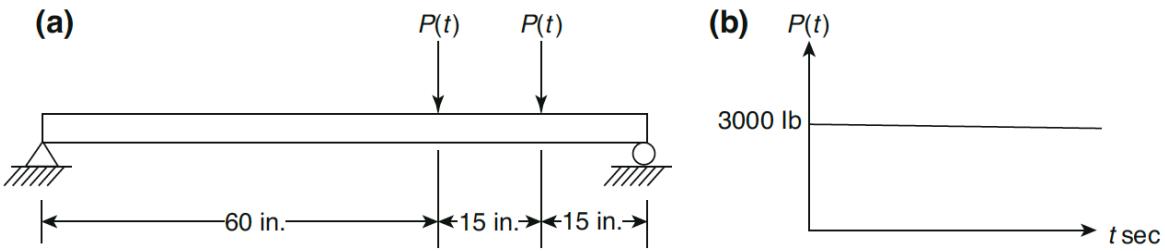
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{29.83} = 0.21$$

$$(DLF)_{max} = \frac{z_{max,1}}{z_{st,1}}$$

$$z_{st,1} = \frac{P_0}{k} = \frac{1.073}{29.83^2} = \frac{1}{829}$$

$$\begin{aligned}z_{max,1} &= (DLF)_{max} z_{st,1} = 0.7 \cdot \frac{1}{829} = 0.00084 \\ z_{max,1} &= \phi_1(x = L) z_{max,1} = 0.0248 \cdot 0.00084 = 2.1 \cdot 10^{-5} \text{ in.}\end{aligned}$$

17.9



$$\bar{m} = 0.5 \frac{lb \cdot sec^2}{in.^3}$$

$$EI = 10^7 lb \cdot in.^2$$

$$L = 90 in.$$

$$y(x, t) = \frac{2P_0}{\bar{m}L} \sum \left[\frac{1}{\omega_n^2} \sin \frac{n\pi x}{L} (1 - \cos \omega_n t) \left(\sin \frac{n\pi x_1}{L} + \sin \frac{n\pi x_2}{L} \right) \right]$$

$$\omega_n = \pi^2 \sqrt{\frac{EI}{\bar{m}L^4}} = \pi^2 \sqrt{\frac{10^7}{0.5 (90)^4}} = 5.45 rad/sec$$

$$y(x, t) = \frac{2P_0}{\bar{m}L} \sum \left[\frac{1}{\omega_n^2} \sin \frac{n\pi x}{L} (1 - \cos \omega_n t) \left(\sin \frac{n\pi x_1}{L} + \sin \frac{n\pi x_2}{L} \right) \right]$$

$$y(x, t) = \frac{-6000}{0.5(90)} \frac{1}{5.43^2} (1 - \cos 5.45 t) \sin \frac{\pi x}{90} \left(\sin \frac{\pi 60}{90} + \sin \frac{\pi 75}{90} \right)$$

$$= -4.49 (1 - \cos 5.45 t) \sin 0.035x (1.366) = -6.13 (1 - \cos 5.45 t) \sin(0.035x)$$

At mid section:

$$\sin(0.035 \cdot 45) = 1.0$$

$$y(45, t) = -6.13(1 - \cos 5.45 t)$$

17.10

$$\ddot{z}_1(t) + \omega_1^2 z_1(t) = \frac{F_1(t)}{M_1}$$

$$F_1(t) = \int_0^L \phi_1(x) P(x, t) dx$$

First mode:

$$\omega_1 = 5.45 rad/sec \quad \text{From Problem 17.9}$$

$$T_1 = \frac{2\pi}{5.45} = 1.15 sec$$

$$M_1 = \int_0^L \bar{m} \phi_1^2(x) dx, \quad \phi_1(x) = \sin \frac{n\pi}{L}$$

$$M_1 = 0.5 \int_0^{90} \frac{(1 - \cos \frac{2\pi}{L} x)}{2} dx = \frac{0.5}{2} \cdot 90 = 22.5$$

$$\ddot{z}_1(t) + 5.45^2 z_1(t) = \frac{-3000}{22.5}, \quad t \leq 0.1 \text{ sec}$$

$$\ddot{z}_1(t) + 5.45^2 z_1(t) = 0, \quad t > 0.1 \text{ sec}$$

$$\frac{t_d}{T} = \frac{0.01}{1.15} = 0.086$$

Max. response from Fig. 4.4:

$$z_{1,max} = 0.5 z_{st}, \quad z_{st} = \frac{3000}{22.5 \cdot 5.45^2} = 4.5$$

$$z_{1,max} = 0.5(4.5) = 2.25$$

$$y_{max,1}(x, t) = \phi_1(x) z_{max,1} = 2.25 \sin \frac{\pi x}{90}$$

at $x = 60''$

$$z_{max,1}(x = 60'') = 1.95 \text{ in.}$$

at $x = 75''$

$$z_{max,1}(x = 75'') = 1.125 \text{ in}$$

17.11

$$\begin{aligned}\bar{m} &= 1 \frac{lb \cdot sec^2}{in.^3} \\ EI &= 30 \cdot 10^6 lb \cdot in.^2 \\ L &= 180 in. \\ P(x, t) &= 2000 \sin 400t\end{aligned}$$

$$t_d = \frac{\pi}{400} sec$$

Modal differential equation for first modes:

$$\ddot{z}_1 + \omega_1^2 z_1 = \frac{F(t)}{M_1} = \frac{\int P(x, t) \phi_1(x) dx}{\int_0^L \bar{m} \phi_1^2(x) dx}, \quad t \leq t_d$$

$$\omega_1 = 22.3733 \sqrt{\frac{EI}{\bar{m}L^4}} = 22.3733 \sqrt{\frac{30 \cdot 10^6}{1(180)^4}} = 3.78 rad/sec$$

$$\begin{aligned}\ddot{z}_1 + 3.78^2 z_1 &= \frac{P_0 I_1}{\bar{m}} \sin \bar{\omega} t, \quad t \leq t_d, \quad I_1 = 0.8308 \text{ (Table 17.3)} \\ \ddot{z}_1 + 3.78^2 z_1 &= 0, \quad t > t_d\end{aligned}$$

$$\ddot{z}_1 + 14.292 z_1 = \frac{2000(0.8308)}{1.0} \sin 400t = 1662 \sin 400t$$

$$T_1 = \frac{2\pi}{3.78} = 1.66 sec$$

$$\frac{t_d}{T} = \frac{\pi}{400 \cdot 1.66} = 0.0047$$

$$\left(\frac{z}{z_{st}}\right)_{max,1} = 0.025 \quad (\text{From Fig. 5.3})$$

$$z_{st} = \frac{F_0}{k} = \frac{1662}{14.29} = 116.3$$

$$z_{max,1} = 0.025 (116.3) = 2.91$$

$$y_{max,1} = \phi_1(x) z_{max,1}$$

$$\phi_1(x) = \cosh a_1 x - \cos a_1 x - \sigma_1 (\sinh a_1 x - \sin a_1 x)$$

$$\sigma_1 = \frac{\cosh a_1 L + \cos a_1 L}{\sinh a_1 L + \sin a_1 L}$$

$$(a_1 L)^2 = C_1 = 22.3733$$

$$a_1 L = \sqrt{22.3733} = 4.72 \text{ (} C_1 \text{ from Table 17.3)}$$

$$\sigma_1 = 0.983, \quad \frac{a_1 L}{2} = 90 a_1 = \frac{4.27}{2} = 2.36$$

$$\phi_1(x = 90^\circ) = 1.586$$

$$y_{max,1} = 1.586(2.91) = 4.61 \text{ in.}$$

17.12

Solve for second mode:

$$\omega_2 = 61.6728 \sqrt{\frac{EI}{mL^4}} = 10.42 \text{ rad/sec}$$

$$(a_2 L)^2 = C_2$$

$$a_2 L = \sqrt{61.6728} = 7.85 \text{ (} C_2 \text{ from Table 20.3)}$$

$$y_{max,2} = \phi_2(x) z_{max,2}$$

$$T_2 = \frac{2\pi}{10.42} = 0.30 \text{ sec}$$

$$\frac{t_d}{T} = \frac{\pi}{400 \cdot 0.3} = 0.026$$

$$z_{max,2} = 0.1$$

$$\phi_2(x) = \cosh a_2 x - \cos a_2 x - \sigma_2 (\sinh a_2 x - \sin a_2 x)$$

$$\sigma_2 = \frac{\cosh a_2 L + \cos a_2 L}{\sinh a_2 L + \sin a_2 L} = 1.014$$

$$\phi_2(x = 90^\circ) = -0.35$$

$$y_{max,2} = 0.35(0.1) = 0.035 \text{ in.}$$

$$y_{max} = \sqrt{y_{max,1}^2 + y_{max,2}^2} = \sqrt{4.61^2 + 0.035^2} = 4.61 \text{ in.}$$

19.1

Fourier series is given by Eq. 19.2

$$F(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\bar{\omega}t + b_n \sin n\bar{\omega}t)$$

$$\frac{T}{2} = \frac{1}{T} \int_0^T F(t) dt$$

$$F(t) = 30 \quad 0 \leq t \leq \frac{T}{2}$$

$$F(t) = -30 \quad \frac{T}{2} \leq t \leq T$$

$a_0 = 0$ (area under $F(t)$ between 0 and $T=1$ sec).

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\bar{\omega}t dt$$

$$a_n = 2 \int_0^{0.5} 30 \cos n\bar{\omega}t dt + 2 \int_{0.5}^{1.0} -30 \cos n\bar{\omega}t dt$$

$$a_n = \frac{60}{n\bar{\omega}} \left[\sin \frac{n\bar{\omega}}{2} \right] - \frac{60}{n\bar{\omega}} \left[\sin n\bar{\omega} - \sin \frac{n\bar{\omega}}{2} \right]$$

$$a_n = \frac{60}{n\bar{\omega}} \left(2 \sin \frac{n\bar{\omega}}{2} - \sin n\bar{\omega} \right), \quad \bar{\omega} = \frac{2\pi}{T} = 2\pi$$

$$a_n = \frac{60}{2\pi n} (2 \sin \pi n - \sin 2\pi n)$$

$$a_n = 0 \quad \text{for all } n.$$

$$b_n = \frac{2}{T} \int_0^T F(t) \sin n\bar{\omega}t dt$$

$$b_n = 2 \int_0^{0.5} 30 \sin n\bar{\omega}t dt + 2 \int_{0.5}^{1.0} -30 \sin n\bar{\omega}t dt$$

$$b_n = \frac{60}{n\bar{\omega}} \left[1 - 2 \cos \frac{n\bar{\omega}}{2} \right] - \frac{60}{n\bar{\omega}} \left[\cos \frac{n\bar{\omega}}{2} - \cos n\bar{\omega} \right]$$

$$b_n = \frac{60}{n\bar{\omega}} \left(1 - 2 \cos \frac{n\bar{\omega}}{2} + \cos n\bar{\omega} \right), \quad \bar{\omega} = \frac{2\pi}{T} = 2\pi$$

$$b_n = \frac{60}{\pi n} (1 - 2 \cos n\pi + \cos 2n\pi)$$

for $n = 1, 3, 5, 7..$ $b_n = \frac{60}{\pi n} (1 - 2(-1) + 1), \quad b_n = \frac{240}{\pi n}$

for $n = 2, 4, 6, 8..$ $b_n = \frac{60}{\pi n} (1 - 2(1) + 1), \quad b_n = 0$

$$\therefore F(t) = \frac{240}{\bar{\omega}} \left(\frac{\sin \bar{\omega}t}{1} + \frac{\sin 3\bar{\omega}t}{3} + \frac{\sin 5\bar{\omega}t}{5} \dots \dots \right)$$

or $F(t) = \frac{120}{\pi} \left(\frac{\sin 2\pi t}{1} + \frac{\sin 6\pi t}{3} + \frac{\sin 10\pi t}{5} \dots \dots \right)$

19.2

From eq. 19.10, the steady-state response of a damped single degree-of-freedom system is:

$$u(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{a_n 2r_n \xi + b_n (1 - r_n^2)}{(1 - r_n^2)^2 + (2r_n \xi)^2} \sin n\bar{\omega}t + \frac{a_n (1 - r_n^2) - b_n 2r_n \xi}{(1 - r_n^2)^2 + (2r_n \xi)^2} \cos n\bar{\omega}t \right\}$$

where a_0, a_n, b_n are the Fourier series coefficients of the function $F(t)$

In our case,

$$\begin{aligned} a_0 &= 0, & a_n &= 0 \\ b_n &= \frac{240}{n\bar{\omega}}, & \text{for } n = 1, 3, 5. & \left(b_n = \frac{120}{n\pi} \right) \\ b_n &= 0, & \text{for } n = 2, 4, 6. \\ r_n &= \frac{n\bar{\omega}}{\omega} & \omega &= \sqrt{\frac{k}{m}} \end{aligned}$$

Then,

$$\begin{aligned} u(t) &= \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{b_n (1 - r_n^2)}{(1 - r_n^2)^2 + (2r_n \xi)^2} \sin n\bar{\omega}t - \frac{b_n 2r_n \xi}{(1 - r_n^2)^2 + (2r_n \xi)^2} \cos n\bar{\omega}t \right\} \\ \omega &= \sqrt{\frac{120 \cdot 386}{128.66}} = 18.9742 \frac{\text{rad}}{\text{sec}}, \quad \xi = 0.10 \\ \bar{\omega} &= 2\pi, \quad r_n = \frac{n\bar{\omega}}{\omega} = 0.3311n \\ u(t) &= \frac{1}{120,000} (42.6642 \sin 2\pi t - 3.1731 \cos 2\pi t + 4.2892 \sin 6\pi t \\ &\quad - 63.8030 \cos 6\pi t - 4.2355 \sin 10\pi t - 0.8057 \cos 10\pi t \dots) \end{aligned}$$

or in terms of

$$r_1 = \frac{\bar{\omega}}{\omega} = 0.3311, \text{ and } \bar{\omega} = 2\pi.$$

$$u(t) = \frac{120}{120,000\pi} \left\{ \begin{aligned} &\frac{1}{(1 - r_1^2)^2 + 0.04(r_1)^2} ((1 - r_1^2) \sin \bar{\omega}t - 0.2r_1 \cos \bar{\omega}t) + \\ &\frac{1}{(1 - 9r_1^2)^2 + 0.04(9r_1^2)} ((1 - 9r_1^2) \sin 3\bar{\omega}t - 0.2 \cdot 3r_1 \cos 3\bar{\omega}t) \\ &+ \frac{1}{(1 - 25r_1^2)^2 + 0.04(25r_1^2)} ((1 - 25r_1^2) \sin 5\bar{\omega}t - 0.2 \cdot 5r_1 \cos 5\bar{\omega}t) + \dots \end{aligned} \right\}$$

19.12

a) The Fourier coefficients for the load are found by applying eq. 19.3 with $T = 2$ sec.

$$a_0 = \frac{1}{2} \int_0^1 P_0 \sin \pi t dt = \frac{P_0}{\pi}$$

$$a_n = \frac{2}{2} \int_0^1 P_0 \sin \pi t \cos \pi t dt = \begin{cases} 0 & n = odd \\ \frac{P_0}{\pi} \frac{2}{1 - \pi^2} & n = even \end{cases}$$

$$b_n = \frac{2}{2} \int_0^1 P_0 \sin \pi t \sin \pi t dt = \begin{cases} \frac{P_0}{2} & n = 1 \\ 0 & n > 1 \end{cases}$$

The substitution into eq. 19.1 leads to following expression for the Fourier series expansion of the loading:

$$P(t) = P_0 \left(\frac{1}{\pi} + \frac{1}{2} \sin \bar{\omega}t - \frac{2}{3\pi} \cos 2\pi \bar{\omega}t - \frac{2}{15\pi} \cos 4\pi \bar{\omega}t - \frac{2}{35\pi} \cos 6\pi \bar{\omega}t \dots \dots \right)$$

or

$$P(t) = P_0 (0.3183 + 0.5 \sin \bar{\omega}t - 0.2122 \cos 2\pi \bar{\omega}t - 0.04244 \cos 4\pi \bar{\omega}t - 0.01819 \cos 6\pi \bar{\omega}t \dots \dots) \text{ kips}$$

where

$$\bar{\omega} = \frac{2\pi}{T} = \pi \text{ (rad/sec)}$$

b) The steady state response is found applying eq. 19.6.

$$u(t) = \frac{P_0}{k} \left(\frac{1}{\pi} + \frac{8}{7} \sin \bar{\omega}t - \frac{8}{15\pi} \cos 2\pi \bar{\omega}t - \frac{1}{60\pi} \cos 4\pi \bar{\omega}t + \dots \dots \right) \text{ in.}$$

or

$$u(t) = P_0 (0.01814 + 0.06513 \sin \bar{\omega}t + 0.009675 \cos 2\pi \bar{\omega}t + 0.0000302 \cos 4\pi \bar{\omega}t + \dots \dots) \text{ in.}$$

20.1

$$[K] = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} \quad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.5 & 0.5 \\ 0.7071 & -0.7071 \end{bmatrix}$$

$$\begin{aligned}\omega_1 &= 7.65 \text{ rad/sec} \\ \omega_2 &= 18.48 \text{ rad/sec}\end{aligned}$$

$$C_n = \{\phi\}_n^T [M] \sum_i a_i ([M]^{-1} [K])^i \{\phi\}_n = \sum_i a_i \omega_n^{2i} M_n = 2\xi_n \omega_n M_n$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega_1 & \omega_1^3 \\ \omega_2 & \omega_2^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 7.65 & 7.65^3 \\ 18.48 & 18.48^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{aligned}3.825a_1 + 223.849a_2 &= 0.2 \\ 9.24a_1 + 3155.556a_2 &= 0.1 \\ a_1 &= 0.0608627 \\ a_2 &= -0.0001465\end{aligned}$$

$$\begin{aligned}[M]^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ [M]^{-1} [K] &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} = \begin{bmatrix} 200 & -100 \\ -200 & 200 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\sum_i a_i ([M]^{-1} [K])^i &= 0.0608627 \begin{bmatrix} 200 & -100 \\ -200 & 200 \end{bmatrix} - 0.0001465 \begin{bmatrix} 200 & -100 \\ -200 & 200 \end{bmatrix}^2 \\ &= \begin{bmatrix} 3.38254 & -0.22627 \\ -0.45254 & 3.38254 \end{bmatrix}\end{aligned}$$

$$[C] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3.38254 & -0.22627 \\ -0.45254 & 3.38254 \end{bmatrix} = \begin{bmatrix} 6.765 & -0.453 \\ -0.453 & 3.383 \end{bmatrix}$$

20.2

$$[K] = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} \quad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.5 & 0.5 \\ 0.7071 & -0.7071 \end{bmatrix}$$

$$[A] = [\Phi]^T[C][\Phi] = \begin{bmatrix} 2\xi_1\omega_1 M_1 & 0 & 0 & - \\ 0 & 2\xi_2\omega_1 M_1 & 0 & - \\ 0 & 0 & 2\xi_3\omega_1 M_1 & - \\ - & - & - & - \end{bmatrix}$$

$$[C] = [M] \left(\sum_{n=1}^N \frac{2\xi_n\omega_n}{M_n} \{\phi\}_n \{\phi\}_n^T \right) [M]$$

$[C] = \text{Damping Matrix}$
 $M_n = 1.0 \text{ if normalized}$

$$\begin{aligned} \xi_1 &= 20\% = 0.2 \\ \xi_2 &= 10\% = 0.10 \end{aligned}$$

For $n = 1$

$$\begin{aligned} \frac{2\xi_n\omega_n}{M_n} &= \frac{2(0.2)(7.65)}{1} = 3.06 \\ \frac{2\xi_1\omega_1}{M_1} \{\phi\}_1 \{\phi\}_1^T &= \begin{bmatrix} 0.5 \\ 0.707 \end{bmatrix} [0.5 \quad 0.707](3.06) = \begin{bmatrix} 0.765 & 1.081863 \\ 10.081863 & 1.530 \end{bmatrix} \end{aligned}$$

For $n = 2$

$$\begin{aligned} \frac{2\xi_n\omega_n}{M_n} &= \frac{2(0.1)(18.48)}{1} = 3.696 \\ \frac{2\xi_2\omega_2}{M_2} \{\phi\}_2 \{\phi\}_2^T &= \begin{bmatrix} 0.5 \\ -0.707 \end{bmatrix} [0.5 \quad -0.707](3.696) = \begin{bmatrix} 0.924 & -1.3067208 \\ -1.3067208 & 1.848 \end{bmatrix} \end{aligned}$$

$$\sum_{n=1}^N = \begin{bmatrix} 0.765 & 1.081863 \\ 10.081863 & 1.530 \end{bmatrix} + \begin{bmatrix} 0.924 & -1.3067208 \\ -1.3067208 & 1.848 \end{bmatrix} = \begin{bmatrix} 1.689 & -0.2248578 \\ -0.2248578 & 3.378 \end{bmatrix}$$

$$\begin{aligned} [C] &= [M] \left(\sum_{n=1}^N \frac{2\xi_n\omega_n}{M_n} \{\phi\}_n \{\phi\}_n^T \right) [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.689 & -0.2248578 \\ -0.2248578 & 3.378 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3.378 & -0.44972 \\ -0.2248578 & 3.378 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6.756 & -0.450 \\ -0.450 & 3.378 \end{bmatrix} \end{aligned}$$

20.3

$$[K] = \begin{bmatrix} 7000 & -3000 & 0 \\ -3000 & 5000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.1114 & -0.1968 & -0.1245 \\ 0.2117 & -0.0277 & 0.2333 \\ 0.2703 & 0.2868 & -0.2114 \end{bmatrix}$$

$$\omega_1 = 9.31 \text{ rad/sec}$$

$$\omega_2 = 20.94 \text{ rad/sec}$$

$$\omega_3 = 29.00 \text{ rad/sec}$$

$$\xi_1 = 10\% = 0.10$$

$$[C] = [M] \left(\sum_{n=1}^N \frac{2\xi_n \omega_n}{M_n} \{\phi\}_n \{\phi\}_n^T \right) [M]$$

For $n = 1$

$$\frac{2\xi_1 \omega_1}{M_1} \{\phi\}_1 \{\phi\}_1^T = \begin{bmatrix} 0.114 \\ 0.2117 \\ 0.2703 \end{bmatrix} [0.114 \quad 0.2117 \quad 0.2703] (1.862) = \begin{bmatrix} 0.023 & 0.043 & 0.056 \\ 0.044 & 0.084 & 0.107 \\ 0.056 & 0.107 & 0.136 \end{bmatrix}$$

For $n = 2$

$$\frac{2\xi_2 \omega_2}{M_2} \{\phi\}_2 \{\phi\}_2^T = \begin{bmatrix} -0.1968 \\ 0.0277 \\ 0.2868 \end{bmatrix} [-0.1968 \quad 0.0277 \quad 0.2868] (4.188) = \begin{bmatrix} 0.162 & 0.022 & -0.236 \\ 0.023 & 0.003 & -0.033 \\ -0.236 & -0.033 & 0.3444 \end{bmatrix}$$

For $n = 3$

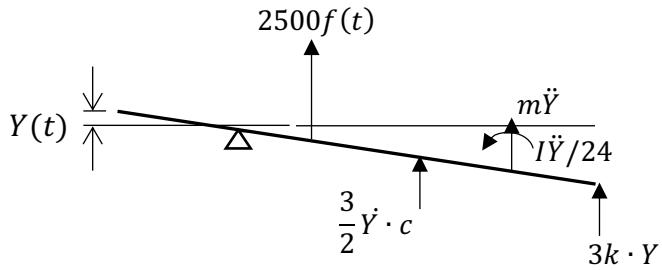
$$\frac{2\xi_3 \omega_3}{M_3} \{\phi\}_3 \{\phi\}_3^T = \begin{bmatrix} -0.1245 \\ 0.2333 \\ -0.2114 \end{bmatrix} [-0.1245 \quad 0.2333 \quad -0.2114] (5.8) = \begin{bmatrix} -0.090 & -0.168 & 0.153 \\ -0.168 & 0.316 & -0.286 \\ 0.153 & -0.286 & 0.259 \end{bmatrix}$$

$$\sum_{n=1}^N = \begin{bmatrix} 0.275 & -0.102 & -0.028 \\ -0.102 & 0.402 & -0.213 \\ -0.028 & -0.213 & 0.740 \end{bmatrix}$$

$$[C] = [M] \left(\sum_{n=1}^N \frac{2\xi_n \omega_n}{M_n} \{\phi\}_n \{\phi\}_n^T \right) [M] = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0.275 & -0.102 & -0.028 \\ -0.102 & 0.402 & -0.213 \\ -0.028 & -0.213 & 0.740 \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4.128 & -1.017 & -0.138 \\ -1.526 & 4.024 & -1.064 \\ -0.415 & -2.128 & 3.699 \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 61.92 & -15.26 & -2.080 \\ -15.26 & 40.24 & -10.64 \\ -2.080 & -10.64 & 18.50 \end{bmatrix}$$

21.1



The figure shows the forces acting in the system for a displaced position of the generalized coordinates, $Y(t)$

Give virtual displacement δY to generalized coord. $Y(t)$ and apply principle of virtual work:

$$-m \cdot \ddot{Y} \cdot \delta Y - \frac{I \cdot \ddot{Y} \cdot \delta Y}{24 \cdot 12} - \frac{3}{2} \dot{Y} \cdot c \frac{3}{2} \delta Y - 3k \cdot Y \cdot 3\delta Y - 2500f(t) \frac{\delta Y}{4} = 0$$

Since $\delta Y \neq 0$

$$\left[\frac{I}{288} + m \right] \ddot{Y} - \frac{9}{4} c \dot{Y} - 9kY = \frac{2500}{4} f(t)$$

Generalized Parameters:

$$M^* = \frac{I}{288} + m = \frac{1}{288} \frac{1}{12} \frac{800}{386} (60)^2 + \frac{800}{386} = 4.23 \frac{lb \cdot sec^2}{in.}$$

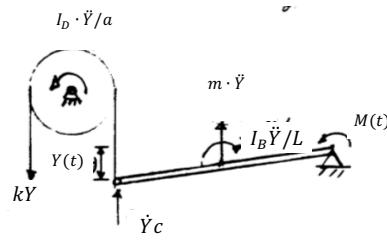
$$C^* = \frac{9}{4} c = \frac{9}{4} 100 = 225 \frac{lb \cdot sec}{in.}$$

$$K^* = 9k = 9(5000) = 45000 \frac{lb}{in.}$$

$$F^*(t) = \frac{2500}{4} f(t) = 625f(t) \frac{lb}{in.}$$

21.2

The figure shows the system in displaced position and corresponding forces:



Give virtual displacement δY and apply principle of virtual work:

$$-\frac{m \cdot \ddot{Y}}{2} \frac{\delta Y}{2} - \frac{I_B \cdot \ddot{Y} \cdot \delta Y}{L \cdot L} - \dot{Y}_c \cdot \delta Y - k \cdot Y \cdot \delta Y - \frac{I_D \cdot \ddot{Y} \cdot \delta Y}{a \cdot a} + M(t) \frac{\delta Y}{L} = 0$$

Since $\delta Y \neq 0$

$$\left(\frac{m}{2} + \frac{I_B}{L^2} + \frac{I_D}{a^2} \right) \ddot{Y} + c \cdot Y + k \cdot Y = \frac{M(t)}{L}$$

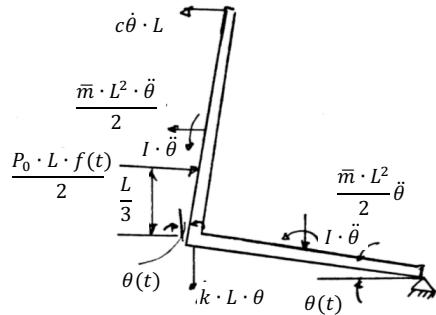
Generalized parameters:

$$M^* = \frac{m}{2} + \frac{I_B}{L^2} + \frac{I_D}{a^2} = \frac{m}{2} + \frac{1}{12} \frac{2mL^2}{L^2} + \frac{ma^2}{2a^2} = \frac{7}{6} m$$

$$\begin{aligned} C^* &= c \\ K^* &= k \\ F^*(t) &= \frac{M(t)}{L} \end{aligned}$$

21.3

The figure shows the system in displaced position for the generalized coordinate $\theta(t)$:



Give virtual displacement $\delta\theta$ and apply Principle of virtual work:

$$-2\left(\frac{\bar{m} \cdot L^2 \cdot \ddot{\theta}}{2} \cdot \frac{\delta\theta L}{2}\right) - 2(I \cdot \ddot{\theta} \cdot \delta\theta) - c\dot{\theta} \cdot L \cdot L \cdot \delta\theta - k \cdot \theta \cdot L \delta\theta - \frac{P_0 \cdot L \cdot f(t)}{2} \cdot \frac{L}{3} \delta\theta = 0$$

Since $\delta\theta \neq 0$

$$\left(\frac{\bar{m} \cdot L^2}{2} + \frac{I}{L}\right) \ddot{\theta} + c \cdot L \cdot \dot{\theta} - k \cdot L \cdot \theta = \frac{P_0 \cdot L \cdot f(t)}{6}$$

Generalized parameters:

$$M^* = \frac{\bar{m} \cdot L^2}{2} + \frac{1}{12} \frac{\bar{m} \cdot L \cdot L^2}{L} = \frac{7}{12} \bar{m} L^2$$

$$\begin{aligned} C^* &= cL \\ K^* &= kL \\ F^*(t) &= \frac{P_0 \cdot L \cdot f(t)}{6} \end{aligned}$$

21.4

Generalized mass:

$$\begin{aligned}
 M^* &= m + \int_0^L \left(\frac{m}{L}\right) \phi^2(x) dx \\
 M^* &= m + \frac{m}{L} \int_0^L \left(1 - \cos \frac{\pi x}{2L}\right)^2 dx \\
 M^* &= m + \frac{m}{L} \int_0^L \left(1 - 2 \cos \frac{\pi x}{2L} + \cos^2 \frac{\pi x}{2L}\right) dx \\
 M^* &= m + \frac{m}{L} \left[x - \frac{4L}{\pi} \sin \frac{\pi x}{2L} + \frac{x}{2} + \frac{L}{2\pi} \sin \frac{\pi x}{L} \right]_0^L \\
 M^* &= m + \frac{m}{L} \left(\frac{3}{2}L - \frac{4L}{\pi} \right) \\
 M^* &= \frac{m}{2\pi} (5\pi - 8)
 \end{aligned}$$

Generalized stiffness:

$$\begin{aligned}
 K^* &= \int_0^L EI \phi''^2(x) dx \\
 \phi(x) &= 1 - \cos \frac{\pi x}{2L}, \quad \phi''(x) = \left(\frac{\pi}{2L}\right)^2 \cos \frac{\pi x}{2L} \\
 K^* &= \int_0^L EI \left(\frac{\pi}{2L}\right)^4 \cos^2 \frac{\pi x}{2L} dx \\
 K^* &= EI \frac{\pi^4}{16L^4} \left[\frac{2L}{\pi} \left(\frac{\pi x}{4L}\right) + \frac{2L}{\pi} \left(\frac{\pi x}{4L}\right) \sin \frac{\pi x}{L} \right]_0^L = \frac{EI\pi^4}{32L^4}
 \end{aligned}$$

Generalized force:

$$\begin{aligned}
 F^*(t) &= \int_0^L p(x, t) \phi(x) dx \\
 F^*(t) &= F_0 f(t) \phi\left(\frac{L}{2}\right) \\
 F^*(t) &= F_0 \left(1 - \cos \frac{\pi}{4}\right) f(t) \\
 F^*(t) &= 0.2929 \cdot F_0 f(t)
 \end{aligned}$$

21.5

The generalized geometric stiffness:

$$K_G^* = N \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx$$

$$\begin{aligned}\phi(x) &= 1 - \cos \frac{\pi x}{2L}, \quad \frac{d\phi}{dx} = \frac{\pi}{2L} \sin \frac{\pi x}{2L} \\ K_G^* &= -N \int_0^L \left(\frac{\pi}{2L} \right)^2 \sin^2 \frac{\pi x}{2L} dx, \quad (\text{negative sign because } N \text{ is tension}) \\ K_G^* &= -N \left(\frac{\pi}{2L} \right)^2 \frac{2L}{\pi} \left[\left(\frac{\pi x}{4L} \right) - \frac{1}{4} \sin \frac{\pi x}{L} \right]_0^L = -N \left(\frac{\pi}{2L} \right) \frac{\pi}{4} = -\frac{N\pi^2}{8L} \\ K_C^* &= K^* - K_G^* \\ K^* &= \frac{EI\pi^4}{32L^3} \quad \text{from Problem 21.4} \\ K_C^* &= \frac{EI\pi^4}{32L^3} + \frac{N\pi^2}{8L}\end{aligned}$$

21.6

The generalized mass:

$$\begin{aligned}
 M^* &= \int_0^L m(x) \phi^2(x) dx \\
 m(x)dx &= A_x \frac{\gamma}{g} dx = \frac{\gamma \pi d^2}{g 4L^2} x^2 dx \\
 \phi(x) &= 1 - \cos \frac{\pi x}{2L} \\
 M^* &= \frac{\gamma \pi d^2}{g 4L^2} \int_0^L x^2 \left(1 - \cos \frac{\pi x}{2L}\right)^2 dx \\
 M^* &= \frac{\gamma \pi d^2}{g 4L^2} \left[\frac{L^3}{3} - 2 \left(\frac{2L}{\pi}\right)^3 \left(\left(\frac{\pi}{2}\right)^2 - 2\right) + \left(\frac{L^3}{6} - \frac{L^3}{\pi^2}\right) \right]_0^L \\
 M^* &= \frac{\gamma \pi d}{g 4} \left(\frac{1}{2} - \frac{4}{\pi} + \frac{32}{\pi^3} - \frac{1}{\pi^2} \right) = 0.1237 \frac{\gamma d}{g}
 \end{aligned}$$

The generalized stiffness:

$$\begin{aligned}
 K^* &= \int_0^L E_c I(x) \phi''^2(x) dx, \quad I(x) = \frac{\pi d^2 x^4}{4 16 L^4} \\
 K^* &= E_c \frac{\pi d^2}{64 L^2} \left(\frac{\pi}{2L}\right)^2 \int_0^L x^4 \cos \frac{\pi x}{2L} dx, \quad \phi''(x) = \left(\frac{\pi}{2L}\right)^2 \cos \frac{\pi x}{2L} \\
 K^* &= \frac{E_c \pi d^2}{128 L^2}
 \end{aligned}$$

The generalized force is

$$\begin{aligned}
 F^*(t) &= \int_0^L p(x, t) \phi(x) dx \\
 F^*(t) &= \int_0^L \frac{xd}{L} p_0(t) \left(1 - \cos \frac{\pi x}{2L}\right) dx \\
 F^*(t) &= \frac{d}{L} p_0(t) \left[\frac{L^2}{2} - \frac{4L^2}{\pi^2} \left(\frac{\pi}{2} - 1\right) \right] = \frac{p_0(t)L \cdot d}{2\pi^2 L} [9 - 4\pi] = -0.1807 p_0(t) L \cdot d
 \end{aligned}$$

21.7

The deflection occurred corresponding to the left half a simply supported beam with a concentrated load at its center is

$$y = \frac{y_0}{L^3} (3L^2x - 4x^3)$$

Where y_0 is the deflection at the center.

The maximum potential energy is calculated as

$$\begin{aligned} V_{max} &= \frac{1}{2} F y_0 \\ V_{max} &= \frac{24EI}{L^3} y_0^2, \quad \text{since } y_0 = \frac{FL^3}{48EI} \end{aligned}$$

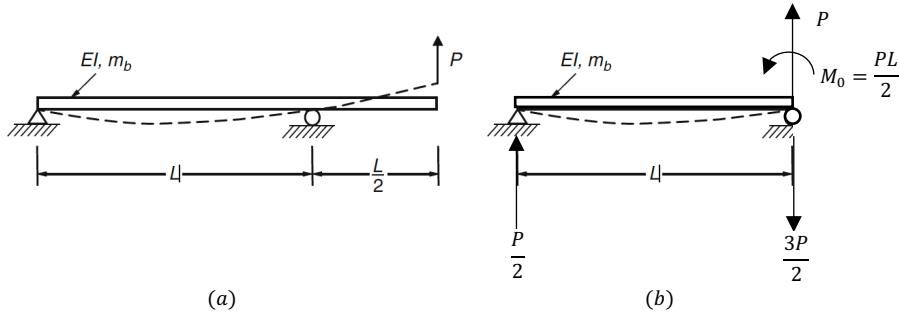
and the maximum Kinetic Energy

$$\begin{aligned} T_{max} &= 2 \int_0^{L/2} \frac{1}{2} \left(\frac{m_b}{L} dx \right) (\omega y)^2 + \frac{1}{2} m (\omega y_0)^2 \\ T_{max} &= \frac{m_b (\omega y_0)^2}{L} \int_0^{L/2} (3L^2x - 4x^3) dx + \frac{1}{2} m (\omega y_0)^2 \\ T_{max} &= \frac{m_b (\omega y_0)^2}{L} \left[3L^4 \left(\frac{L}{2} \right)^3 + \frac{16}{7} \left(\frac{L}{2} \right)^4 - \frac{24L^2}{5} \left(\frac{L}{2} \right)^5 \right] + \frac{1}{2} m (\omega y_0)^2 \\ T_{max} &= \frac{m_b (\omega y_0)^2}{7 \cdot 8 \cdot 5} [105 + 5 - 42] + \frac{1}{2} m (\omega y_0)^2 \end{aligned}$$

$$\begin{aligned} V_{max} &= T_{max} \\ \frac{24EI}{L^3} y_0^2 &= (\omega y_0)^2 \left(\frac{m}{2} + \frac{17}{70} m_b \right) \\ \omega &= \sqrt{\frac{48EI}{L^3 \left(m + \frac{17}{35} m_b \right)}} \end{aligned}$$

21.8

The beam loaded as shown in Fig. (a), between supports is equivalent to beam shown in Fig. (b)



Moment a section x in Fig (b):

$$M_x = \frac{Px}{2}; \quad \frac{d^2y}{dx^2} = \frac{Px}{2EI}; \quad \frac{dy}{dx} = \frac{Px^2}{2EI} + C_1$$

$$y = \frac{Px^3}{12EI} + C_1x + C_2, \quad \text{at } x = 0, \ y = 0, \ C_2 = 0$$

$$x = L, \ y = 0 = \frac{PL^3}{12EI} + C_1L$$

$$C_1 = -\frac{PL^2}{12EI}, \quad \left. \frac{dy}{dx} \right|_{x=L} = \frac{PL^2}{6EI}$$

$$y = \frac{Px^3}{12EI} - \frac{PL^2}{12EI}x = \frac{P}{12EI}[x^3 - L^2x]$$

Deflection of overhang:

$$\frac{d^2y}{dx^2} = \frac{P}{EI}\left(\frac{L}{2} - x\right)$$

$$\frac{dy}{dx} = \frac{P}{EI}\left(\frac{L}{2}x - \frac{x^2}{2}\right) + C_3$$

$$y = \frac{P}{EI}\left(\frac{L}{4}x^2 - \frac{x^3}{6}\right) + C_3x + C_4$$

at

$$x = 0, \quad \frac{dy}{dx} = \frac{PL^2}{6EI} = C_3, \quad x = 0, \ y = 0, \ C_4 = 0$$

$$y = \frac{P}{12EI}(3Lx^2 - 2x^3 + 2L^2x), \quad \text{in overhang}$$

Max. potential:

$$V_{max} = \frac{Py_0}{2}$$

$$y_0 = \frac{P}{12EI}\left(3L\frac{L^2}{4} - 2\frac{L^3}{8} + 2L^2\frac{L}{2}\right)$$

$$V_{max} = \frac{P^2L^3}{16EI}, \quad y_0 = \frac{PL^3}{8EI}$$

Max. Kinetic Energy:

$$T_{max} = \frac{1}{2} \frac{m_b \omega^2}{\frac{3L}{2}} \int_0^L \frac{P^2}{144(EI)^2} (x^6 + L^4x^2 - 2L^2x^4) dx + \frac{1}{2} \frac{m_b \omega^2}{\frac{3L}{2}} \int_0^{L/2} y^2 dx = \frac{m_b \omega^2 P^2 L^7}{36 \cdot 144(EI)^2} [0.076 + 0.3615]$$

$$T_{max} = V_{max}$$

$$\frac{m_b \omega^2 P^2 L^7}{987(EI)^2} = \frac{PL^3}{8EI}$$
$$\omega = 7.854 \sqrt{\frac{EI}{L^4 m_b}}$$

21.9

The deflection at section x is:

$$y = Y \sin \frac{\pi x}{L} \sin \omega t$$

Maximum velocity is:

$$\dot{y}_{max} = Y \omega \sin \frac{\pi x}{L}$$

and Max kinetic Energy:

$$T_{max} = \frac{1}{2} m Y^2 \omega^2 + \int_0^{L/2} \frac{1}{2} \frac{m_b}{L} Y^2 \omega^2 \sin^2 \frac{\pi x}{L} dx = \frac{1}{2} m Y^2 \omega^2 + \frac{1}{2} \frac{m_b}{L} Y^2 \frac{L}{2} = \frac{\omega^2 Y^2}{4} (2m + m_b) = 5\omega^2 Y^2$$

Maximum Potential Energy:

$$V_{max} = \frac{EI}{2} \int_0^{L/2} \left(\frac{d^2 y}{dx^2} \right)_{max}^2 dx$$

$$\frac{dy}{dx} = Y \frac{\pi}{L} \cos \frac{\pi x}{L} \sin \omega t, \quad \frac{d^2 y}{dx^2} = -Y \frac{\pi^2}{L^2} \sin \frac{\pi x}{L} \sin \omega t$$

$$V_{max} = \frac{EI}{2} \int_0^{L/2} Y^2 \frac{\pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx = \frac{EI}{2} Y^2 \frac{\pi^4 L}{L^4 2} = \frac{10^8 Y^2 \pi^4}{4 \cdot 100^3} = 2435 Y^2$$

$$T_{max} = V_{max}$$

$$5\omega^2 Y^2 = 2435 Y^2$$

$$\omega^2 = 487$$

$$\omega = 22.07 \text{ rad/sec}$$

$$f = \frac{22.07}{2\pi} = 3.51 \text{ cps}$$

21.10

Use Eq. 21.83:

$$\omega = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}}$$

In this equation the deflection y_i at the floor levels is due the static load W applied horizontally at each story.

The shear force in the first story is:

$$V_1 = \frac{12EI}{L^3} y_1$$

And in the second story:

$$V_2 = \frac{12EI}{L^3} (y_2 - y_1)$$

Since $V_1 = 2W$ and $V_2 = W$

$$y_1 = 2 \frac{wL^3}{12EI}$$

$$y_2 = 2 \frac{wL^3}{12EI} + \frac{wL^3}{12EI} = \frac{3wL^3}{12EI}$$

$$\sum w_i y_i = w \left[2 \frac{wL^3}{12EI} + \frac{3wL^3}{12EI} \right] = \frac{5w^2 L^3}{12EI}$$

$$\sum w_i y_i^2 = w \left[4 \frac{w^2 L^6}{(12EI)^2} + \frac{9w^2 L^6}{(12EI)^2} \right] = \frac{13w^3 L^6}{(12EI)^2}$$

$$\omega^2 = \frac{g \frac{5w^2 L^3}{12EI}}{\frac{13w^3 L^6}{(12EI)^2}} = \frac{g(12EI)5}{13wL^3} = \frac{4.62EI}{wL^3}$$

$$\omega = 2.15 \sqrt{\frac{gEI}{wL^3}}$$

21.11

As in problem 21.6 load the frame with horizontal force W at each story.

For fixed column the shear force is

$$V = \frac{AG\Delta}{L}$$

Then

$$\Delta = \frac{VL}{AG}$$

For the first story: $V = 2W$, $\Delta = y_1$

$$\begin{aligned} y_1 &= \frac{2WL}{AG} \\ y_2 &= y_1 + \frac{WL}{AG} = \frac{3WL}{AG} \\ \omega &= \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \\ \sum w_i y_i &= w \left[2 \frac{wL}{AG} + \frac{3wL}{AG} \right] = \frac{5w^2 L}{AG} \\ \sum w_i y_i^2 &= w \left[4 \frac{w^2 L^2}{(AG)^2} + \frac{9w^2 L^2}{(AG)^2} \right] = \frac{13w^3 L^2}{(AG)^2} \\ \omega^2 &= \frac{g \frac{5w^2 L}{AG}}{\frac{13w^3 L^2}{(AG)^2}} = \frac{59gAG}{13wL} \\ \omega &= 0.62 \sqrt{\frac{gAG}{wL}} \end{aligned}$$

21.12

The deflection shape of a cantilever beam with a concentrated load at its end is:

$$y(x) = \frac{Y_{max}}{2L^3} (3Lx^2 - x^3)$$

where Y_{max} is the maximum deflection at the free end

$$\begin{aligned}\frac{dy}{dx} &= \frac{Y_{max}}{2L^3} (6Lx - 3x^2) \\ \frac{d^2y}{dx^2} &= \frac{Y_{max}}{2L^3} (6L - 6x) = \frac{3Y_{max}}{L^3} (L - x)\end{aligned}$$

Maximum Potential Energy:

$$V_{max} = \frac{EI}{2} \int_0^{L/2} \left(\frac{d^2y}{dx^2} \right)^2 dx = \frac{EI}{2} \frac{9Y_{max}^2}{L^6} \int_0^{L/2} (L^2 - 2Lx + x^2) dx = \frac{EI}{2} \frac{9Y_{max}^2}{L^6} \left(1 - 1 + \frac{1}{3} \right) = 5581.65Y_{max}^2$$

Maximum Potential Energy:

$$\begin{aligned}T_{max} &= \int_0^L \frac{1}{2} \bar{m} \dot{y}^2 dx + \frac{m}{2} [\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2] \\ T_{max} &= \frac{10\omega^2 Y_{max}^2}{8L^6} \int_0^L (9L^2 x^4 - 6Lx^5 + x^6) dx + \frac{1000\omega^2 L^3}{8L^6} \left[\frac{8}{27} + \frac{28}{27} + 2 \right] Y_{max}^2 \\ T_{max} &= \frac{10\omega^2 Y_{max}^2 L}{8L^6} \left[\frac{9}{5} - \frac{6}{6} + \frac{1}{7} \right] + \frac{1000\omega^2 L^3}{8L^6} 3.33 Y_{max}^2 \\ &= 509.14\omega^2 Y_{max}^2 + 41.67\omega^2 Y_{max}^2 = 505.81\omega^2 Y_{max}^2\end{aligned}$$

$$T_{max} = V_{max}$$

$$505.81\omega^2 Y_{max}^2 = 5581.65Y_{max}^2$$

$$\omega^2 = 10.13$$

$$\omega = 3.18 \text{ rad/sec}$$

$$f = 0.507 \text{ cps}$$

21.3

The deflection shape of a cantilever beam with a concentrated load at its end is:

$$\phi(x) = \frac{1}{3} \left(\frac{x^4}{L^4} - \frac{4}{L^3} x^3 + \frac{6}{L^2} x^2 \right)$$

and the deflection at a section x by:

$$\begin{aligned} y(x) &= Y_{max} \phi(x) \sin \omega t \\ \frac{dy}{dx} &= Y_{max} \frac{1}{3} \left(\frac{4x^3}{L^4} - \frac{12}{L^3} x^2 + \frac{12}{L^2} x \right) \sin \omega t \\ \frac{d^2y}{dx^2} &= Y_{max} \frac{1}{3} \left(\frac{12x^2}{L^4} - \frac{24}{L^3} x + \frac{12}{L^2} \right) \sin \omega t \end{aligned}$$

Maximum Potential Energy:

$$\begin{aligned} V_{max} &= \frac{EI}{2} \int_0^L \left(\frac{d^2y}{dx^2} \right)_{max}^2 dx = \frac{Y_{max}^2 EI}{18} \int_0^L \left(\frac{144x^4}{L^8} + \frac{576}{L^6} x^2 + \frac{144}{L^4} - \frac{576x^3}{L^7} + \frac{288}{L^6} x^2 - \frac{576x}{L^5} \right) dx \\ V_{max} &= \frac{144Y_{max}^2 EI}{18} \left[\frac{1}{5} + \frac{4}{3} + 1 - \frac{4}{4} + \frac{2}{3} - \frac{4}{2} \right] = \frac{144EIY_{max}^2}{90} = 5953.74Y_{max}^2 \\ \frac{EI}{L^3} &= \frac{3 \cdot 10^{11}}{(36 \cdot 12)^3} = 3721.1 \end{aligned}$$

Max. Kinetic Energy:

$$\begin{aligned} T_{max} &= \int_0^L \frac{1}{2} \bar{m} \dot{y}^2 dx + \frac{m}{2} [\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2] \\ T_{max} &= \frac{10\omega^2 Y_{max}^2}{18L^6} \int_0^L \left(\frac{x^8}{L^8} - \frac{16x^6}{L^6} + \frac{36x^4}{L^4} - \frac{8x^7}{L^7} + \frac{12x^6}{L^6} - \frac{48x^5}{L^5} \right) dx + \frac{100\omega^2}{18} 12.803Y_{max}^2 \\ T_{max} &= \frac{10\omega^2 LY_{max}^2}{18L^6} \left[\frac{1}{9} - \frac{16}{7} + \frac{36}{5} - \frac{8}{8} + \frac{12}{7} - \frac{48}{6} \right] + 71.13\omega^2 Y_{max}^2 \\ T_{max} &= (240 + 71.13)\omega^2 Y_{max}^2 \end{aligned}$$

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