CONFIGURATION SPACES IN ALGEBRAIC TOPOLOGY: LECTURE 8

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In this lecture, we complete our third and final proof of the Arnold relation, following [Sin06].

Recollection. One version of Poincaré duality for oriented, connected, boundaryless, possibly non-compact n-manifolds of finite type is the isomorphism

$$\widetilde{H}_i(M^+) \cong H^{n-i}(M),$$

where M^+ denotes the one-point compactification of M and we reduce with respect to the point at infinity. In particular, such a manifold has a fundamental class $[M] \in \widetilde{H}^n(M^+)$, defined as the preimage of $1 \in H^0(M)$ under this isomorphism. This duality can sometimes be interpreted geometrically.

(1) If $N \subseteq M$ is a proper submanifold of dimension r and $P \subseteq M$ is a compact submanifold of dimension $n-r$, we may contemplate the composite

$$\widetilde{H}_r(N^+) \otimes H_{n-r}(P) \longrightarrow \widetilde{H}_r(M^+) \otimes H_{n-r}(M) \cong H^r(M) \otimes H_{n-r}(M) \xrightarrow{\langle -,- \rangle} \mathbb{Z}.$$

(note that the existence of the first map uses the fact that N is properly embedded). If N and P intersect transversely, then the value of this composite on $[N] \otimes [P]$ is the signed intersection number of N and P.

(2) Since cohomology is a ring, we may likewise contemplate the composite

$$\widetilde{H}_r(N_1^+) \otimes \widetilde{H}_s(N_2^+) \longrightarrow H^{n-r}(M) \otimes H^{n-s}(M) \xrightarrow{\smile} H^{2n-r-s}(M) \cong \widetilde{H}^{r+s-n}(M^+),$$

where N_1 and N_2 are proper submanifolds of dimension r and s, respectively. If N_1 and N_2 intersect transversely, then the value of this composite on $[N_1] \otimes [N_2]$ is $[N_1 \cap N_2]$.

Now, consider the submanifold of $\mathrm{Conf}_3(\mathbb{R}^n)$ defined by requiring that x_1, x_2, and x_3 be collinear. This manifold has three connected components, and we let C_a denote the component in which x_a lies between x_b and x_c. Then the map

$$\begin{aligned} C_a &\to \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times S^{n-1} \\ (x_1, x_2, x_3) &\mapsto \left(x_a,\, |x_b - x_a|,\, |x_c - x_a|,\, \frac{x_c - x_b}{|x_c - x_b|} \right) \end{aligned}$$

is a homeomorphism. In particular, $\dim C_a = 2n+1$. Note that C_a is closed as a subspace of $\mathrm{Conf}_3(\mathbb{R}^n)$ and hence proper as a submanifold.

Sinha's proof of the Arnold relation. Pushing forward $[C_1]$ and applying Poincaré duality as above, we obtain an element of $H^{n-1}(\mathrm{Conf}_3(\mathbb{R}^n))$. By our homology calculation, this class is determined by evaluating it on $P_{(12)}$ and $P_{(13)}$. These values are given by the respective intersection numbers with C_1, which are ± 1 with opposite signs. Thus, with the appropriate choice of orientation, C_1

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is Poincaré dual to $\alpha_{12} - \alpha_{13}$. Similar remarks apply to C_2, and, since $C_1 \cap C_2 = \varnothing$, we conclude that

$$\begin{aligned}0 &= (\alpha_{12} - \alpha_{13})(\alpha_{23} - \alpha_{21}) \\ &= \alpha_{12}\alpha_{23} - \alpha_{12}\alpha_{21} - \alpha_{13}\alpha_{23} + \alpha_{13}\alpha_{21} \\ &= \alpha_{12}\alpha_{23} + (-1)^{n(n-1)}\alpha_{23}\alpha_{31} + (-1)^{2n}\alpha_{31}\alpha_{12} \\ &= \alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}.\end{aligned}$$

□

The Arnold relation has its reflection in homology. For trees T_1 and T_2, we write $[T_1, T_2]$ for the tree obtained by grafting the roots of T_1 and T_2 to the leaves of (12), in this order.

Proposition (Jacobi identity)**.** *The relation* $[[T_1, T_2], T_3] + [[T_2, T_3], T_1] + [[T_3, T_1], T_2] = 0$ *holds in* $H_*(\mathrm{Conf}_k(\mathbb{R}^n))$*. More generally, if* R *is any tree, then the trees resulting from grafting the roots of* $[[T_1, T_2], T_3]$*,* $[[T_2, T_3], T_1]$*, and* $[[T_3, T_1], T_2]$ *to any fixed leaf of* R *sum to zero.*

It is possible to give a geometric derivation of the Jacobi identity—see [Sin06]—but we will pursue in an alternate route. We begin by observing that the most basic case of the identity, in which T_1, T_2, T_3, and R are all trivial trees with no internal vertices, is essentially immediate from what we have already done.

Proof of proposition, trivial case. We calculate that

$$\begin{aligned}\langle((23)1), \alpha_{12}\alpha_{23}\rangle &= \langle((23)1), -\alpha_{23}\alpha_{31} - \alpha_{31}\alpha_{12}\rangle \\ &= -\langle((23)1), \alpha_{23}\alpha_{31}\rangle + (-1)^{1+2n+(n-1)^2}\langle((23)1), \alpha_{21}\alpha_{13}\rangle \\ &= -\langle((13)2), \alpha_{13}\alpha_{32}\rangle + (-1)^n\langle((13)2), \alpha_{12}\alpha_{23}\rangle \\ &= -1,\end{aligned}$$

where we have applied the permutation τ_{12} in going from the second to the third line, and the last equality follows from the perfect pairing between tall trees and the corresponding cohomology classes. A similar calculation shows that $\langle((23)1), \alpha_{31}\alpha_{12}\rangle = -1$, and it follows that

$$((23)1) = -((31)2) - ((12)3),$$

as desired. □

The general form of the Jacobi identity follows from this basic case once we are assured that grafting of trees is linear. In order to see why this linearity might hold, we turn to an alternative model for the homotopy types of configuration spaces. For original references, see [BV73, May72].

Definition. A *little n-cube* is an embedding $f : (0,1)^n \to (0,1)^n$ of the form $f(x) = Dx + b$, where $b \in (0,1)^n$ and D is a diagonal matrix with positive eigenvalues.

We write $\mathcal{C}_n(k)$ for the space of k-tuples of little n-cubes with pariwise disjoint images, topologized either as a subspace of $\mathrm{Map}(\amalg_k (0,1)^n, (0,1)^n)$. Note that little cubes are closed under composition, we have a collection of maps of form

$$\mathcal{C}_n(m) \times \mathcal{C}_n(k_1) \times \cdots \mathcal{C}_n(k_m) \to \mathcal{C}_n(k)$$

whenever $k_1 + \cdots + k_m = k$. These maps furnish the collection $\{\mathcal{C}_n(k)\}_{k \geq 0}$ of spaces with the structure of an *operad* [May72], but we will not need to make use of the full strength of this notion.

Proposition. *The map* $\mathcal{C}_n(k) \to \mathrm{Conf}_k((0,1)^n) \cong \mathrm{Conf}_k(\mathbb{R}^n)$ *given by evaluation at* $(1/2, \ldots, 1/2)$ *is a homotopy equivalence.*

Sketch proof. A section of the map in question is defined by sending a configuration x to the unique k-tuple of little cubes $(f_1, \ldots, f_k)$ with the following properties:

(1) $f_i(1/2, \ldots, 1/2) = x_i$ for $1 \leq i \leq k$;
(2) all sides of each f_i have equal length, and all f_i have equal volume;
(3) the images of the f_i do not have pairwise disjoint closures.

One checks that this map is continuous, so that we may view the configuration space as a subspace of $\mathcal{C}_n(k)$. Scaling defines a deformation retraction onto this subspace. □

For further details, see [May72, 4.8].

Proof of the Jacobi identity, general case. By considering planetary systems of little cubes rather than configurations, one obtains the dashed lifts depicted in the diagram

$$\begin{tikzcd} & \mathcal{C}_n(k) \arrow[d] \\ (S^{n-1})^{V(F)} \arrow[ur, dashed, "P_F^{\square}"] \arrow[r, "P_F"] & \mathrm{Conf}_k(\mathbb{R}^n). \end{tikzcd}$$

With these maps in hand, the combinatorics of grafting trees becomes the combinatorics of composing little cubes; that is, the tree $[[T_1, T_2], T_3]$ is the image of $(((12)3), T_1, T_2, T_3)$ under the composition map

$$\mathcal{C}_n(3) \times \mathcal{C}_n(3) \times \mathcal{C}_n(k_1) \times \mathcal{C}_n(k_2) \times \mathcal{C}_n(k_3) \to \mathcal{C}_n(k),$$

and similar remarks pertain to grafting roots of trees onto a fixed leaf of a tree R. Thus, grafting, as the map induced on homology by a map of spaces, is linear, and the general identity follows from the basic case proven above. □

A similar argument as in our earlier cohomology calculation, using the Jacobi identity to rebracket forests into sums of long forests, proves the following.

Theorem (Cohen)**.** *The graded Abelian group $H_*(\mathrm{Conf}_k(\mathbb{R}^n))$ is isomorphic to the quotient of the free Abelian group with basis the set of k-forests by the Jacobi relations and signed antisymmetry.*

Remark. This isomorphism may be promoted to an isomorphism of the operad $\{H_*(\mathrm{Conf}_k(\mathbb{R}^n))\}_{k\geq 0}$ with the operad controlling $(n-1)$-shifted Poisson algebras.

We close with a calculation in the unordered case.

Proposition. *For $k \geq 2$ and $n \geq 1$, there is an isomorphism*

$$H_i(B_k(\mathbb{R}^n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if either } i = 0 \text{ or } i = n-1 \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Note the vast difference in size and complexity between the rational homology of $B_k(\mathbb{R}^n)$ and that of $\mathrm{Conf}_k(\mathbb{R}^n)$. This disparity, which may at first seem surprising, is characteristic of the relationship between ordered and unordered configuration spaces in characteristic zero. In finite characteristic, as we will see, this relationship is reversed, and it is the homology in the unordered case that is by far more complex.

One obvious indicator of the rational difference between ordered and unordered is the fact that ith Betti number of $\mathrm{Conf}_k(\mathbb{R}^n)$ tends to infinity with k, while that of $B_k(\mathbb{R}^n)$ quickly stabilizes to a fixed value. This observation is a simple example of the general phenomenon of *homological stability* for configuration spaces of manifolds [Chu12, RW13]. Although the Betti numbers in the ordered case do not stabilize, the analogous of *representation stability*, which takes the action of Σ_k into account, does [Far].

In making this calculation, we will use the following basic fact.

Lemma. *Let $\pi : E \to B$ be a finite regular cover with deck group G. If $\mathbb{F}$ is a field in which $|G|$ is invertible, then the natural map*

$$\bar{\pi}_* : H_*(E;\mathbb{F})_G \to H_*(B;\mathbb{F})$$

is an isomorphism.

This result is a consequence of the existence and basic properties of the *transfer map.* Recall that the transfer is a wrong-way map on homology

$$\mathrm{tr} : H_*(B) \to H_*(E)$$

defined by sending a singular chain to the sum over its $|G|$ lifts to E, which is clearly a chain map. It is obvious from the definition that $\pi_*(\mathrm{tr}(\alpha)) = |G|\alpha$.

Proof of lemma. We claim that the composite

$$f : H_*(B;\mathbb{F}) \xrightarrow{\frac{1}{|G|}\mathrm{tr}} H_*(E;\mathbb{F}) \longrightarrow H_*(E;\mathbb{F})_G$$

is an inverse isomorphism to $\bar{\pi}_*$. Note that we have used the assumption that $|G|$ is invertible in $\mathbb{F}$ in defining f. In one direction, we compute that

$$\bar{\pi}_*(f(\alpha)) = \pi_*\left(\frac{1}{|G|}\mathrm{tr}(\alpha)\right) = \frac{1}{|G|}\pi_*(\mathrm{tr}(\alpha)) = \alpha,$$

and in the other we have

$$f(\bar{\pi}_*([\beta])) = f(\pi_*(\beta)) = \frac{1}{|G|}\left[\mathrm{tr}(\pi_*(\beta))\right] = \frac{1}{|G|}\left[\sum_{g\in G} g\cdot\beta\right] = \frac{1}{|G|}\left[\sum_{g\in G}\beta\right] = \beta.$$

□

With the identification $H_*(B_k(\mathbb{R}^n);\mathbb{Q}) \cong H_*(\mathrm{Conf}_k(\mathbb{R}^n);\mathbb{Q})_{\Sigma_k}$ in hand, we proceed by first identifying the coinvariants in top degree.

Lemma. *For $k > 1$, there is an isomorphism*

$$H_{(n-1)(k-1)}(\mathrm{Conf}_k(\mathbb{R}^n);\mathbb{Q})_{\Sigma_k} \cong \begin{cases} \mathbb{Q} & k = 2 \text{ and } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If n is odd, then any tall tree T is equal to the additive inverse of the tree obtained by switching the labels of the first two leaves of T. Since this operation may be achieved by the action of the symmetric group, it follows that $2[T] = 0$ in the coinvariants, whence $[T] = 0$. Since tall trees span the top homology, their images span its coinvariants, and the claim follows in this case.

Assume that n is even. If $k \geq 3$, then the Jacobi identity applied to the bottom three leaves of a tall tree T shows that $3[T] = 0$, and so $[T] = 0$, and we conclude as before. In the remaining case $k = 2$, we note that $H_{n-1}(\mathrm{Conf}_2(\mathbb{R}^n)) \cong \mathbb{Z}\langle P_{(12)}\rangle$, and that Σ_2 acts trivially. □

Proof of proposition. As a consequence of our description in terms of tall forests, we have the following calculation:

$$\begin{aligned} H_*(\mathrm{Conf}_k(\mathbb{R}^n)) &\cong \bigoplus_{\text{partitions of [k]}} \bigotimes_i H_{(n-1)(k_i-1)}(\mathrm{Conf}_{k_i}(\mathbb{R}^n)) \\ &\cong \bigoplus_{r\geq 0}\left(\bigoplus_{k_1+\cdots+k_r=k}\bigotimes_{i=1}^{r} H_{(n-1)(k_i-1)}(\mathrm{Conf}_{k_i}(\mathbb{R}^n)) \otimes_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_r}} \mathbb{Z}[\Sigma_k]\right)_{\Sigma_r}. \end{aligned}$$

Thus, tensoring with $\mathbb{Q}$, forming the Σ_k-coinvariants, and using that $k!$ is invertible, we find that

$$H_*(B_k(\mathbb{R}^n);\mathbb{Q}) \cong \bigoplus_{r\geq 0}\left(\bigoplus_{k_1+\cdots+k_r=k}\bigotimes_{i=1}^{r} H_{(n-1)(k_i-1)}(\mathrm{Conf}_{k_i}(\mathbb{R}^n);\mathbb{Q})_{\Sigma_{k_i}}\right)_{\Sigma_r}.$$

The claim now follows easily from the previous lemma, since the only nonvanishing terms up to the action of Σ_r are $(k_1,\ldots,k_m)=(1,\ldots,1)$ and possibly $(k_1,\ldots,k_m)=(2,1,\ldots,1)$. □

With a few more definitions in hand, this calculation may be packaged in a more succint form.

Definition. A *symmetric sequence* of graded Abelian groups is a collection $\{V_k\}_{k\geq 0}$ where $V(k)$ is a graded Abelian group equipped with an action of Σ_k.

Thus, a symmetric sequence is equivalent to the data of a functor from the category Σ of finite sets and bijections to graded Abelian groups. There is a notion of tensor product of symmetric sequences, which is given by the formula

$$(V\otimes^\Sigma W)_k = \bigoplus_{i+j=k} V_i\otimes W_j\otimes_{\Sigma_i\times\Sigma_j}\mathbb{Z}[\Sigma_k].$$

Defining a symmetric sequence by $H_*(\mathrm{Conf}(\mathbb{R}^n))_k = H_*(\mathrm{Conf}_k(\mathbb{R}^n))$, we now recognize the identification

$$H_*(\mathrm{Conf}(\mathbb{R}^n))\cong \mathrm{Sym}^\Sigma(H_{\mathrm{top}}(\mathrm{Conf}(\mathbb{R}^n)))$$

with the symmetric algebra for this tensor product.

Now, a symmetric sequence V determines a bigraded Abelian group V_Σ by the formula

$$V_\Sigma = \bigoplus_{k\geq 0}(V_k)_{\Sigma_k},$$

and it is immediate from the formula that

$$(V\otimes W)_\Sigma \cong V_\Sigma\otimes W_\Sigma.$$

Thus, we have an isomorphism of bigraded vector spaces

$$\begin{aligned}\textstyle\bigoplus_{k\geq 0} H_*(B_k(\mathbb{R}^n);\mathbb{Q}) &\cong H_*(\mathrm{Conf}(\mathbb{R}^n))_\Sigma\\ &\cong \mathrm{Sym}^\Sigma(H_{\mathrm{top}}(\mathrm{Conf}(\mathbb{R}^n)))_\Sigma\\ &\cong \mathrm{Sym}(H_{\mathrm{top}}(\mathrm{Conf}(\mathbb{R}^n))_\Sigma)\\ &\cong \mathrm{Sym}(\mathbb{Q}[0,1]\oplus\mathbb{Q}[n-1,2]).\end{aligned}$$

Remark. From the operadic point of view, this bigraded Abelian group is the free shifted Poisson algebra on one generator.

This calculation illustrates a valuable lesson, namely that configuration spaces tend to exhibit more structure when taken all together. This insight will be indispensable to us in our future investigations. Before pursuing this direction, however, we will need to invest in some new tools.

References

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