MA 530 Complex Analysis: Review

Yingwei Wang [*]

Department of Mathematics, Purdue University, West Lafayette, IN, USA

Contents

[*]This is based on the lecture notes of Prof Buzzard (Fall 2010) and Prof Bell (Spring 2011).

1 How to prove analytic (holomorphic, complex differentiable)?

Note: let Ω be an open set in $\mathbb{C}$ and f be a complex-valued function on Ω.

1.1 Definition (Difference Quotient)

Definition 1.1. *Say f is complex differentiable (holomorphic) at $z_0 \in \Omega$, if*

$$DQ = \frac{f(z_0+h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. Call the limit $f'(z_0)$.

If f is complex differentiable at all points in Ω, then call f holomorphic on Ω.

Remark 1.1. *It should be emphasized that in the above limit, h is a complex number that may approach 0 from any direction.*

Remark 1.2. *A holomorphic function will actually be infinitely many times complex differentiable, that is, the existence of the first derivative will guarantee the existence of derivatives of any order. For proof, see Section 1.4.3 and Theorem 2.4.*

For more application of this method, see Lemma 2.3, Theorem 2.2, Theorem 2.4.

1.2 Cauchy-Riemann Equations

Theorem 1.1 (C-R $\Rightarrow$ analytic)**.** *Suppose $u, v \in C^1(\Omega)$ and satisfy the C-R equations*

$$\begin{cases} u_x = v_y, \\ u_y = -v_x, \end{cases} \tag{1.1}$$

then $f(x,y) = u(xy) + iv(x,y)$ is analytic on Ω.

Remark 1.3. *Actually, by Looman-Monchoff Theorem, we just need that u, v are continuous and all their first partial derivatives exit (may be not continuous) and satisfy the C-R equations, then $f = u + iv$ is analytic.*

1.3 Integration along closed curves equals zeros

Theorem 1.2 (Morera)**.** *Suppose f is continuous on an open set Ω and for any triangle T contained in Ω,*

$$\int_T f(z)\mathrm{d}z = 0,$$

then f is holomorphic.

1.4 Power series

1.4.1 From holomorphic to power series

Lemma 1.1. *Let*

$$S_N(z) = 1 + z + \cdots + z^N, \tag{1.2}$$

$$E_N(z) = \frac{z^{N+1}}{1-z}, \tag{1.3}$$

then

$$\frac{1}{1-z} = S_N(z) + E_N(z).$$

Furthermore, if $|z| < \rho < 1$, *then* $|E_N(z)| \leq \frac{\rho^{N+1}}{1-\rho}$.

Theorem 1.3 (holomorphic $\Rightarrow$ analytic)**.** *Suppose f is holomorphic on Ω and $\overline{D_r(z_0)} \subset \Omega$. Then f has a power series expansion in $D_r(z_0)$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall z \in D_r(z_0),$$

with $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w$.

Proof. Without loss of generality, we can take $z_0 = 0$ and $\rho < r$. By Cauchy formula (Thm 2.3) and Lemma 1.1,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\frac{1}{1-\frac{z}{w}}\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w} S_N(z/w)\,\mathrm{d}w + \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w} E_N(z/w)\,\mathrm{d}w \\
&= \sum_{n=0}^{N}\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}\mathrm{d}w\right) z^n + \varepsilon_N(z),
\end{aligned}$$

where

$$|\varepsilon_N(z)| \leq \frac{1}{2\pi}\,\frac{\sup_{C_r}|f|}{r}\,\frac{(\rho/r)^{N+1}}{1-\rho/r}\,(2\pi r) \to 0.$$

as $N \to \infty$.

Besides,

$$a_n = \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}\mathrm{d}w = \frac{f^{(n)}(0)}{n!}.$$

$\square$

1.4.2 From power series to holomorphic

Definition 1.2. *Say $\{f_n\}$ converges uniformly on compact subsets of Ω to f, if for any compact subsect $K \subset \Omega$, and $\forall \varepsilon > 0$, there is an N such that $|f_n(z) - f(z)| < \varepsilon$, $\forall z \in K$, $n > N$.*

Remark 1.4. *Power series converge uniformly on compact subsects inside the circle.*

Theorem 1.4 (uniformly limit $\Rightarrow$ analytic)**.** *Suppose $\{f_n\}$ analytic on Ω and converges uniformly on compact subsets of Ω to f. Then f is analytic.*

Consequently, power series converge to analytic functions.

Proof. Let D be any disc whose closure is contained in Ω and T be any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem (Lemma 2.4) implies

$$\int_T f_n(z)\ \mathrm{d}z = 0, \quad \forall n.$$

By assumption, $f_n \to f$ uniformly on $\bar{D}$, so f is continuous and

$$\int_T f_n(z)\mathrm{d}z \to \int_T f(z)\ \mathrm{d}z.$$

As a result, we find $\int_T f(z)\mathrm{d}z = 0$ for $\forall T \subset D$. By Morera theorem (Thm 1.2), we conclude that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω. $\square$

1.4.3 Differentiate

Theorem 1.5 (Can differentiate power series term by term)**.** *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with radius of convergence R. Then power series for f' has the same radius of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

Theorem 1.6 (Derivative convergence)**.** *Suppose $\{f_n\}$ analytic on Ω and converges uniformly on compact subsets of Ω to f. Then $\{f_n^{(k)}\}$ converges uniformly on compact subsets of Ω to $f^{(k)}$.*

2 Integration along curves

2.1 Preliminaries

Definition 2.1. *Given a curve $\gamma \in \mathbb{C}$ with para $z(t) : [a,b] \to \mathbb{C}$. Suppose $f : \gamma \to \mathbb{C}$ is continuous. Then*

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t = \lim_{|\Delta z|\to 0} \sum_{j=0}^{N-1} f(z_j)(z_{j+1} - z_j),$$

where $\Delta z_j = z_{j+1} - z_j$, $|\Delta z| = \max_j |\Delta z_j|$.

Lemma 2.1 (Basic estimate)**.**

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \text{ length } (\gamma) \sup_{z\in\gamma} |f(z)|$$

where length $(\gamma) = \int_a^b |z'(t)|\ \mathrm{d}t$.

Lemma 2.2 (Reverse orientation)**.** *If γ^- is γ with reverse orientation, then*

$$\int_\gamma f(z)\mathrm{d}z = -\int_{\gamma^-} f(z)\mathrm{d}z.$$

Proof. Let γ be parameterized by $z(t) : [a,b] \to \mathbb{C}$ and γ^- parameterized by $z^-(t) : [a,b] \to \mathbb{C}$. The relationship between $z(t)$ and $z^-(t)$ is $z^-(t) = z(a+b-t)$.

Let $i = 0, \cdots, n, \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, x_0 = a, x_n = b$, and $y_i = a+b-x_i, \Delta y = \Delta x, y_0 = b, y_n = a$. By the definition of integration along curves (Def 2.1),

$$\begin{aligned}
\int_{\gamma^-} f(z)\mathrm{d}z &= \int_a^b f(z^-(t))(z^-)'(t)\mathrm{d}t \\
&= \lim_{n\to\infty} \sum_{i=1}^n f(z^-(x_i))(z^-)'(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n f(z(y_i))\left(-z'(y_i)\right)\Delta y \\
&= -\int_a^b f(z(t))z'(t)\mathrm{d}t \\
&= -\int_\gamma f(z).
\end{aligned}$$

□

2.2 Fundamental Theorem of Calculus

Definition 2.2. *A primitive for f on Ω is a function F that is holomorphic on Ω and such that*

$$F'(z) = f(z), \quad \forall z \in \Omega.$$

Theorem 2.1 (Fundamental Theorem of Calculus #2)**.** *Let Ω be an open set in $\mathbb{C}$ and γ be a curve (or path) in Ω that begins at w_1 and ends at w_2.*

Version I: *If f is analytic on Ω, then*

$$\int_\gamma f'(z)\mathrm{d}z = f(w_2) - f(w_1).$$

Version II: *If f has a primitive in Ω, then*

$$\int_\gamma f(z)\mathrm{d}z = F(w_2) - F(w_1).$$

Proof. Chain Rule + Fundamental Theorem of Calculus # 1. □

Corollary 2.1. *If γ is a closed curve and f is holomorphic, then $\int_\gamma f(z)\mathrm{d}z = 0$.*

Corollary 2.2. *Any two primitives of f (if they exist) differ by a constant.*

Proof. Suppose both F and G are the primitives of function f. According to Thm 2.1 (version II), we know that if γ is a curve in Ω from w_0 to w, then

$$\int_\gamma f(z)\mathrm{d}z = F(w) - F(w_0) = G(w) - G(w_0).$$

Fix w_0, then $\forall w \in \Omega, F(w) - G(w) = F(w_0) - G(w_0) =$ constant. □

Corollary 2.3. *If Ω is a region (open+connected), f is complex differentiable at each point in Ω, and $f'(z) = 0$ for all $z \in \Omega$, then f is a constant.*

Proof. **Method I:** Path connected + Theorem 2.1.

Fix a point $w_0 \in \Omega$. It is suffices to show that $f(w) = f(w_0)$ for all $w \in \Omega$.

Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w. By Thm 2.1 (version I),

$$\int_\gamma f'(z)\mathrm{d}z = f(w) - f(w_0).$$

By assumption, $f' = 0$ so the integral on the left is 0 and we conclude that $f(w) = f(w_0)$ as desired.

Method II: Definition + C-R equations.

$$
\begin{aligned}
& f'(z) = 0, \\
\Rightarrow \quad & \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) = 0, \\
\Rightarrow \quad & (u_x + iv_x) - i(u_y + iv_y) = 0, \quad \text{since } f_x = u_x + iv_x, f_y = u_y + iv_y, \\
\Rightarrow \quad & (u_x + v_y) + i(v_x - u_y) = 0, \\
\Rightarrow \quad & u_x + v_y = 0, \quad v_x - u_y = 0.
\end{aligned}
$$

Besides, by C-R Eqn.(1.1), we can get that

$$
\begin{aligned}
& u_x = u_y = v_x = v_y = 0 \\
\Rightarrow \quad & u(x,y) = \text{constant}, \quad v(x,y) = \text{constant} \\
\Rightarrow \quad & f = \text{constant}.
\end{aligned}
$$

□

2.3 Cauchy Theorem

Lemma 2.3 (Deferential under integral)**.** *Suppose that $\varphi(z)$ is continuous function on the trace of a path γ. Prove that the function*

$$
f(z) = \int_\gamma \frac{\varphi(w)}{w - z}\, \mathrm{d}w,
$$

is analytic on $\mathbb{C} \setminus \gamma$.

Idea: just need to show that

$$
f'(z) = \int_\gamma \frac{\varphi(w)}{(w - z)^2}\, \mathrm{d}w. \tag{2.1}
$$

Proof. Recall the DQ method, for $\forall z_0 \in \mathbb{C} \setminus \gamma$,

$$
\begin{aligned}
& \frac{f(z) - f(z_0)}{z - z_0} - \int_\gamma \frac{\varphi(w)}{(w - z)^2}\, \mathrm{d}w \\
= \quad & \int_\gamma \varphi(w) \left[\frac{1}{(w - z)(w - z_0)} - \frac{1}{(w - z_0)^2}\right] \mathrm{d}w \\
= \quad & \int_\gamma \varphi(w) \left[\frac{z - z_0}{(w - z)(w - z_0)^2}\right] \mathrm{d}w \\
= \quad & (z - z_0) \int_\gamma \left[\frac{\varphi(w)}{(w - z)(w - z_0)^2}\right] \mathrm{d}w.
\end{aligned}
$$

Let

$$
\begin{aligned}
E(z) &= \int_\gamma \left[\frac{\varphi(w)}{(w-z)(w-z_0)^2}\right] \mathrm{d}w, \\
D &= \text{dist }(z_0, \mathrm{tr}(\gamma)) = \min_{w\in \mathrm{tr}\ (\gamma)} |z_0 - w|, \\
M &= \max_{w\in \mathrm{tr}(\gamma)} |\varphi(w)|.
\end{aligned}
$$

Choose a disc $D_{D/2}(z_0)$, $\forall z \in D_{D/2}(z_0)$, $\forall w \in \gamma$, we have

$$
|z-w| \geq D/2, \quad |z_0 - w| \geq D/2.
$$

By Lemma 2.1, we have

$$
\begin{aligned}
& |E(z)| \leq M\frac{1}{(D/2)(D/2)^2} \text{ length } (\gamma), \\
\Rightarrow \quad & \left|\frac{f(z)-f(z_0)}{z-z_0} - \int_\gamma \frac{\varphi(w)}{(w-z)^2} \, \mathrm{d}w\right| \\
& \leq |z-z_0|\frac{M}{(D/2)(D/2)^2} \text{ length } (\gamma) \to 0, \quad \text{as } z \to z_0.
\end{aligned}
$$

It implies Eq.(2.1). $\square$

Lemma 2.4 (Goursat). *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$
\int_T f(z)\mathrm{d}z = 0,
$$

whenever f is continuous on Ω and analytic on $\Omega\backslash\{p\}$.

Theorem 2.2 (Cauchy's theorem on a convex open set). *Suppose Ω is convex and open, $p \in \Omega$. If f is continuous on Ω and analytic on $\Omega\backslash\{p\}$, then*

$$
\int_\gamma f \mathrm{d}z = 0,
$$

for any closed γ in Ω.

Idea: Construct F holomorphic with $F' = f$, Then f is continuous so F is continuous and by Thm 2.1,

$$
\int_\gamma f \mathrm{d}z = \int_\gamma F' \mathrm{d}z = F(\text{end}) - F(\text{start}) = 0. \tag{2.2}
$$

Proof. Fix $a \in \Omega$. Let L_a^z be the line segement from a to z. Ω is convex indicates that L_a^z is contained in Ω.

Define

$$f(z) = \int_{L_a^z} f(w)\mathrm{d}w.$$

Fix $z_0 \in \Omega$ and consider z near z_0. By Lemma 2.4,

$$\begin{aligned}
&\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \int_{-L_a^z}\right) f(w)\mathrm{d}w = 0, \\
\Rightarrow \quad & F(z_0) + \int_{L_{z_0}^z} f(w)\mathrm{d}w - F(z) = 0, \\
\Rightarrow \quad & \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{L_{z_0}^z} f(w)\mathrm{d}w.
\end{aligned}$$

Since f is continuous at z_0, then

$$f(z) = f(z_0) + R(z),$$

with $R(z) \to 0$, as $z \to z_0$. So for $\varepsilon > 0$, $\exists \delta > 0$ so that if $|z - z_0| < \delta$ then $|R(z)| < \varepsilon$.

Also,

$$\int_{L_{z_0}^z} f(z_0)\mathrm{d}w = \int_{L_{z_0}^z} \frac{\mathrm{d}}{\mathrm{d}w}[f(z_0)w]\mathrm{d}w = f(z_0)(z - z_0).$$

Hence,

$$\begin{aligned}
& \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{L_{z_0}^z} (f(z_0) + R(w))\mathrm{d}w = f(z_0) + \frac{1}{z - z_0} \int_{L_{z_0}^z} R(w)\mathrm{d}w, \\
\Rightarrow \quad & \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \sup_{w \in L_{z_0}^z} |R(w)| \to 0, \quad \text{as } z \to z_0.
\end{aligned}$$

It indicates that $F'(z_0) = f(z_0)$. Then $F' = f$. By Eq.(2.2), we can get the conclusion. $\square$

2.4 Cauchy Integral Formula

Theorem 2.3 (Cauchy Integral Formula on a disc). *Suppose f is holomorphic on $D_R(z_0)$ and $0 < r < R$. Then $\forall a \in D_r(z_0)$,*

$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - a}\mathrm{d}z. \tag{2.3}$$

Proof. Let

$$G(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a, \\ f'(a) & z = a. \end{cases} \tag{2.4}$$

It is easy to know that $G(z)$ is continuous and $G(z)$ is holomorphic on $D_r(z_0)\backslash\{a\}$. By Thm 2.2, we have

$$\begin{aligned} & \int_{C_r(z_0)} G(z)\mathrm{d}z = 0, \\ \Rightarrow \quad & \int_{C_r(z_0)} \left(\frac{f(z)}{z-a} - \frac{f(a)}{z-a}\right) \mathrm{d}z = 0, \\ \Rightarrow \quad & \int_{C_r(z_0)} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{C_r(z_0)} \frac{f(a)}{z-a}\,\mathrm{d}z = 2\pi i f(a). \end{aligned}$$

□

Theorem 2.4 (Cauchy Integral Formula with derivatives)**.** *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $\overline{D_r(z_0)} \subset \Omega$, then $\forall z \in D_r(z_0)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w. \tag{2.5}$$

Proof. Here, we give three methods to prove this.

Method I Induction on n and by Def 1.1.

$n = 0$, by Thm 2.3,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\mathrm{d}w.$$

Suppose f has desideratives $0, 1, \cdots, n-1$ and the formula (2.5) is ture. Then

$$DQ = \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{h}\left(\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}\right)\mathrm{d}w$$

Let $A = \frac{1}{w-z-h}, B = \frac{1}{w-z}$, then

$$\begin{aligned} A - B &= \frac{h}{(w-z-h)(w-z)}, \\ A^n - B^n &= (A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1}). \end{aligned}$$

So we have

$$\begin{aligned}
& \lim_{h\to 0} DQ \\
= \quad & \lim_{h\to 0}\frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z-h)(w-z)}(A^{n-1}+A^{n-2}B+\cdots+B^{n-1})\mathrm{d}w, \\
= \quad & \frac{(n-1)!}{2\pi i}\int_C \lim_{h\to 0}\frac{f(w)}{(w-z-h)(w-z)}(A^{n-1}+A^{n-2}B+\cdots+B^{n-1})\mathrm{d}w, \\
= \quad & \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\ \mathrm{d}w.
\end{aligned}$$

Method II Induction on n and differential under integral (Lemma 2.3).

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}z}f^{(n-1)}(z) \\
= \quad & \frac{(n-1)!}{2\pi i}\int_C \left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{f(w)}{(w-z)^n}\right]\mathrm{d}w \\
= \quad & \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\ \mathrm{d}w.
\end{aligned}$$

Method III Power series expansion By Eq.(2.3),

$$f(z)=\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(w)}{w-z}\mathrm{d}w.$$

Do power expansion

$$\begin{aligned}
& \frac{1}{w-z} \\
= \quad & \frac{1}{(w-z_0)-(z-z_0)} \\
= \quad & \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\
= \quad & \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n.
\end{aligned}$$

Then

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\mathrm{d}w \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w\right)(z-z_0)^n.
\end{aligned}$$

It implies Eq.(2.5).

□

2.5 Liouville's theorem

Theorem 2.5 (Cauchy inequality). *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Theorem 2.6 (Liouville's theorem). *If f is entire and bounded, then f is constant.*

2.6 Fundamental Theorem of Algebra

Lemma 2.5 (Basic polynomial estimate). *Suppose*

$$p(z) = a_N z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0,$$

is a complex polynomial of degree N. Then there exist constants $0 < A < B$ and a radius R such that

$$A|z|^N \leq |p(z)| \leq B|z|^N, \quad \text{if } |z| > R. \tag{2.6}$$

Remark 2.1. *Here, $0 < A < |a_N| < B$ and A, B can be as close to a_N as desired.*

Theorem 2.7 (Fundamental Theorem of Algebra). *Every non-constant polynomial $p(z)$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. Assume that $p(z) \neq 0$, $\forall z \in \mathbb{C}$. By Eq.(2.6), we know that $f(z) = \frac{1}{p(z)}$ is bounded entire. And by Liouville Theorem (Thm 2.6), f is constant, so $p(z)$ is constant, which is a contradiction. $\square$

Corollary 2.4. *Every polynomial $p(z)$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, w_2, \cdots, w_n$, then $p(z)$ can be factored as*

$$p(z) = a_n(z - w_1)(z - w_2)\cdots(z - w_n).$$

3 Useful properties of holomorphic functions

3.1 Isolated zeros

Theorem 3.1 (Zero theorem). *Suppose f is holomorphic on Ω and $f(z_0) = 0, z_0 \in \Omega$. If $\exists r > 0$ such that $D_r(z_0) \in \Omega$ and $f(z)$ is not identically 0 on $D_r(z_0)$, then $\exists n_0 \in \mathbb{Z}^+$ and a function $h(z)$ which is holomorphic on $D_r(z_0)$ so that $h(z_0) \neq 0$ and*

$$f(z) = (z - z_0)^{n_0} h(z), \quad \forall z \in D_r(z_0).$$

Proof. From Theorem 1.3, we know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \in D_r(z_0),$$

with $a_n = \frac{f^{(n)}(z_0)}{n!}$.

Since f is not identically 0 on $D_r(z_0)$, we know that $\exists n$ such that $a_n \neq 0$. Also, $f(z_0) = 0 \Rightarrow a_0 = 0$.

Let $n_0 = \min\{n \in \mathbb{Z}^+ : f^{(n)}(z_0) \neq 0\}$, then

$$f(z) = (z - z_0)^{n_0} \sum_{n=n_0}^{\infty} a_n (z - z_0)^{n-n_0}.$$

Let $h(z) = \sum_{n=n_0}^{\infty} a_n (z - z_0)^{n-n_0}$ which is holomorphic on $D_r(z_0)$. Also, $h(z_0) = a_{n_0} = \frac{f^{n_0}(z_0)}{n!} \neq 0$. Then we get the conclusion. □

Corollary 3.1. *Suppose f and g are analytic on a domain Ω and that $f^2 = g^2$ on Ω. Prove that either $f = g$ or $f = -g$ on Ω.*

Proof. Choose $F = f + g$, $G = f - g$. Assume both of F and G do not vanish on Ω. It means that F and G are not identically zero on Ω. It also means that F and G only have isolated zeros in Ω.

$$\begin{aligned} & f^2 = g^2 \\ \Rightarrow \quad & F(z)G(z) = 0 \\ \Rightarrow \quad & \exists z_0 \text{ such that either } F(z_0) = 0 \text{ or } G(z_0) = 0. \end{aligned}$$

Without loss of generality, let $F(z_0) = 0$. Since z_0 is an isolated zero of F, $\exists r_1 > 0$ such that $F(z) \neq 0$ on $\hat{D_{r_1}}(z_0) = D_{r_1}(z_0) \setminus \{z_0\}$. Besides, G also only has isolated zeros, so we can choose $a \in \hat{D_{r_1}}(z_0)$, such that $G(a) \neq 0$. $\exists r_2 > 0$ such that $G(z) \neq 0$ on $D_{r_2}(a)$ and $D_{r_2}(a) \subset \hat{D_{r_1}}(z_0)$. Then

$$F(z)G(z) \neq 0, \quad \forall z \in D_{r_2}(a),$$

which contradicts with the assumption. □

3.2 Identity theorem

Theorem 3.2 (Identity theorem)**.** *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $Z_f = \{z \in \Omega : f(z) = 0\}$. Then either $Z_f = \Omega$ or Z_f has no limit points in Ω.*

Proof. **Step 1:** Disc version.

Step 2: Let U be the interior of Z_f. Then U is open and nonempty.

Step 3: Let $V = \Omega \backslash U$, then V is also open.

Step 4: Since Ω is connected and $\Omega = U \cup V$, U is not empty, we know that $V = \emptyset$ and $\Omega = Z_f$. □

Corollary 3.2. *Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some non-empty open subsets of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ throughout Ω.*

Corollary 3.3. *Only one way to extend e^x and trig functions to $\mathbb{C}$. Besides, any trig identity holds for complex angles.*

Remark 3.1. *An analytic function may have infinitely many (at most countable) zeros on a bounded domain as soon as the limit point of these zeros is not in this domain. For example, $f(z) = \sin\left(\frac{1}{1-z}\right)$ has infinitely many zeros on the open unit disc $D_1(0)$, i.e. $z_k = 1 - \frac{1}{k\pi}$, but $z_k \to 1 \bar{\in} D_1(0)$. So each $\{z_k\}$ is also isolated zero. Besides, $z = 1$ is the essential singularity of $f(z)$.*

3.3 Averaging property

Lemma 3.1 (Averaging property). *f is holomorphic on Ω and $\overline{D_r(z_0)} \subset \Omega$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\mathrm{d}t \tag{3.1}$$

Proof. Let C_r be a circle in Ω, centered at z_0 with radius r. If we parameterize C_r by $z = z_0 + r\mathrm{e}^{it}, 0 \leq t \leq 2\pi$, then by Cauchy integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0}\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{it})}{r\mathrm{e}^{it}}\, ir\mathrm{e}^{it}\ \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\ \mathrm{d}t \end{aligned}$$

□

3.4 Maximum principle

Theorem 3.3 (Maximum principle #1). *Suppose f is holomorphic on a domain Ω. If $|f|$ attains a local maximum at a point in Ω, then $f \equiv$ constant.*

Proof. **Method I By Open mapping theorem**

Suppose $|f|$ has a local max at $z_0 \in \Omega$. Let $w_0 = f(z_0)$. Then $\exists r > 0$ such that $D_r(z_0) \in \Omega$ and $|f(z)| \leq |f(z_0)|$ for $z \in D_r(z_0)$.

By Open Mapping Theorem (Thm 6.3), if f is nonconstant, $f(D_r(z_0))$ contains a disc about $f(z_0)$. But there are points in such a disc with modulus bigger than $|w_0|$. This is a contradiction.

Method II By averaging property

Suppose $z_0 \in \Omega$ and that for all $z \in \Omega$, $|f(z)| \leq |f(z_0)|$. By Eq.(3.1), we know that

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{it})| \ \mathrm{d}t. \tag{3.2}$$

However, the assumption $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$ implies that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{it})| \ \mathrm{d}t \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| \ \mathrm{d}t = |f(z_0)|. \tag{3.3}$$

From (3.2)-(3.3), we can get

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{it})| \ \mathrm{d}t. \tag{3.4}$$

It follows that

$$\begin{aligned} 0 &= |f(z_0)| - \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{it})| \ \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} (|f(z_0)| - |f(z_0 + r\mathrm{e}^{it})|) \ \mathrm{d}t, \end{aligned}$$

which means $|f(z_0)| = |f(z_0 + r\mathrm{e}^{it})|$ for all $t \in [0, 2\pi]$. That is to say $|f(z)| = |f(z_0)|$ for all $z \in C_r$. Since r is arbitrary, it follows that $|f(z)| = |f(z_0)|$ for all $z \in \Omega$. Since f is holomorphic and $|f(z)|$ is a constant, f is also a constant (by Chapter One, # 13(c), Page 28). □

Theorem 3.4 (Maximum principle #2)**.** *Suppose Ω is a bounded domain. f is continuous on $\bar{\Omega}$ and holomorphic on Ω. Then $|f|$ assumes its maximum value on the boundary of Ω.*

3.5 Rouche's Theorem

Theorem 3.5 (Rouche's Theorem)**.** *Suppose f and g are meromorphic functions on a connected open $G \subset \mathbb{C}$ and γ is a piecewise C^1 closed curve in G with*

(i) $Ind_\gamma(w) = 0$ for $\forall w \in \mathbb{C} \setminus G$;

(ii) no zeros or poles of f or g on γ;

(iii) $|f(z) - g(z)| < |f(z)|$ for all $z \in \gamma$ (that is, the difference is strictly smaller than one of the functions $|f|$ on γ).

Then,

$$N_f - P_f = N_g - P_g, \tag{3.5}$$

where

$$N_f = \sum_{a \in G, f(a)=0} mult_a \ Ind_\gamma(a),$$
$$P_f = \sum_{b \in G, f(b)=\infty} order_b \ Ind_\gamma(b),$$

and similarly for N_g and P_g.

Here are some applications.

Corollary 3.4. *There are no such sequence of polynomials that uniformly converges to* $f(z) = \frac{1}{z}$ *on the circle* $\{z : |z| = 1\}$.

Proof. Suppose $\exists\ p_n(z) \to \frac{1}{z}$ uniformly on $C_1(0)$, then

$$|zp_n(z) - 1| = |p_n(z) - \frac{1}{z}| \to 0, \quad \forall z \in C_1(0).$$

Then $\exists$ sufficient large N such that

$$|zp_n(z) - 1| < 1, \quad \forall n > N,\ \forall z \in C_1(0).$$

By Rouche's Theorem, $N_{zp_n(z)} = N_1$. But $zp_n(z)$ has at least one zero, while 1 has no zeros, which is a contradiction. □

Corollary 3.5. *There are no such sequence of polynomials that uniformly converges to* $f(z) = (\bar{z})^2$ *on the circle* $\{z : |z| = 1\}$.

Proof. Suppose $\exists\ p_n(z) \to (\bar{z})^2$ uniformly on $C_1(0)$, then

$$|z^2 p_n(z)| \to |z^2\, (\bar{z})^2\, | = |z|^4 = 1, \quad \forall z \in C_1(0).$$

Then do the similar thing as last corollary. □

3.6 Argument Principle

Theorem 3.6 (Argument principle)**.** *Let* C *be a simple closed path. Suppose that* $f(z)$ *is analytic and nonzero on* C *and meromorphic inside* C. *List the zeros of* f *inside* C *as* $z_1, z_2, \cdots, z_k$ *with multiplicities* $N_1, \cdots, N_k$, *and* $Z_C = \sum_{i=1}^{k} N_i$. *List the poles of* f *inside* C *as* $w_1, w_2, \cdots, w_l$ *with orders* $M_1, \cdots, M_l$, *and* $P_C = \sum_{j=1}^{l} M_j$. *Then*

$$Z_C - P_C = \frac{1}{2\pi} \triangle_C\ arg\ f(z) \tag{3.6}$$

$$= \frac{1}{2\pi i} \int_C \frac{f'(\zeta)}{f(\zeta)}\, \mathrm{d}\zeta. \tag{3.7}$$

Remark 3.2. *From Eqs.(3.6)-Eq.(3.7), we know that*

$$\triangle_C\ arg\ f(z) = \frac{1}{i} \int_C \frac{f'(\zeta)}{f(\zeta)}\, \mathrm{d}\zeta. \tag{3.8}$$

This formula is always true even if the curve C *is not a closed path.*

Corollary 3.6. *How many zeros does the polynomial*

$$f(z) = z^{1998} + z + 2001$$

have in the first quadrant?

Proof. Chose a closed curve:

$$\gamma = L_0^R + C_R^+ + L_{iR}^0,$$

where C_R^+ here means the circle in the first quadrant and R is sufficient large.

On one hand, on L_0^R and L_{iR}^0, we know that

$$\triangle_{L_0^R} \arg f(z) = 0, \tag{3.9}$$

$$\triangle_{L_{iR}^0} \arg f(z) = -\pi. \tag{3.10}$$

Hint for Eq.(3.10): if $z = iR$, then $f(z) = -R^1 998 + iR + 2001$. And consider R from ∞ to 0.

On the other hand, on C_R^+, we know that

$$\begin{aligned} \int_{C_R^+} \frac{f'(\zeta)}{f(\zeta)} \, \mathrm{d}\zeta &= \int_{C_R^+} \frac{1998\zeta^{1997} + 1}{\zeta^{1998} + \zeta + 2001} \, \mathrm{d}\zeta \\ &\approx \int_{C_R^+} \frac{1998}{\zeta} \, \mathrm{d}\zeta \\ &= 1998 \, \frac{2\pi i}{4} = 999\pi i. \end{aligned} \tag{3.11}$$

By Thm. 3.6 and Eqs.(3.9)-(3.11), we know that the number of zeros of $f(z)$ in the first quadrant is

$$\frac{999\pi - \pi}{2\pi} = 499.$$

□

4 Harmonic functions

4.1 Definition

The following definitions of harmonic functions are equivalent:

Definition 4.1. *Say $u : \Omega \to \mathbb{R}$ is harmonic if $u \in C^\infty$ and $\Delta u \equiv 0$.*

Definition 4.2. *Say $u : \Omega \to \mathbb{R}$ is harmonic if $u \in C^2$ and $\Delta u \equiv 0$.*

Definition 4.3. *Say $u : \Omega \to \mathbb{R}$ is harmonic if u is locally the real (imaginary) part of a holomorphic function.*

Definition 4.4. *Say $u : \Omega \to \mathbb{R}$ is harmonic if u is continuous on Ω and u_x, u_y, u_{xx}, u_{yy} exist and $\Delta u = 0$ on Ω.*

Remark 4.1. *Here, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called* Laplacian operator *and $\Delta u \equiv 0$ is called* Laplacian equation.

4.2 Harmonic conjugate

Theorem 4.1. *A harmonic function on a simply connected domain has a global harmonic conjugate.*

Remark 4.2. *Suppose u be harmonic and f be the analytic function with $Ref = u$. Then, $f' = u_x - iu_y$ is also analytic.*

4.3 Poisson integral formula

Definition 4.5 (Poisson kernel)**.**

$$P(z,t) = \frac{1}{2\pi}\frac{1-|z|^2}{|\mathrm{e}^{it}-z|^2} = \frac{1}{2\pi}\ Re\ \left(\frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\right), \tag{4.1}$$

$$P(r\mathrm{e}^{i\theta},t) = \frac{1}{2\pi}\ \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}. \tag{4.2}$$

Theorem 4.2 (Dirichlet problem)**.** *Suppose u is continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$. Then*

$$u(z) = \int_0^{2\pi} P(z,t)u(\mathrm{e}^{it})\mathrm{d}t.$$

Lemma 4.1. *Some properties about Poisson kernel:*
- *(i)* $P(a,t) > 0$ *if* $a \in D_1(0), t \in [0,2\pi]$.
- *(ii)* $\int_0^{2\pi} P(a,t)\mathrm{d}t = 1$ *for any* $a \in D_1(0)$.
- *(iii)* $P(z,t) \to 0$ *as* $|z| \to 1$.
- *(iv)* $P(z,t)$ *is harmonic in* z.

Theorem 4.3 (Convergence)**.** *Suppose $\{u_m\}_{m=1}^{\infty}$ is a sequence of harmonic functions on Ω such that u_m converges uniformly to a function u on each compact subset of Ω. Then u is harmonic on Ω.*

Moreover, for every multi-index α, $D^{\alpha}u_m$ converges uniformly on each compact subset of Ω.

Proof. Given $\overline{D_r(a)} \subset \Omega$, we need only show that u is harmonic on $D_r(a)$. Without loss of generality, we assume $D_r(a) = D_1(0)$.

By Thm 4.2, we know that

$$u_m(z) = \int_0^{2\pi} P(z,t)u_m(\mathrm{e}^{it})\mathrm{d}t,$$

for $\forall z \in D_1(0)$ and $\forall m$. Taking the limit of both sides, we obtain

$$u(z) = \int_0^{2\pi} P(z,t)u(\mathrm{e}^{it})\mathrm{d}t,$$

for $\forall z \in D_1(0)$. Thus, u is harmonic on $D_1(0)$. ∎

4.4 Mean Value Property and Maximum Principle

Theorem 4.4 (Mean Value Property)**.** *Since holomorphic $f = u + iv$ satisfies the averaging property, take the real part of Eq.(3.1), we can get the mean value property of harmonic u. More precisely,*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r\mathrm{e}^{i\theta})\mathrm{d}\theta.$$

An important sequence of the mean value property is the following maximum principle for harmonic functions.

Theorem 4.5 (Maximum Principle)**.** *Suppose Ω is connected, u is real valued and harmonic on Ω, and u has a maximum or minimum in Ω. Then u is a constant.*

The following corollary is frequently useful. Note that the connectivity of Ω is not needed here.

Corollary 4.1. *Suppose Ω is bounded, u is a continuous real valued function on $\bar{\Omega}$ that is harmonic on Ω. Then u attains its maximum and minimum values over $\bar{\Omega}$ on $\partial\Omega$.*

The next corollary is a version of maximum principle for complex valued functions.

Corollary 4.2. *Let Ω be connected and u be harmonic on Ω. If $|u|$ has a maximum in Ω, then u is a constant.*

Remark 4.3. *For holomorphic cases, see Thm 1.4 and Thm 1.6.*

Theorem 4.6 (Converse of the mean value property #1)**.** *A continuous function that satisfies the mean value property must be harmonic.*

Proof. **Step I: Poisson Kernel**

Let

$$v(z) = \int_0^{2\pi} P(z,t)u(\mathrm{e}^{it})\mathrm{d}t,$$

then $v(z)$ is harmonic and $v(\mathrm{e}^{it}) = u(\mathrm{e}^{it})$.

Step II: Maximum Principal

Since $u(z)$ satisfies the mean value property, then $u(z)$ also satisfies the maximum principal. So $u - v$ satisfies the maximum principal. Besides, $u - v \equiv 0$ on the boundary, so $u - v \leq 0$ for all $z \in D_1(0)$.

Repeat the argument, we can get $v - u \leq 0$. Hence, $u \equiv v$. □

Definition 4.6. *Say f satisfies the weak mean value property if for each $z_0 \in \Omega$, $\exists \varepsilon > 0$, such that*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho\mathrm{e}^{it})\mathrm{d}t,$$

for all ρ with $0 < \rho < \varepsilon$.

Theorem 4.7 (Converse of the mean value property #2)**.** *A continuous function that satisfies the weak mean value property must be harmonic.*

4.5 Zeros of harmonic functions

Theorem 4.8. *Let u be real-valued and harmonic function in the open set Ω.*
(1) Let $A = \{z \in \Omega : u(z) = 0\}$. show that A cannot be isolated.
(2) Let $B = \{z \in \Omega : \nabla u = 0\}$. Show that either $B = \Omega$ or B is isolated.

Proof. (1) Suppose $z_0 \in A$ and $\overline{D_{r_n}(z_0)} \subseteq \Omega$, $r_n = \frac{1}{n}$.
By Mean value property (Thm 4.4),

$$0 = u(z_0) = \frac{1}{2\pi} \int_{C_{r_n}} u(z) \mathrm{d}z.$$

For each n, since $u(z)$ is continuous on $C_{r_n}(z_0)$, there exists $z_n \in C_{r_n}(z_0)$ such that $u(z_n) = 0$.
Then we found a sequence $\{z_n\}_{n=1}^{\infty} \subseteq A$ such that $z_n \to z_0$. So A cannot be isolated.
(2) Since u is harmonic, there exists a holomorphic function $f = u + iv$ on Ω.
$\nabla u = 0$ means $u_x = 0, u_y = 0$. So we have $f'(z) = u_x - iu_y = 0$, $\forall z \in B$.
Let $g(z) = f'(z)$, then g is also holomorphic on Ω, and A is the set of zeros of g. So either $g(z) \equiv 0$ or z_0 is isolated for $\forall z_0 \in B$.
It follows that either $B = \Omega$ or B is isolated. □

Remark 4.4. *Suppose u is a non-constant real-valued function on the whole complex plane. Then the zero set $\{z \in \mathbb{C} : u(z) = 0\}$ is an unbounded set.*

5 Isolated singularity

Note: let *punctured domain* $\hat{\Omega} = \Omega \backslash \{z_0\}$, and *punctured disk* $\hat{D}_r(z_0) = D_r(z_0) \backslash \{z_0\}$, where Ω is an open and connected domain and $D_r(z_0)$ is an open disc centered at z_0 with radius r.

5.1 Definition

A *point singularity* of a function f is a complex number z_0 such that f is defined in a neighborhood of z_0 but not at the point z_0 itself. We also call such points *isolated singularities.*

5.1.1 Removable singularity

Let f be holomorphic in $\hat{\Omega}$. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a *removable singularity* of f.

5.1.2 Pole

We say that a function f defined in $\hat{D}_r(z_0)$ has a *pole* at z_0, if the function $\frac{1}{f}$, defined to be zero at z_0, is holomorphic in $D_r(z_0)$.

5.1.3 Essential singularity

f oscillates and may grow faster than any power at z_0, which is called *essential singularity.*

Remark 5.1. *f has an isolated singularity at ∞ means $f(\frac{1}{z})$ has an isolated singularity at* 0.

Remark 5.2. *f is* meromorphic *on Ω means that at each $z \in \Omega$, either f is holomorphic or f has a pole.*

5.1.4 From the view of power series

Consider the Laurent expansion of a function in the *punctured disk* $\hat{D}_r(z_0)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)} \mathrm{d}\zeta.$$

- If
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$
then $f(z)$ has a *removable singularity* at $z = z_0$, which means $f(z)$ may be extended by defining $f(z_0) = a_0$, and the resulting function is analytic in the open disk $D_r(z_0)$.

- If
$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n, \quad N > 0, a_N \neq 0,$$
then $f(z)$ has a *zero of multiply* N at $z = z_0$. Near z_0, $f(z) = (z - z_0)^N g(z)$, where $g(z)$ is analytic in $D_r(z_0)$, $g(z_0) \neq 0$.

- If
$$f(z) = \sum_{n=-M}^{\infty} a_n (z - z_0)^n, \quad M > 0, a_{-M} \neq 0,$$
then $f(z)$ has a *pole of order* M at $z = z_0$. Near z_0, $f(z) = (z - z_0)^{-M} g(z)$, where $g(z)$ is analytic in $D_r(z_0)$, $g(z_0) \neq 0$.

- If
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad a_n \neq 0 \text{ for infinitely many negative } n,$$
then $f(z)$ has a *essential singularity* at $z = z_0$.

- The coefficient of $(z-z_0)^{-1}$ is called the *residue* of $f(z)$ at z_0. Suppose z_0 is the m order pole of $f(z)$, then

$$\mathrm{Res}_{z_0} f(z) = (m-1)! \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z-z_0)^m f(z) \right].$$

Remark 5.3. *Here, from this point of view, we make the summary of Thm.1.3, Thm.3.1, Thm.5.1, Thm.5.2 and Thm.5.3.*

5.2 Riemann removable singularities theorem

Theorem 5.1. *Suppose f is holomorphic and bounded in $\hat{\Omega}$, then f has a removable singularity at z_0.*

That is to say, if $\exists M > 0$ such that $|f(z)| < M, \forall z \in \hat{D}_r(z_0)$, then $\exists h(z)$ holomorphic in $D_r(z_0)$ and $h = f$ on $\hat{D}_r(z_0)$.

Proof: Here we give too methods to prove this.

5.2.1 Method I: integral formulas + estimates

We shall prove that for $\forall z \in \hat{D}_r(z_0)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta. \tag{5.1}$$

By Cauchy theorem, we have

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta + \int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta + \int_{\gamma'_\epsilon} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = 0, \tag{5.2}$$

where γ_ϵ and γ'_ϵ are small circles of radius ϵ with negative orientation and centered at z and z_0 respectively.

On one hand,

$$\int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = -2\pi i f(z). \tag{5.3}$$

On the other hand, since f is bounded and ϵ is small, ζ stays away from z, we have

$$\left| \int_{\gamma'_\epsilon} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \right| \leq C\epsilon. \tag{5.4}$$

By (5.2)-(5.4) and letting ϵ tend to 0, then we can get (5.1).

Now it is OK to choose $h(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$.

5.2.2 Method II: construct function

Consider the function $g(z)$ defined by

$$g(z) = \begin{cases} 0, & \text{if } z = z_0, \\ (z-z_0)^2 f(z), & \text{if } z \neq z_0. \end{cases} \tag{5.5}$$

By assumption, $g(z)$ is holomorphic on $\hat{D}_r(z_0)$. Next to find $g'(z_0)$.

On one hand,

$$g(z) = g(z_0) + 0(z-z_0) + [(z-z_0)f(z)](z-z_0). \tag{5.6}$$

Note that $|(z-z_0)f(z_0)| \leq |z-z_0|M \to 0$ as $z \to z_0$.

On the other hand, consider the Taylor series

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + R(z)(z-z_0), \tag{5.7}$$

where $R(z) \to 0$ as $z \to z_0$.

By (5.6) and (5.7), we can know that $g'(z_0) = 0$ and $g(z)$ is holomorphic on $D_r(z_0)$.

So we have

$$g(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^2 \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n-2}. \tag{5.8}$$

Let $h(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n-2}$, which is holomorphic on $D_r(z_0)$. Besides, by (5.5) and (5.8), we can know that $h(z) = f(z)$ for $\forall z \in \hat{D}_r(z_0)$.

5.3 Pole Theorem

Lemma 5.1. *z_0 is a pole of f $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to z_0$.*

Theorem 5.2 (Pole theorem)**.** *Suppose $f : \hat{\Omega} \to \mathbb{C}$ is holomorphic and $\lim_{z \to z_0} |f(z)| = \infty$. Then $\exists n \in \mathbb{Z}^+$ and $h(z)$ satisfying $h(z_0) \neq 0$ and $f(z) = \frac{h(z)}{(z-z_0)^n}$*

Proof. Since $|f(z)| = \infty$ as $z \to z_0$, $\exists r > 0$ such that $D_r(z_0) \subseteq \Omega$ and $|f(z)| > 1$ on $\hat{D}_r(z_0)$.

Let $g(z) = \frac{1}{f(z)}$, then $g(z)$ is holomorphic and bounded on $\hat{D}_r(z_0)$. By Theorem 5.1, $g(z)$ can extend to be holomorphic on $D_r(z_0)$ by defining

$$g(z_0) = \lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

Since z_0 is a isolated zero of $g(z)$, by Theorem 3.1, there exits $n \in \mathbb{Z}^+$ and $H(z)$ which is holomorphic on $D_r(z_0)$ with $H(z_0) \neq 0$ such that $g(z) = (z-z_0)^n H(z)$.

Then $h(z) = \frac{1}{H(z)}$ is holomorphic in $D_r(z_0)$ and

$$f(z) = \frac{1}{g(z)} = \frac{h(z)}{(z-z_0)^n}.$$

$\square$

5.4 Casorati-Weierstrass Theorem

Theorem 5.3 (Casorati-Weierstrass)**.** *Suppose $f : \hat{D}_R(z_0) \to \mathbb{C}$ has an essential singularity at z_0. Then $\forall r \in (0, R)$, $f(\hat{D}_r(z_0))$ is dense in $\mathbb{C}$.*

Proof. Assume that $\exists r > 0$ such that $f(\hat{D}_r(z_0))$ is not dense, then there exits $a \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - a| > \delta, \quad \text{for } \forall z \in \hat{D}_r(z_0).$$

Consider $g(z) = \frac{1}{f(z)-a}$. Since $g(z)$ is holomorphic and bounded on $\hat{D}_r(z_0)$, by theorem (5.1), there exits $h(z)$ which is holomorphic on $D_r(z_0)$ and $h(z) = g(z)$ on $\hat{D}_r(z_0)$.

Then $f(z) = \frac{1}{h(z)} + a$. If $h(z_0) = 0$, then f has a pole at z_0; if $h(z_0) \neq 0$, f has a removable singularity at z_0. This contradicts that f has an essential singularity at z_0. □

Remark 5.4. *In order to prove that z_0 is the essential singularity of $f(z)$, just need to show that $\exists$ two distinguishable sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ such that*

$$z_n \to z_0, \quad w_n \to z_0, \quad \text{as } n \to \infty,$$

but

$$f(z_n) \to z, \quad f(w_n) \to w, \quad \text{as } n \to \infty,$$

where $z \neq w$.

5.5 Meromorphic functions

5.5.1 Polynomials

Lemma 5.2. *Suppose f is an entire function that satisfies an estimate of the form*

$$|f(z)| \leq C|z|^N, \quad \text{if } |z| > R, \tag{5.9}$$

for some positive integer N and positive real constants C and R. Then f must be a polynomial with degree N or less.

Proof. **Method I: Cauchy inequality + power series**

Let $r > R$, $M_r = \sup_{|z|=r} |f(z)| \leq Cr^N$. By Cauchy inequality (Thm 2.5),

$$|f^{(n)}(0)| \leq \frac{n!M_r}{r^n} \leq Cn!r^{N-n} \to 0,$$

as $r \to \infty$ if $n > N$. It implies that $f^{(n)}(0) = 0$ if $n > N$.

By Theorem 1.3,

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n,$$

which is a polynomial with degree N or less.

Method II: Principal parts + Liouville theorem

By Theorem 1.3,

$$
\begin{aligned}
& f(z) = \sum_{n=0}^{\infty} a_n z^n, \\
\Rightarrow \quad & \frac{f(z)}{z^N} = \left(\frac{a_0}{z^N} + \frac{a_1}{z^{N-1}} + \cdots + \frac{a_{N-1}}{z} \right) + \sum_{k=0}^{\infty} a_{N+k} z^k.
\end{aligned} \tag{5.10}
$$

Let $R(z) = \frac{a_0}{z^N} + \frac{a_1}{z^{N-1}} + \cdots + \frac{a_{N-1}}{z}$ which is called the principal part of $f(z)/z^N$ at point $z = 0$, $g(z) = \sum_{k=0}^{\infty} a_{N+k} z^k = \frac{f(z)}{z^N} - R(z)$ which is entire.

Besides, claim that $g(z)$ is bounded since

$$
\lim_{z\to\infty} |g(z)| \leq \lim_{z\to\infty} \left| \frac{f(z)}{z^N} \right| + \lim_{z\to\infty} |R(z)| = C.
$$

By Liouville theorem (Thm 2.6), $g(z) =$ constant. Denote $g(z) = a_N$, then by Eq.(5.10),

$$
f(z) = \sum_{n=0}^{N} a_n z^n.
$$

□

Lemma 5.3. *Suppose f is an entire function that satisfies an estimate of the form*

$$
|f(z)| \geq C|z|^N, \quad \textit{if } |z| > R, \tag{5.11}
$$

for some positive integer N and positive real constants C and R. Then f must be a polynomial with degree N or more.

Proof. **Method I: Cauchy inequality + power series**

Consider the function $f(1/z)$. By assumption,

$$
\begin{aligned}
& f(1/z) \geq \frac{C}{z^N}, \quad if\ |z| < R, \\
\Rightarrow \quad & f(1/z) = \frac{g(z)}{z^m}, \quad m \geq N,\ g(z) \text{ entire}, \\
\Rightarrow \quad & f(z) = z^m g(1/z).
\end{aligned}
$$

Since $\lim_{z\to\infty} g(1/z) = g(0)$, we can choose $r > R$ such that

$$
\sup_{|z|=r} |g(z)| \leq |g(0) + 1|.
$$

By Cauchy formula,

$$\begin{aligned}
|f^n(0)| &= \left|\frac{n!}{2\pi i}\int_{C_r}\frac{w^m g(1/w)}{w^{n+1}}\mathrm{d}w\right| \\
&\le \frac{n!}{2\pi}\ 2\pi r\ r^{m-(n+1)}\ (|g(0)|+1) \\
&\le Cr^{m-n} \to 0, \quad \text{as } r\to\infty, \text{ if } n>m\ge N.
\end{aligned}$$

By Theorem 1.3,

$$f(z) = \sum_{n=0}^{m}\frac{f^{(n)}(0)}{n!}z^n,$$

which is a polynomial with degree N or more.

Method II: Principal parts + Liouville theorem

Eq.(5.11) implies that f has finitely many zeros.

Consider

$$g(z) = \frac{1}{f(z)} - \sum \text{ principal parts at finitely zeros of } f(z),$$

then $g(z)$ is bounded entire. By Liouville theorem (Thm 2.6), $g(z)$ is constant. Thus $f(z)$ is a rational function. Furthermore, since $f(z)$ is entire, f should be a polynomial.

Denote $N_f = \text{degree} f(z)$, by Lemma 2.5, $\exists$ positive constants c_1, c_2, R_0 such that

$$c_1|z|^{N_f} \le |f(z)| \le c_2|z|^{N_f}, \quad |z|>R_0.$$

Compared with Eq.(5.11), we know that $N_f \ge N$. $\square$

Theorem 5.4. *Suppose f is entire and with a pole of order m at ∞. Then f is a polynomial of degree m.*

Proof. f is entire means

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z\in\mathbb{C}. \tag{5.12}$$

$$\Rightarrow \quad f(1/z) = \sum_{n=0}^{\infty}\frac{a_n}{z^n}, \quad \forall z\ne 0. \tag{5.13}$$

That $f(z)$ has a pole of order m at at ∞ means $f(1/z)$ has a pole of order m at 0. Hence, Eq.(5.13) only has finitely many terms; more precisely, $\forall n>m$, $a_n=0$. It indicates that $f(z)$ is a polynomial of order m. $\square$

Corollary 5.1. *Suppose f is entire and one-to-one, then f must be a linear function*

$$f(z) = az+b, \quad a\ne 0.$$

Proof. **Step I**: Claim that f has a pole at $z = \infty$.

If removable, then f is bounded entire and must be a constant; if essential, then f can not be one-to-one since $f(z)$ is dense in any neighborhood of ∞.

Step II: Claim that f must be a polynomial.

See the proof of Theorem 5.4.

Step III: Claim that the degree of f must be one.

If the degree of f is bigger than one, then f' at least has one zero in $\mathbb{C}$, which contradicts with Theorem 7.3. □

5.5.2 Rational functions

Theorem 5.5. *Suppose $P(z)$ and $Q(z)$ are functions with no common factors and $N_Q = deg\ (Q) > N_p = \ deg\ P$. Let $\{a_k\}_{k=1}^M$ denote the zeroes of $Q(z)$ with order m_k, then*

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{M} R_k(z),$$

where

$$R_k(z) = \frac{A_0}{(z-a_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-a_k},$$

is the principal part of $\frac{P(z)}{Q(z)}$ at a_k.

Proof. **Step I:** By Thm 3.1, near a_k,

$$Q(z) = (z-a_k)^{m_k} q_k(z),$$

where $q_k(z)$ is a polynomial and $q_k(a_k) \neq 0$.

Step II:

$$\begin{aligned}
\frac{P(z)}{Q(z)} &= \frac{1}{(z-a_k)^{m_k}}\left[\frac{P(z)}{q_k(z)}\right] \\
&= \frac{1}{(z-a_k)^{m_k}}[A_0 + A_1(z-a_k) + \cdots] \\
&= \frac{A_0}{(z-a_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-a_k} + \text{ holomorphic function} \\
&= R_k(z) + \text{holomorphic function.}
\end{aligned}$$

Step III: Consider $f(z) = \frac{P(z)}{Q(z)} - \sum_{k=1}^M R_k(z)$. Claim that $f(z)$ is bounded entire.

First, need to show that $a_k's$ are removable singularity. Near a_j,

$$\frac{P(z)}{Q(z)} - \sum_{k=1}^{M} R_k(z) = \left[\frac{P(z)}{Q(z)} - R_j(z)\right] - \sum_{k\neq j}^{M} R_k(z),$$

which is holomorphic and hence bounded.

Second,

$$
\begin{aligned}
& \lim_{|z|\to\infty} |R_k(z)| = 0, \quad \lim_{|z|\to\infty} \left|\frac{P(z)}{Q(z)}\right| = 0, \\
\Rightarrow \quad & \lim_{|z|\to\infty} |f(z)| = 0. \qquad (5.14)
\end{aligned}
$$

Step IV: By Thm 2.6, $f(z)$ is a constant. Besides, by Eq.(5.14), we know that the constant should be 0. Then we can get the conclusion. □

Theorem 5.6. *f is meromorphic $\Leftrightarrow$ f is a rational function.*

Proof. "$\Leftarrow$" is obviously.

"$\Leftarrow$" : **Step I:** Claim that $f(z)$ has only finitely many singularities in $\mathbb{C}$.

$f(1/z)$ has either removable singularity or a pole at 0,

$\Rightarrow \exists r > 0$ such that $f(1/z)$ is holomorphic on $D_r(0) \setminus \{0\}$

$\Rightarrow f(z)$ is holomorphic on $\mathbb{C} \setminus D_{1/r}(0)$.

Since $\overline{D_{1/r}(0)}$ is compact and each singularity is isolated, so $f(z)$ has only finitely many singularities in $\mathbb{C}$, say $z_1, z_2, \cdots, z_n$, with order $m_1, m_2, \cdots, m_n$.

Step II: There are two methods.

Method 1: Let $Q(z) = \prod_{k=1}^n (z - z_k)^{m_k}$, then near z_j, $\exists$ a holomorphic function $h_j(z)$ such that

$$
\begin{aligned}
& f(z) = \frac{h_j(z)}{(z - z_j)^{m_j}}, \quad h_j(z_j) \neq 0, \\
\Rightarrow \quad & f(z)Q(z) = h_j(z) \prod_{k=j} (z - z_k)^{m_k},
\end{aligned}
$$

which is bounded near z_j and holomorphic everywhere else, hence extent to be entire.

Let $P(z) = f(z)Q(z)$ which is entire. Besides, $P(z)$ has a pole or removable singularity at ∞. By Thm 5.4, $P(z)$ is a polynomial and then $f(z) = \frac{P(z)}{Q(z)}$ is a rational function.

Method 2: Let $R_k(z)$ be the principal part of $f(z)$ at z_k and $R_\infty(z)$ be the principal part of $f(1/z)$ at 0.

Let $H(z) = f(z) - R_\infty(z) - \sum_{k=1}^n R_k(z)$. Then $H(z)$ is holomorphic on $\bar{\mathbb{C}} \setminus \{z_1, \cdots, z_m, \infty\}$ and has removable singularity at these points. Hence $H(z)$ can be extent to be holomorphic on $\bar{\mathbb{C}}$, which is bounded entire. By Thm 2.6, $H(z)$ is a constant. So $f(z) = H(z) + R_\infty(z) + \sum_{k=1}^n R_k(z)$ is a rational function. □

6 Uniform convergence

6.1 Hurwitz' Theorem

Theorem 6.1. *Suppose $f_n : \Omega \to \mathbb{C}$ is holomorphic with $f_n \neq 0$ on Ω, and $f_n \to f$ uniformly on compact subsets of Ω. Then either $f \equiv 0$ or $f(z) \neq 0$ or all $z \in \Omega$.*

Proof. By Thm 1.4, f is holomorphic.

Suppose f not identically 0 but $f(z_0) = 0$.

On one hand, by Identity theorem 3.2, $\exists r > 0$ so that $f(z) \neq 0$ for all $z \in \hat{D}_r(z_0)$ and

$$|f(z)| > \delta > 0, \quad \forall z \in C_r(z_0). \tag{6.1}$$

On the other hand, since $f_n \to f$ uniformly on $C_r(z_0)$, $\exists n_0$ such that $\forall n > n_0$,

$$|f_n(z) - f(z)| < \delta, \quad \forall z \in C_r(z_0). \tag{6.2}$$

By (6.1), (6.2) and Rouche's theorem (Thm 3.5), we know that f and $f + (f_n - f) = f_n$ has the same number of zeros. But this contradicts with $f(z_0) = 0$ and $f_n \neq 0$ on $D_r(z_0)$. □

6.2 Montel Theorem

6.2.1 Normal Family

Let $\mathcal{F}$ be a set of holomorphic functions on Ω. $\mathcal{F}$ is a normal family means any sequence $\{f_n\} \subseteq \mathcal{F}$, $\exists$ a subsequence $\{f_{n_k}\}$ such that $\forall$ compact set $K \subseteq \Omega$, f_{n_k} converges uniformly on K.

6.2.2 Uniform boundedness

The family $\mathcal{F}$ is said to be uniformly bounded on compact subsets of Ω if for each compact set $K \subseteq \Omega$, $\exists M = M(K) > 0$ such that

$$|f(z)| < M, \text{ for } \forall z \in K, \ \forall f \in \mathcal{F}.$$

6.2.3 Equicontinuity

The family $\mathcal{F}$ is said to be equicontinuous on a compact set K if for $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall z, w \in K$ with $|z - w| < \delta$, then

$$|f(z) - f(w)| < \varepsilon, \ \forall f \in \mathcal{F}.$$

6.2.4 Montel Theorem

Theorem 6.2. *Let $\mathcal{F}$ be a set of holomorphic functions on Ω. If $\mathcal{F}$ is uniformly bounded on compact subsets of Ω, then*

(i) $\mathcal{F}$ is equicontinuous on any compact subset of Ω.

(ii) $\mathcal{F}$ is a normal family.

Proof:

(i) Use the Cauchy estimates on small circles.

(ii) Use pointwise convergence on a dense set plus equicontinuity and diagonalization.

6.3 Open mapping theorem

Theorem 6.3. *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, then f maps open sets to open sets.*

Proof. Let $w_0 = f(z_0)$ for some $z_0 \in \Omega$. For w near w_0, let

$$\begin{aligned} g(z) &= f(z) - w, \\ F(z) &= f(z) - w_0, \\ G(z) &= w_0 - w, \end{aligned}$$

then $g(z) = F(z) + G(z)$.

Since z_0 is an isolated zero for F and for w near w_0, $|G|$ is small, we can find $r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$ and $\exists \varepsilon > 0$,

$$|F(z)| > \varepsilon > |G(z)|, \quad \forall z \in C_r(z_0).$$

By Rouche Theorem, $g = F + G$ has a zero in $D_r(z_0)$, say $\exists z_1 \in D_r(z_0)$ such that $g(z_1) = 0$, i.e. $f(z_1) = w$. Hence $w \in f(D_r(z_0))$ and $D_\varepsilon(w_0) \subseteq f(D_r(z_0))$. ☐

7 Univalence

7.1 Local univalence

Theorem 7.1. *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $\exists z_0 \in \Omega$ with $f'(z_0) \neq 0$. Then $\exists r > 0$ such that f is univalent in $D_r(z_0)$.*

Proof. Since $f'(z_0) \neq 0$, then $f(z) - f(z_0)$ has a zero of order 1 at z_0. So

$$f(z) - f(z_0) = h(z)(z - z_0),$$

where $h(z)$ is holomorphic in Ω and $h(z_0) \neq 0$.

Then

$$\begin{aligned} & \exists R > 0, \text{ s.t. } |h(z)| > \frac{1}{2}|h(z_0)|, \quad \text{for } \forall z \in \overline{D_R(z_0)}. \\ \Rightarrow \quad & |f(z) - f(z_0)| > \frac{1}{2}|h(z_0)|R, \quad \text{for } \forall z \in C_R(z_0). \\ \Rightarrow \quad & \exists r > 0, \text{ s.t. } |f(z) - f(a)| > \frac{1}{4}|h(z_0)|R, \quad \text{for } \forall z \in C_R(z_0),\ a \in D_r(z_0). \end{aligned}$$

For fixed $a \in D_r(z_0)$, let $g_a(z) = f(z) - f(a)$ and define

$$F(a) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{g_a'(z)}{g_a(z)} \mathrm{d}z.$$

On one hand, since $g_a(z)$ is continuous in (z, a) and uniformly bounded, F should be continuous. On the other hand, by Rouche Theorem, $F(a) \in \mathbb{Z}$ and $F(z_0) = 1$. So $F(a) = 1$ for all $a \in D_r(z_0)$. It implies that for $\forall a \in D_r(z_0)$, the equation $f(z) = f(a)$ has unique solution, which means f is univalent in $D_r(z_0)$. ☐

Corollary 7.1. Definition: *A holomorphic mapping $f : U \to V$ is a local bijection on U if for $\forall z \in U$, there exists an open disc $D \subset U$ centered at z such that $f : D \to f(D)$ is a bijection.*

Question: *A holomorphic mapping $f : U \to V$ is a local bijection on U $\Leftrightarrow$ $f'(z) \neq 0$ for $\forall z \in U$.*

Proof. "$\Leftarrow$" is obvious by the Theorem 7.1.

"$\Rightarrow$": Suppose $f'(z_0) = 0$ for some $z_0 \in U$. Then $\exists r > 0$, such that

$$f(z) - f(z_0) = a(z - z_0)^k + G(z), \quad \text{for } \forall z \in D_r(z_0), \tag{7.1}$$

with $a \neq 0, k \geq 2$ and $G(z)$ vanishing to order $k+1$ at z_0.

Besides, since f is bijective in $D_r(z_0)$, $z = z_0$ should be an isolated zero of $f'(z)$. Thus, for sufficiently small r, we have

$$f'(z) \neq 0, \quad \text{for } \forall z \in D_r(z_0) \backslash \{z_0\}. \tag{7.2}$$

Let's choose $w \in \mathbb{C}$ sufficiently small such that for $\forall z \in C_r(z_0)$, we have

$$|w| < |a(z - z_0)^k|, \tag{7.3}$$

$$|G(z)| < |a(z - z_0)^k - w|. \tag{7.4}$$

Let $F(z) = a(z - z_0)^k - w$. By (7.3), (7.4) and Rouche's Theorem, in the disk $D_r(z_0)$, the function $a(z - z_0)^k$ has at least two zeros, then so does $F(z)$ and further does $f(z) - f(z_0) = F(z) + G(z)$. It contradicts with the fact that $f(z)$ is bijective in $D_r(z_0)$. □

Theorem 7.2. *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $\exists z_0 \in \Omega$ with $f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0, f^{(n)}(z_0) \neq 0$. Then $\exists V \ni z_0$ with $V \subset \Omega$ and φ holomorphic on V such that*

(i) $\varphi(z_0) = 0, \varphi'(z) \neq 0$, $\forall z \in V$.

(ii) $f(z) = f(z_0) + [\varphi(z)]^n$, $\forall z \in V$.

(iii) φ is a univalent map from V onto $D_r(0)$ for some $r > 0$.

7.2 Global univalence

Theorem 7.3. *Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and univalent. Then $f'(z) \neq 0$, $\forall z \in \Omega$.*

Proof. $\forall z_0 \in \Omega$, let $f(z_0) = w_0$ and

$$g(z) = f(z) - w_0, \quad \forall z \in \Omega.$$

The fact that f is univalent implies that $z = z_0$ is a simple root of $g(z)$, by Theorem 3.1, $\exists h(z)$ holomorphic on Ω and $h(z_0) \neq 0$, such that

$$g(z) = (z - z_0)h(z) \tag{7.5}$$

$$\Rightarrow \quad g'(z) = h(z) + (z - z_0)h'(z) \tag{7.6}$$

$$\Rightarrow \quad f'(z_0) = g'(z_0) = h(z_0) \neq 0. \tag{7.7}$$

□

7.3 Limits univalence

Theorem 7.4. *Suppose $f_n : \Omega \to \mathbb{C}$ is holomorphic and univalent on Ω, and $f_n \to f$ uniformly on compact subsets of Ω. Then f is either constant or univalent in Ω.*

Proof. Suppose f is not constant in Ω. Suppose $z_1 \neq z_2$ but $f(z_1) = f(z_2) = w$. Let $F_n = f_n - w$, $F = f - w$.

$\exists r_1, r_2$ such that $\overline{D_{r_1}(z_1)} \cap \overline{D_{r_2}(z_2)} = \emptyset$.

$\exists N$ such that F_N and F have the same number of zeros in $\overline{D_{r_1}(z_1)}$ and $\overline{D_{r_2}(z_2)}$.

Then we get two points $\xi_1^N \in D_{r_1}(z_1)$ and $\xi_2^N \in D_{r_2}(z_2)$ in two discs satisfying $f_N(\xi_1^N) = f_N(\xi_2^N) = w$. It is a contradiction.

□

8 Conformal mappings

Let $\mathbb{D} = D_1(0)$, Aut($\mathbb{D}$) denote the set of all automorphism of $\mathbb{D}$; UHP denote the upper half plane.

8.1 Schwartz Lemma

Theorem 8.1 (Schwartz Lemma). *Let $f \in Aut(\mathbb{D})$ with $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ for $\forall z \in \mathbb{D}$;

(ii) $|f'(0)| \leq 1$;

(iii) If either the equality in (i) holds for some $z \neq 0$ or the equality in (ii) holds, then $f(z)$ is a rotation.

Proof. Define

$$F(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0, \\ f'(0), & \text{if } z = 0. \end{cases} \tag{8.1}$$

Then $F(z)$ is holomorphic on $\mathbb{D}$.

By MMP (Thm 3.3),

$$\max_{|z|\leq r} |F(z)| = \max_{|z|=r} |F(z)| = \max_{|z|=r} \left| \frac{|f(z)|}{|z|} \right| < \frac{1}{r}.$$

Let $r \to 1$, we can get (i) and (ii).

For (iii), if either the equality in (i) holds for some $z \neq 0$ or the equality in (ii) holds, then $F(z)$ assumes maximum modulus at point inside the disc $\mathbb{D}$, so $F(z) \equiv$ constant. □

Corollary 8.1. *Let $f \in$ Aut($\mathbb{D}$). If $f(z)$ has zeros of order N, then*

$$|f(z)| \leq |z|^N.$$

Proof. Let $f(z) = z^N g(z)$, $g(0) \neq 0$. Define

$$F(z) = \begin{cases} \frac{f(z)}{z^N}, & \text{if } z \neq 0, \\ g(0), & \text{if } z = 0. \end{cases} \tag{8.2}$$

Then do the similar things as we just did. □

8.2 Automorphism of the disc

Lemma 8.1 (Blashke factor)**.** *Let $a \in \mathbb{D}$, then*

$$\varphi_a = \frac{a-z}{1-\bar{a}z} \in \ Aut\ (\mathbb{D}).$$

Furthermore,
(i) $\varphi_a^{-1} = \varphi_a$;
(ii) $\varphi_a(0) = a, \ \varphi_a(a) = 0$;
(iii) $\varphi_a'(z) = \frac{|a|^2-1}{(1-\bar{a}z)^2}$, $\varphi_a(0) = |a|^2 - 1$, $\varphi_a(a) = \frac{1}{|a|^2-1}$.

Theorem 8.2. *If $f \in \ Aut(\mathbb{D})$, then $\exists \theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = \mathrm{e}^{i\theta} \frac{a-z}{1-\bar{a}z}.$$

Corollary 8.2. *If $f \in \ Aut(\mathbb{D})$, then $|f'(0)| \leq 1 - |f(0)|^2$.*

Proof. Suppose $f(0) = a$, then construct $\varphi_a = \frac{a-z}{1-\bar{a}z}$. Let $F(z) = (\varphi_a \circ f)(z)$, then $F(0) = 0$. By Schwartz Lemma (Thm 8.1) and Lemma 8.1,

$$\begin{aligned} & |F'(0)| \leq 1 \\ \Rightarrow \quad & |\varphi_a(a)||f'(0)| \leq 1, \\ \Rightarrow \quad & |f'(0)| \leq 1 - |a|^2. \end{aligned}$$

□

8.3 From upper half plan to the unit disc

Theorem 8.3. *The conformal mapping from UPH to $\mathbb{D}$ has the form*

$$f(z) = \mathrm{e}^{i\theta} \frac{z-a}{z-\bar{a}}, \quad Im\ (a) > 0, \quad \theta \in \mathbb{R}.$$

9 Roots of functions

Theorem 9.1 (Log roots on a convex open set)**.** *Given analytic and non-vanishing function $F(z)$ on a convex open set Ω, there is an analytic function $G(z)$ on Ω such that $F(z) = \mathrm{e}^{G(z)}$. Given a positive integer N, then $H(z) = \mathrm{e}^{G(z)/N}$ is an N-th root of $F(z)$, i.e. $F(z) = H(z)^N$.*

Proof. Fix a point $a \in \Omega$, and define the function

$$G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)} \mathrm{d}w,$$

where L_a^z is the line from a to z. Then we have

$$\begin{aligned} & G' = \frac{F'}{F} && (9.1)\\ \Rightarrow\quad & \frac{\mathrm{d}}{\mathrm{d}z}(F\mathrm{e}^{-G}) = F'\mathrm{e}^{-G} - FG'\mathrm{e}^{-G} = 0, && (9.2)\\ \Rightarrow\quad & F\mathrm{e}^{-G} \equiv c, && (9.3) \end{aligned}$$

where c is a constant.

Pick up $\alpha \in \log(c)$, then

$$F(z) = c\mathrm{e}^G = \mathrm{e}^\alpha \mathrm{e}^G = \mathrm{e}^{G+\alpha}.$$

Define $\tilde{G} = G + \alpha$, and $H(z) = \mathrm{e}^{\tilde{G}/N}$, then we have $F = H^N$. $\square$

Corollary 9.1. *Suppose that f is non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = \mathrm{e}^{it}$ where $0 \le t \le 2\pi$. Suppose that*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} \setminus \{0\}$.

Proof. **Step I:** Fix $a \in \Omega$ and define the function

$$F(z) = \exp\left(\frac{1}{3} \int_{\gamma_a^z} \frac{f'(w)}{f(w)} \mathrm{d}w\right),$$

where γ_a^z is a curve from a to z.

Need to show that $F(z)$ is well defined. Choose two curves γ_a^z and $\tilde{\gamma}_a^z$, then

$$\begin{aligned} & \int_{\gamma_a^z} \frac{f'(w)}{f(w)} \mathrm{d}w - \int_{\tilde{\gamma}_a^z} \frac{f'(w)}{f(w)} \mathrm{d}w = \int_{\gamma_a^z \cup (-\tilde{\gamma}_a^z)} \frac{f'(w)}{f(w)} \mathrm{d}w = N\ 3m\ 2\pi i,\\ \Rightarrow\quad & \exp\left(\frac{1}{3} \int_{\gamma_a^z} \frac{f'(w)}{f(w)} \mathrm{d}w - \int_{\tilde{\gamma}_a^z} \frac{f'(w)}{f(w)} \mathrm{d}w\right) = \exp(2N\pi i) = 1,\\ \Rightarrow\quad & \frac{F_\gamma(z)}{F_{\tilde{\gamma}}(z)} = 1, \end{aligned}$$

which implies that $F(z)$ is independent of the curve γ.

Step II: Claim that $F' = \frac{1}{3}\frac{f'}{f}F$.

Restrict attention to a disk.

$$F(z) = \exp\left[\frac{1}{3}\left(\int_{\gamma_a^{z_0}} \frac{f'(w)}{f(w)}\mathrm{d}w + \int_{L_{z_0}^z} \frac{f'(w)}{f(w)}\mathrm{d}w\right)\right].$$

Use chain rule and the fact that on convex disc that

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_{L_{z_0}^z} \frac{f'(w)}{f(w)}\mathrm{d}w = \frac{f'(z)}{f(z)}.$$

Step III: Claim that $(f/F^3)' = 0 \Rightarrow f/F^3 \equiv C$.

$$\left(\frac{f}{F^3}\right)' = \frac{f'F^3 - f3F^2\frac{1}{3}\frac{f'}{f}F}{F^6} = 0.$$

$$\Rightarrow \quad \frac{f}{F^3} = C.$$

Choose $\alpha = \log(C)$, and define

$$\tilde{F}(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\mathrm{d}w + \alpha\right).$$

It is easy to know that $f = \tilde{F}^3$.

□

Remark 9.1. *Let $f(z)$ be holomorphic on $\Omega \setminus \{p_0\}$, which has a zero at $z = z_0$ with multiplicity n and has a pole at $z = p_0$ with order m. Choose $a \in \Omega \setminus \{p_0\}$, define a new function as*

$$g(z) = \exp\left(\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\mathrm{d}w\right).$$

Then we know that:

- $h(z) = \frac{f'(z)}{f(z)}$ *has a simple pole at $z = z_0$ with residue equal to a positive integer.*
- $g(z)$ *is a well defined analytic function on $\Omega \setminus \{p_0, z_0\}$.*
- $g(z)$ *has a removable singularity at $z = z_0$. More precisely, if $g(z)$ is redefined at $z = z_0$ as $g(z_0) = 0$, then $z = z_0$ is also a zero of $g(z)$ with multiplicity n.*
- $g(z)$ *also has a pole at $z = p_0$ with order m.*